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RANDOM PERTURBATIONS OF EXPONENTIAL RIESZ BASES IN $L^2(-\pi,\pi)$

by G. CHISTYAKOV (*) and Yu. LYUBARSKII (**)

1. Introduction.

a. We study the following question. Let a sequence of points $\Lambda = \{\lambda_n\} \subset \mathbb{R}$ be given such that the exponential system

$$\mathcal{E}(\Lambda) = \{\exp(i\lambda_n x);\ \lambda_n \in \Lambda\}$$

forms a Riesz basis in the space $L^2(-\pi,\pi)$. Consider the system $\{\exp(i(\lambda_n + \xi_n)x)\}$, obtained by means of a random perturbation of the exponents $\{\lambda_n\}$ by a sequence $\{\xi_n\}$ of independent real random variables with expectation zero:

$$\mathbb{E}(\xi_n) = 0,\ n = 0, \pm 1, \pm 2, \dots. \tag{1.1}$$

In contrast to the well-developed theory of classical Fourier series with random coefficients (see the remarkable book [1]) only a few facts are known in the case of random exponents. We mention [2] which considers $\Lambda = \mathbb{Z}$ and equaly distributed ξ_n and which studies the dependence of the frame property of the corresponding exponential system on the distribution of ξ_n.

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Let the random variables $\{\xi_n\}$ be defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\xi_n(\omega)$ be the value of ξ_n at the point $\omega \in \Omega$. We set

$$\lambda_n(\omega) = \lambda_n + \xi_n(\omega), \quad \Lambda_\omega = \{\lambda_n(\omega)\}, \tag{1.2}$$

and also

$$\mathcal{E}_\omega(\Lambda) = \{x^l \exp(i\lambda_n(\omega)x)\}_{l=0,\ n\in\mathbb{Z}}^{p_n-1}. \tag{1.3}$$

Here, as usual, $p_n (= p_n(\omega))$ denotes the multiplicity of the point $\lambda_n(\omega)$ in the sequence Λ_ω and, if $p_n > 1$, we replace the coinciding exponentials by the functions $\{x^l \exp(i\lambda_n(\omega)x)\}_{l=0}^{p_n-1}$. In what follows we suppose that the points $\lambda_n(\omega)$ are pairwise distinct, if this does not essentially alter our reasoning.

Recall that a system of elements $\{h_k\}$ in a Hilbert space H is said to form a Riesz basis if each $h \in H$ admits a unique representation of the form

$$h = \sum c_k(h) h_k, \qquad c_k(h) \in \mathbb{C},$$

and in addition

$$c\|h\|^2 \leqslant \sum |c_k(h)|^2 \leqslant C\|h\|^2$$

with some constants $c, C > 0$ independent of h.

In particular, each Riesz basis in H is a complete and minimal system in H, i.e. it ceases to be complete after removing at least one element from the system $\{h_k\}$.

The study of completeness and minimality as well as the Riesz basis property of a system of exponentials has been started in the classical books [3], [4]. The necessary and sufficient conditions for a sequence $\Lambda \subset \mathbb{R}$ to generate a Riesz basis $\mathcal{E}(\Lambda)$ have been obtained in [5] see also [6] and [7]. We refer the reader to [8], [6] also for the history and full account of materials concerning Riesz bases from exponentials in L^2-space on a segment.

In this article, we are interested in determining which basic properties of $\mathcal{E}(\Lambda)$ as a system in $L^2(-\pi, \pi)$ are typical, i.e. they are almost surely inherited by the perturbed system $\mathcal{E}_\omega(\Lambda)$. Specifically we study

a) basis property; *b)* completeness; *c)* minimality.

We also study the reconstruction of functions in $L^2(-\pi, \pi)$ via their expansions in the system $\mathcal{E}_\omega(\Lambda)$.

b. If the values of the random variables ξ_n belong to sufficiently small intervals around zero, the stability theorems for a Riesz basis (see for example [6], §2, p. III) provide that property *a)* (and, hence, *b), c)*) still remains valid. The situation changes drastically when the stability theorems can no longer be applied. We shall see that, for almost all $\omega \in \Omega$, the system $\mathcal{E}_\omega(\Lambda)$ no longer forms a Riesz basis, whereas properties *b)* and *c)* are fulfilled under natural restrictions on the decay rate of the "tails" of the distribution functions of the ξ_n's.

Therefore under these restrictions, for almost all $\omega \in \Omega$, to each function $f \in L^2(-\pi,\pi)$ there corresponds a series with respect to the system $\mathcal{E}_\omega(\Lambda)$:

$$f(x) \sim \sum c_k(f,\omega)\exp(i\lambda_k(\omega)x). \tag{1.4}$$

In the typical situation there are two kinds of obstacles for this series to be convergent: first, the points $\lambda_k(\omega)$ no longer are separated, for almost all $\omega \in \Omega$ there are couples (and even n-bunches for any finite n) of arbitrarily close (maybe coinciding) points $\lambda_k(\omega)$; second, as it will be clear below, even the "separated part" of the sequence Λ_ω generates an exponential system that does not form a Riesz basis in its own linear span.

To bypass these obstacles we first consider a block summation procedure taking as one block all the terms that correspond to close exponents $\lambda_k(\omega)$; second, we prove that (with such summation procedure) the series (1.4) converges in $L^2(-\pi,\pi)$ for all functions f satisfying the Lipschitz condition with a positive exponent.

We shall also see that the random perturbations change the local convergence properties of exponential series very slightly: at each point of the interval $(-\pi,\pi)$ and for each function $f \in L^2(-\pi,\pi)$ the series (1.4) is equiconvergent (with respect to the corresponding summation procedure) with the usual Fourier series.

c. Another reason, which attracts our attention to the subject is its (natural) connection with signal analysis problems. Consider the Paley-Wiener space PW_π, i.e. the space of all entire functions of exponential type (*e.f.e.t.*) not greater than π such that

$$\|F\|^2_{PW_\pi} := \int_{-\infty}^{\infty} |F(t)|^2 dt < \infty.$$

Each function $F \in PW_\pi$ admits the representation

$$F(t) = \int_{-\pi}^{\pi} e^{itx} f(x)dx, \ f \in L^2(-\pi, \pi).$$

One may treat the functions from this space as signals with bandlimited (located on $[-\pi, \pi]$) spectrum (see e.g. surveys [9], [10], [11], [12]).

The (now) classical duality reasoning (see, for example, [6]) shows that the system $\mathcal{E}(\Lambda)$ forms a Riesz basis in the space $L^2(-\pi, \pi)$ iff it possesses a *generating function*, i.e. there exists the limit

$$S(\lambda) = \lim_{R\to\infty} \prod_{|\lambda_n|<R} \left(1 - \frac{\lambda}{\lambda_n}\right) \tag{1.5}$$

representing an *e.f.e.t.* π and also each function $F \in PW_\pi$ may be reconstructed via its samples $\{F(\lambda_n)\}$ by means of the Lagrange interpolation series

$$F(t) = \sum F(\lambda_n) \frac{S(t)}{S'(\lambda_n)(t - \lambda_n)}, \tag{1.6}$$

which converges both compactwise and in $L^2(-\infty, \infty)$. Thus we reformulate the questions of this article as follows. Let the sequence $\{\lambda_n\} \in \mathbb{R}$ determine the bandlimited signals by (1.5),(1.6). What can one say (almost surely) about its random perturbations $\{\lambda_n + \xi_n\}$, where $\{\xi_n\}$ is a sequence of random variables as above? We prove that (under natural restrictions on the decay rate of the "tails" of the distribution functions of ξ_n's) the sequence of values $\{F(\lambda_n + \xi_n)\}$ still determines the function $F \in PW_\pi$ uniquely, though the representation by interpolating series is not valid, generally speaking. Nevertheless we shall see that this representation is still valid under an additional decay condition: $(1 + |t|)^\beta F(t) \in L^2(\mathbb{R})$ for some $\beta > 0$.

d. One of our main tools and, partly, a purpose of this article is the study of entire functions with random zeroes:

$$S_\omega(\lambda) = \lim_{R\to\infty} \prod_{|\lambda_n|<R} \left(1 - \frac{\lambda}{\lambda_n(\omega)}\right).$$

$S_\omega(\lambda)$ is the generating function for the perturbed sequence Λ_ω. In order to estimate such functions we develop specific techniques which combine methods of function theory and probability theory and (we hope) are of independent interest themselves.

It appears that the obtained estimates match very precisely with results from function theory such as the Helson-Szegö (or (A_2)- Muckenhoupt) condition. Starting with these estimates and operating with the methods that stem from the works [3], [4], [13], [14] and further [5], [6], [15], [16], [17], [18] we study convergence and summability properties of random series with respect to the system $\mathcal{E}_\omega(\Lambda)$.

e. The article is organized as follows. The next section contains the precise description of the problems and the statements of the main theorems, and in this way it gives a complete outline of the article. In Section 3 we collect the preliminary results on Riesz bases and probability theory and also prove some preliminary statements. Section 4 contains estimates on the imaginary axis of entire functions with random zeroes as well as a simple estimate from above on a horizontal line – these estimates provide the completeness and minimality theorems for exponential systems with random exponents. Section 5 gives a precise two-sided estimate on a horizontal line. This estimate is a base of the block summability procedure for series (1.4), which is constructed in Section 6. In Section 7 we study the local convergence of this series.

2. Statements of the main results.

a. Through the whole article we assume that the sequence $\Lambda = \{\lambda_n\} \subset \mathbb{R}$ generates an exponential Riesz basis in $L^2(-\pi, \pi)$ and the sequence of real independent random variables $\{\xi_n\}$ satisfies (1.1). We also keep notations (1.2)-(1.5).

For the sake of simplicity we formulate the no-go theorem under additional restrictions on the sequence $\{\xi_n\}$.

THEOREM 1. — *Let each random variable ξ_n have the distribution density $p_n(t)$ and, for each $T > 0$,*

$$\inf_n \inf_{|t|<T} p_n(t) > 0. \tag{2.1}$$

Then

a. *The system $\mathcal{E}_\omega(\Lambda)$ does not form a Riesz basis in $L^2(-\pi, \pi)$;*

b. *For almost all $\omega \in \Omega$ there exists a sequence $\{n_k\}(= \{n_k(\omega)\}) \subset \mathbb{Z}$ with the following properties:*

$$\inf_{k \neq m} |\lambda_{n_k}(\omega) - \lambda_{n_m}(\omega)| > 0; \tag{2.2}$$

the system $\{\exp(i\lambda_{n_k}(\omega)x)\}$ does not form a Riesz basis in the closure of its linear span.

This theorem is proved in items **f - h** of the next section.

b. In Section 4 we prove the theorems which grant existence of the generating function $S_\omega(\lambda)$ for the perturbed sequence of exponents Λ_ω, compare the behavior of the functions $S_\omega(\lambda)$ and $S(\lambda)$ and, finally, yield the completeness and minimality theorems in $L^2(-\pi,\pi)$ (as well as the corresponding uniqueness theorem in the space PW_π).

THEOREM 2. — *Let condition (1.1) be fulfilled and, for some $\delta > 0$,*

$$\sup_n \mathbb{E}(|\xi_n|^{1+\delta}) < \infty. \tag{2.3}$$

Then, for almost all $\omega \in \Omega$, there exists the limit

$$S_\omega(\lambda) = \lim_{R\to\infty} \prod_{|\lambda_n|<R} \left(1 - \frac{\lambda}{\lambda_n(\omega)}\right) \tag{2.4}$$

representing an e.f.e.t. π.

In addition the following relation holds:

$$\frac{1}{r}|\log|\frac{S_\omega(r\exp(i\theta))}{S(r\exp(i\theta))}|| \to 0 \quad as \quad r\to\infty,\ \theta\in[0,2\pi),\ \theta\neq 0,\ \pi. \tag{2.5}$$

One can give a stronger estimate on the imaginary axis.

THEOREM 3. — *Let condition (1.1) be fulfilled and, for some $\delta > 0$,*

$$\sup_n \mathbb{E}(|\xi_n|^{2+\delta}) < \infty. \tag{2.6}$$

Then, for almost all $\omega \in \Omega$, the following relation holds:

$$c < \frac{|S_\omega(i\eta)|}{|S(i\eta)|} < C,\ \eta\in\mathbb{R}. \tag{2.7}$$

Here the constants $c, C > 0$ depend on $\omega \in \Omega$, generally speaking.

In the case

$$\sup_n \mathbb{E}(|\xi_n|^2) < \infty, \tag{2.8}$$

a weaker estimate takes place:

(2.9)
$$c\exp(-(\log(1+|\eta|))^{3/4}) < \frac{|S_\omega(i\eta)|}{|S(i\eta)|} < C\exp((\log(1+|\eta|))^{3/4}),\ \eta\in\mathbb{R}.$$

THEOREM 4. — *Let conditions (1.1) and* (2.8) *be fulfilled. Then the following statements hold (for almost all $\omega\in\Omega$):*

a. *the system $\mathcal{E}_\omega(\Lambda)$ is complete in $L^2(-\pi,\pi)$;*

b. *if a function $F\in PW_\pi$ vanishes on Λ_ω (with account of possible multiplicities) then $F\equiv 0$.*

c. In order to study the minimality of $\mathcal{E}_\omega(\Lambda)$ we need to estimate the function $|S_\omega(\lambda)|$ from above on a horizontal line. This demands additional conditions on the perturbations ξ_n.

THEOREM 5. — *Let a sequence $\{\xi_n\}$ of independent random variables satisfy (1.1) and also*

$$\sup_n \mathbb{E}(|\xi_n|^m) < \infty,\ m=2,3,\dots\ . \tag{2.10}$$

Then the system $\mathcal{E}_\omega(\Lambda)$ is complete and minimal in $L^2(-\pi,\pi)$ for almost all $\omega\in\Omega$.

Under the hypothesis of this theorem, for almost all $\omega\in\Omega$, one can correspond to each function $f\in L^2(-\pi,\pi)$ its Fourier series in the system $\mathcal{E}_\omega(\Lambda)$:

$$f\sim\sum_k\sum_{l=0}^{p_k-1} c_{k,l}(\omega,f)x^l\exp(i\lambda_k x). \tag{2.11}$$

d. To construct a summability procedure for such a series we also need to estimate the function $S_\omega(\lambda)$ from below which, in turn demands that we impose some additional restrictions on the distribution functions of the ξ_n's. In order to formulate these restrictions we introduce the following definition.

DEFINITION. — *A sequence $\{\xi_n\}$ of independent random variables satisfying (1.1), belongs to the class $B_{\alpha,\gamma}$, $\alpha,\gamma>0$, if*

$$\mathbb{E}(\exp(|\xi_n|t)) \leqslant \exp(\alpha(\exp(\gamma t)-1)),\ t>0,\ n=0,\pm1,\pm2,\dots\ .$$

We also denote $B_0 = \bigcap_{\gamma>0} \bigcup_{\alpha>0} B_{\alpha,\gamma}$.

Example. — Let $\{\xi_n\}$ be a sequence of independent centered Gaussian random variables with uniformly bounded variances. Then $\{\xi_n\} \in B_0$.

It should also be mentioned that a sequence of centered non-degenerate Poisson-type random variables with uniformly bounded variances and Poisson spectrum does not belong to the class B_0. Nevertheless it belongs to the class $B_{\alpha,\gamma}$ for appropriate values of α and γ.

The theorem below (proven in Sect. 5) compares the perturbed function $S_\omega(\lambda)$ on a horizontal line (as well as on some vertical segments in the strip $|\Im z| < 1$) with the original function $S(\lambda)$.

THEOREM 6. — *Let* $\{\xi_n\} \in B_{\alpha,\gamma}$. *Then there exists a constant* a *independent of* γ *and* α *such that for almost all* $\omega \in \Omega$, *the following relations hold:*

a.

$$[C(\omega,\alpha,\gamma)|x+i|^{a\gamma}]^{-1} \leqslant \frac{|S_\omega(x+i)|}{|S(x+i)|} \leqslant C(\omega,\alpha,\gamma)|x+i|^{a\gamma}, \; x \in \mathbb{R} \tag{2.12}$$

with some constant $C(\omega,\alpha,\gamma) \in (0,\infty)$.

b. *If*

$$\operatorname{dist}(x, \Lambda_\omega \cup \Lambda) \geqslant \frac{1}{\log|x|}, \; |x| > 10, \tag{2.13}$$

then

$$[C(\omega,\alpha,\gamma)|x+i|^{a\gamma}]^{-1} \leqslant \frac{|S_\omega(x+iy)|}{|S(x+iy)|} \leqslant C(\omega,\alpha,\gamma)|x+i|^{a\gamma}, \; y \in [-1,1]. \tag{2.14}$$

One can choose the constant $C(\omega,\alpha,\gamma)$ *such that both (2.12) and (2.14) are fulfilled. Therefore we do not specify notation of the constant in these relations.*

c. *For all but finitely many* $m \in \mathbb{Z}$ *there exists* $x \in [m, m+1]$, *satisfying (2.13).*

Assuming $\{\xi_n\} \in B_0$ we wish to describe the summability procedure for the series (2.11).

