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BMO AND LIPSCHITZ APPROXIMATION BY SOLUTIONS OF ELLIPTIC EQUATIONS

by J. MATEU, Y. NETRUSOV, J. OROBITG and J. VERDERA

0. INTRODUCTION

Let L be a homogeneous elliptic operator of order r in $\mathbb{R}^n$ with constant coefficients. Given a non-negative real number s and a compact subset X of $\mathbb{R}^n$, let $H^s(X)$ be the set of those functions on X which are limits, in the norm of $\mathrm{Lip}(s, X)$, of sequences (f_n) such that $L(f_n) = 0$ on some neighbourhood (depending on n) of X. Here $\mathrm{Lip}(s, X)$ denotes the space of functions on X satisfying a Lipschitz condition of order s. The precise definition of $\mathrm{Lip}(s, X)$ is rather technical and will be postponed to section 1. The reader should only keep in mind that for a non-integer s convergence in $\mathrm{Lip}(s, X)$ is equivalent, at least for reasonable X, to uniform convergence on X of all derivatives up to order $[s]$ (the integer part of s) and convergence of the derivatives of order $[s]$ in the standard Lipschitz (Hölder) seminorm of order $s - [s]$. Bounded mean oscillation is involved in the definition of $\mathrm{Lip}(s, X)$ for $s = 0$ and the Zygmund class for positive integers s.

We are interested in describing functions in $H^s(X)$. One finds readily two simple necessary conditions: if f is in $H^s(X)$ then $f \in \mathrm{lip}(s, X)$, the closure of $C^\infty(\mathbb{R}^n)_{|X}$ in $\mathrm{Lip}(s, X)$, and $L(f) = 0$ on the interior $\overset{\circ}{X}$ of X. If

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we set $h^s(X) = \{f \in \operatorname{lip}(s,X) : L(f) = 0 \text{ on } \overset{\circ}{X}\}$, then $H^s(X) \subset h^s(X)$, for all X. The main problem of qualitative approximation by solutions of the equation $L(f) = 0$ in the Lipschitz space of order s consists in describing those X for which $H^s(X) = h^s(X)$. A complete solution has been found in the range $r - 2 < s$, $s \neq r$, thanks to the contributions of several authors ([MV], [MO], [OF1], [OF2], [V1], [V3]). In particular, the known results cover the case of analytic functions in the plane ($L = \bar{\partial}$) and the case of harmonic functions ($L = \Delta$) in any dimension, with the only exception of the limiting case $s = 0$ in dimensions $n \geq 3$. Other samples of the cases left open up to now are $L = \bar{\partial}^3$ in the plane and $0 < s < 1$, or $L = \Delta^2$ and $0 < s \leq 2$. The reader is referred to the recent survey [V2] for a detailed description of the work done in the subject in the last twenty years.

In this paper we present a fairly complete picture of the situation in the range $0 \leq s \leq r - 2$. We show that the standard necessary condition for the approximation on a compact X, namely

(1) $\quad M_*^{n-r+s}(B \setminus \overset{\circ}{X}) \leq \operatorname{Const} M^{n-r+s}(B \setminus X)$, for each open ball B,

where M and M_* denote Hausdorff content, and lower Hausdorff content, respectively, is not always sufficient when $s < r - 2$. Our examples evolve from one due to Hedberg in the L^p context [H] (see also [M]).

In the opposite sense, we find a finite set of capacitary conditions (including (1)) expressed in terms of Hausdorff contents, which are sufficient for the approximation on X (see Theorem 1 in the next section). When $r - 2 < s < r$ this sufficient condition reduces to (1), and hence we recover the known theorem for that range. The examples of section 4 show that none of the capacitary conditions can be dispensed with.

It is worth mentioning that, following the work of Bagby in the L^p context [B], it is not difficult to find necessary and sufficient conditions for the approximation on X involving families of capacitary conditions (see also [AH], 11.5.10, [N1] and [TA]). However, most of the capacities involved cannot be described in terms of Hausdorff content and, in fact, they are very difficult to handle. Thus the conditions one would get would not be really satisfactory.

Our method of proof is well known: one uses duality to reduce matters to an appropriate spectral synthesis problem. In the case at hand we proceed as follows.

The homogeneous Lipschitz spaces on $\mathbb{R}^n$ are particular examples of Lizorkin-Triebel spaces and thus their duality theory is well understood.

Duality arguments reduce our approximation problem to proving a weak $*$ spectral synthesis theorem for the Lizorkin-Triebel space $\overset{\circ}{F}{}^{\ell}_{1,q}$, $1 \leq q < \infty$; the latter does not follow from the (strong) spectral synthesis theorem for the same space proved by the second named author in [N1] (see also Chapter 10 of the book [AH]). We show that weak $*$ spectral synthesis for $\overset{\circ}{F}{}^{\ell}_{1,q}$ is a consequence of strong spectral synthesis in a certain generalized Lizorkin-Triebel space $\overset{\circ}{F}{}^{\omega}_{1,q}$. The function $\omega(t)$ satisfies $\omega(t)t^{-\ell} \to \infty$ as $t \to 0$, which means that functions in $\overset{\circ}{F}{}^{\omega}_{1,q}$ are not as smooth as those in $\overset{\circ}{F}{}^{\ell}_{1,q}$. However ω satisfies certain regularity conditions, that turn out to be good enough so that the proof of strong spectral synthesis described in [AH], Chapter 10, can be adapted without difficulty to the more general case we are forced to consider. One of our main contributions is precisely to find a way to make the transition from $\overset{\circ}{F}{}^{\ell}_{1,q}$ to $\overset{\circ}{F}{}^{\omega}_{1,q}$ without loosing too much regularity on ω.

In section 1 we set some notational conventions and state our results. In section 2 we describe the reduction to spectral synthesis and in section 3 we prove the spectral synthesis result we need. Section 4 contains two examples of failure of approximation, showing that our main theorem is, in some sense, sharp.

1. BACKGROUND AND STATEMENT OF RESULTS

1.1. Lipschitz spaces and BMO.

Given $0 < s < 1$ the space $\Lambda^s(\mathbb{R}^n)$ consists of those functions f satisfying

$$\|f\|_s \equiv \sup\{\omega_s(f,\delta) : \delta > 0\} < \infty,$$

where $\omega_s(f,\delta) = \sup\{|f(x) - f(y)|\,|x-y|^{-s} : |x-y| \leq \delta\}$. Clearly $\|\ \ \|_s$ is, modulo constants, a Banach space norm on $\Lambda^s(\mathbb{R}^n)$.

Let f be a locally integrable function on $\mathbb{R}^n$. The mean oscillation of f on a ball B is

$$\Omega(f,B) = \frac{1}{|B|}\int_B |f(x) - f_B|\,dx,$$

where $|B|$ is the Lebesgue measure of B and $f_B = |B|^{-1} \int_B f(x)\,dx$ is the mean value of f on B. One says that f has bounded mean oscillation and writes $f \in \mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_0 \equiv \sup\{\omega_0(f,\delta) : \delta > 0\} < \infty,$$

where $\omega_0(f,\delta) = \sup\{\Omega(f,B) : \text{radius } B \leq \delta\}$. A well-known result of Campanato [CA] and Meyers [ME] states that $f \in \Lambda^s(\mathbb{R}^n)$, $0 < s < 1$, if and only if for some constant C and all balls B one has

$$\Omega(f,B) \leq C \text{ radius } (B)^s. \tag{2}$$

Since for $s = 0$ (2) becomes the BMO condition, we set $\Lambda^0(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ and we think of BMO as the limit of Λ^s as s tends to zero.

For $s = 1$ we let $\Lambda^1(\mathbb{R}^n)$ stand for the Zygmund class, that is, for the set of continuous functions f such that

$$\|f\|_1 \equiv \sup\{\omega_1(f,\delta) : \delta > 0\} < \infty,$$

where $\omega_1(f,\delta) = \sup\{|f(x+h) + f(x-h) - 2f(x)|\,|h|^{-1} : |h| \leq \delta\}$. The quantity $\|\ \ \|_1$ is, modulo first degree polynomials, a Banach space norm on $\Lambda^1(\mathbb{R}^n)$.

There is a well established fact that explains our choice of the end-point spaces Λ^0 and Λ^1, namely, that the family of spaces Λ^s, $0 \leq s \leq 1$, is well behaved under Riesz potentials and Calderón-Zygmund operators. The apparently more natural choices $\Lambda^0 = L^\infty$ and $\Lambda^1 = \mathrm{Lip}\,1$ (the class of standard Lipschitz functions of order 1) lead to much harder removability and approximation problems, most of which are still unsolved (see [V2]).

Given a positive real number s we write $s = m + \sigma$, with m integer and $0 < \sigma \leq 1$. If $s > 1$ the space $\Lambda^s(\mathbb{R}^n)$ consists of those functions with continuous partial derivatives up to order m, satisfying $\partial^\alpha f \in \Lambda^\sigma(\mathbb{R}^n)$, $|\alpha| = m$.

The quantity

$$\|f\|_s \equiv \sup_{|\alpha|=m} \|\partial^\alpha f\|_\sigma$$

defines a Banach space norm on $\Lambda^s(\mathbb{R}^n)$, modulo polynomials of degree m if $\sigma < 1$ and of degree $m+1 = s$ if $\sigma = 1$. For $0 \leq s$, we denote by λ^s the closure in Λ^s of $C_0^\infty(\mathbb{R}^n)$, the class of all infinitely differentiable functions with compact support on $\mathbb{R}^n$. If $f \in \lambda^s(\mathbb{R}^n)$, then $\omega_\sigma(\partial^\alpha f,\delta) \to 0$, as $\delta \to 0$, for $|\alpha| = m$.

The (non-homogeneous) Banach space version of Λ^s, which we proceed to define next, is in some sense more natural. A function f is in

$\operatorname{Lip}(s,\mathbb{R}^n)$, $s > 0$, if it has bounded continuous derivatives up to order m and $\partial^\alpha f \in \Lambda^\sigma(\mathbb{R}^n)$ for $|\alpha| = m$. If moreover $\omega_\sigma(\partial^\alpha f, \delta) \to 0$ as $\delta \to 0$, $|\alpha| = m$, then one writes $f \in \operatorname{lip}(s,\mathbb{R}^n)$. The classes $\operatorname{Lip}(s,\mathbb{R}^n)$ and $\operatorname{lip}(s,\mathbb{R}^n)$ become Banach spaces when endowed with the norm

$$\sum_{|\alpha|\le m} \|\partial^\alpha f\|_\infty + \sum_{|\alpha|=m} \|\partial^\alpha f\|_\sigma.$$

Given a compact $X \subset \mathbb{R}^n$ set

$$\operatorname{Lip}(s,X) = \operatorname{Lip}(s,\mathbb{R}^n)/I(X),$$

where $I(X) = \{f \in \operatorname{Lip}(s,\mathbb{R}^n) : f(x) = 0,\ x \in X\}$, and

$$\operatorname{lip}(s,X) = \operatorname{lip}(s,\mathbb{R}^n)/I(X) \cap \operatorname{lip}(s,\mathbb{R}^n),$$

both endowed with the quotient norm. It turns out that, as sets of functions, $\operatorname{Lip}(m,X) = \operatorname{Lip}(m,\mathbb{R}^n)_{|X}$ and $\operatorname{lip}(m,X) = \operatorname{lip}(m,\mathbb{R}^n)_{|X}$ and that $\operatorname{lip}(m,X)$ is the closure in $\operatorname{Lip}(m,X)$ of $C^\infty(\mathbb{R}^n)_{|X}$. The reader is referred to [St] and [JW] for descriptions of the functions in $\operatorname{Lip}(m,X)$ and $\operatorname{lip}(m,X)$ related to variants of the Whitney extension theorem.

Define $H^s(X)$ as the closure in $\operatorname{Lip}(s,X)$ of the set of restrictions to X of functions g satisfying $Lg = 0$ on some neighbourhood (depending on g) of X. Clearly

$$H^s(X) \subset \operatorname{lip}(s,X) \cap \{f : Lf = 0 \text{ on } \overset{\circ}{X}\} \equiv h^s(X).$$

We wish to define $H^s(X)$ and $h^s(X)$ also for $s = 0$. To avoid technical complications we will deal with classes which are Banach spaces modulo constant functions. Set $\Lambda^0(X) = \Lambda^0(\mathbb{R}^n)/I(X)$, where

$$I(X) = \{f \in \Lambda^0(\mathbb{R}^n) : f = 0 \text{ a.e. on } X\},$$

and $\lambda^0(X) = \lambda^0(\mathbb{R}^n)/I(X) \cap \lambda^0(\mathbb{R}^n)$. Functions in $\Lambda^0(X)$ and $\lambda^0(X)$ can be described in terms of their values on X, as shown in [GR], p. 440 and [Ho]. We define $H^0(X)$ as the closure in $\Lambda^0(X)$ of the set of functions g on X satisfying the equation $Lg = 0$ on some neighbourhood of X. Let $h^0(X)$ denote the set of functions f in $\lambda^0(X)$ such that $Lf = 0$ on $\overset{\circ}{X}$. Then $H^0(X) \subset h^0(X)$.

For $0 \le s < r$ there are no other obvious necessary conditions for $f \in H^s(X)$ besides $Lf = 0$ on $\overset{\circ}{X}$ and $f \in \lambda^s(X)$. *The problem of* Lip s (BMO *if* $s = 0$) *approximation for the operator* L, consists, for $s < r$, in describing those X for which $H^s(X) = h^s(X)$. For $s \ge r$ the reader is referred to [OF1], [V1] and [V2].

There is an interesting variant of the above problem arising when one considers jets instead of functions. For $s > 0$ set

$$J(X) = \{f \in \mathrm{Lip}(s, \mathbb{R}^n) : \partial^\alpha f(x) = 0,\ x \in X,\ |\alpha| \le m\},$$
$$\mathrm{Lip}_{\mathrm{jet}}(s, X) = \mathrm{Lip}(s, \mathbb{R}^n)/J(X)$$

and

$$\mathrm{lip}_{\mathrm{jet}}(s, X) = \mathrm{lip}(s, \mathbb{R}^n)/J(X) \cap \mathrm{lip}(s, \mathbb{R}^n).$$

See [St], Chapter VI for general information about jets and Whitney extension Theorem, and [V2] for a quick description of their role in approximation. As before, we define $H^s_{\mathrm{jet}}(X)$ as the closure in $\mathrm{Lip}_{\mathrm{jet}}(s, X)$ of the set of equivalence classes $\tilde{g}$, where $Lg = 0$ on some neighbourhood of X. We have

$$H^s_{\mathrm{jet}}(X) \subset \mathrm{lip}_{\mathrm{jet}}(s, X) \cap \{\tilde{f} : Lf = 0 \text{ on } \overset{\circ}{X}\} \equiv h^s_{\mathrm{jet}}(X).$$

For $0 < s < r$, the problem of $\mathrm{Lip}\, s$-jet approximation for L consists in characterizing those X for which $H^s_{\mathrm{jet}}(X) = h^s_{\mathrm{jet}}(X)$. It will be shown in section 2 that the $\mathrm{Lip}\, s$ and the $\mathrm{Lip}\, s$-jet problems are equivalent. This was implicitly stated in [V2] but no proof was provided.

One can also define

$$\Lambda^s(X) = \Lambda^s(\mathbb{R}^n)/I(X)$$

and

$$\lambda^s(X) = \lambda^s(\mathbb{R}^n)/I(X) \cap \lambda^s(\mathbb{R}^n),$$

where now

$$I(X) = \{f \in \Lambda^s(\mathbb{R}^n) : f = 0 \text{ on } X\}.$$

As before, one can formulate a Λ^s-approximation (resp. Λ^s_{jet}) problem for L, but it turns out that it is equivalent to the $\mathrm{Lip}\, s$ (resp. $\mathrm{Lip}\, s_{\mathrm{jet}}$) problem. A simple argument to prove that can be found in [V1], p. 184. This technical remark will be used in section 2.

1.2. Hausdorff content.

A measure function is a non-decreasing function $h(t)$, $t > 0$, such that $\lim_{t \to 0} h(t) = 0$. The Hausdorff content M^h related to a mesure function h is defined for $A \subset \mathbb{R}^n$ by

$$M^h(A) = \inf \sum_i h(\rho_i),$$

where the infimum is taken over all countable coverings of A by open balls $B(x_i, \rho_i)$. When $h(t) = t^\alpha$, $\alpha > 0$, $M^h(A) = M^\alpha(A)$ is called the α-dimensional Hausdorff content of A. The lower α-dimensional Hausdorff content of A is defined by

$$M_*^\alpha(A) = \sup M^h(A),$$

the supremum being taken over all measure functions which satisfy $h(t) \leq t^\alpha$ and $\lim_{t\to 0} h(t)t^{-\alpha} = 0$.