DEFINITION. — *Let a sequence $\mathcal{R} = \{R_k\}_{-\infty}^{\infty}$, $R_k \to \pm\infty$ as $k \to \pm\infty$ be given. Denote $I_k = [R_k, R_{k+1})$. We say that series (2.11) is $\mathcal{R}$-summable to the function f if the limit*

$$f(x) = \lim_{M,N\to\infty} \sum_{k=-M}^{N} \sum_{\lambda_n \in I_k} \sum_{l=0}^{p_n-1} c_{n,l}(\omega; f) x^l \exp(i\lambda_n(\omega)x) \tag{2.15}$$

*exists. Explicit expressions for the coefficients $c_{n,l}(\omega)$ are given below in Sec.6***a***. In order to observe the class of functions for which the series (2.11) is $\mathcal{R}$-summable we need additional definitions and notation.*

We use the Fourier transform in the form

$$\mathcal{F} : f \to \mathcal{F}(f)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \exp(-itx) dx \tag{2.16}$$

and consider $PW_{\pi,\alpha}$ – the space of all e.f.e.t. π such that

$$\|F\|^2_{PW_{\pi,\alpha}} = \int_{-\infty}^{\infty} |F(t)|^2 (1 + |t|)^{2\alpha} dt < \infty, \tag{2.17}$$

and also the corresponding Sobolev space

$$L^2_\alpha(-\pi, \pi) = \mathcal{F}^{-1} PW_{\pi,\alpha}. \tag{2.18}$$

Here the Fourier transform is considered in the distributional sense, generally speaking. If it does not cause misunderstanding, we denote the Sobolev space by L^2_α or, if $\alpha = 0$, simply by L^2.

It should be mentioned (see for example [19]), that if a function f satisfies the Lipschitz condition with exponent $\beta > 0$ on $[-\pi, \pi]$, then $f \in L^2_{\beta/2}(-\pi, \pi)$, say.

In Section 6 we study $\mathcal{R}$-summability of the series (2.11).

THEOREM 7. — *Let $\{\xi_n\} \in B_0$. Then, for almost all $\omega \in \Omega$, the following statement holds:*

If a sequence $\mathcal{R} = \{R_k\}$ satisfies condition (2.13) with R_k instead of x for each $k \in \mathbb{Z}$, then (2.11) is $\mathcal{R}$-summable in $L^2(-\pi, \pi)$ for each function $f \in L^2_\alpha(-\pi, \pi)$ for some $\alpha(= \alpha(f)) > 0$.

The following statement on reconstruction of bandlimited signals is, in essence, a combination of Theorems 5–7.

THEOREM 8. — *Let $\{\xi_n\} \in B_0$. Then, for almost all $\omega \in \Omega$, the following statements hold:*

a. *The function $S_\omega(\lambda)(1+|\lambda|)^{-1} \in L^2(-\pi,\pi)$;*

b. *If a sequence $\mathcal{R} = \{R_k\}$ satisfies condition (2.13) with R_k instead of x for each $k \in \mathbb{Z}$, then*

$$F(\lambda) = \lim_{M,N\to\infty} \sum_{k=-M}^{N} \sum_{\lambda_n \in I_k} F(\lambda_n(\omega)) \frac{S_\omega(\lambda)}{S'_\omega(\lambda_n(\omega))(\lambda-\lambda_n(\omega))} \tag{2.19}$$

for each function $F \in PW_{\pi,\alpha}$ with some $\alpha(=\alpha(F)) > 0$. Here lim *is considered in the $L^2(\mathbb{R})$-sense.*

Remark. — Here expression (2.19) is written for the case when all points $\{\lambda_n\}$ are distinct. In the case when some of them coincide, one should consider the Lagrange-Hermite interpolation series instead of (2.19). [See details in Sec.6.]

e. The last theorem (see Sect. 7) describes the local convergence properties of (2.11). For each $f \in L^2(-\pi,\pi)$ denote

$$a_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \exp(-ikx)dt \tag{2.20}$$

and

$$S(f;R,T)(x) = \sum_{R\leqslant k<T} Ta_k(f)\exp(ikx).$$

THEOREM 9. — *Let $\{\xi_n\} \in B_0$. Then, for almost all $\omega \in \Omega$, the following statement holds:*

For each function $f \in L^2(-\pi,\pi)$ there exists a sequence $\mathcal{R}(=\mathcal{R}(f,\omega)) = \{R_k\}$, $R_k \to \pm\infty$, as $k \to \pm\infty$ satisfying condition (2.13) with R_k instead x for each $k \in \mathbb{Z}$ and such that

$$S(f;R_{-M},R_{N+1})(x) - \sum_{k=-M}^{N} \sum_{\lambda_n\in I_k} \sum_{l=0}^{p_n-1} c_{n,l}(\omega;f)x^l \exp(i\lambda_n(\omega)x) \to 0, \quad M,N\to\infty \tag{2.21}$$

for each $x \in (-\pi,\pi)$.

Remark. — The proofs of Theorems 7–9 will show that for each particular choice of the sequence Λ, these theorems can be strengthenen to some (depending on Λ) extent. If say, $\Lambda = \mathbb{Z}$, one can replace the condition $\{\xi_n\} \in B_0$ by $\{\xi_n\} \in B_{\alpha,\gamma}$ with $\alpha\gamma < 1/2$. So in this case the statement of Theorems 7–9 holds for a sequence of independent perturbations, having Poisson-type distributions with sufficiently small parameters.

f. Question: How far can be strengthenen Theorems 3-9 in the whole class of sequences Λ which generate exponential Riesz bases in $L^2(-\pi, \pi)$?

3. Preliminary results.

In subsections **a** and **b** we formulate preliminary results on exponential systems in $L^2(-\pi, \pi)$ and related theorems on *e.f.e.t.*

a. Recall that the exponential system $\mathcal{E}(\Lambda)$ is a Riesz basis in $L^2(-\pi, \pi)$ if each function $f \in L^2(-\pi, \pi)$ admits a unique representation

$$f(t) = \sum c_k \exp(i\lambda_k t), \tag{3.1}$$

convergent in $L^2(-\pi, \pi)$-norm and satisfying

$$\sum |c_k|^2 \asymp \|f\|^2_{L^2(-\pi,\pi)}. \tag{3.2}$$

Here and in what follows the sign $\asymp$ means that the ratio of the lefthand and righthand sides lies between two positive constants.

In this case we say that the sequence Λ possesses *the Riesz basis property* (or, briefly, Λ is a *Riesz basis sequence*).

In order to get a criterion for a sequence $\Lambda \subset \mathbb{R}$ to possess the Riesz basis property we need the following definition.

DEFINITION ([20]). — *A non-negative function $w(x)$, $x \in \mathbb{R}$ satisfies the Helson-Szegö condition (briefly $w \in$ (HS)) if there are real-valued functions $u, v \in L^\infty(\mathbb{R})$ such that*

$$\|v\|_{L^\infty(\mathbb{R})} < \frac{\pi}{2} \tag{3.3}$$

and

$$w(x) = \exp(u(x) + \mathcal{H}v(x)),\ x \in \mathbb{R}. \tag{3.4}$$

Here $\mathcal{H}v$ denotes the Hilbert transform of $v \in L^\infty(\mathbb{R})$:

$$\mathcal{H}v(x) = \frac{1}{\pi}\,\text{V.P.} \int_{-\infty}^{\infty} v(t) \left\{ \frac{1}{x-t} + \frac{t}{t^2+1} \right\} dt.$$

THEOREM A ([5]). — *For a sequence $\Lambda = \{\lambda_k\} \subset \mathbb{R}$ to be a Riesz basis sequence it is necessary and sufficient that*

1.

$$d = \inf_{k \neq m} |\lambda_k - \lambda_m| > 0; \tag{3.5}$$

2. *the limit*

$$S(\lambda) = \lim_{R\to\infty} \prod_{|\lambda_n|<R} \left(1 - \frac{\lambda}{\lambda_n}\right) \tag{3.6}$$

exists uniformly on each compact set in $\mathbb{C}$, represents an e.f.e.t.π. and

3.

$$|S(x+i)|^2 \in (\mathrm{HS}). \tag{3.7}$$

One can describe the Riesz basis property of the sequence Λ in terms of the space PW_π. Following [21] we say that a sequence $M = \{\mu_k\} \subset \mathbb{R}$ is *interpolating* in PW_π, if for each sequence $\{\alpha_k\} \in l^2$ there exists a function $F \in PW_\pi$ solving the interpolation problem

$$F(\mu_k) = \alpha_k, \ k = 0, \pm 1, \pm 2, \dots. \tag{3.8}$$

The sequence M is *sampling* in PW_π if

$$\|\{F(\mu_k)\}\|_{l^2} \asymp \|F\|_{PW_\pi}, \ F \in PW_\pi. \tag{3.9}$$

Simple duality reasoning (see e.g. [6]) yields the following statement:

PROPOSITION 3.1. — *For the system $\mathcal{E}(\Lambda) = \{\exp(i\lambda_k t)\}$ to be a Riesz basis in $L^2(-\pi,\pi)$ it is be necessary and sufficient that the sequence Λ is both sampling and interpolating in PW_π. In this case the interpolation problem (3.8) has a unique solution. This solution admits the representation*

$$F(\lambda) = \sum \alpha_k \frac{S(\lambda)}{S'(\lambda_k)(\lambda - \lambda_k)}, \tag{3.10}$$

the series converges with respect to the PW_π-norm.

b. We need some additional definitions.

If $F \subset \mathbb{R}$ and $\alpha > 0$ are given, we denote $F_\alpha = \{x \in \mathbb{R} : \mathrm{dist}(x,F) < \alpha\}$. For $F, G \subset \mathbb{R}$ we set

$$\rho(F,G) = \inf\{\alpha : F \subset G_\alpha \text{ and } G \subset F_\alpha\}.$$

Following [21] we say that a sequence of sets $\{F_n\}$ approaches a set $F \in \mathbb{R}$ weakly (denoted $F_n \rightharpoonup F$), if for any compact set $K \subset \mathbb{R}$ $\rho(F_n \cap K, F \cap K) \to 0$, as $n \to \infty$. One can assure (see, say, [21]) that the family of sets $\{F+t\}_{t\in\mathbb{R}}$ is relatively compact in the topology of weak convergence. Denote its weak closure by $W(F)$.

Now let $M = \{\mu_k\} \in \mathbb{R}$ be a separated sequence.

THEOREM B. — *If M is a Riesz basis sequence, then each sequence from $W(M)$ also possesses the Riesz basis property.*

This theorem involves ideas from seminal articles [21], [22] and may be proved in the same pattern. In these articles the more complicated case of the non-Hilbertian norm $\|\cdot\|_{L^\infty}$ was considered, see also [23] for modification of this reasoning for the Hilbertian case (different from PW_π). In the present setting one can also give a direct proof of this fact.

COROLLARY. — *Assume the sequence $\Lambda = \{\lambda_n\}$ to be increasing. Then*

$$\Delta := \sup\{\lambda_{n+1} - \lambda_n\} < \infty. \tag{3.11}$$

Proof. — Let the contrary be true, i.e. there exists a sequence $\{k_l\}$ such that $\lambda_{k_l+1} - \lambda_{k_l} \to \infty$ as $l \to \infty$. Then $\left(\Lambda - \frac{1}{2}(\lambda_{k_l+1} + \lambda_{k_l})\right) \rightharpoonup \varnothing$ which contradicts Theorem B.

c. Properties of the class (HS).

THEOREM C (see [20], [24]). — *Let a measurable function $w(x) \geqslant 0$, $x \in \mathbb{R}$ be given. The following statements are equivalent:*

a. *$w \in$ (HS);*

b. *w satisfies the A_2-Muckenhoupt condition (briefly $w \in (A_2)$):*

$$\sup_I \left\{ \frac{1}{|I|} \int_I w dx \, \frac{1}{|I|} \int_I \frac{1}{w} dx \right\} < \infty, \tag{3.12}$$

where supremum is taken over all intervals.

c. *For any bounded finite function f we have*

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^2 w(x) dx \leqslant \text{Const} \int_{-\infty}^{\infty} |f(x)|^2 w(x) dx, \tag{3.13}$$

where Const *does not depend on* f. *Here* $\tilde{f}$ *denotes the usual Hilbert transform:*

$$\tilde{f}(x) = \frac{1}{\pi} V.P. \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt.$$

We also need the following propositions.

PROPOSITION 3.2. — *Let* $w \in$ *(HS). Then, for some* β_0 *(depending on* w*) we have* $(1+|x|)^{\pm\beta} w(x) \in$ *(HS) for all* $\beta \in (0, \beta_0)$.

Proof. — Consider representation (3.4) of the function w and take some $\beta_0 \in (0, \pi/2 - \|v\|_\infty)$. Denote $\sigma(t) = \operatorname{sign}(t)$. Clearly, $w(x)\exp(\pm\beta\tilde{\sigma}(x)) \in (HS)$ for any β, $0 \leqslant \beta \leqslant \beta_0$. On the other hand direct calculation shows that $\exp(\pm\beta\tilde{\sigma}(x)) \asymp (1+|x|)^{\pm\beta}$, $x \in \mathbb{R}$.

PROPOSITION 3.3. — *If* $w \in$*(HS), then* $w^{-1} \in$*(HS).*

Obvious.

PROPOSITION 3.4. — *Let* $w \in$*(HS). Then*

$$\int_{-\infty}^{\infty} \frac{w(x)}{1+x^2} dx < \infty.$$

This statement follows from (3.13) if one takes $f(t) = 1 + \operatorname{sign}(1-t^2)$, say, and mention that $|\tilde{f}(x)| \asymp (1+|x|)^{-1}$, $x \to \pm\infty$.

d. To prove the completeness theorem below we need a bound for e.f.e.t. satisfying (3.7).

LEMMA 3.1. — *Let* $S(\lambda)$ *be an e.f.e.t.* π *satisfying* (3.7). *Then there exists a number* $\beta > 0$ *such that*

$$|S(i\eta)| \geqslant \text{Const}\, |\eta|^{-\frac{1}{2}+\beta} \exp(\pi|\eta|), \quad |\eta| > 1. \tag{3.14}$$

Proof. — Suppose, for simplicity, $\Im\lambda > 0$ and take $\beta > 0$ so that $b^2 \in$ (HS), where $b(x) = |x+i|^{-\beta}|S(x)|$. The function $(\lambda+i)^{-\beta} S(\lambda)\exp(i\pi\lambda)$ is of bounded characteristic in the upper half-plane $\mathbb{C}^+$. Therefore one can factorize it as

$$(\lambda+i)^{-\beta} S(\lambda) = \text{Const}\, \exp(Pb(\lambda))\exp(-i\pi\lambda), \ \lambda \in \mathbb{C}^+,$$

where, as usual, (see e.g. [25])

$$Pb(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{x-\lambda} - \frac{x}{x^2+1} \right\} \log b(x) dx.$$

(The singular factor does not appear in this factorization since the function $(\lambda+i)^{-\beta}S(\lambda)$ admits an analytic continuation through the real line, (see [26], Theorem 6.2)). Hence

$$\frac{1}{S(\lambda)} = \text{Const}\,(\lambda+i)^{-\beta}\exp(-Pb(\lambda))\exp(i\pi\lambda),\ \lambda\in\mathbb{C}^+$$

and in order to obtain the estimate

$$\frac{1}{|S(i\eta)|} \leqslant \text{Const}\,|\eta|^{\frac{1}{2}-\beta}\exp(-\pi|\eta|),\ |\eta|>1,$$

which is equivalent to the desired estimate (3.14), it now suffices to prove

$$|\exp(-Pb(i\eta))| \leqslant |\eta|^{1/2},\ |\eta|>1. \tag{3.15}$$

The standard estimates of the Poisson integral (see e.g. [27]) show that the function $H(\lambda)=\exp(-Pb(\lambda))$ possesses zero exponential type in $\mathbb{C}^+$. Since b^{-2} $\in$(HS), we can apply Proposition 3.4 to claim that $|(x+i)^{-1}H(x)| \asymp ((1+|x|)b(x))^{-1}) \in L^2(\mathbb{R})$. Applying a Phragmén-Lindelöf type theorem for L^2-norm (see e.g. [27]) we obtain that $(\lambda+i)^{-1}H(\lambda)\in H^2(\mathbb{C}^+)$ and, in particular

$$H(i\eta)=\frac{\eta+1}{2\pi}\int_{\mathbb{R}}\frac{H(\zeta)}{\zeta-i\eta}\frac{d\zeta}{\zeta+i}.$$

Now the Cauchy-Bunyakovskii inequality yields (3.15).

e. In this section we present the results from probability theory that we need for our constructions. We refer to [28] for complete proofs and comments.