1.3. The main result.

We proceed now to state our main result. The number $d = n - r + s$ in the statement below is the most relevant index in determining the possibility of approximation.

THEOREM 1. — *Let $X \subset \mathbb{R}^n$ be compact, $0 \leq s < r$ and set $d = n - r + s$.*

(i) *If $H^s(X) = h^s(X)$ and $d > 0$ then there exists a constant C such that*

$$M_*^d(B\setminus \overset{\circ}{X}) \leq CM^d(B\setminus X), \text{ for each open ball } B.$$

(ii) *Define a set of indexes J as follows: $J = \{0\}$ if $0 < n - 1 \leq d$ and*

$$J = \{j \in \mathbb{Z} : j \geq 0 \text{ and } 0 < d + j < n - 1\}$$

in the remaining cases (that is, $d < n - 1$ or $n = 1$ and $d = 0$). If for each $j \in J$ there exists a constant C such that

$$M_*^{d+j}(B\setminus \overset{\circ}{X}) \leq CM^{d+j}(B\setminus X), \text{ for each open ball } B,$$

then $H^s(X) = h^s(X)$.

Some remarks on the above statement are in order.

1. In part (ii) we understand that if J is the empty set, then $H^s(X) = h^s(X)$. For instance, this holds for all X if $n = 1$ and $d \leq 0$ or if $n = 2$, s is integer and $d \leq 0$. In particular, for $L = \Delta$, $n = 2$ and $s = 0$ we recover a theorem proved in [MV] by methods not relying on spectral synthesis.

It is a remarkable fact that our examples in section 4 show that the cases listed above are the only ones in which $H^s(X) = h^s(X)$ for all X. For example, in dimension 2, for an integer $s \geq 0$ and $0 < \varepsilon < 1$, there exist compact sets X_0 and X_1 such that $H^{s-\varepsilon}(X_0) \neq h^{s-\varepsilon}(X_0)$ and $H^{s+\varepsilon}(X_1) \neq h^{s+\varepsilon}(X_1)$, while $H^s(X) = h^s(X)$ for all X (for $s = 0$ only the claim about X_1 has to be considered).

2. For $L = \Delta$, $n \geq 3$ and $s = 0$, Theorem 1 gives the following result, which confirms a conjecture formulated in [MV].

1.3.1. COROLLARY. — *Let $X \subset \mathbb{R}^n$ be compact, $n \geq 3$, and assume $L = \Delta$. Then the following are equivalent:*

(i) $H^0(X) = h^0(X)$.

(ii) *For some positive constant C,*

$$M_*^{n-2}(B \setminus \overset{\circ}{X}) \leq C M^{n-2}(B \setminus X), \quad \textit{for each open ball } B.$$

3. Assume now that $r - 2 \leq s < r$. Then $J = \{0\}$ (at least if $n \geq 2$) and thus Theorem 1 gives that $H^s(X) = h^s(X)$ if and only if $M_*^{n-r+s}(B \setminus \overset{\circ}{X}) \leq C M^{n-r+s}(B \setminus X)$, for some constant C and each open ball B. This was proved in [V1] for $r - 1 < s < r$ and in [MO] for $r - 2 < s \leq r - 1$.

4. For $L = \bar{\partial}$ in the plane and $0 \leq s < 1$, Theorem 1 contains the main results of [OF2], [V3].

1.4. Lizorkin-Triebel spaces.

General information on Lizorkin-Triebel spaces can be found in [T]. Here we recall only some basic facts and definitions we need in the sequel.

Let $\mathcal{S}$ stand for the class of rapidly decreasing functions on $\mathbb{R}^n$ and $\mathcal{S}'$ for the set of tempered distributions. Denote by Z the set of functions $\varphi \in \mathcal{S}$ satisfying $\partial^\alpha \widehat{\varphi}(0) = 0$, for all α, where $\widehat{\varphi}$ is the Fourier transform of φ. Then Z' can be identified to $\mathcal{S}'/\mathcal{P}$, $\mathcal{P}$ being the set of polynomials. For the sake of notational convenience we will not distinguish between a tempered distribution and its equivalence class in Z'. Then, by $f \in Z'$ we understand the class of the tempered distribution f in $\mathcal{S}'/\mathcal{P}$.

Let ω be a positive function on $[0, \infty)$. Given $1 \leq p$ and $1 \leq q$ the homogeneous Lizorkin-Triebel space $\overset{\circ}{F}{}^{\omega}_{p,q}$ associated to the weight ω

consists of those $f \in Z'$ admitting an expansion of the form

$$f = \sum_{i=-\infty}^{\infty} f_i, \tag{3}$$

where the Fourier transform of f_i has compact support contained in $B(0,2^{i+1})\backslash B(0,2^{i-1})$, $i \in \mathbb{Z}$, the series is weak $*$ convergent in Z' and

$$\Big\| \Big(\sum_{i=-\infty}^{\infty} \Big(\frac{|f_i(x)|}{\omega(2^{-i})} \Big)^q \Big)^{1/q} \Big\|_p < \infty. \tag{4}$$

When $p = \infty$ or $q = \infty$ we understand that L^∞-norms are used in (4).

The infimum of (4), taken over all representations (3) described above, is a norm on $\mathring{F}^{\omega}_{p,q}$. If $\omega(t) = t^\ell$, $t \in \mathbb{R}$, we get the classical Lizorkin-Triebel spaces and we write $\mathring{F}^{\ell}_{p,q}$ instead of $\mathring{F}^{\omega}_{p,q}$. We also let $\mathring{F}^{\omega}_{p,q,0}$ stand for the closure in $\mathring{F}^{\omega}_{p,q}$ of $\mathcal{S}_{00}$, the set of $f \in \mathcal{S}$ such that $\hat{f}$ is compactly supported with support disjoint from the origin. In fact, $\mathring{F}^{\omega}_{p,q} = \mathring{F}^{\omega}_{p,q,0}$ if $1 \leq p$, $q < \infty$.

We recall that $\mathring{F}^{0}_{p,2} = L^p$, $1 < p < \infty$, and $\mathring{F}^{0}_{1,2} = H^1$, the Fefferman-Stein real Hardy space. Let $I_\ell(f)$ be the Riesz potential of order $\ell \in \mathbb{R}$ of f, defined on the Fourier transform side by

$$(I_\ell f)\hat{}(\xi) = |\xi|^{-\ell} \hat{f}(\xi).$$

Then $\mathring{F}^{\ell}_{p,2} = I_\ell(L^p)$, $1<p<\infty$, and $\mathring{F}^{\ell}_{1,2} = I_\ell(H^1)$. If $s>0$, $\mathring{F}^{s}_{\infty,\infty} = \Lambda^s$ and $\mathring{F}^{s}_{\infty,\infty,0} = \lambda^s$; likewise, $\mathring{F}^{0}_{\infty,2} = \mathrm{BMO}$ and $\mathring{F}^{0}_{\infty,2,0} = \lambda^0$.

Our next task is to introduce a special class of functions ω for which $\mathring{F}^{\omega}_{p,q}$ enjoys most of the nice properties of $\mathring{F}^{\ell}_{p,q}$. Recall that a function $\rho(t)$ is called C-increasing (C-decreasing) if for some positive constant C one has $\rho(t) < C\rho(s)$ ($\rho(t) > C\rho(s)$) whenever $t < s$.

DEFINITION. — *For a positive real number ℓ we denote by $\Omega(\ell)$ the set of functions ω defined on $[0,\infty)$ satisfying*

(i) $\omega(t)t^{-\ell}$ *is C-decreasing,*

(ii) *for each $m < \ell$, $\omega(t)t^{-m}$ is C_m-increasing,*

(iii) $\omega(t) = at^\ell$, $t \geq 1$, *for some positive constant a.*

When $\omega \in \Omega(\ell)$, properties 1.4.1–1.4.4 below hold. The proof of this fact is a lengthy, routine adaptation of the known arguments used to prove the special case $\omega(t) = t^\ell$ in [FJ], [T], [N1] and [N4].

1.4.1 THEOREM. — *Let $\omega \in \Omega(\ell)$, $\ell > 0$, $1 \le p$, $q \le \infty$.*

(i) *If $\ell > 1$ and $f \in \overset{\circ}{F}{}^{\omega}_{p,q}$ then $\partial f/\partial x_j \in \overset{\circ}{F}{}^{\omega(t)/t}_{p,q}$, $1 \le j \le n$.*

(ii) *The Riesz potential I_α of order $\alpha > 0$ maps isomorphically $\overset{\circ}{F}{}^{\omega}_{p,q}$ onto $\overset{\circ}{F}{}^{\omega(t)t^\alpha}_{p,q}$.*

(iii) *The space $\overset{\circ}{F}{}^{\omega}_{p,q}$ is invariant under classical Calderón-Zygmund operators with regular kernel.*

1.4.2. DUALITY THEOREM. — *Let $\omega \in \Omega(\ell)$, $\ell > 0$, and $1 \le q \le \infty$. If $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, then*

(i) $\left(\overset{\circ}{F}{}^{\omega}_{1,q} \right)^* = \overset{\circ}{F}{}^{1/\omega}_{\infty,q'}$

((ii) $\left(\overset{\circ}{F}{}^{1/\omega}_{\infty,q',0} \right)^* = \overset{\circ}{F}{}^{\omega}_{1,q}$.

The exterior capacity of a set $A \subset \mathbb{R}^n$ relative to $\overset{\circ}{F}{}^{\omega}_{p,q}$ is the quantity

$$\operatorname{cap}(A, \overset{\circ}{F}{}^{\omega}_{p,q}) = \inf \|f\|^p_{\overset{\circ}{F}{}^{\omega}_{p,q}},$$

where the infimum is taken over all open sets $G \supset A$ and functions $f \in \overset{\circ}{F}{}^{\omega}_{p,q} \cap L^p$ such that $f \ge 1$ a.e. on G. The set function $\operatorname{cap}(\cdot, \overset{\circ}{F}{}^{\omega}_{p,q})$ does not depend on q [N3]. When $p = 1$ there is a relation between capacity and Hausdorff content given by the next result (see [N4], Theorem 2.3(b)).

1.4.3. THEOREM. — *Let $\omega \in \Omega(\ell)$, $0 < \ell \le n$ and $1 \le q \le \infty$. Then for any set $A \subset \mathbb{R}^n$,*

$$C^{-1} M^{t^n/\omega(t)}(A) \le \operatorname{cap}(A, \overset{\circ}{F}{}^{\omega}_{1,q}) \le C M^{t^n/\omega(t)}(A),$$

where C is a constant independent of A.

As in the theory of Sobolev spaces the capacity associated to $\overset{\circ}{F}{}^{\omega}_{p,q}$ can be used to describe the continuity properties of elements in $\overset{\circ}{F}{}^{\omega}_{p,q}$. When $\omega \in \Omega(\ell)$, $\ell > n$, each f in $\overset{\circ}{F}{}^{\omega}_{1,q}$ is a continuous function (actually, we should say that any representative of the class of f in $\mathcal{S}'/P$ is a continuous function). If $\ell \le n$ and $f \in \overset{\circ}{F}{}^{\omega}_{1,q}$ then a much weaker statement remains true: f is in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\int f^*(x)\, dM^h(x) \le C \|f\|_{\overset{\circ}{F}{}^{\omega}_{1,q}}, \tag{5}$$

where $h(t) = t^n/\omega(t)$, f^* is the Hardy-Littlewood maximal operator of f and C is independent of f. Thus f admits an M^h-quasicontinuous representative, that is, f can be modified outside a set of zero Lebesgue measure so that, given $\varepsilon > 0$, there exists an open set G with $M^h(G) < \varepsilon$ such that the restriction of f to $\mathbb{R}^n \backslash G$ is continuous on $\mathbb{R}^n \backslash G$ (see [A] and [N2]).

In particular, taking into account 1.4.1, if $f \in \overset{\circ}{F}{}^{\omega}_{1,q}$ then $\partial^\alpha f$ is continuous whenever $0 \le |\alpha| < \ell - n$ and $\partial^\alpha f$ can be taken to be $M^{t^{n+|\alpha|}/\omega(t)}$-quasicontinuous whenever $\ell - n \le |\alpha| \le [\ell]$, $[\ell]$ being the integer part of ℓ. We can now state the "strong" spectral synthesis theorem for $\overset{\circ}{F}{}^{\omega}_{1,q}$ [N1].

1.4.4. SPECTRAL SYNTHESIS THEOREM. — *Let $\omega \in \Omega(\ell)$, $\ell > 0$, $1 \le q < \infty$ and let F be a closed subset of $\mathbb{R}^n$. Given $f \in \overset{\circ}{F}{}^{\omega}_{1,q}$, the following are equivalent.*

(i) *There exists a sequence (φ_j) of compactly supported C^∞ functions on $\mathbb{R}^n$ such that* $\operatorname{spt}\varphi_j \cap F = \emptyset$ *for all j and $\varphi_j \to f$ in $\overset{\circ}{F}{}^{\omega}_{1,q}$.*

(ii) *Let σ be the largest integer less than ℓ and let j be the smallest non-negative integer larger than or equal to $\ell - n$. Then $\nabla^k f = 0$ on F, $0 \le k < j$, and $\nabla^k f = 0$ $M^{t^{n+k}/\omega(t)}$ almost everywhere on F, $j \le k \le \sigma$.*

Remark. — Recall that we do not distinguish between a tempered distribution and its equivalence class in Z'. We emphasize that in (i) of the statement above f is a class in $\overset{\circ}{F}{}^{\omega}_{1,q}$ and in part (ii) we assert the *existence of some specific representative* of the given class in $\overset{\circ}{F}{}^{\omega}_{1,q}$ satisfying the vanishing conditions stated there.

The next result is the basic tool in our proof of Theorem 1. We warn the reader that the latest remark applies also to the statement below.

1.4.5. THEOREM 2 (Weak $*$ spectral synthesis). — *Let $\ell > 0$, $1 \le q < \infty$ and let F be a closed subset of $\mathbb{R}^n$. Let τ be the maximum between 1 and the largest integer less than ℓ, and let j be the minimum between τ and the smallest non-negative integer larger than $\ell - n$. If $f \in \overset{\circ}{F}{}^{\ell}_{1,q}$ satisfies*

$$\nabla^k f = 0 \text{ on } F, \quad 0 \le k < j,$$

and

$$\nabla^k f = 0 \ M_*^{n-\ell+k} \text{ almost everywhere on } F, \quad j \le k \le \tau - 1,$$

then f is in the weak $*$ closure in $\overset{\circ}{F}{}^{\ell}_{1,q}$ of the set of compactly supported C^∞ functions on $\mathbb{R}^n$ with support disjoint from F.

Some remarks on Theorem 2 are in order.

1. One can prove that the sufficient conditions in Theorem 2 are also necessary. The proof we know is too technical to be included here.

2. Observe that in condition (ii) of 1.4.4 there are σ vanishing conditions. Instead in Theorem 2, at least in most cases, one has $\tau - 1$ conditions. Thus, with the same ℓ, Theorem 2 involves one condition less than 1.4.4. For instance, take $n = 3$, $\ell = 2$, $\omega(t) = t^2$ in 1.4.4. The vanishing conditions are then $f = 0 \ M^1$ a.e. on F and $\nabla f = 0 \ M^2$ a.e. on F. If $n = 3$, $\ell = 2$ in Theorem 2 the only vanishing hypothesis is $f = 0 \ M_*^1$ a.e. on F.

2. PROOF OF THEOREM 1

2.1. Proof of Theorem 1, part (i).

LEMMA. — *Let $X \subset \mathbb{R}^n$ be compact and $0 < s < r$. Then $H^s(X) = h^s(X)$ if and only if $H^s_{\text{jet}}(X) = h^s_{\text{jet}}(X)$.*

Proof. — Consideration of the natural mapping from $\mathrm{Lip}^s_{\text{jet}}(X)$ onto $\mathrm{Lip}^s(X)$ shows that the condition is sufficient.

Assume now that $H^s(X) = h^s(X)$ and let $f \in \mathrm{lip}(x, \mathbb{R}^n)$, $Lf = 0$ on $\overset{\circ}{X}$. We have to show that the jet $\tilde{f}$ of f on X belongs to $H^s_{\text{jet}}(X)$. We will use the following two well-known facts.

First, $H^s_{\text{jet}}(X)$ is defined by local conditions, that is $\tilde{f} \in H^s_{\text{jet}}(X)$ if (and only if) each $x \in X$ has an open neighbourhood U such that $\tilde{f} \in H^s_{\text{jet}}(X \cap \bar{U})$. This is proved in [W], p. 515 (see also [OF3]).