By $\mathbb{P}$ we denote a probability measure on a given probability space. If ξ is a random variable, $\mathbb{E}(\xi)$ denotes the expectation of ξ and $D(\xi)$ its variance.

THEOREM D (Chebyshev inequality). — *Let ξ be a real-valued random variable and $f(x)$ be a non-decreasing positive continuous function. Then, for each $a>0$,*

$$\mathbb{P}(|\xi|>a)\leqslant\frac{\mathbb{E}(f(|\xi|))}{f(a)}. \tag{3.16}$$

THEOREM E (Borel-Cantelli lemma). — *Let $A_1,\dots,A_n,\dots$ be a sequence of events from a probability space and $B=\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}A_n$.*

1. *If* $\sum \mathbb{P}(A_n) < \infty$, *then* $\mathbb{P}(B) = 0$;

2. *If the events* A_n *are independent and* $\sum \mathbb{P}(A_n) = \infty$, *then* $\mathbb{P}(B) = 1$.

THEOREM F (On two series). — *Let* ξ_n *be a sequence of real-valued independent random variables. In order for the series* $\sum \xi_n$ *to converge for almost all* $\omega \in \Omega$ *it suffices that both numerical series* $\sum \mathbb{E}(\xi_n)$ *and* $\sum D(\xi_n)$ *converge.*

THEOREM G (Kolmogorov law of large numbers). — *Let* ξ_n *be independent random variables with finite second moments and* $\{b_n\}$ *be a sequence of positive numbers,* $b_n \nearrow \infty$ *such that* $\sum D(\xi_n) b_n^{-2} < \infty$. *Then*

$$\frac{S_n - \mathbb{E}(S_n)}{b_n} \to 0 \quad \textit{for almost all} \quad \omega \in \Omega,$$

here $S_n = \sum\limits_{k=1}^{n} \xi_k$.

THEOREM H ([29], [30] and also [31]). — *Let* $\xi_1, \dots \xi_n$ *be a sequence of independent random variables such that* $\mathbb{E}(\xi_k) = 0$, $k = 1, \dots n$. *For* $p \geqslant 2$ *denote*

$$M_{p,n} = \sum_{k=1}^{n} \mathbb{E}(|\xi_k|^p).$$

Then

$$\mathbb{E}(|\xi_1 + \xi_2 + \dots + \xi_n|^p) \leqslant c(M_{p,n} + (M_{2,n})^{p/2}),$$

here the constant $c(= c(p))$ *is independent of* n.

f. In the remained items of this section we prove Theorem 1, which being a (simple) negative statement is considered as a preliminary result.

Thus until the end of this section we suppose that the variables ξ_n have continuous distribution densities $p_n(t)$ satisfying (2.1).

Then up to some translation and ϵ-shivering, we can imitate any finite pattern by points from Λ_ω:

LEMMA 3.2. — *Let* $\epsilon > 0$ *and a finite number of intervals* $\{I_j\}_{j=1}^{N}$, $I_j = (a_j, a_j + \epsilon)$ *of length* ϵ *be given (they need not be disjoint). Let* $T > 0$. *Then for almost all* $\omega \in \Omega$, *there exists a number* $t > T$ *and pairwise distinct indices* $n_1, n_2, \dots, n_N$ *such that* $\lambda_{n_j}(\omega) \in I_j + t, j = 1, 2, \dots N$.

Proof. — Let Δ be the quantity from relation (3.11). Take $M > 0$ so that the set $\bigcup_1^N I_j$ is contained in an interval of length M. Now take $L > 2M + 3N\Delta$ and consider the translations $\{I_j + kL\}_{j=1}^N$ $k = 0, 1, \dots$ of the original set of intervals. For each k choose the pairwise distinct indices $n_1^{(k)}, n_2^{(k)}, \dots, n_N^{(k)}$ to minimize the distance $\rho\left(\bigcup_1^N (I_j + kL), \{\lambda_{n_j^{(k)}}\}_{j=1}^N\right)$. Denote by A_k the events $A_k = \{\omega : \lambda_{n_j^{(k)}}(\omega) \in I_j + kL, j = 1, 2, \dots, N\}$. Since the sets $\{I_j + kL\}_{j=1}^N$ are fairly distant for various values of k the sets of indices $\{n_j^{(k)}\}_{j=1}^N$ are disjoint and, hence, the events A_k are independent. Besides relation (3.11) provides $\rho\left(\bigcup_{j=1}^N (I_j + kL), \{\lambda_{n_j^{(k)}}\}_{j=1}^N\right) < \text{Const}$ uniformly with respect to k. The latter with of (2.1) yields $\inf_k \mathbb{P}(A_k) > 0$. Now it remains to apply the Borel-Cantelli lemma: infinitely many events A_k hold.

We are now in a position to verify statement a. of the theorem: it suffices to take $I_j = (a, a + \epsilon)$ for all $j = 1, 2, \dots, N$. Then, for almost all $\omega \in \Omega$, there exist $t > 0$ and pairwise distinct indices $n_1, n_2, \dots, n_N$ such that $\lambda_{n_j} \in (a + t, a + t + \epsilon), j = 1, 2, \dots, N$. This means we will observe N-bunches of closed exponents from the set Λ_ω. In particular $\mathcal{E}_\omega(\Lambda)$ does not form a Riesz basis in $L^2(-\pi, \pi)$.

g. In order to prove statement *b.*[1] we mention that a set of exponentials $\{\exp(i\mu_n t)\}$ forms a Riesz basis in the closure of its own linear span in $L^2(-\pi, \pi)$ iff there exist constants $C, c > 0$ such that

$$c\|\{c_n\}\|_{l^2} \leqslant \|\sum c_n \exp(i\mu_n t)\|_{L^2(-\pi,\pi)} \leqslant C\|\{c_n\}\|_{l^2}$$

for any finite sequence $\{c_n\}$. For each (either finite or infinite) sequence $\mu = \{\mu_k\} \in \mathbb{R}$ we denote

$$C(\mu) = \sup_{\{c_k\}} \left\{ \frac{\|\{c_k\}\|_{l^2}}{\|\sum c_k \exp(i\mu_k t)\|_{L^2(-\pi,\pi)}} \right\}, \tag{3.17}$$

where *sup* is taken over all finite sequences $\{c_n\}$. Thus we need to extract a separated subsequence $\{\lambda_{n_k}(\omega)\}$ so that

$$C(\{\lambda_{n_k}(\omega)\}) = \infty. \tag{3.18}$$

[1] The proof below was suggested by A.M. Olevskii and is published here according to his kind permission. The authors' original proof was more complicated.

The latter is straightforward if we combine Lemma 3.2 with the following propositions.

PROPOSITION 3.5. — *For any* $C > 0$ *there exists a finite sequence* $\mu = \{\mu_k\}_1^N, N = N(C)$ *such that* $\inf_{k \neq l} |\mu_k - \mu_l| > 1/4$ *and* $C(\mu) > C$.

PROPOSITION 3.6. — *For any* $N \in \mathbb{Z}^+$ *the mapping* $\mu = \{\mu_k\}_1^N \mapsto C(\mu)$ *is a continuous mapping from* $\mathbb{R}^N$ *into* $\mathbb{R}$.

PROPOSITION 3.7. — *For any* $t \in \mathbb{R}, \mu = \{\mu_k\}_1^N$ *denote* $\mu + t = \{\mu_k + t\}_1^N$. *Then* $C(\mu) = C(\mu + t)$.

After these propositions have been proved it remains to apply a direct diagonalization procedure to construct, for almost all $\omega \in \Omega$, a separated subsequence $\{\lambda_{n_k}(\omega)\} \subset \Lambda_\omega$ satisfying (3.18).

h. Propositions 3.6 and 3.7 are straightforward. In order to verify Proposition 3.5 it suffices to construct a collection of infinite sequences $\mu^{(\alpha)} = \{\mu_k^{(\alpha)}\}_1^\infty$ which still possess the Riesz basis property and also $C(\mu^{(\alpha)}) \nearrow \infty$ as $\alpha \nearrow 1/4$ (in the next lines we shall see that $1/4$ is the most convenient choice of the critical value) and then take finite truncations of the sequences $\mu^{(\alpha)}$.

Below is an example of such a collection.

For $\alpha \in (0, 1)$ denote

$$\mu_k^{(\alpha)} = k - \alpha \text{ sign } k, k \in \mathbb{Z}; \quad S_\alpha(\lambda) = \lambda \lim_{R \to \infty} \prod \left(1 - \frac{\lambda}{\mu_k^{(\alpha)}}\right).$$

A direct estimate based on the Stirling formula gives

$$|S_\alpha(x + i)| \asymp (1 + |x|)^{2\alpha}, \ x \in \mathbb{R} \tag{3.19}$$

uniformly with respect to α.

Therefore the exponential system $\{\exp(i\mu_k^{(\alpha)}t)\}$ is complete and minimal in $L^2(-\pi, \pi)$ for $\alpha \in (0, 1/4)$ (actually if one applies Theorem A one can readily verify it forms a Riesz basis) and one may obtain its biorthogonal system $h_{k,\alpha}(t)$ from the relations

$$\int_{-\pi}^{\pi} \exp(i\lambda t) h_{k,\alpha}(t) dt = \frac{S_\alpha(\lambda)}{S'_\alpha(\mu_k^{(\alpha)})(\lambda - \mu_k^{(\alpha)})} =: F_{k,\alpha}(\lambda).$$

Relation (3.19) immediately yields (it suffices to estimate the L^2-norm of the function $F_{k,\alpha}(\lambda)$ on a horizontal line) that $\|h_{k,\alpha}(t)\|_{L^2(-\pi,\pi)} \to \infty$, as $\alpha \to 1/4$, which is equivalent to (3.18).

4. Completeness of exponential systems with random exponents.

In this section we prove Theorems 2–5.

a. The proof of Theorem 2 consists of several steps:

First, let us switch to truncated random variables. For a fixed $\epsilon \in (0,1)$ (value to be specified later) consider the random variables

$$\xi_n^* = \begin{cases} \xi_n, & \text{if } |\xi_n| \leqslant \epsilon|\lambda_n|; \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

LEMMA 4.1. — *For almost all $\omega \in \Omega$ the following relation holds:*

$$\#\{n : \xi_n(\omega) \neq \xi_n^*(\omega)\} < \infty.$$

Proof. — We use condition (2.3). The Chebyshev inequality yields:

$$\mathbb{P}(\xi_n \neq \xi_n^*) = \mathbb{P}(|\xi_n| > \epsilon|\lambda_n|) \leqslant \frac{\mathbb{E}(|\xi_n|^{1+\delta})}{\epsilon^{1+\delta}|\lambda_n|^{1+\delta}} \leqslant \frac{\text{Const}}{|\lambda_n|^{1+\delta}}.$$

Relation (3.5) gives $\sum|\lambda_n|^{-(1+\delta)} < \infty$. So $\sum\limits_n \mathbb{P}(\xi_n \neq \xi_n^*) < \infty$, and by the Borel-Cantelli lemma we have

$$\mathbb{P}\Big(\bigcap_{k=0}^{\infty} \bigcup_{|n|>k} \{\xi_n \neq \xi_n^*\}\Big) = 0,$$

which is equivalent to the statement of the lemma.

Denote $\lambda_n^*(\omega) = \lambda_n + \xi_n^*(\omega)$. Since we assumed $0 \notin \Lambda$, we have $\lambda_n^*(\omega) \neq 0, n \in \mathbb{Z}$, $\omega \in \Omega$.

COROLLARY 1. — *The product $S_\omega(\lambda)$ defined in (2.4) and the product*

$$S_\omega^*(\lambda) = \lim_{R\to\infty} \prod_{|\lambda_n|<R} \left(1 - \frac{\lambda}{\lambda_n^*(\omega)}\right) \tag{4.2}$$

are simultaneously convergent (or divergent) for almost all $\omega \in \Omega$ and, if convergent, represent entire functions of the same exponential type.

When proving Theorem 3 we will consider the cases $\delta = 0$ and $\delta > 0$ simultaneously. In order to compactify the formulae we introduce the switch s_δ: in what follows we assume $s_\delta = 0$ for $\delta > 0$ and $s_\delta = 1$ for $\delta = 0$.

COROLLARY 2. — *For almost all $\omega \in \Omega$ estimates (2.7) and (2.9) are equivalent to*

(4.3)
$$c\exp(-s_\delta(\log(|1+|\eta|))^{3/4}) < \frac{|S^*_\omega(i\eta)|}{|S(i\eta)|} < C\exp(s_\delta(\log(|1+|\eta|))^{3/4}), \ \eta \in \mathbb{R}.$$

b. Convergence of random series. I.

LEMMA 4.2. — *The following statement is valid:*

$$\sum_n \frac{|\xi^*_n(\omega)|}{|\lambda_n||\lambda^*_n(\omega)|} < \infty \ \textit{for almost all}\ \omega \in \Omega. \tag{4.4}$$

Proof. — This follows immediately from the convergence of the series

$$\sum_n \frac{\mathbb{E}(|\xi^*_n|)}{(|\lambda_n|+1|)^2} \quad \text{and} \quad \sum_n \frac{\mathbb{E}((\xi^*_n)^2)}{(|\lambda_n|+1|)^4} < \sum_n \frac{\epsilon|\lambda_n|\mathbb{E}(|\xi^*_n|)}{(|\lambda_n|+1|)^4}$$

and Theorem F on two series.

c. Convergence and an estimate of the canonical product. Denote $n^*_\omega(r) = \#\{n: \ |\lambda^*_n(\omega)| < r\}$. Since $|\xi^*_n| \leqslant \epsilon|\lambda_n|$, we have

$$\limsup_{r\to\infty} \frac{n^*_\omega(r)}{r} \leqslant 1+\epsilon.$$

Therefore (see for example [32]) the canonical product

$$F^*_\omega(\lambda) = \prod \left(1 - \frac{\lambda}{\lambda^*_n(\omega)}\right) \exp \frac{\lambda}{\lambda^*_n(\omega)}$$

converges uniformly on each compact set in $\mathbb{C}$ and the function $M_{F^*_\omega}(r) := \max\{|F^*_\omega(\lambda)|; |\lambda| < r\}$ admits the estimate:

$$\log M_{F^*_\omega}(r) \leqslant r \sum_{|\lambda^*_n|\leqslant r} \frac{1}{\lambda^*_n} + O(r). \tag{4.5}$$

Since $|S|^2 \in$(HS), Proposition 3.4 yields $S(\lambda)(\lambda-\lambda_0)^{-1} \in L^2(\mathbb{R})$ and, hence, (see e.g. [32])

$$\text{the limit} \quad A_0 = \lim_{r\to\infty} A(r), \ \text{where}\ A(r) = \sum_{|\lambda_n|\leqslant r} \frac{1}{\lambda_n} \quad \text{exists.} \tag{4.6}$$

This allows one to improve the convergence in the first term in the right-hand side of (4.5). We have

$$\sum_{|\lambda_n^*|\leqslant r} \frac{1}{\lambda_n^*} = \sum_{|\lambda_n|\leqslant \frac{r}{1+\epsilon}} \frac{1}{\lambda_n^*} + \sum_{|\lambda_n^*|\leqslant r; |\lambda_n|> \frac{r}{1+\epsilon}} \frac{1}{\lambda_n^*} = B_1(r) + B_2(r),$$

where

$$\begin{aligned} |B_2(r)| &\leqslant \sum_{\frac{r}{1+\epsilon}<|\lambda_n|\leqslant r(1+\epsilon)} \frac{1}{|\lambda_n^*|} \leqslant \frac{1}{1-\epsilon} \sum_{\frac{r}{1+\epsilon}<|\lambda_n|\leqslant r(1+\epsilon)} \frac{1}{|\lambda_n|} \\ &\leqslant \frac{2}{d(1-\epsilon)} \sum_{\frac{r}{1+\epsilon}<kd\leqslant r(1+\epsilon)} \frac{1}{k} \leqslant \text{Const}\,. \end{aligned}$$

Here d is the constant which participates in the separateness condition (3.5).