On the other hand, if X has zero (n-dimensional Lebesgue) measure then $H^s_{\text{jet}}(X) = \mathrm{lip}^s_{\text{jet}}(X)$. See [W] (actually, in [W] one deals with an integer s and another topology, but the argument goes through to cover the situation we are envisaging).

Let $\bar{f}$ denote the class of f in $\mathrm{lip}(s,X)$, so that $\bar{f} \in H^s(X)$. Given $\varepsilon > 0$ there exists $g \in \mathrm{lip}(s,\mathbb{R}^n)$ such that $Lg = 0$ on some neighbourhood of X and $\|\bar{f} - \bar{g}\|_{\mathrm{Lip}(s,X)} < \varepsilon$. Thus $\|f - g - h\|_{\mathrm{Lip}(s,\mathbb{R}^n)} < \varepsilon$ for some $h \in \mathrm{lip}(s,\mathbb{R}^n)$, $h = 0$ on X. Set $s = m + \sigma$, with m integer and $0 < \sigma \leq 1$. Define

$$X_0 = \{x : \partial^\alpha h(x) = 0,\ |\alpha| \leq m\},$$

and for $1 \leq j \leq m$,

$$X_j = \big\{x : \partial^\alpha h(x) = 0,\ |\alpha| \leq m - j, \\ \text{and } \partial^\alpha h(x) \neq 0 \text{ for some } \alpha,\ |\alpha| = m - j + 1\big\}.$$

It turns out that $|X \backslash X_0| = 0$, the bars indicating Lebesgue measure. This is a consequence of the fact that $X \backslash X_0 = \bigcup_{j=1}^{m} X_j$ and each X_j, $j \geq 1$, is locally contained in a finite union of C^1-hypersurfaces.

It is an elementary fact that h can be approximated in $\mathrm{Lip}(s,\mathbb{R}^n)$ by functions vanishing on some neighbourhood of X_0. Then, without loss of generality, we can assume that $F \equiv g + h$ satisfies the equation $LF = 0$ on some neighbourhood of X_0. We must show now that $\widetilde{F} \in H^s_{\text{jet}}(X)$.

Fix $x \in X$. If $x \in X_0$, then there is an open ball B centered at x such that $LF = 0$ on some neighbourhood of $\bar{B}$. Thus $\widetilde{F} \in H^s_{\text{jet}}(X \cap \bar{B})$.

If $x \in X \backslash X_0$, then there is an open ball B centered at x such that $|X \cap \bar{B}| = 0$. Hence $\widetilde{F} \in H^s_{\text{jet}}(X \cap \bar{B})$. Therefore $\widetilde{F} \in H^s_{\text{jet}}(X)$, as desired.□

Now, it is clear that the argument given in [V1], p. 185 proves part (i) of Theorem 1.

Notice that the statement of Theorem 2 in [V1] involves "functions"but the proof in p. 185 works only for "jets". Thus the preceding lemma fills the gap.

2.2. Proof of Theorem 1, part (ii).

By the last remark in subsection 1.1 of section 1 it is sufficient to perform our approximation in the context of the homogeneous spaces $\lambda^s(X)$. Recall that if $s > 0$, $s = m+\sigma$, with m integer and $0 < \sigma \leq 1$, then

$$\lambda^s_{\text{jet}}(X) = \lambda^s(\mathbb{R}^n)/J(X),$$

where

$$J(X) = \{f \in \lambda^s(\mathbb{R}^n) : \partial^\alpha f(x) = 0,\, x \in X,\, |\alpha| \le m\}.$$

If $s = 0$ we simply set

$$\lambda^0_{\text{jet}}(X) = \lambda^0(X).$$

The advantage of $\lambda^s_{\text{jet}}(X)$ on $\text{lip}_{\text{jet}}(s, X)$ is that its dual space can be described very simply in terms of Lizorkin-Triebel spaces. In fact, if $s > 0$

$$(\lambda^s(\mathbb{R}^n))^* = \left(\overset{\circ}{F}{}^{s}_{\infty,\infty,0} \right)^* = \overset{\circ}{F}{}^{-s}_{1,1},$$

and if $s = 0$

$$(\lambda^0(\mathbb{R}^n))^* = \left(\overset{\circ}{F}_{\infty,2,0} \right)^* = H^1(\mathbb{R}^n),$$

and therefore

$$\lambda^s_{\text{jet}}(X)^* = \left\{ h \in \overset{\circ}{F}{}^{-s}_{1,1} : \operatorname{spt} h \subset X \right\}, \quad s > 0,$$

and

$$\lambda^0(X)^* = \{h \in H^1(\mathbb{R}^n) : \operatorname{spt} h \subset X\}.$$

The proof of (ii) in Theorem 1 proceeds by duality. We must prove that if $h \in \lambda^s_{\text{jet}}(X)^*$ ($\lambda^0(X)^*$ if $s = 0$) and h annihilates all jets $\tilde{f}$ such that $Lf = 0$ on some neighbourhood of X, then h annihilates also all jets $\tilde{f} \in \lambda^s_{\text{jet}}(X)$ such that $Lf = 0$ on $\overset{\circ}{X}$. Set $g = h * E$, E being the fundamental solution of the operator L. We claim that

$$g \in \overset{\circ}{F}{}^{r-s}_{1,1} \text{ if } s > 0, \text{ and } g \in \overset{\circ}{F}{}^{r}_{1,2} \text{ if } s = 0.$$

The Fourier transform of E has the form $\widehat{E}(\xi) = cL(\xi)^{-1}$, where $L(\xi)$ is an homogeneous polynomial of degree r. Set $m(\xi) = |\xi|^r L(\xi)^{-1}$, so that $\widehat{E}(\xi) = cm(\xi)|\xi|^{-r}$. The operator defined on the Fourier transform side by $(Tf)\hat{} = m\hat{f}$ is a Calderón-Zygmund operator [St], Theorem 6, p. 75 and hence preserves Lizorkin-Triebel spaces. On the other hand convolution with the Riesz kernel I_r of order r gives a bounded operator from $\overset{\circ}{F}{}^{-s}_{1,1}$ into $\overset{\circ}{F}{}^{r-s}_{1,1}$ (from $\overset{\circ}{F}{}^{0}_{1,2}$ into $\overset{\circ}{F}{}^{r}_{1,2}$ if $s = 0$).

Consequently $g = E * h = cT(I_r * h)$, and the claim is proved.

The orthogonality assumption on h implies that g vanishes on X^c. The next lemma indicates how to get from the hypothesis of (ii) some information about the vanishing of g on ∂X.

2.2.1. LEMMA. — *Let $0 < t < n$, and assume that $g \in \overset{\circ}{F}{}^{t}_{1,1}$ or $g \in \overset{\circ}{F}{}^{t}_{1,2}$. Let $X \subset \mathbb{R}^n$ be a compact such that $M^{n-t}_*(B \setminus \overset{\circ}{X}) \le CM^{n-t}(B \setminus X)$,* *for all open balls B. If $g = 0$ M^{n-t} almost everywhere on X^c, then $g = 0$ M_*^{n-t} almost everywhere on $(\overset{\circ}{X})^c$.*

Proof. — By the elementary inclusion relations between Lizorkin-Triebel spaces, $\overset{\circ}{F}{}^t_{1,1} \subset \overset{\circ}{F}{}^t_{1,2} = I_t * H^1(\mathbb{R}^n)$. For $I_t * H^1(\mathbb{R}^n)$ the lemma is proved in [MO], p. 724–727. □

Set $\ell = r - s$. Assume first $\ell \geq n$. Let j be the largest integer such that $j \leq \ell - n + 1$, and k the largest integer such that $k < \ell$. If $0 \leq i \leq j-1$, then $\ell - i \geq n$. Since $\partial^\alpha g \in \overset{\circ}{F}{}^{\ell-i}_{1,1}$, ($\partial^\alpha g \in \overset{\circ}{F}{}^{\ell-i}_{1,2}$ if $s = 0$) whenever $|\alpha| = i$, $\partial^\alpha g$ is continuous for $|\alpha| \leq j-1$. Therefore

$$\nabla^i g = 0 \text{ on } \overset{\circ}{X}{}^c, \quad 0 \leq i \leq j-1.$$

If $j \leq i \leq k-1$, then $0 < \ell - i < n$. Applying Lemma 2.2.1 to the i-th order derivatives of g we get

$$\nabla^i g = 0, \quad M_*^{n-\ell+i} \text{ almost everywhere on } (\overset{\circ}{X})^c, \quad j \leq i \leq k-1.$$

Let now $\tilde{f} \in \lambda^s_{\text{jet}}(X)$ satisfy $Lf = 0$ on $\overset{\circ}{X}$. Since

$$f \in \lambda^s(\mathbb{R}^n) = \overset{\circ}{F}{}^s_{\infty,\infty,0} \quad \left(\overset{\circ}{F}{}^0_{\infty,2,0} \text{ if } s = 0\right),$$

we have $Lf \in \overset{\circ}{F}{}^{s-r}_{\infty,\infty,0}$ $\left(\overset{\circ}{F}{}^{-r}_{\infty,2,0} \text{ if } s = 0\right)$. By the weak $*$ spectral synthesis theorem applied to g and the set $(\overset{\circ}{X})^c$, given $\varepsilon > 0$, there exists $\varphi \in C_0^\infty(\overset{\circ}{X})$ such that $|\langle g - \varphi, Lf\rangle| < \varepsilon$. Thus

$$|\langle h, f\rangle| = |\langle h, Lf * E\rangle| = |\langle g, Lf\rangle| = |\langle g - \varphi, Lf\rangle| < \varepsilon,$$

where the last identity follows from $Lf = 0$ on $\overset{\circ}{X}$. Hence $\langle h, f\rangle = 0$.

Assume now that $\ell < n$. Let k be the maximum between 1 and the largest integer less than ℓ. Using again Lemma 2.2.1 we get

$$\nabla^i g = 0 \quad M_*^{n-\ell+i} \text{ almost everywhere on } (\overset{\circ}{X})^c, \quad 0 \leq i \leq k-1,$$

and so the weak $*$ spectral synthesis theorem can be applied as above to complete the proof. □

To finish this section we point out an interesting corollary (of the proof) showing that one can solve the Lip s-approximation problem for sets with empty interior.

2.2.2. COROLLARY. — *Let $X \subset \mathbb{R}^n$ be compact, $0 \leq s < r$ and set $d = n - r + s$.*

If $d > 0$ then the following are equivalent:

(i) $H^s(X) = \operatorname{lip}(s, X)$ ($\lambda^0(X)$ *for* $s = 0$).

(ii) $M^d(B(x,r) \backslash X) \geq Cr^d$, $x \in X$, $r > 0$.

If $d \leq 0$ then $H^s(X) = \operatorname{lip}(s, X)$ ($\lambda^0(X)$ if $s = 0$) for each X with empty interior.

Remark. — Notice that any of the conditions (i) or (ii) implies that X has empty interior.

Proof of the corollary. — Condition (ii) follows from (i) because of Theorem 1 part (i) and $M_*^d(B(x,r)) = r^d$. To prove (ii) $\Rightarrow$ (i) we keep the notation used in the proof of Theorem 1 part (ii). We proved there that $g = 0$, M_*^d almost everywhere on $(\overset{\circ}{X})^c$. Since now $\overset{\circ}{X}$ is empty we conclude that $g = 0$ as a distribution, and thus $h = Lg = 0$. $\square$

3. WEAK $*$ SPECTRAL SYNTHESIS

In this section we prove Theorem 2.

Let q' be the dual exponent of q, that is $\dfrac{1}{q'} + \dfrac{1}{q} = 1$. We must show that given $\varepsilon > 0$ and $G_1, \ldots, G_N \in \overset{\circ}{F}{}^{-\ell}_{\infty,q',0}$, there exists a function $g \in C^\infty(\mathbb{R}^n)$ with support disjoint from F such that

$$|\langle f - g, G_j \rangle| < \varepsilon, \quad 1 \leq j \leq N,$$

where f is the function in the statement of Theorem 2.

We would like to apply Theorem 1.4.4 ((b) $\Rightarrow$ (a)) and for that we need to overcome two difficulties. First, we have to show that for some $\omega \in \Omega(\ell)$, $G_j \in \overset{\circ}{F}{}^{1/\omega}_{\infty,q'}$, $1 \leq j \leq N$. This is given by the next lemma.

3.1. LEMMA. — *Let ℓ and p be positive real numbers. For any $G \in \overset{\circ}{F}{}^{-\ell}_{\infty,p,0}$ there exists $\omega \in \Omega(\ell)$, satisfying $t^\ell/\omega(t) \to 0$ as $t \to 0$, such that $G \in \overset{\circ}{F}{}^{1/\omega}_{\infty,p}$.*

The second difficulty is related to the fact that condition (b) in Theorem 1.4.4 involves $\sigma + 1$ vanishing assumptions on f, while the hypothesis in Theorem 2 contains only σ vanishing assumptions (except for the exceptional case $\ell \leq 1$). Our second lemma provides the additional vanishing condition we need.

3.2. LEMMA. — *Let $F \subset \mathbb{R}^n$ be closed, $q \geq 1$ and $1 < t \leq 2$. If $f \in \mathring{F}^{t}_{1,q}$ and $f = 0$, M_*^{n-t} almost everywhere on F, then $\nabla f = 0$, M_*^{n-t+1} almost everywhere on F.*

We postpone the proof of Lemmas 3.1 and 3.2 and we proceed to complete the proof of Theorem 2.

Applying Lemma 3.1 with $p = q'$ to each G_j we find $\omega_j \in \Omega(\ell)$ such that $G_j \in \mathring{F}^{1/\omega_j}_{\infty,q'}$, $1 \leq j \leq N$. Set $\omega = \min(\omega_1, \ldots, \omega_N)$. Then $\omega \in \Omega(\ell)$ and $G_j \in \mathring{F}^{1/\omega}_{\infty,q'}$, $1 \leq j \leq N$.

On the other hand, since $t^\ell \leq C\omega(t)$ we have $f \in \mathring{F}^{\omega}_{1,q}$. We are going to show that the conditions in part (ii) of Theorem 1.4.4 are fulfilled. Assume, for the sake of convenience, that $\ell < n$ (the case $\ell \geq n$ is dealt with similarly). By hypothesis

$$\text{(6)} \qquad \nabla^j f = 0,\ M_*^{n-\ell+j} \text{ almost everywhere on } F, \quad 0 \leq j \leq \sigma - 1.$$

Excluding the exceptional case $\ell \leq 1$, we have $1 < \ell - \sigma + 1 \leq 2$. Lemma 3.2 can be applied to each $\partial^\alpha f$, $|\alpha| = \sigma - 1$, with $t = \ell - \sigma + 1$, to get $\nabla^\sigma f = 0$, $M_*^{n-\ell+\sigma}$ almost everywhere on F. Since $t^{n+j}/\omega(t) = t^{n-\ell+j}\varphi(t)$ with $\varphi(t) \to 0$ as $t \to 0$, (6) implies

$$\nabla^j f = 0,\ M^{t^{n+j}/\omega(t)} \text{ almost everywhere on } F, \quad 0 \leq j \leq \sigma.$$

By Theorem 1.4.4 given $\varepsilon > 0$ there exists a function $g \in C^\infty$ with support disjoint from F such that

$$\|f - g\|_{\mathring{F}^{\omega}_{1,q}} \leq P\varepsilon, \text{ where } P^{-1} = \max_{1 \leq j \leq N} \|G_j\|_{\mathring{F}^{1/\omega}_{\infty,q'}}.$$

Finally, using duality (1.4.2)

$$|\langle f - g, G_j \rangle| \leq C \|f - g\|_{\mathring{F}^{\omega}_{1,q}} \|G_j\|_{\mathring{F}^{\omega}_{\infty,q'}} < C\varepsilon.$$

Proof of Lemma 3.1. — Let $G \in \mathring{F}^{-\ell}_{\infty,q',0}$ be given. Then there exist functions G_m, $m = 1, 2, \ldots$ such that

(a) $G = \sum_{m=1}^{\infty} G_m$,

(b) the Fourier transform of G_m has compact support contained in $B(0, 2^{j(m)+1}) \backslash B(0, 2^{-j(m)-1})$, where $j(m)$ is an increasing sequence of positive integers,

(c) $\|G_m\|_{\overset{\circ}{F}{}^{-\ell}_{\infty,q'}} \leq C 2^{-m} \|G\|_{\overset{\circ}{F}{}^{-\ell}_{\infty,q'}}$.

The construction of the G_m is easy. For each m there is $g_m \in \mathcal{S}_{00}$ such that $\|G - g_m\|_{\overset{\circ}{F}{}^{-\ell}_{\infty,q'}} \leq 2^{-m} \|G\|_{\overset{\circ}{F}{}^{-\ell}_{\infty,q'}}$. The functions $G_1 = g_1$ and $G_m = g_m - g_{m-1}$, $m \geq 2$, clearly satisfy the above conditions.