Further

$$B_1(r) = A\left(\frac{r}{1+\epsilon}\right) + \sum_{|\lambda_n|\leqslant \frac{r}{1+\epsilon}} \left(\frac{1}{\lambda_n^*} - \frac{1}{\lambda_n}\right) = A\left(\frac{r}{1+\epsilon}\right) - \sum_{|\lambda_n|\leqslant \frac{r}{1+\epsilon}} \frac{\xi_n^*(\omega)}{\lambda_n \lambda_n^*(\omega)}.$$

Now relations (4.6), (4.4) imply

$$\sup_r |B_1(r)| < \infty,$$

and by inequality (4.5) we obtain $\log M_{F_\omega^*}(r) < C_\omega r$ for almost all $\omega \in \Omega$. Thus $F_\omega^*(\lambda)$ is an entire function of exponential type, the latter depends on $\omega \in \Omega$, generally speaking.

d. Convergence of random series. II.

To estimate the generating function for the sequence $\{\lambda_n^*(\omega)\}$ we need the following lemma.

LEMMA 4.3. — *The following statement holds:*

$$\sum \left| \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)} \right|^2 < \infty \ \textit{ for almost all } \omega \in \Omega.$$

Proof. — One can obtain the conclusion of the lemma by the theorem on two series if using (2.3) and noting that

$$\sum_n \frac{\mathbb{E}(\xi_n^*(\omega)^2))}{\lambda_n^2+1} \leqslant \sum_n \frac{\mathbb{E}(|\xi_n(\omega)|^{1+\delta})(\epsilon|\lambda_n|)^{1-\delta}}{\lambda_n^2+1} < \infty$$

and

$$\sum_n \frac{\mathbb{E}(\xi_n^*(\omega)^4)}{(\lambda_n^2+1)^2} \leqslant \epsilon^2 \sum_n \frac{\lambda_n^2 \mathbb{E}(\xi_n^*(\omega)^2)}{(\lambda_n^2+1)^2} < \infty.$$

e. Properties of the generating function.

LEMMA 4.4. — *For almost all $\omega \in \Omega$ the product $S_\omega^*(\lambda)$ defined by relation (4.2) converges uniformly on each compact set in $\mathbb{C}$ to an entire function of exponential type $\pi + C\epsilon$, where the constant $C > 0$ is independent of ϵ. In addition the following relation holds for each fixed $\theta \neq 0, \pi$:*

$$\limsup_{r\to\infty} \frac{1}{r} \left| \log \left| \frac{S_\omega^*(r\exp(i\theta))}{S(r\exp(i\theta))} \right| \right| \leqslant C(\theta)\epsilon \tag{4.7}$$

for sufficiently small $\epsilon > 0$ with some $C(\theta) \in (0, \infty)$ independent of ϵ.

Proof. — We have

$$\begin{aligned} F_\omega^*(\lambda) &= \lim_{R\to\infty} \prod_{|\lambda_n|<R} \left(1 - \frac{\lambda}{\lambda_n^*(\omega)}\right) \exp\left(\frac{\lambda}{\lambda_n^*(\omega)}\right) \\ &= \lim_{R\to\infty} \exp\{\lambda A(R)\} \exp\left\{-\lambda \sum_{|\lambda_n|<R} \frac{\xi_n^*(\omega)}{\lambda_n \lambda_n^*(\omega)}\right\} \prod_{|\lambda_n|<R} \left(1 - \frac{\lambda}{\lambda_n^*(\omega)}\right). \end{aligned}$$

It follows from Lemma 4.2 and relation (4.6) that, for almost all $\omega \in \Omega$, the limit

$$S_\omega^*(\lambda) = \lim_{R\to\infty} \prod_{|\lambda_n|<R} \left(1 - \frac{\lambda}{\lambda_n^*(\omega)}\right)$$

exists and, along with $F_\omega^*(\lambda)$, is an entire function of exponential type.

To evaluate this type we compare the function $S_\omega^*(\lambda)$ with the unperturbed function $S(\lambda)$ in the angle $D_\gamma = \{\lambda = x + iy : |x| < \gamma|y|\}$, where $\gamma > 0$ is a fixed number. We have

$$\frac{S_\omega^*(\lambda)}{S(\lambda)} = \lim_{R\to\infty} \prod_{|\lambda_n|<R} G_{n,\omega}^*(\lambda), \tag{4.8}$$

where

$$G_{n,\omega}^*(\lambda) = \left(1 - \frac{\lambda}{\lambda_n^*(\omega)}\right)\left(1 - \frac{\lambda}{\lambda_n}\right)^{-1} = 1 + \frac{\lambda}{\lambda_n - \lambda} \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)}.$$

Since $\lambda \in D_\gamma$, we have

$$\left| \frac{\lambda}{\lambda_n - \lambda} \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)} \right| \leqslant \epsilon \text{ Const } < \frac{1}{2}, \quad n = 0, \pm 1, \pm 2, \dots$$

for sufficiently small ϵ. Therefore

$$\left| \log |G_{n,\omega}^*(\lambda)| - \Re \frac{\lambda}{\lambda_n - \lambda} \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)} \right| \leqslant \text{Const} \left| \frac{\lambda}{\lambda_n - \lambda} \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)} \right|^2 .$$

Since $\left|\dfrac{\lambda}{\lambda_n - \lambda}\right| < \text{Const}$ for $\lambda \in D_\gamma$, Lemma 4.3 yields

$$\left| \sum_n \log |G_{n,\omega}^*(\lambda)| - \Re \sum_n \frac{\lambda}{\lambda_n - \lambda} \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)} \right| \leqslant C_\omega, \quad \lambda \in D_\gamma, \tag{4.9}$$

and (since $\lambda = x + iy$)

$$\left| \Re \sum_n \frac{\lambda}{\lambda_n - \lambda} \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)} \right| \leqslant \sum_n \left| \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)} \frac{(\lambda_n - x)x - y^2}{(\lambda_n - x)^2 + y^2} \right|$$

$$\leqslant C_1 \left[\sum_{|\lambda_n| < 2\gamma|y|} \left| \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)} \right| + \sum_{|\lambda_n| \geqslant 2\gamma|y|} \left| \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)} \right| \frac{|y|}{|\lambda_n|} \right]$$

$$\leqslant C_2 \left[\epsilon|y| + |y| \sum_{|\lambda_n| > 2\gamma|y|} \left| \frac{\xi_n^*(\omega)}{\lambda_n \lambda_n^*(\omega)} \right| \right] . \tag{4.10}$$

Since series (4.4) converges, the last summand in the rightmost inequality does not exceed $\epsilon|y|$ for all sufficiently large $|y|$ (depending on $\omega \in \Omega$).

Therefore, combining relations (4.9), (4.10), we obtain, for almost all $\omega \in \Omega$,

$$\sum \log |G_{n,\omega}^*(\lambda)| \leqslant C_3 \epsilon |y|, \quad \lambda = x + iy \in D_\gamma, \quad |y| > y(\epsilon, \omega), \tag{4.11}$$

where the constant C_3 is independent of ϵ.

Turning our attention to (4.8) and using the estimate $|S(\lambda)(1+|\lambda|)^{-1}| < \text{Const}\ \exp(\pi|y|)$, which follows, say, from the inclusion $S(x)(1+|x|)^{-1} \in L^2(\mathbb{R})$, we get

$$|S_\omega^*(\lambda)| \leqslant \text{Const}\ \exp\{(\pi + C_4\epsilon)|y|\}, \quad \lambda \in D_\gamma, |y| > y(\epsilon, \omega).$$

The quantity $C_4(= C_4(\gamma))$ as well as $C_3(= C_3(\gamma))$ above is *independent of* ϵ. Now the Phragmén-Lindelöf theorem makes this estimate uniform with respect to γ:

$$|S_\omega^*(\lambda)| \leqslant \text{Const}\ \exp\{(\pi + C_5\epsilon)|y|\}$$

in the whole complex plane, i.e. $S_\omega^*(\lambda)$ is an entire function of exponential type no greater than $\pi + C_5\epsilon$. Now (4.7) follows from (4.9) and (4.10). This completes the proof of Lemma 4.4.

Since, for any $\epsilon > 0$, the functions $S(\lambda)$ and $S_\omega(\lambda)$ differ at most by a finite number of factors, Theorem 2 follows at once from Lemma 4.4.

Remark. — Close reasoning shows that condition (2.3) can be weakened. In particular, if all independent random variables ξ_n have the same distribution it suffices to demand $\mathbb{E}(|\xi_0|) < \infty$.

f. To prove **Theorem 3** which delivers the exact estimates (2.7), (2.9) on the imaginary axis we need exact bounds for the sum of a random series.

Taking into account relations (4.8) and (4.9) we see that Theorem 3 follows from

LEMMA 4.5. — *Under the hypothesis of Theorem 3 the inequality*

$$\left|\Re \sum_n \frac{iy}{\lambda_n - iy} \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)}\right| = \left|\sum_n \frac{y^2}{\lambda_n^2 + y^2} \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)}\right| \leqslant C_\omega + s_\delta(\log(1+|y|))^{3/4},$$

$y \in \mathbb{R}$, holds for almost all $\omega \in \Omega$ with some $C_\omega \in (0,\infty)$ (we recall that $s_\delta = 0$, if $\delta > 0$ and $s_\delta = 1$, if $\delta = 0$).

Proof. — Set

$$\sum_n \frac{y^2}{\lambda_n^2 + y^2} \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)} = \sum_{|\lambda_n| \leqslant |y|+1} + \sum_{|\lambda_n| > |y|+1} = \sigma_y + R_y \tag{4.12}$$

and estimate each summand separately.

We have

$$\sigma_y = \sum_{|\lambda_n|\leqslant|y|+1} \frac{\xi_n^*(\omega)}{\lambda_n^*(\omega)} - \sum_{|\lambda_n|\leqslant|y|+1} \frac{\lambda_n^2\xi_n^*}{(\lambda_n^2+y^2)(\lambda_n+\xi_n^*)} = \sigma_y^{(1)} - \sigma_y^{(2)}.$$

g. The quantity $\sigma_y^{(1)}$ is a partial sum of the series

$$\sum \frac{\xi_n^*}{\lambda_n+\xi_n^*} = \sum \left(\frac{\xi_n^*}{\lambda_n} - \frac{(\xi_n^*)^2}{\lambda_n(\lambda_n+\xi_n^*)} \right),$$

whose sum is independent of y. That the series $\sum(\xi_n^*)^2(\lambda_n(\lambda_n+\xi_n^*))^{-1}$ converges is in the main proved in Lemma 4.3.

According to the Borel-Cantelli lemma convergence of the series $\sum \xi_n^*(\lambda_n)^{-1}$ is equivalent to convergence of the series $\sum \xi_n(\lambda_n)^{-1}$, which in turn can be obtain by applying the theorem on two series: $\sum \mathbb{E}(\xi_n)(\lambda_n)^{-1}$ is convergent because $\mathbb{E}(\xi_n) = 0,\ n \in \mathbb{Z}$ and, obviously, $\sum \mathbb{E}(\xi_n^2)(\lambda_n^2)^{-1} < \infty$. Thus

$$\sup_y |\sigma_y^{(1)}| < \infty \text{ for almost all } \omega \in \Omega. \tag{4.13}$$

The quantity $\sigma_y^{(2)}$ admits the estimate

$$|\sigma_y^{(2)}| \leqslant \frac{1}{1-\epsilon}\frac{1}{|y|} \sum_{|\lambda_n|\leqslant|y|+1} |\xi_n^*|.$$

Apply the law of large numbers (Theorem G) to the random variables $\{|\xi_n^*|\}$. Together with the inequality $n_\omega^*(|y|+1) \geqslant \text{Const}\,|y|$ (the value $n_\omega^*(\cdot)$ was defined in the beginning of n°. **c**) this yields

$$\sup_y |\sigma_y^{(2)}| < \infty \quad \text{for almost all} \quad \omega \in \Omega, \tag{4.14}$$

and taking (4.13) into account we obtain

$$\sup_y |\sigma_y| < \infty \quad \text{for almost all} \quad \omega \in \Omega. \tag{4.15}$$

h. The remainder term R_y in (4.12) consists of two parts:
(4.16)

$$R_y = \left(\sum_{|y|+1<|\lambda_n|\leqslant(|y|+1)^2} + \sum_{|\lambda_n|>(|y|+1)^2} \right) \frac{y^2}{\lambda_n^2+y^2}\frac{\xi_n^*(\omega)}{\lambda_n+\xi_n^*(\omega)} = R_y^{(1)} + R_y^{(2)}.$$

We use the rough estimate

$$\left| \frac{y^2}{\lambda_n^2 + y^2} \frac{\xi_n^*(\omega)}{\lambda_n + \xi_n^*(\omega)} \right| \leqslant \frac{y^2}{\lambda_n^2}$$

for $|\lambda_n| > (|y|+1)^2$, whence

(4.17)
$$|R_y^{(2)}| \leqslant y^2 \sum_{|\lambda_n|>(|y|+1)^2} \lambda_n^{-2} \leqslant \mathrm{Const}\, y^2 \sum_{kd>(|y|+1)^2} \frac{1}{k^2 d^2} = O(1), \quad y \to \infty.$$

Here, as earlier, d is the constant from (3.5).

Standard (now) arguments show that instead of estimating $R_y^{(1)}$ one may consider

(4.18)
$$r_y^{(1)} = \left(\sum_{1+|y|<\lambda_n \leqslant (|y|+1)^2} + \sum_{1+|y|<-\lambda_n \leqslant (|y|+1)^2} \right) \frac{y^2}{\lambda_n^2 + y^2} \frac{\xi_n^*(\omega)}{\lambda_n} = r_y^+ + r_y^-.$$

We shall prove that

$$\sup_y |r_y^+| < \infty, \tag{4.19}$$

one can obtain the estimate of r_y^- in a similar way.

i. In order to obtain (4.19) we apply Theorem H. For definiteness, let $y > 0$. Consider the set of indices

$$N_y = \{n : 1 + y < \lambda_n \leqslant (1+y)^2\} = \{n_y, n_y + 1, \dots, n_y + k_y\}$$

and represent the value r_y^+ in the form

$$r_y^+ = (\zeta_y^{(0)} - \mathbb{E}(\zeta_y^{(0)})) + (\zeta_y^{(1)} - \mathbb{E}(\zeta_y^{(1)})) \dots$$
$$\dots + (\zeta_y^{(n_y-1)} - \mathbb{E}(\zeta_y^{(n_y-1)})) + \sum_{k=0}^{n_y-1} \mathbb{E}(\zeta_y^{(k)}),$$

where

$$\zeta_y^{(k)} = \sum_{l\in\mathbb{N};\, ln_y \leqslant (1+|y|)^2-k} \frac{y^2}{\lambda_{ln_y+k}^2 + y^2} \frac{\xi_{ln_y+k}^*}{\lambda_{ln_y+k}}.$$

Now apply Theorem H twice, first for the random values

$$X_l = \frac{y^2}{\lambda_{ln_y+k}^2 + y^2} \frac{\xi_{ln_y+k}^* - \mathbb{E}(\xi_{ln_y+k}^*)}{\lambda_{ln_y+k}}, \quad l = 1, 2, \dots, \tag{4.20}$$

for each $k = 0, 1, \dots, n_y - 1$, and then for random values $\zeta_y^{(k)} - \mathbb{E}(\zeta_y^{(k)})$, $k = 0, 1, \dots, n_y - 1$.

First we mention that, by (3.5),

$$\lambda_m \geqslant md \tag{4.21}$$

and also (since $\lambda_n n^{-1} \to 1$ as $n \to \infty$ (see e.g. [32])), we have

$$\frac{1}{2} n_y < y < 2n_y \tag{4.22}$$

for all sufficiently large y.