To construct ω we start considering a continuous non-decreasing function $h(t)$ on the interval $[0, \infty)$ satisfying $h(0) = 0$, $h(t) = 1$, $1 \leq t$, and $h(2^{-i}) \geq 2^{-m/2}$, $1 \leq i \leq j(m) + 2$, $m = 1, 2, \ldots$ Unfortunately $t^\ell / h(t)$ does not necessarily belong to $\Omega(\ell)$, because $t^\varepsilon / h(t)$ could fail to be C_ε-increasing for some $\varepsilon > 0$. Set

$$\varphi(t) = \sup_{t \leq u \leq 1} h(u) \Big(\frac{t}{u} \Big)^{1/\log\log(e/t)}, \quad 0 < t < 1,$$

and $\varphi(t) = 1$, $1 \leq t$. A non-difficult but tedious computation shows that φ has the following properties:

1. φ is continuous on $[0, \infty)$, increasing on $[0, 1]$, $\varphi(0) = 0$, $\varphi(t) = 1$, $t \geq 1$.

2. $\varphi(t) \geq h(t)$ for all t. In particular

$$\varphi(2^{-i}) \geq 2^{-m/2}, \quad 1 \leq i \leq j(m) + 2, \quad m = 1, 2, \ldots .$$

3. $t^\ell / \varphi(t)$ belongs to $\Omega(\ell)$.

Take $\omega(t) = t^\ell / \varphi(t)$. We have to ascertain that $G \in \overset{\circ}{F}{}^{1/\omega}_{\infty,q'}$. For each m one can find a decomposition of type (3)

$$G_m = \sum_{i=-j(m)-2}^{j(m)+2} f_{m,i},$$

such that

$$2\|G_m\|_{\overset{\circ}{F}{}^{-\ell}_{\infty,q'}} \geq \sup_{x \in \mathbb{R}^n} \Big(\sum_{i=-j(m)-2}^{j(m)+2} (|f_{m,i}(x)| 2^{-i\ell})^{q'} \Big)^{1/q'}.$$

This follows using the definition of the Lizorkin-Triebel spaces given in [T], p. 238. Therefore

$$\begin{aligned}\|G_m\|_{\overset{\circ}{F}{}^{1/\omega}_{\infty,q'}} &\le \sup_{x\in\mathbb{R}^n}\Big(\sum_{i=-j(m)-2}^{j(m)+2}(|f_{m,i}(x)|\omega(2^{-i}))^{q'}\Big)^{1/q'}\\ &\le 2^{m/2}\sup_{x\in\mathbb{R}^n}\Big(\sum_{i=-j(m)-2}^{j(m)+2}(|f_{m,i}(x)|2^{-i\ell})^{q'}\Big)^{1/q'}\\ &\le C2^{m/2}\|G_m\|_{\overset{\circ}{F}{}^{-\ell}_{\infty,q'}} \le C2^{-m/2}\|G\|_{\overset{\circ}{F}{}^{-\ell}_{\infty,q'}},\end{aligned}$$

where the last inequality comes from (c).

Finally

$$\|G\|_{\overset{\circ}{F}{}^{1/\omega}_{\infty,q'}} \le \sum_{m=1}^{\infty}\|G_m\|_{\overset{\circ}{F}{}^{1/\omega}_{\infty,q'}} \le C\|G\|_{\overset{\circ}{F}{}^{-\ell}_{\infty,q'}}\sum_{m=1}^{\infty}2^{-m/2}. \qquad \square$$

Proof of Lemma 3.2. — The case $1 < t < 2$ is proven by a minor variation of an argument used to prove Lemmas 3.2 and 3.3 in [MO]. Set $D = \{x \in F : f(x) = 0 \text{ and } \nabla f(x) \neq 0\}$. We wish to show that $M_*^{n-t+1}(D) = 0$. From (5), as in Lemma 3.2 in [MO], one gets that M^{n-t+1} almost all $a \in \mathbb{R}^n$ satisfy

$$\text{(7)}\quad \lim_{r\to 0}\int_{B(a,r)}|f(x)-f(a)-\nabla f(a)\cdot(x-a)|r^{n-t+1}|x-a|\,dM^{n-t+1}(x) = 0.$$

As in Lemma 3.3 in [MO] one proves $M^{n-t+1}(D)=0$, and so $M_*^{n-t+1}(D)=0$.

When $t = 2$ the argument is a little trickier. Using (7) one obtains that for M^{n-1} almost all $a \in D$ there exists a straight line L through the origin for which

$$\text{(8)}\qquad \lim_{r\to 0}\frac{M^{n-1}(D\cap B(a,r)\backslash X(a,L,\eta))}{r^{n-1}} = 0,$$

for all η, $0 < \eta < 1$, where

$$X(a,L,\eta) = \{x \in \mathbb{R}^n : \operatorname{dist}(x-a,L) < \eta|x-a|\}.$$

If moreover one has that

$$\limsup_{r\to 0}\frac{M^{n-1}(D\cap B(a,r))}{r^{n-1}} > 0,$$

then one says that $a + L$ is an approximate tangent for D at a. Because of the density theorem for Hausdorff content [MP], 6.3.(4), p. 92 and (8) we conclude that M^{n-1} almost all points in D have an approximate tangent. Thus, as in [MP], 15.22.(1), p. 214, one can show that D is $(n-1)$-rectifiable. Thus $M_*^{n-1}(D) = 0$. $\square$

4. EXAMPLES

In this section we give two examples showing that none of the sufficient conditions in Theorem 1, part (ii), can be dispensed with.

The first one, introduced by Hedberg in [H] (see also [GT]), shows that when $n \geq 3$ and $d = n - r + s \leq 0$ or when $n = 2$, $d < 0$ and $d \notin \mathbb{Z}$, there exists a compact X (necessarily with non-empty interior, in view of the corollary in section 2) such that $H^s(X) \neq h^s(X)$. In particular this means that at least one of the sufficient conditions listed in (ii) of Theorem 1 fails for this X.

We start by describing the example in the case of a non-integer s. Then $d < 0$.

Let m be the positive integer satisfying $0 < \alpha \equiv d + m < 1$. Consider the interval in $\mathbb{R}^n$

$$I = \left\{(x_1, \ldots, x_n) : -\frac{1}{2} \leq x_1 \leq \frac{1}{2} \text{ and } x_i = 0,\ 2 \leq i \leq n\right\},$$

and take a sequence of disjoint open balls $B_k = B(a_k, \rho_k)$, $a_k \in I$, $k = 1, 2, \ldots$, such that the closure of $\cup B_k$ contains I, $\sum \rho_k^\alpha < \infty$ and $\sum \rho_k \leq 1/2$. Set $F = I \setminus \cup B_k$. Then the length of F is larger than or equal to $1/2$, and so $M_*^\alpha(F) > 0$. Define $X = \bar{B} \setminus \cup B_k^*$, where $B = B(0,1)$ and $B_k^* = B(a_k, \rho_k/2)$.

Our goal is to construct a distribution $T \in \overset{\circ}{F}{}_{1,1}^{-s}$, supported on X, such that T annihilates $H^s(X)$ but T does not annihilate $h^s(X)$. In fact T will be of the form $T = L(\varphi)$ for some φ appropriately chosen.

Take $\varphi_k \in C_0^\infty(B_k)$, $\varphi_k = 1$ on B_k^*, $|\nabla^j \varphi_k| \leq C \rho_k^{-j}$, $0 \leq j \leq r$ and $\varphi_0 \in C_0^\infty(B)$, $\varphi_0 \equiv 1$ on $B(0, 3/4)$. Set $\varphi(x) = x_n^m \varphi_0(x) - \sum\limits_{k=1}^{\infty} x_n^m \varphi_k(x)$. Since φ vanishes on $\cup B_k^*$ and on $\mathbb{R}^n \setminus B$, the support of φ is contained in X. On the other hand, $\varphi(x) = x_n^m$ on $B(0,3/4) \setminus \cup B_k$, and hence

$$\partial^m \varphi / (\partial x_n)^m = m! \quad \text{on} \quad F.$$

We claim that $T \equiv L(\varphi) \in \lambda^s(X)^* = \left\{f \in \overset{\circ}{F}{}_{1,1}^{-s} : \operatorname{spt} f \subset X\right\}$. A simple computation shows that

$$\|L(x_n^m \varphi_k)\|_\infty \leq C |B_k|^{-1/p} R_k^\alpha,$$

where $s = n\left(\dfrac{1}{p} - 1\right)$. Therefore $x_n^m \varphi_k(x) R_k^{-\alpha}$ is a (p, ∞)-atom and consequently [FJ], Theorem 7.4

$$\|T\|_{\overset{\circ}{F}{}_{1,1}^{-s}} \leq C \sum \rho_k^\alpha < \infty.$$

Clearly $\varphi = E * T$. Since φ vanishes outside X, T annihilates $H^s(X)$.

Applying Frostman's Lemma [C], p. 7 we find a positive measure μ supported on F, $\mu(F) > 0$, such that $\mu(B(x,r)) \leq \varepsilon(r)r^\alpha$, for all $x \in \mathbb{R}^n$ and $r > 0$, with $\varepsilon(r) \to 0$ as $r \to 0$. We have $\partial^m E/(\partial x_n)^m * \mu \in h^s(X)$ [C], p. 91. Since

$$\begin{aligned}\langle T, \partial^m E/(\partial x_n)^m * \mu\rangle &= (-1)^r \langle \varphi, \partial^m \mu/(\partial x_n)^m\rangle \\ &= (-1)^{r+m} \int \partial^m \varphi/(\partial x_n)^m \, d\mu \\ &= (-1)^{r+m} m! \mu(F) \neq 0,\end{aligned}$$

T does not annihilate $h^s(X)$.

We now briefly indicate the changes needed to deal with the case of integer s (recall that in this case $n \geq 3$). We choose m such that $\alpha \equiv d + m = 1$. The above construction is not useful because $M_*^1(F) = 0$. We replace I by the square

$$\left\{(x_1, \ldots, x_n) : -\frac{1}{2} \leq x_i \leq \frac{1}{2},\ i = 1, 2, \text{ and } x_i = 0,\ 3 \leq i \leq n\right\},$$

and we take balls B_k as before requiring now $\sum \rho_k < \infty$ and $\sum \rho_k^2 \leq 1/2$. Then $M_*^1(F) > 0$ and the argument goes on as above.

Our second example concerns the case $d = n - r + s > 0$.

PROPOSITION. — *Assume $0 < d = n - r + s < n - 2$. Then there exists a compact $X \subset \mathbb{R}^n$ such that $H^s(X) \neq h^s(X)$ and*

$$M_*^d(B \setminus \overset{\circ}{X}) \leq C M^d(B \setminus X), \text{ for all open balls } B. \tag{9}$$

Remark. — Notice that the hypothesis on d implies $n \geq 3$. It can be shown that if $0 < d < n - 2$ then, given a positive integer j_0 with $d + j_0 < n - 1$, one can construct a compact X such that $H^s(X) \neq h^s(X)$ and

$$M_*^{d+j}(B \setminus \overset{\circ}{X}) \leq C M^{d+j}(B \setminus X), \text{ for all open balls } B,$$

for all non-negative integers $j \neq j_0$ satisfying $d + j < n - 1$. This follows from a slight modification of the argument given below.

Proof of the Proposition. — For the sake of expository clarity we consider only the case $n = 3$ and s non-integer. Then $0 < d < 1$. Our construction starts with any arc γ of diameter 1 in the square

$$\left\{(x_1, x_2, x_3) : -\frac{1}{2} \leq x_i \leq \frac{1}{2},\ i = 1, 2,\ x_3 = 0\right\}$$

such that $M_*^{1+d}(\gamma) > 0$. Consider a sequence of polygonal arcs P_j such that $P_j \cap P_k = \emptyset$, $j \neq k$, $P_j \cap \gamma = \emptyset$, for all j, and $P_j \to \gamma$ in the Hausdorff distance. We can now associate to each j a finite sequence of open balls (in $\mathbb{R}^3$) $B_{j_k} = B(a_{j_k}, \rho_j)$, $a_{j_k} \in P_j$ with the following properties:

(i) each $x \in \mathbb{R}^3$ belongs to at most N balls B_{j_k}, where N is a numerical constant.

(ii) Set $S_j = \bigcup\limits_k B_{j_k}$. Then $S_j \supset P_j$ and $S_j \to \gamma$ in the Hausdorff distance.

(iii) Set $\widetilde{S}_j = \bigcup\limits_k B(a_{j_k}, 2\rho_j)$. Then $\widetilde{S}_j \cap \widetilde{S}_k = \emptyset$, $j \neq k$, and $\widetilde{S}_j \cap \gamma = \emptyset$ for all j.

(iv) $\sum\limits_k \rho_j^{1+d} < 2^{-j}$, for each j.

Define $X = \bar{B}(0,1) \backslash \bigcup\limits_j S_j$. According to [MPO], Th. 4 (9) follows from

$$M_*^d(B(x,\rho)\backslash X) \geq C\rho^d,\ x \in \partial X,\ 0 < \rho < \rho(x). \tag{10}$$

Clearly (10) is satisfied for $x \in \partial X \backslash \gamma$ because X satisfies an exterior cone condition at those x. When $x \in \gamma$ we use the fact that $\operatorname{diam}(B(x,\rho) \cap \gamma) \geq \rho$. Thus $B(x,\rho) \cap P_j$ contains a continuum K_j of diameter larger than $\rho/2$ if j is large enough. Therefore, since $0 < d < 1$,

$$\begin{aligned} M^d(B(x,\rho)\backslash X) = M^d\Big(\Big(\bigcup_j S_j\Big) \cap B(x,\rho)\Big) &\geq M^d(K_j) \\ &\geq (\operatorname{diam}(K_j))^d \geq 2^{-d}\rho^d. \end{aligned}$$

We must show now that $H^s(X) \neq h^s(X)$. For this we follow closely the argument in our first example.

Take $\varphi_{j_k} \in C_0^\infty(B(a_{j_k} 2\rho_j))$ such that $\sum\limits_k \varphi_{j_k} = 1$ on S_j, $|\nabla^\ell \varphi_{j_k}| \leq C\rho_j^{-\ell}$, $0 \leq \ell \leq r$, and $\varphi_0 \in C_0^\infty(B(0,1))$, $\varphi_0 = 1$ on $B(0,3/4)$. Set $\varphi(x) = x_3\varphi_0(x) - x_3 \sum\limits_{j,k} \varphi_{j_k}(x)$. Clearly φ vanishes on $\bigcup\limits_j S_j$ and $\mathbb{R}^3 \backslash B(0,1)$, and so φ is supported on X. On the other hand, $\varphi(x) = x_3$ on $B(0,3/4) \backslash \bigcup\limits_j \widetilde{S}_j$. Hence $\partial\varphi/\partial x_3 = 1$ on γ.

Set $T = L(\varphi)$. Let us prove that $T \in \lambda^s(X)^* = \big\{f \in \overset{\circ}{F}{}_{1,1}^{-s} : \operatorname{spt} f \subset X\big\}$. A simple computation shows that

$$\|L(x_3\varphi_{j_k})\|_\infty \leq C|B_{j_k}|^{-1/p}\rho_j^{d+1},$$

where $s = 3\Big(\dfrac{1}{p} - 1\Big)$. Therefore $x_3\varphi_{j_k}(x)\rho_j^{-(d+1)}$ is a (p,∞)-atom and consequently [FJ], Theorem 7.4

$$\|T\|_{\overset{\circ}{F}{}^{-s}_{1,1}} \le C \sum_{j,k} \rho_j^{d+1} \le C \sum_j 2^{-j} = C$$

because of property (iv). Then $T \in \lambda^s(X)^*$.

Since $\varphi = E * T$ and φ vanishes outside X, T annihilates $H^s(X)$. Take a positive measure μ supported on γ such that $\mu(\gamma) > 0$ and $\mu(B(x,\rho)) \le \varepsilon(\rho)\rho^{d+1}$, for all $x \in \mathbb{R}^3$ and $\rho > 0$, where $\varepsilon(\rho) \to 0$ as $\rho \to 0$. Then $(\partial E/\partial x_3) * \mu \in h^s(X)$ [C]. We have

$$\begin{aligned}\langle T, (\partial E/\partial x_3) * \mu\rangle &= (-1)^r \langle \varphi, \partial\mu/\partial x_3\rangle \\ &= (-1)^{r+1} \int (\partial\varphi/\partial x_3)\, d\mu = (-1)^{r+1}\mu(\gamma) \ne 0.\end{aligned}$$

Hence T does not annihilate $h^s(X)$. $\square$

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