Therefore

$$|\mathbb{E}(\zeta_y^{(k)})| \leqslant \sum_{l=1}^{\infty} \frac{y^2}{d^2(ln_y + k)^2 + y^2} \frac{\mathbb{E}(|\xi^*_{ln_y+k}|)}{d(ln_y + k)}$$

$$\leqslant \text{Const} \sum_{l=1}^{\infty} \frac{1}{l^3 n_y} \mathbb{E}(|\xi^*_{ln_y+k}|) \leqslant \frac{\text{Const}}{n_y} \tag{4.23}$$

and, hence,

$$|\mathbb{E}(r_y^+)| \leqslant \sum_{k=0}^{n_y} |\mathbb{E}(\zeta_y^{(k)})| \leqslant \text{Const}\,. \tag{4.24}$$

j. In this item we consider the cases when (2.6) holds, and, when only (2.8) holds, simultaneously. Therefore below we set either $\nu = \delta$ or $\nu = 0$ depending on what case takes place. Applying Theorem H to sequence (4.20) we have

$$\begin{aligned}
&\mathbb{E}(|\zeta_y^{(k)} - \mathbb{E}(\zeta_y^{(k)})|^{2+\nu}) \\
&\quad \leqslant c_{2+\nu} \left[\sum_{l=1}^{\infty} \left(\frac{y^2}{\lambda^2_{ln_y+k} + y^2} \right)^{2+\nu} \frac{\mathbb{E}(|\xi^*_{ln_y+k} - \mathbb{E}(\xi^*_{ln_y+k})|^{2+\nu})}{|\lambda_{ln_y+k}|^{2+\nu}} \right. \\
&\qquad \left. + \left(\sum_{l=1}^{\infty} \left(\frac{y^2}{\lambda^2_{ln_y+k} + y^2} \right)^2 \frac{\mathbb{E}(|\xi^*_{ln_y+k} - \mathbb{E}(\xi^*_{ln_y+k})|^2)}{\lambda^2_{ln_y+k}} \right)^{1+\nu/2} \right].
\end{aligned}$$

Applying the inequalities

$$\mathbb{E}((\xi^*_{ln_y+k} - \mathbb{E}(\xi^*_{ln_y+k}))^2) \leqslant \mathbb{E}((\xi^*_{ln_y+k})^2) \leqslant \text{Const}\,;$$

$\mathbb{E}(|\xi^*_{ln_y+k}-\mathbb{E}(\xi^*_{ln_y+k})|^{2+\nu}) \leqslant 2^{2+\nu}(\mathbb{E}(|\xi^*_{ln_y+k}|^{2+\nu})+|\mathbb{E}(\xi^*_{ln_y+k})|^{2+\nu}) \leqslant \mathrm{Const}$ and also inequalities (4.21), (4.22) we obtain

$$\mathbb{E}(|\zeta_y^{(k)} - \mathbb{E}(\zeta_y^{(k)})|^{2+\nu}) \leqslant \mathrm{Const} \left[\sum_{l=1}^{\infty} \left(\frac{y^2}{ln_y^2+y^2}\right)^{2+\nu} \frac{1}{(ln_y)^{2+\nu}} + \left(\sum_{l=1}^{\infty} \left(\frac{y^2}{(ln_y^2+y^2)}\right)^2 \frac{1}{(ln_y)^2}\right)^{1+\nu/2}\right] \leqslant \frac{\mathrm{Const}}{y^{2+\nu}}.$$

Similar reasoning yields

$$\mathbb{E}(|\zeta_y^{(k)} - \mathbb{E}(\zeta_y^{(k)})|^2) \leqslant \frac{\mathrm{Const}}{y^2}.$$

Applying Theorem H to the random variable $r_y^+ - \mathbb{E}(r_y^+)$ we obtain

$$\mathbb{E}(|r_y^+ - \mathbb{E}(r_y^+)|^{2+\nu}) \leqslant \mathrm{Const} \left(\frac{n_y}{n_y^{2+\nu}} + \left(\frac{n_y}{n_y^2}\right)^{1+\nu/2}\right) \leqslant \frac{\mathrm{Const}}{n_y^{1+\nu/2}}.$$

k. Consider the events $A_m = \{|r_m^+ - \mathbb{E}(r_m^+)| > 1 + s_\delta(\log m)^{3/4}\}$, $m = 1, 2, \dots$ The Chebyshev inequality yields

$$P(A_m) \leqslant \frac{\mathbb{E}(|r_m^+ - \mathbb{E}(r_m^+)|^{2+\delta})}{(1+s_\delta(\log m)^{3/4})^{2+\delta}} \leqslant \frac{\mathrm{Const}}{m(1+\log m))^{3/2}}.$$

Since $\sum P(A_m) < \infty$ almost everywhere, only a finite number of events A_m hold and, taking into account (4.24), we obtain

$$\limsup_{m\to\infty} |r_m^+| < \infty$$

for almost all $\omega \in \Omega$.

So we have estimated the quantities r_y^+ for $y \in \mathbb{Z}_+$. To obtain the desired relation (4.19) for all y one should mention that, for $t \in (0,1)$,

$$|r_m^+ - r_{m+t}^+| = \left|\sum_{1+m<\lambda_n\leqslant(1+m)^2} \left(\frac{m^2}{\lambda_n^2+m^2} - \frac{(m+t)^2}{\lambda_n^2+(m+t)^2}\right)\frac{\xi_n^*(\omega)}{\lambda_n}\right| \leqslant \mathrm{Const} \sum \frac{|\xi_n^*(\omega)|}{\lambda_n^2}$$

and again apply Lemma 4.3. Boiling together (4.19), (4.17), (4.16), (4.15) we obtain the statement of Lemma 4.5 and thus Theorem 3.

1. Proof of the completeness theorem (Theorem 4). Suppose that the system $\mathcal{E}_\omega$ is not complete in $L^2(-\pi,\pi)$. Then there exists a function $f \in L^2(-\pi,\pi)$, $f \neq 0$ which annihilates it, i.e. the function $F \in PW_\pi$ defined by the relation

$$F(\lambda) = \int_{-\pi}^{\pi} \exp(i\lambda t) f(t)dt \tag{4.25}$$

vanishes on Λ_ω. The relation

$$\Phi(\lambda) = \frac{F(\lambda)}{S_\omega(\lambda)}$$

defines an entire function Φ also of exponential type.

Denote by $h_\Phi(\theta), h_F(\theta), h_S(\theta), h_{S_\omega}(\theta)$ and $I_\Phi, I_F, I_S, I_{S_\omega}$ the indicator functions and indicator diagrams of the entire functions Φ, F, S, S_ω respectively. (For definitions and properties of indicator functions, indicator diagrams as well as other related properties of entire functions of exponential type see e.g. [32]).

We have

$$h_F(\theta) = \limsup_{r\to\infty} \frac{\log|F(r\exp(i\theta)|}{r} \leqslant \pi|\sin\theta|. \tag{4.26}$$

Since $(1+|x|)^{-1}S(x) \in L^2(\mathbb{R})$, S with real zeroes, we also have

$$h_S(\theta) = \limsup_{r\to\infty} \frac{\log|S(r\exp(i\theta)|}{r} = \pi|\sin\theta|$$

and, in addition, one can replace lim sup in the later relation by lim for all $\theta \neq 0, \pi$. Now, by (2.5) we obtain that, for $\theta \neq 0, \pi$, there exists the limit

$$h_{S_\omega}(\theta) = \lim_{r\to\infty} \frac{\log|S_\omega(r\exp(i\theta)|}{r} = \pi|\sin\theta|. \tag{4.27}$$

Applying the Cauchy-Bunyakovski inequality to (4.25) we see

$$|F(i\eta)| \leqslant |\eta|^{-1/2}\exp(\pi|\eta|),\ |\eta| > 1,$$

and combining this estimate with (2.9),(3.14), we have

$$\Phi(i\eta) = O(|\eta|^{-\beta/2}), |\eta| \to \pm\infty, \tag{4.28}$$

which in particular implies that $h_\Phi\left(\pm\frac{\pi}{2}\right) = 0$, i.e. I_Φ coincides with a segment $[a,b] \in \mathbb{R}$ and

$$h_\Phi(\theta) = \begin{cases} b\cos\theta, & |\theta| \leqslant \pi/2 \\ -a\cos\theta, & |\theta| \in [\pi/2, \pi]. \end{cases}$$

We claim $a = b = 0$. Indeed, if, say, $b > 0$, then we have

$$h_F(\theta) = \limsup_{r\to\infty} \frac{\log|\Phi(r\exp(i\theta))S_\omega(\exp(i\theta))|}{r} = h_\Phi(\theta) + h_{S_\omega}(\theta) > \pi|\sin\theta|,$$

for $\theta \in (0, \pi/2)$. This contradicts (4.26).

Thus Φ is an entire function of zero exponential type and, according to (4.28), decays along the imaginary axis. By the Phragmén-Lindelöf theorem (see e.g. [27]) we obtain $\Phi \equiv 0$ and, finally $f \equiv 0$, which completes the proof of the theorem.

m. Minimality of the system $\mathcal{E}_\omega(\Lambda)$ (Proof of **Theorem 5**). In view of Theorem 4 we need to prove the minimality only. Using (1.1),(2.10) for $k = 2$ and the theorem on two series (which grants convergence) we see

$$\left|\frac{S_\omega(\lambda)}{S(\lambda)}\right| = \left|\frac{\lim_{R\to\infty}\prod_{|\lambda_n|<R}\frac{\lambda_n + \xi_n - \lambda}{\lambda_n - \lambda}}{\lim_{R\to\infty}\prod_{|\lambda_n|<R}(1 + \frac{\xi_n}{\lambda_n})}\right|.$$

The denominator does not depend on λ, thus consider

$$\Pi(x) = \left(\prod_n\left|\frac{\lambda_n + \xi_n - \lambda}{\lambda_n - \lambda}\right|^2\right)_{\lambda = x+i} = \prod_n Y_n(x),$$

where

$$Y_n(x) = 1 + \frac{2(\lambda_n - x)\xi_n + \xi_n^2}{1 + (\lambda_n - x)^2}.$$

Let $m \in \mathbb{Z}, m > 0$ (its value will be determined later). Using independence of the random variables ξ_n and also the Fatou lemma we have

$$\mathbb{E}(\Pi^m(x)) \leqslant \liminf_{N\to\infty}\prod_{-N}^{N}\mathbb{E}((Y_n(x))^m)$$

and (with account (1.1) and then (2.10))

$$\mathbb{E}((Y_n(x))^m) = 1 + m\frac{\mathbb{E}(\xi_n^2)}{1+(\lambda_n - x)^2} + \sum_{j=2}^{m} C_m^j \mathbb{E}\left(\left(\frac{2(\lambda_n - x)\xi_n + \xi_n^2}{1+(\lambda_n - x)^2}\right)^j\right) \leqslant 1 + C(m)\frac{1}{1+(\lambda_n - x)^2},$$

whence (with account of (3.5)) $\mathbb{E}(\Pi^m(x)) \leqslant C_1(m)$, the constants $C(m), C_1(m)$ are independent of x.

Now the Chebyshev inequality yields $\mathbb{P}(\Pi(x) > |x|^{2\alpha}) \leqslant C_1(m)|x|^{-2\alpha m}$ for any $\alpha > 0, x \in \mathbb{R}$. In particular, if $2\alpha m > 1$ we have $\sum\limits_k \mathbb{P}(\Pi(kh) > (|k|h)^{2\alpha}) < \infty$ for any $h > 0$ and, by the Borel-Cantelli lemma, for almost all $\omega \in \Omega$ the inequality

$$\Pi(kh) \leqslant (|k|h)^{2\alpha} \tag{4.29}$$

holds for all but a finite number values of k.

By the M. Cartwright-type inequality for the integral norms (see for example [33]) we have, if $h < 1$

$$\int_{-\infty}^{\infty} \frac{|F(x+i)|^2}{(1+x^2)^\beta}dx \asymp \sum_k \frac{|F(kh)|^2}{(1+k^2)^\beta} \tag{4.30}$$

in the class of all e.f.e.t.π for each $\beta > 0$. To complete the proof it remains to choose the proper values of all the parameters in the case. By Proposition 3.2 there exists $\alpha > 0$ such that $\int |S(x+i)|^2(1+x^2)^{-1+2\alpha}dx < \infty$ and, hence, $\sum |S(kh)|^2(1+k^2)^{-1+2\alpha} < \infty$ for each $h \in (0,1)$. With this α, taking $m \in \mathbb{Z}$ to satisfy $2\alpha m > 1$ and using (4.29) we get $\sum |S_\omega(kh)|^2(1+k^2)^{-1} < \infty$ and, again by (4.30), $\int |S_\omega(x+i)|^2(1+x^2)^{-1}dx < \infty$, for almost all $\omega \in \Omega$. This already yields the minimality of $\mathcal{E}_\omega$: the elements of the biorthogonal system are the Fourier transforms of the functions $S_\omega(\lambda)(\lambda - \lambda_n(\omega))^{-1} \in L^2(\mathbb{R})$.

Remark. — It follows from the proof that the statement of Theorem 5 is still valid if one demands fulfillment of the inequalities $\mathbb{E}(|\xi_n|^k) < \infty, n \in \mathbb{Z}$ only for a finite number $k = 1, 2, \ldots, m, m$ depending upon S. Namely, if β_0 is the constant from Proposition 3.2, then one may take $m > (\beta_0)^{-1}$, say. In particularly, if $\log|S(x+i)|$ is bounded, (this happens, say, for $\Lambda = \mathbb{Z}$), one can take $m = 3$.

5. Estimates of entire functions with random zeroes on a horizontal line.

In this section we prove **Theorem 6** on estimates on a horizontal line of entire functions with random real zeroes.

a. This (new) setting demands construction of truncated random variables to be more precise if compared with the previous section.

Take an integer $n_0 > 0$ (its value to be specified later) and denote

$$t_n = \frac{1}{\gamma} \log \left(\frac{1}{\alpha} \log |\lambda_n| \right), \ n \in \mathbb{Z}, \ |n| > n_0; \tag{5.1}$$

$$l_n = \frac{2}{t_n} \log |\lambda_n| + 2\gamma, \ n \in \mathbb{Z}, \ |n| > n_0. \tag{5.2}$$

We suppose n_0 is chosen so that $t_n > 1, l_n > 2$ for all $|n| > n_0$.

Now consider the random variables

$$\xi_n^* = \begin{cases} \xi_n, & \text{if } |\xi_n| \leqslant l_n \text{ and } |n| > n_0; \\ 0, & \text{otherwise.} \end{cases} \tag{5.3}$$

The Chebyshev inequality yields

$$\mathbb{P}(|\xi_n| > l_n) \leqslant \exp\{-t_n l_n\} \mathbb{E}(\exp\{t_n |\xi_n|\}), \ |n| > n_0,$$

and now by (5.2) we have

$$\begin{aligned} \mathbb{P}(|\xi_n| > l_n) &\leqslant \exp(\alpha e^{\gamma t_n} - l_n t_n) \\ &= \exp(\log|\lambda_n| - 2\log|\lambda_n| - 2\log\log|\lambda_n| + 2\log\alpha) \leqslant \frac{\text{Const}}{|\lambda_n| \log^2 |\lambda_n|} . \end{aligned}$$

Using once more the fact that $S(\lambda)(\lambda - \lambda_0)^{-1} \in L^2(\mathbb{R})$ we obtain (see e.g. [32])

$$\frac{\lambda_{[x]}}{x} \to 1, \text{ as } x \to \pm\infty. \tag{5.4}$$

Therefore $\sum \mathbb{P}(|\xi_n| > l_n) < \infty$ and by the Borel-Cantelli lemma we almost surely have $\xi_n^* = \xi_n$ for all but a finite number of n's, the latter depends on $\omega \in \Omega$.

So we replace the function $S_\omega(\lambda)$ by

$$S_\omega^*(\lambda) = \lim_{R \to \infty} \prod_{|\lambda_n| < R} \left(1 - \frac{\lambda}{\lambda_n + \xi_n^*} \right). \tag{5.5}$$

Since $\{\xi_n\} \in B_{\alpha,\gamma}$, we have

$$\mathbb{E}(|\xi_n^k|) \leqslant c_k,\ k = 0, 1, 2, \ldots,\ n \in \mathbb{Z}, \tag{5.6}$$

for a sequence $\{c_k\}$ independent of n. The Chebyshev inequality yields

$$\mathbb{P}(|\xi_n| \geqslant x) \leqslant \frac{c_3}{x^3},\ x > 0. \tag{5.7}$$

Let P_n be the probability distribution of ξ_n. Then

$$\left|\int_{|x|>l_n} x dP_n\right| \leqslant l_n \mathbb{P}(|\xi_n| \geqslant l_n) + \int_{l_n}^{\infty} \mathbb{P}(|\xi_n| \geqslant x) dx \leqslant \frac{c_3}{l_n^2} + \int_{l_n}^{\infty} \frac{c_3}{x^3} dx \leqslant \frac{2c_3}{l_n^2}.$$

Since $\mathbb{E}(\xi_n) = 0$, we obtain $|\mathbb{E}(\xi_n^*)| \leqslant 2c_3 l_n^{-2}$. and, in particular

$$\sum \frac{|\mathbb{E}(\xi_n^*)|}{|\lambda_n|} < \infty. \tag{5.8}$$

This inequality will be used when applying the theorem on two series.

b. We have

$$\left|\frac{S_\omega^*(\lambda)}{S_\omega(\lambda)}\right| = \left|\frac{\lim_{R\to\infty} \prod_{|\lambda_n|<R} \dfrac{\lambda_n + \xi_n^* - \lambda}{\lambda_n - \lambda}}{\lim_{R\to\infty} \prod_{|\lambda_n|<R} \left(1 + \dfrac{\xi_n^*}{\lambda_n}\right)}\right|.$$

Relations (5.8), (5.6) for $k = 2$, and the theorem on two series provide for almost all $\omega \in \Omega$ the compactwise convergence of each product in the right-hand side of this relation rather than simply convergence as mean values. Besides, the denominator does not depend on λ. Therefore to obtain estimate (2.12) we restrict our attention to the products

$$\Pi(x) = \left(\prod_n \left|\frac{\lambda_n + \xi_n^* - \lambda}{\lambda_n - \lambda}\right|^2\right)_{\lambda=x+i} = \prod_n \frac{(\lambda_n + \xi_n^* - x)^2 + 1}{(\lambda_n - x)^2 + 1}$$

$$= \prod_{\lambda_n \leqslant x-N} \prod_{|\lambda_n - x|<N} \prod_{\lambda_n \geqslant x+N} \left(1 + \frac{2(\lambda_n - x)\xi_n^* + (\xi_n^*)^2}{1 + (\lambda_n - x)^2}\right)$$

$$\Pi_1(x)\Pi_2(x)\Pi_3(x) \tag{5.9}$$

only. Define

$$N = N_x = 8 l_{[x]+1} \tag{5.10}$$

and study each factor for sufficiently large x, $x > 0$, say.

c. *Estimate of the product* $\Pi_2(x)$.

First describe the behavior of the quantities t_n, l_n and N_x. By (5.4) we have

$$t_n = \frac{1}{\gamma} \log\log|n| + O(1), \tag{5.11}$$

$$l_n = 2\gamma \frac{\log|n|}{\log\log|n|} + O(1), \tag{5.12}$$

$$N_x = 16\gamma \frac{\log|x|}{\log\log|x|} + O(1). \tag{5.13}$$

Now notice that the number of factors in $\Pi_2(x)$ does not exceed $2N/d$, where d is the constant from (3.5). Since, for $|\lambda_n - x| < N$ we, obviously, have

$$\frac{1}{1+N^2} \leqslant \frac{(\lambda_n + \xi_n^* - x)^2 + 1}{(\lambda_n - x)^2 + 1} = 1 + \frac{2(\lambda_n - x)\xi_n^* + (\xi_n^*)^2}{(\lambda_n - x)^2 + 1} \leqslant 1 + N^2.$$

Therefore for some absolute constant c and sufficiently large x

$$e^{-cd^{-1}N\log N} \leqslant \Pi_2(x) \leqslant e^{cd^{-1}N\log N}, \; N = N_x.$$

It follows from relation (5.13), that there exists $x_0 = x_0(\alpha, \gamma)$ such that $N_x \log N_x \asymp \gamma \log x$, $x > x_0$. Here as usually the sign $\asymp$ means that the ratio of the both parts of the relation lies between two positive constants.

Taking into account this relation we obtain

$$|x|^{-C_0\gamma} \leqslant \Pi_2(x) \leqslant |x|^{C_0\gamma}, \; |x| > x_0(\alpha, \gamma) \tag{5.14}$$

for a constant $C_0 \in (0, \infty)$ independent of α, γ, and $\omega \in \Omega$.

d. To estimate the products $\Pi_1(x), \Pi_3(x)$ we mention that relations (5.12), (5.13) imply $l_n/|x - \lambda_n| < 1/4$ for $|\lambda_n - x| \geqslant N_x$. Therefore, when dealing with $\Pi_1(x), \Pi_3(x)$, one can use the inequalities

$$\frac{2|\lambda_n - x||\xi_n^*|}{(\lambda_n - x)^2 + 1} + \frac{(\xi_n^*)^2}{(\lambda_n - x)^2 + 1} \leqslant \frac{2l_n}{|\lambda_n - x|} + \left(\frac{l_n}{\lambda_n - x}\right)^2 \leqslant \frac{3}{4}$$

and, hence,

$$\log \Pi_1(x) = \sum_{\lambda_n \leqslant x - N} \left\{ \frac{2(\lambda_n - x)\xi_n^*}{(\lambda_n - x)^2} + \frac{(\xi_n^*)^2}{(\lambda_n - x)^2} \right\} + O(1); \tag{5.15}$$

$$\log \Pi_3(x) = \sum_{\lambda_n \geqslant x+N} \left\{ \frac{2(\lambda_n - x)\xi_n^*}{(\lambda_n - x)^2} + \frac{(\xi_n^*)^2}{(\lambda_n - x)^2} \right\} + O(1). \tag{5.16}$$

We have

$$\sum_{\lambda_n \geqslant x+N} \frac{(\xi_n^*)^2}{(\lambda_n - x)^2} \leqslant \text{Const} \sum_{x+N \leqslant \lambda_n \leqslant 2x} \frac{l_{2[x]+1}^2}{(\lambda_n - x)^2} + \sum_{\lambda_n > 2x} \frac{l_n^2}{(\lambda_n - x)^2} = \sigma_1 + \sigma_2.$$

Simple estimates show

$$\sigma_1 \leqslant \text{Const} \, \frac{l_{2[x]+1}^2}{N_x} \leqslant \text{Const} \, l_{2[x]}, \qquad \sigma_2 \leqslant \text{Const},$$

and finally

$$\sum_{\lambda_n \geqslant x+N} \frac{(\xi_n^*)^2}{(\lambda_n - x)^2} \leqslant \text{Const} \, l_{[x]+1}.$$

Similarly

$$\sum_{\lambda_n \leqslant x-N} \frac{(\xi_n^*)^2}{(\lambda_n - x)^2} \leqslant \text{Const} \, l_{[x]+1}$$

and

$$\left| \log \Pi_1(x)\Pi_3(x) - \sum_{|\lambda_n - x| \geqslant N} \frac{2\xi_n^*}{\lambda_n - x} \right| = o(\log|x|), \; x \to \infty. \tag{5.17}$$

e. Therefore it suffices to estimate the sum of the series

$$s^*(x) = \sum_{|\lambda_n - x| \geqslant N} \frac{\xi_n^*}{\lambda_n - x}. \tag{5.18}$$

It is more convenient here to consider the sum

$$s(x) = \sum_{|\lambda_n - x| \geqslant N} \frac{\xi_n}{\lambda_n - x} \tag{5.19}$$

this will alter only nonessential constants in our estimates.

We need the following auxiliary statement:

LEMMA 5.1. — *There exist numbers $\epsilon > 0$ and $c > 0$ such that*

$$\mathbb{E}(\exp\{\tau \xi_n\}) \leqslant \exp\{c|\tau|^2\} \quad \textit{for} \quad |\tau| < \epsilon, \; n = 0, \pm 1, \ldots. \tag{5.20}$$

Proof. — This statement follows from (1.1) and the Taylor formula applied to the functions $\phi_n(\tau) = \mathbb{E}(\exp(\tau\xi_n))$. Here when estimating the second derivative of $\phi_n(\tau)$ near zero we use the bounds from the definition of the classes $B_{\alpha,\gamma}$.

Now split the series $s(x)$ into two parts

$$s(x) = \sum_{\lambda_n - x \geqslant N} + \sum_{\lambda_n - x \leqslant -N} = s^+(x) + s^-(x)$$

and confine ourselves to considering $s^+(x)$ only; the estimate of $s^-(x)$ is similar.

In order to estimate the "tail" of the distribution function of the random variable $s^+(x)$ we begin with the partial sum

$$s_M^+(x) = \sum_{x+N \leqslant \lambda_n \leqslant M} \frac{\xi_n}{\lambda_n - x}$$

and evaluate its characteristic function. Since the random variables ξ_n are independent, we have

$$\mathbb{E}(\exp\{ts_M^+(x)\}) = \prod_{x+N \leqslant \lambda_n \leqslant M} \mathbb{E}\left(\exp\left\{t\frac{\xi_n}{\lambda_n - x}\right\}\right), \quad t > 0.$$

Fix an ϵ satisfying the conditions of Lemma 5.1 and denote $a = \min(x + t/\epsilon, M)$. We have[(2)]

$$\begin{aligned}\mathbb{E}(\exp\{ts_M^+(x)\}) &= \prod_{x+N \leqslant \lambda_n \leqslant a} \prod_{a < \lambda_n \leqslant M} \\ &\leqslant \prod_{x+N \leqslant \lambda_n \leqslant a} \exp\left\{\alpha \exp\left\{\gamma\frac{t}{\lambda_n - x}\right\}\right\} \prod_{a < \lambda_n \leqslant M} \mathbb{E}\left(\exp\left\{\xi_n\frac{t}{\lambda_n - x}\right\}\right).\end{aligned}$$

The number of factors in the first product does not exceed $C_1 t$, for some $C_1 > 0$; in order to estimate the factors in the second product one can use (5.20). So

$$\mathbb{E}(\exp\{ts_M^+(x)\}) \leqslant \exp\left\{\alpha C_1 t \exp\left\{\frac{\gamma t}{N}\right\}\right\} \exp\left\{ct^2 \sum_{a < \lambda_n \leqslant M} \frac{1}{(\lambda_n - x)^2}\right\}$$

(2) Some of the factors below may be absent.

and

$$\mathbb{E}(\exp\{ts_M^+(x)\}) \leqslant \exp\left\{\alpha C_1 t \exp\left\{\frac{\gamma t}{N}\right\} + C_2 t\right\}.$$

Since the right-hand side is independent of M, we obtain

$$\mathbb{E}(\exp\{ts^+(x)\}) \leqslant \exp\left\{\alpha C_1 t \exp\left\{\frac{\gamma t}{N}\right\} + C_2 t\right\}$$

and, similarly,

$$\mathbb{E}(\exp\{-ts^+(x)\}) \leqslant \exp\left\{\alpha C_1 t \exp\left\{\frac{\gamma t}{N}\right\} + C_2 t\right\}.$$

So, finally

$$\mathbb{E}(\exp\{t|s^+(x)|\}) \leqslant 2\exp\left\{\alpha C_1 t \exp\left\{\frac{\gamma t}{N}\right\} + C_2 t\right\}. \tag{5.21}$$

Next we estimate the values $s^+(m)$ for integer (and still positive) m's. Denote $\tau_m = N_m/\gamma$, $Q_m = \gamma \log m + C_1\alpha e + C_2$, where C_1, C_2, α are the constant from relation (5.21), and consider the events $A_m = \{|s^+(m)| > Q_m\}$. The Chebyshev inequality yields

$$\begin{aligned}\mathbb{P}(A_m) &\leqslant \exp\{-\tau_m Q_m\}\mathbb{E}(\exp\{\tau_m|s^+(m)|\}) \\ &\leqslant 2\exp\{-\tau_m Q_m + C_1\alpha e\tau_m + C_2\tau_m\} = 2\exp\{-\tau_m\gamma\log m\}.\end{aligned}$$

Since $\tau_m \to \infty$, we have $\sum \mathbb{P}(A_m) < \infty$ and by the Borel-Cantelli lemma, for almost all $\omega \in \Omega$, only a finite number of events A_m may happen, i.e. almost surely

$$|s^+(m)| \leqslant \gamma \log m + C_1\alpha e + C_2 \tag{5.22}$$

for all sufficiently large m.

For arbitrary $u \in (0,1)$ we now have

$$\begin{aligned} s^+(m+u) - s^+(m) &= \sum_{\lambda_n \geqslant m+N_m} \xi_n(\omega)\left(\frac{1}{\lambda_n - u - m} - \frac{1}{\lambda_n - m}\right) \\ &= \sum_{\lambda_n \geqslant m+N_m} u\frac{\xi_n^*(\omega)}{(\lambda_n - u - m)(\lambda_n - m)} + O(1). \end{aligned}$$

It is obvious that the right-hand side of this expression is bounded uniformly with respect to m and u.

Therefore, almost everywhere with respect to ω and for all sufficiently big $x > 0$ we have

$$|s^+(x)| \leqslant \gamma \log x + C_4(\alpha, \gamma, \omega).$$

The bound for $s^-(x)$ is the same. By combining these estimates with relation (5.17) we obtain statement *a.* of Theorem 6.

h. One can prove statement *b.* of the theorem (relation 2.14) in the same pattern by taking into account the separateness condition (2.13). That each segment $[m.m+1]$, m large enough, contains points x, satisfying (2.13) (this is statement *c.* of the theorem) follows at once, if we mention that the number of point from Λ^*_ω in $[m, m+1]$ is $o(\log|m|)$ and also the sets Λ^*_ω and Λ_ω differ only by a finite number of points.

6. Series expansions in random exponential systems.

a. In this section we assume $\{\xi_n\} \in B_0$, in particulary the hypothesis of Theorem 6 to be fulfilled. In this case the system $\mathcal{E}_\omega(\Lambda)$ is complete and minimal in $L^2(-\pi, \pi)$ (for almost all $\omega \in \Omega$) and to each function $f \in L^2(-\pi, \pi)$, one can correspond the series

$$f \sim \sum_k \sum_{l=0}^{p_k - 1} c_{k,l}(\omega, f) t^l \exp(i\lambda_k(\omega)t). \tag{6.1}$$

We describe the summability procedure for such series.

The coefficients of series (6.1) are

$$c_{k,l} = \int_{-\pi}^{\pi} f(t) h_{k.l}(t) dt, \tag{6.2}$$

where

$$\mathcal{H}_\omega = \{h_{k.l}(t)\}_{l=0;\ k\in\mathbb{Z}}^{p_k-1} \tag{6.3}$$

is the system, biorthogonal to $\mathcal{E}_\omega$.

Following [13] we give the explicit expression for $h_{k,l}$ to make the exposition self-contained. Let

$$G_k\left(\frac{1}{\lambda - \lambda_k(\omega)}\right) = \sum_{j=1}^{p_k} \frac{C_{k,j}}{(\lambda - \lambda_k(\omega))^j}$$

be the principal part of the Laurent expansion of the function $[S_\omega(\lambda)]^{-1}$ near the point $\lambda_k(\omega)$. Consider the entire functions

$$H_{k,l}(\lambda) = \frac{S_\omega(\lambda)}{l!} \sum_{j=1}^{p_k - l} \frac{C_{k,j+l}}{(\lambda - \lambda_k(\omega))^j} \tag{6.4}$$

and notice that the Taylor expansions of the functions $H_{k,l}$ have the form

$$H_{k,l}(\lambda) = \frac{(\lambda - \lambda_k(\omega))^l}{l!} \{1 + a_k(\lambda - \lambda_k(\omega))^{p_k - l} + \dots\}.$$

We have

$$H_{k,l}^{(m)}(\lambda_j(\omega)) = \delta_{j,k}\delta_{l,m}, \quad j, k \in \mathbb{Z}, \quad l = 0, 1, \dots, p_k - 1, \ m = 0, 1, \dots, p_j - 1$$

and the relations

$$h_{k,l}(t) = \mathcal{F}^{-1}(H_{k,l})(t) \tag{6.5}$$

define the biorthogonal system $\mathcal{H}_\omega$.

b. We proceed to the **proof of Theorem 7**. By Proposition 3.2 we can fix a number $\alpha > 0$ such that the functions

$$F_{\pm\beta}(x) = (1 + |x|)^{\pm\beta} |S(x + i)| \tag{6.6}$$

still satisfy the condition

$$|F_{\pm\beta}|^2 \in (\mathrm{HS}) \tag{6.7}$$

for all $\beta \in (0, \alpha)$. According to Theorem 6 for almost all $\omega \in \Omega$ the following property holds:

for each $\beta > 0$ there exists a constant $C_{\omega,\beta}$ such that

$$C_{\omega,\beta}^{-1} F_{-\beta}(x) < |S_\omega(x + i)| < C_{\omega,\beta} F_\beta(x). \tag{6.8}$$

Everywhere below we assume ω satisfies this condition. We also assume that $0 < \beta < \alpha$, so the outer terms of inequality (6.8), being squared, belong to (HS), in contrast to the intermediate one (otherwise $\mathcal{E}_\omega(\Lambda)$ would form a Riesz basis in $L^2(-\pi, \pi)$).

c. Now take $\gamma < \alpha/10$, $\gamma > 0$ and assume that $f \in L^2_\gamma$ and the sequence $\mathcal{R} = \{R_n\}$ satisfies (2.13) with R_n instead of x.

Define the operator $P_{M,N}$ by the relation:

$$P_{M,N} : f \mapsto \sum_{-M}^{N} \Big\{ \sum_{\lambda_k \in I_n} \sum_{l=0}^{p_k-1} c_{k,l} t^l \exp(i\lambda_k(\omega)t) \Big\}. \tag{6.9}$$

One can read the statement of the theorem as

$$P_{M,N}(L^2_\gamma \to L^2) \longrightarrow Id(L^2_\gamma \to L^2),\ M, N \to \infty,$$

here the notations in brackets mean that we consider operators acting from L^2_γ into L^2 and Id denotes the embedding operator.

d. It suffices to prove the following statements.

LEMMA. 6.1. — *The system $\mathcal{E}_\omega$ is complete in L^2_γ .*

LEMMA 6.2.. — *The following relation holds:*

$$\sup_{M,N} \{ \|P_{M,N}\|_{L^2_\gamma \to L^2} \} < \infty. \tag{6.10}$$

Indeed, the sequence $\{P_{M,N}\}$ approaches the embedding operator on finite linear combinations of elements from $\mathcal{E}_\omega(\Lambda)$ which, by Lemma 6.1, are dense in L^2_γ. Now Lemma 6.2 allows us to extend this relation to the whole space L^2_γ.

Let us use the quasiscalar product:

$$\langle f, g \rangle = (2\pi)^{-1} \int_{-\pi}^{\pi} f(t) g(t) dt. \tag{6.11}$$

The spaces L^2_γ and $L^2_{-\gamma}$ are mutually conjugate with respect to this product and the general form of a linear functional in L^2_γ is as follows:

$$\mathcal{L}_f(g) = (2\pi)^{-1} \int_{-\pi}^{\pi} f(t) g(t) dt,\ g \in L^2_\gamma, f \in L^2_{-\gamma}. \tag{6.12}$$

The proof of Lemma 6.1 repeats with a natural modification the proof of the completeness Theorem 4. One can also prove it in a simpler way involving exact estimates (2.12).

e. We split the **proof of Lemma 6.2** into several steps. First, consider the conjugate operator. Relations (6.4)-(6.8) together with Proposition 3.4 show that the biorthogonal system $\mathcal{H}_\omega$ belongs to $L^2_{-\beta}$. For a function $g \in L^2$ denote

$$(6.13)\quad b_{k,l} = b_{k,l}(g) = (2\pi)^{-1} \int_{-\pi}^{\pi} t^l \exp(i\lambda_k t) g(t)dt,\ l \in \overline{0, p_k - 1},\ k \in \mathbb{Z}$$

and define an operator $T_{M,N} : L^2 \longrightarrow L^2_{-\gamma}$ by the relation

$$(6.14)\qquad T_{M,N} : f \mapsto \sum_{n=-M}^{N} \Big\{ \sum_{\lambda_k \in I_n} \sum_{l=0}^{p_k-1} b_{k,l} h_{k,l}(t) \Big\}.$$

The operator $T_{M,N}$ is conjugate to $P_{M,N}$ with respect to the quasiscalar product (6.11). Therefore Lemma 6.2 is equivalent to

LEMMA 6.3. — *The following relation holds:*

$$\sup_{M,N} \{ \|T_{M,N}\|_{L^2 \to L^2_{-\gamma}} \} < \infty.$$

f. One can reformulate the later lemma as an interpolation problem. Let a function $g \in L^2$ be given with $b_{k,l}$ as in (6.13). Denote $G(\lambda) = (\mathcal{F}g)(\lambda)$ (here $G(\lambda)$ means *the extension* of the Fourier transform from the real line onto $\mathbb{C}$) and consider the meromorphic function

$$(6.15)\qquad \Phi(\lambda) = \frac{G(\lambda)}{S_\omega(\lambda)}.$$

Denote by Γ_n the rectangle with vertices $R_n \pm i,\ R_{n+1} \pm i$. Direct calculation (see e.g. [13]) shows that, for $\lambda \notin \Gamma_n$,

$$\begin{aligned}\mathcal{F}\Big\{ \sum_{\lambda_k \in I_n} \sum_{l=0}^{p_k-1} b_{k,l} h_{k,l}(t) \Big\}(\lambda) &= \sum_{\lambda_k \in I_n} \sum_{l=0}^{p_k-1} b_{k,l} H_{k,l}(\lambda) \\ &= \frac{S_\omega(\lambda)}{2\pi i} \int_{\Gamma_n} \Phi(\zeta) \frac{d\zeta}{\zeta - \lambda}.\end{aligned}$$

Now set $\mathcal{T}_{M,N} = \mathcal{F} T_{M,N} \mathcal{F}^{-1}$. For $\lambda \notin \{|\Im\lambda| < 1\}$ we have

$$(6.16)\qquad \mathcal{T}_{M,N} : G(\lambda) \longmapsto S_\omega(\lambda) \sum_{-M}^{N} \frac{1}{2\pi i} \int_{\Gamma_n} \Phi(\zeta) \frac{d\zeta}{\zeta - \lambda}.$$

The poles of the meromorphic function Φ coincide (accounting for their multiplicities) with zeroes of $S_\omega(\lambda)$. Therefore the right-hand side of the last relation represents an entire function and Lemma 6.3 is equivalent to

LEMMA 6.4. — *The following relation holds:*

$$\sup_{M,N}\{\|\mathcal{T}_{M,N}\|_{PW_\pi \to PW_{\pi,-\gamma}}\} < \infty.$$

g. Proof of Lemma 6.4 — reduction to a problem in weighted Hardy spaces.

When evaluating the norm in the space $PW_{\pi,-\gamma}$ one can integrate along any horizontal line. Therefore to prove the lemma it suffices to obtain the relation

$$\int_{-\infty}^{\infty} \frac{|S_\omega(x+2i)|^2}{|x+2i|^{2\gamma}} \left|\sum_{-M}^{N} \int_{\Gamma_n} \Phi(\zeta)\frac{d\zeta}{\zeta-(x+2i)}\right|^2 dx \leqslant \text{Const},$$

where the constant is independent of $G \in PW_\pi$, $\|G\|_{PW_\pi \leqslant 1}$.

In turn, it suffices to prove the relation

$$\int_{-\infty}^{\infty} \frac{|S(x+2i)|^2}{|x+2i|^{\gamma}} \left|\sum_{-M}^{N} \int_{\Gamma_n} \Phi(\zeta)\frac{d\zeta}{\zeta-(x+2i)}\right|^2 dx \leqslant \text{Const}, \tag{6.17}$$

which can be viewed as an estimate in a weighted Hardy space. To make this reasoning clear we need the following notation.

Let a function $a(x) \geqslant 0$, $-\infty < x < \infty$ satisfy the condition

$$\int_{-\infty}^{\infty} \frac{|\log a(x)|}{1+x^2}\,dx < \infty. \tag{6.18}$$

Denote by $f_a^{\pm}(z)$ the outer functions in the half-planes $\mathbb{C}_2^+ = \{z : \Im z > 2\}$, $\mathbb{C}_2^- = \{z : \Im z < 2\}$, respectively with boundary values $|f_a^{\pm}(x+2i)| = a(x)$.

One can construct these functions as follows:

$$f_a^{\pm}(z) = \exp\{\pm\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{t+2i-z} - \frac{t}{t^2+1}\right)\log a(t)dt\},\ \pm(\Im z-2) > 0.$$

It should be mentioned that

$$f_{a^{-1}}^{\pm} = (f_a^{\pm})^{-1}. \tag{6.19}$$

For detailed explanation about outer functions see, for instance, [25].

Let H^2_+ be the usual Hardy space in the half-plane $\mathbb{C}_2^+$. Consider the weighted Hardy space

$$H^2_{+,a} = f^+_{a^{-1}}(z)H^2_+. \tag{6.20}$$

This is the space of all holomorphic in $\mathbb{C}_2^+$ functions $\Psi(z)$ such that

$$\|\Psi\|^2_{H^2_{+,a}} = \sup_{y>2}\int_{-\infty}^{\infty} |f^+_a(x+iy)\Psi(x+iy)|^2 dx$$

$$= \int_{-\infty}^{\infty} |a(x)\Psi(x+2i)|^2 dx < \infty.$$

The weighted Hardy space in the half-plane $\mathbb{C}_2^-$ may be defined in a similar way.

Denote

$$a_\gamma(x) = \frac{|S(x+2i)|}{|x+2i|^{\gamma/2}}.$$

The function

$$\Psi_{M,N}(z) = \sum_{-M}^{N}\int_{\Gamma_n}\Phi(\zeta)\frac{d\zeta}{\zeta - z}$$

is holomorphic in $\mathbb{C}_2^+$. Therefore one can read the estimate (6.17) as

$$\sup_{M,N}\{\|\Psi_{M,N}\|_{H^2_{+,a_\gamma}}\} \leqslant \text{Const} \tag{6.21}$$

uniformly with respect to all functions $G \in PW_\pi$, $\|G\|_{PW_\pi} \leqslant 1$.

h. The dual of weighted Hardy spaces. To evaluate the norm of a function from a weighted Hardy space we need the following.

LEMMA 6.5. — *Let the weight function $a(x) \geqslant 0$ belong to (HS) and $b(x) = (a(x))^{-1}$. Then the space dual to $H^2_{+,a}$ admits the representation $(H^2_{+,a})^* = H^2_{-,b}$. The functional $\mathcal{L}_\Theta$ corresponding to $\Theta \in H^2_{-,b}$ has the form*

$$\mathcal{L}_\Theta(\Psi) = \int_{-\infty}^{\infty}\Theta(x+2i)\Psi(x+2i)dx, \ \Psi \in H^2_{+,a}$$

and $\|\mathcal{L}_\Theta\|_{(H^2_{+,a})^*} \asymp \|\Theta\|_{H^2_{-,b}}$.

Proof. — Consider the space

$$L_a^2(\mathbb{R}+2i) = \{f(\cdot+2i) : \int_{-\infty}^{\infty} (a(x))^2 |f(x+2i)|^2 dx < \infty\}.$$

Its dual $(L_a^2(\mathbb{R}+2i))^*$ admits the natural realization $(L_a^2(\mathbb{R}+2i))^* = L_b^2(\mathbb{R}+2i)$. The functional $\mathcal{L}_g$ corresponding to a function $g \in L_b^2(\mathbb{R}+2i)$, has the form

$$\mathcal{L}_g(f) = \int_{-\infty}^{\infty} g(x+2i) f(x+2i) dx, \ f \in L_a^2(\mathbb{R}+2i)$$

and $\|\mathcal{L}_g\|_{(L_a^2(\mathbb{R}+2i))^*} \asymp \|g\|_{L_b^2(\mathbb{R}+2i)}$.

Besides $a \in$(HS) and, hence, $b \in (HS)$. Therefore the Hilbert transform is bounded in each space $L_a^2(\mathbb{R}+2i), L_b^2(\mathbb{R}+2i)$. The Sokhotski-Plemelj formula leads one to the decompositions

$$L_a^2(\mathbb{R}+2i) = H_{+,a}^2 \dot{+} H_{-,a}^2, \quad L_b^2(\mathbb{R}+2i) = H_{+,b}^2 \dot{+} H_{-,b}^2.$$

Thus $(H_{+,a}^2)^* = L_b^2(\mathbb{R}+2i)/A$, where A is the annihilator of $H_{+,a}^2$. To complete the proof it remains to mention that $A = \{g \in L_b^2(\mathbb{R}+2i) : \mathcal{L}_g(f) = 0, \forall f \in H_{+,a}^2\} = H_{-,b}^2$.

i. We use this lemma for $a = a_\gamma$. Let, as earlier, $b = a^{-1}$. For $\Theta \in H_{-,b}^2$ denote

$$A_{M,N}(\Theta) = \int_{-\infty}^{\infty} \Theta(x+2i) \Psi_{M,N}(x+2i) dx.$$

and mention that relation (6.21) is equivalent to

$$\sup\{\|A_{M,N}(\Theta)\| : \Theta \in H_{-,b}^2, \|\Theta\|_{H_{-,b}^2} \leqslant 1\} < \infty. \tag{6.22}$$

We have

$$A_{M,N}(\Theta) = \sum_{-M}^{N} \int_{\Gamma_n} \Phi(\zeta) \int_{-\infty}^{\infty} \frac{\Theta(x+2i)}{\zeta - (x+2i)} dx d\zeta = \int_{\cup_{-M}^{N} \partial\Gamma_n} \Phi(\zeta)\Theta(\zeta) d\zeta. \tag{6.23}$$

The integrals over all the vertical segments, but $[R_{-M}+i, R_{-M}-i]$ and $[R_{N+1}+i, R_{N+1}-i]$ disappear. Therefore in (6.23) one may integrate over the curve $\Upsilon_{M,N}$ which is the boundary of the rectangle with the vertices $R_{-M} \pm i, R_{N+1} \pm i$.

Let $f_{a_\gamma}^-$ be the outer function in $\mathbb{C}_-$ with the boundary values $|f_{a_\gamma}^-(x+2i)| = a_\gamma(x)$. Then

$$\Theta(\zeta) = f_{a_\gamma}^-(\zeta)\Theta_0(\zeta); \ \Theta_0 \in H_-^2, \|\Theta_0\|_{H_-^2} \leqslant 1.$$

Combining this with (6.15) one obtains

$$A_{M,N}(\Theta) = \sum_{-M}^{N} \int_{\Upsilon_{M,N}} \frac{G(\zeta)}{S_\omega(\zeta)} f^-_{a_\gamma}(\zeta)\Theta_0(\zeta)d\zeta$$

and we need to estimate $S_\omega(\zeta)$ from below on $\Upsilon_{M,N}$. The function $F_\omega(z) = S_\omega(z)\exp(-i\pi z)$ belongs to the Nevanlinna class in $\mathbb{C}_-$ and the factorization theorem (see e.g.[25]) yields

$$S_\omega(z) = \exp(i\pi z) f^-_{|S_\omega(x+2i)|}(z) B_\omega(z),$$

where $B_\omega(z)$ is the Blaschke product for the half-plane $\mathbb{C}_-$, corresponding to the zero set $\{\lambda_n(\omega)\}$. (Here the singular part of the decomposition vanishes because the function $S_\omega(z)$ admits an analytic continuation at each point of the boundary of $\mathbb{C}_-$, see e.g. [25]).

With account of (6.19) we obtain

$$\begin{aligned} A_{M,N}(\Theta) &= \int_{\Upsilon_{M,N}} \frac{G(\zeta)}{B_\omega(\zeta)} (f^-_{|S_\omega(x+2i)|}(\zeta))^{-1} f^-_{a_\gamma}(\zeta)\Theta_0(\zeta)\exp(-i\pi\zeta)d\zeta \\ &= \int_{\Upsilon_{M,N}} \frac{G(\zeta)}{B_\omega(\zeta)} f^-_d(\zeta)\Theta_0(\zeta)\exp(-i\pi\zeta)d\zeta, \end{aligned} \tag{6.24}$$

where

$$d(x) = d_\gamma(x) = \frac{|S(x+2i)|}{|S_\omega(x+2i)|} \frac{1}{(1+|x|^{\gamma/2})}.$$

By Theorem 6 there exists a constant $C = C(\omega, \gamma)$ such that, for almost all $\omega \in \Omega$,

$$\frac{|S(x+2i)|}{|S_\omega(x+2i)|} \leqslant C(1+|x|^{\gamma/4})$$

and, finally, $d_\gamma(x) \leqslant \text{Const}\,(1+|x|^{-\gamma/4})$, whence (evidently)

$$|f^-_d(\zeta)| \leqslant \text{Const}\,|f^-_{(1+|x|)^{-\gamma/4}}| \asymp |\zeta - 3i|^{-\gamma/4}, \ \zeta \in \mathbb{C}_-.$$

Returning to (6.24) we have

$$|A_{M,N}(\Theta)| \leqslant \text{Const} \int_{\Upsilon_{M,N}} |G(\zeta)||\Theta_0(\zeta)| \frac{|d\zeta|}{|B_\omega(\zeta)||\zeta - 3i|^{\gamma/4}}.$$

j. To complete our construction we need the following lemma.

LEMMA 6.6. — *For almost all $\omega \in \Omega$ there exists a constant $k_0 = k_0(\omega, \gamma)$ such that*

$$|B_\omega(\zeta)||\zeta - 3i|^{\gamma/4} \geqslant k_0,\ \zeta \in \Upsilon_{M,N}, M, N \in \mathbb{Z}.$$

The constant k_0 is independent of $M, N \in \mathbb{Z}$.

We postpone the proof until the next subsection. Assuming this lemma to be already proved, it remains only to use the inequalities

$$\sup_{M,N} \int_{\Upsilon_{M,N}} |G(\zeta)|^2 |d\zeta| \leqslant \text{Const}\, \|G\|^2_{PW_\pi};$$

$$\sup_{M,N} \int_{\Upsilon_{M,N}} |\Theta_0(\zeta)|^2 |d\zeta| \leqslant \text{Const}\, \|\Theta_0\|^2_{H^2_-};$$

this follows from the uniform square integrability of the functions G, Θ_0 along each horizontal line $\mathbb{R} + iy$, $|y| < 2$.

We have

$$\begin{aligned} |A_{M,N}(\Theta)| &\leqslant \text{Const} \int_{\Upsilon_{M,N}} |G(\zeta)||\Theta_0(\zeta)||d\zeta| \\ &\leqslant \Big(\int_{\Upsilon_{M,N}} |G(\zeta)|^2|d\zeta|\Big)^{1/2} \Big(\int_{\Upsilon_{M,N}} |\Theta_0(\zeta)|^2|d\zeta|\Big)^{1/2} \leqslant \text{Const} \end{aligned}$$

uniformly with respect to $M, N \in \mathbb{Z}$. This completes the proof of Theorem 7 (subject to the lemma above).

k. Proof of Lemma 6.6. Taking into account that all the zeroes $\lambda_k(\omega)$ are real we have

$$|B_\omega(\zeta)| = \prod_{-\infty}^{\infty} \left| \frac{\zeta - \lambda_k(\omega)}{\zeta - \lambda_k(\omega) - 4i} \right| = \prod_{-\infty}^{\infty} \left|1 - \frac{\zeta - 2i}{\lambda_k(\omega) - 2i}\right| \left|1 - \frac{\zeta - 2i}{\lambda_k(\omega) + 2i}\right|^{-1}.$$

We compare this quantity with the similar one corresponding to the unperturbed zeroes:

$$|B(\zeta)| = \prod_{-\infty}^{\infty} \left|1 - \frac{\zeta - 2i}{\lambda_k - 2i}\right| \left|1 - \frac{\zeta - 2i}{\lambda_k + 2i}\right|^{-1}.$$

We have

$$
\begin{aligned}
&|B_\omega(\zeta)/B(\zeta)| \\
&\quad = \left| \left\{ \prod \left(1 - \frac{\zeta - 2i}{\lambda_k(\omega) - 2i}\right) \left(1 - \frac{\zeta - 2i}{\lambda_k - 2i}\right)^{-1} \right\} \right. \\
&\qquad \left. \left\{ \prod \left(1 - \frac{\zeta - 2i}{\lambda_k(\omega) + 2i}\right) \left(1 - \frac{\zeta - 2i}{\lambda_k + 2i}\right)^{-1} \right\}^{-1} \right| .
\end{aligned}
$$

After estimating the quantities in the braces from below and from above (one can do it exactly in the same way as when proving Theorem 6), we see that, for almost all $\omega \in \Omega$

$$
\left| \frac{B_\omega(\zeta)}{B(\zeta)} \right| \geqslant \text{Const}\, |\zeta|^{-\gamma/4}, \zeta \in \Upsilon_{M,N}, M, N \in \mathbb{Z},
$$

where the constant depends only on γ and ω.

It remains to prove that

$$
|B(\zeta)| > k_1, \zeta \in \Upsilon_{M,N}, M, N \in \mathbb{Z}.
$$

Estimates of this type are standard, see e.g. [4]. We include the necessary arguments here in order to make the exposition self-contained. In fact we prove a statement little bit more general:

Let $d = \inf\{\lambda_{k+1} - \lambda_k\}$. For each $\epsilon > 0$ there exists a constant $k_1 = k_1(\epsilon, d)$ such that

$$
|B(\zeta)| > k_1, \text{ for } |\Im\zeta| < 2, \ \operatorname{dist}(\zeta, \Lambda) > \epsilon.
$$

Indeed, without loss of generality one can assume $0 < \Re\zeta < 1$, (otherwise we can shift the sequence Λ). Then we have

$$
\begin{aligned}
|B(\zeta)| &= \left(\prod_{|\lambda_k| \leqslant 20} \prod_{|\lambda_k| > 20} \right) \left| \left(1 + \frac{4i}{\lambda_k - 2i}\right) \left(1 - \frac{4i}{\lambda_k + 4i - \zeta}\right) \right| \\
&= \Pi_1(\zeta)\Pi_2(\zeta).
\end{aligned}
$$

The first product contains at most $20/d$ factors and, clearly, is bounded (from both sides) by constants depending on d and ϵ only.

Since $|(\lambda_k - 2i)^{-1}|, |(\lambda_k + 4i - \zeta)^{-1}| < 1/8$ for $|\lambda_k| > 20$, we have

$$|\log|\Pi_2(\zeta)|| = \left|\Re \sum_{|\lambda_k|>20} \left[\log\left(1+\frac{4i}{\lambda_k - 2i}\right) + \log\left(1 - \frac{4i}{\lambda_k + 4i - \zeta}\right)\right]\right|$$

$$\asymp \left|\Re \sum_{|\lambda_k|>20} 4i\left(\frac{1}{\lambda_k - 2i} - \frac{1}{\lambda_k + 4i - \zeta}\right)\right| \leqslant 10 \sum_{|\lambda_k|>20} \frac{1}{\lambda_k^2}.$$

The last term is independent of λ. Q.E.D.

1. Now we outline **the proof of Theorem 8**. Since statement *a.* is already proved, it remains to consider statement *b.* only. Direct calculation shows that the operator $\mathcal{T}_{M,N}$ defined in (6.16), being a partial sum of the Lagrange-Hermite interpolation series, admits the representation

$$\mathcal{T}_{M,N} : G(\lambda) \longmapsto \sum_{n=-M}^{N} \sum_{\lambda_k(\omega)\in\Gamma_n} \sum_{l=0}^{p_k-1} \frac{G^{(l)}(\lambda_k(\omega))}{l!} H_{k,l}(\lambda), \tag{6.25}$$

where the functions $H_{k,l}$ are defined in relation (6.4).

It is clear that the righthand side of relation (6.25) approaches the function $G(\lambda)$ as $M, N \to \infty$ if the latter decays fast enough when $\lambda \to \pm\infty$. Since such functions are dense in PW_π, and, by Lemma 6.4, the operators $\mathcal{T}_{M,N}$ are uniformly bounded from PW_π into $PW_{\pi,\gamma}$, it follows that, for each function $G \in PW_\pi$, the series

$$S_\omega(\lambda) \sum_{n=-\infty}^{\infty} \sum_{\lambda_k(\omega)\in\Gamma_n} \sum_{l=0}^{p_k-1} \frac{G^{(l)}(\lambda_k(\omega))}{l!} H_{k,l}(\lambda), \tag{6.26}$$

converges to G in the space $PW_{\pi,-\gamma}$ whose norm is weaker than the norm of PW_π. (We mention that this series coincides with (2.19) in the case when all points $\lambda_n(\omega)$ are distinct).

Theorem 8 says that series (6.26) for each function $G \in PW_{\pi,\gamma}$ converges to G in the space PW_π, whose norm is weaker than the norm of $G \in PW_{\pi,\gamma}$. One can carry the proof of this statement in a very similar way; the main step — the proof of uniform boundedness of the operators $\mathcal{T}_{M,N}$ from $G \in PW_{\pi,\gamma}$ into PW_π is a slight modification of the proof of Lemma 6.4. We omit the details.

7. Local convergence of series expansions in random exponential systems.

In this section we prove Theorem 9.

a. First we obtain a representation for the partial sum

$$\Sigma_\omega(f;R,T) = \sum_{R<\lambda_k<T} \sum_{l=0}^{p_k-1} c_{k,l} t^l \exp(i\lambda_k(\omega)t),$$

assuming, as usually, that all zeroes $\lambda_k(\omega)$ are simple (i.e. $p_k = 1$ for all k). The elements of the biorthogonal system are

$$h_{k,\omega}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_\omega(\lambda)}{S'_\omega(\lambda_k(\omega))(\lambda-\lambda_k(\omega))} \exp(-i\lambda t)d\lambda, \tag{7.1}$$

and the coefficients of series (2.11) have the form

$$c_k(\omega;f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_\omega(\lambda)}{S'_\omega(\lambda_k(\omega))(\lambda-\lambda_k(\omega))} F(\lambda)d\lambda, \tag{7.2}$$

where

$$F(\lambda) = \int_{-\pi}^{\pi} f(t)\exp(-i\lambda t)dt. \tag{7.3}$$

Therefore

$$\Sigma_\omega(f;R,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_\omega(\lambda)F(\lambda) \sum_{R<\lambda_k(\omega)<T} \frac{\exp(i\lambda_k(\omega)t)}{S'_\omega(\lambda_k(\omega))(\lambda-\lambda_k(\omega))} d\lambda.$$

Denote by $\Gamma_{R,T}$ a closed curve whose interior $G_{R,T}$ contains $\Lambda_\omega \cap (R,T)$ and does not contain other points of Λ_ω and define $\chi_{R,T}(\lambda) = 1$, if $\lambda \in G_{R,T}$ and $\chi_{R,T}(\lambda) = 0$ otherwise. We have

$$\begin{aligned}\frac{1}{2\pi i}&\int_{\Gamma_{R,T}} \frac{\exp(i\zeta t)}{S_\omega(\zeta)(\lambda-\zeta)} d\zeta \\ &= \sum_{R<\lambda_k(\omega)<T} \frac{\exp(i\lambda_k(\omega)t)}{S'_\omega(\lambda_k(\omega))(\lambda-\lambda_k(\omega))} - \frac{\exp(i\lambda t)}{S_\omega(\lambda)}\chi_{R,T}(\lambda).\end{aligned}$$

Therefore

$$\begin{aligned}\Sigma_\omega(f;R,T) &= \frac{1}{2\pi}\int_{\mathbb{R}\cap G_{R,T}} \exp(i\lambda t)F(\lambda)d\lambda \\ &\quad + \frac{1}{4\pi^2 i}\int_{-\infty}^{\infty} S_\omega(\lambda)F(\lambda)\int_{\Gamma_{R,T}} \frac{\exp(i\zeta t)}{S_\omega(\zeta)(\lambda-\zeta)} d\zeta d\lambda \\ &= \Sigma_\omega^{(1)}(f;R,T) + \Sigma_\omega^{(2)}(f;R,T). \end{aligned} \tag{7.4}$$

b. The first summand $\Sigma_\omega^{(1)}(f;R,T)$ is independent of ω. It simulates the behavior of the classical partial Fourier sums. Indeed, taking (7.3) into account we have

$$\Sigma_\omega^{(1)}(f;R,T)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{\exp(iT(t-\tau)) - \exp(iR(t-\tau))}{i(t-\tau)}\,d\tau$$

and comparing the obtained kernel with that of Dirichlet we obtain (7.5)

$$\Sigma_\omega^{(1)}(f;R,T)(t) - S(f;R,T)(t) \to 0 \text{ as } R \to -\infty, T \to \infty;\ t \in (-\pi,\pi).$$

c. Therefore we need to construct a sequence $\mathcal{R}(=\mathcal{R}(f,\omega)) = \{R_N\}_{-\infty}^{\infty}; R_N \to \pm\infty$ as $N \to \pm\infty$ so that

$$\Sigma_\omega^{(2)}(f;R_{-M},R_{N+1})(t) \to 0, M,N \to +\infty, t \in (-\pi,\pi).$$

We have (not specifying the contour $\Gamma_{R,T}$)

$$\Sigma^{(2)} - \omega(f;R,T)(t) = \frac{1}{4\pi^2 i}\int_{\Gamma_{R,T}} \frac{\exp(i\zeta t)}{S_\omega(\zeta)}\int_{-\infty}^{\infty}\frac{S_\omega(\lambda)F(\lambda)}{\lambda-\zeta}\,d\lambda d\zeta.$$

Denote $\Gamma_{R,T}^{\pm} = \Gamma_{R,T} \cap \mathbb{C}^{\pm}$ and consider the quantities

$$\sigma_\omega^{\pm}(f;R,T)(t) = \frac{1}{4\pi^2 i}\int_{\Gamma_{R,T}^{\pm}} \frac{\exp(i\zeta t)}{S_\omega(\zeta)}\int_{-\infty}^{\infty}\frac{S_\omega(\lambda)F(\lambda)}{\lambda-\zeta}\,d\lambda d\zeta \tag{7.6}$$

separately. (We recall that here, as earlier, we assume $\mathbb{C}^+ = \{z; \Im z > 2\}$ and $\mathbb{C}^- = \{z; \Im z < 2\}$.)

d. Transformation of $\sigma_\omega^+(f;R,T)$. By Theorem 6 the function S_ω on the line $\mathbb{R}+i$ is comparable with the function S:

$$C_{\omega,\beta}^{-1}(1+|x|)^{-\beta}|S(x+i)| < |S_\omega(x+i)| < C_{\omega,\beta}(1+|x|)^{\beta}|S(x+i)| \tag{7.7}$$

for each $\beta > 0$. Take β to provide that the squares of both the right- and lefthand sides of this inequality satisfy the (HS) condition.

Denote by $e_+(\lambda)$ the outer function in $\{z; \Im z > 1\}$ satisfying the condition $|e_+(x+i)| = (1+|x|^\beta)|S(x+i)|$. We have $|e_+(x+i)|^2 \in$(HS).

All zeroes of the function S are real. Therefore (see the similar reasoning in the previous section)

$$|e_+(\zeta)| = (1+|\zeta|^\beta)|S(\zeta)||\exp(i\pi\zeta)|, \Im\zeta > 1. \tag{7.8}$$

For $\zeta \in \Gamma^+_{R,T}$ one can replace the inner integral in (7.6) by

$$\int_{-\infty+i}^{\infty+i} \frac{S_\omega(\lambda)F(\lambda)}{\lambda-\zeta}\,d\lambda =: \Theta_+(\zeta).$$

As in the previous section,

$$\Theta_+(\zeta) = e_+(\zeta)\Theta^0_+(\zeta),\ \Theta^0_+ \in H^2(\Im\zeta > 1). \tag{7.9}$$

Combining (7.6), (7.8), (7.9) we obtain

$$|\sigma^+_\omega(f;R,T)(t)| \leqslant \text{Const} \int_{\Gamma^+_{R,T}} (1+|\zeta|)^\beta \exp(-(t+\pi)\Im\zeta)|\frac{S(\zeta)}{S_\omega(\zeta)}||\Theta^0_+(\zeta)||d\zeta|.$$

Estimating the function $S_\omega(\zeta)$ from below (via $S(\zeta)$), we, finally, obtain

$$|\sigma^+_\omega(f;R,T)(t)| \leqslant \text{Const} \int_{\Gamma^+_{R,T}} (1+|\zeta|)^\alpha \exp(-(t+\pi)\Im\zeta)|\Theta^0_+(\zeta)||d\zeta|, \tag{7.10}$$

here the value $\alpha(= 2\beta)$ may be chosen arbitrarily small (if we increase the constant before the integral).

e. Now define the contour $\Gamma_{R,T}$ as follows:

$$\Gamma_{R,T} = I^+_R \cup I^-_R \cup I^+_T \cup I^-_T \cup C^+_{R,T} \cup C^-_{R,T},$$

where $I^\pm_a$ denote the segments $[a, a \pm i\log^2|a|] + 2i$ and $C^\pm_{R,T}$ denote the half-circles located in the upper and lower half-planes respectively with diameters $[R \pm i\log^2|R|, T \pm i\log^2|T|] + 2i$. So $\Gamma^+_{R,T} = I^+_R \cup I^+_T \cup C^+_{R,T}$ and

$$|\sigma^+_\omega(f;R,T)(t)| \leqslant \text{Const}\left(\int_{I^+_R} + \int_{I^+_T} + \int_{C^+_{R,T}}\right)(1+|\zeta|)^\alpha \exp(-(t+\pi)\Im\zeta)$$
$$|\Theta^0_+(\zeta)||d\zeta| =: \sigma^+_\omega(f;R)(t) + \sigma^+_\omega(f;T)(t) + \Delta^+_\omega(f;R,T)(t).$$

f. The definition of the contour $\Gamma_{R,T}$ provides, for $t \in (-\pi,\pi)$,

$$\sup_{\zeta\in C^+_{R,T}} \left\{(1+|\zeta|)^\alpha \exp\Big(-\frac{1}{2}(t+\pi)\Im\zeta\Big)\right\} \to 0 \text{ as } R \to -\infty, T \to \infty.$$

Besides $\Theta^0_+(\zeta) \to 0, \zeta \to \infty, \Im\zeta > 1$. The (essentially) same estimates as when proving the Jordan lemma yield

$$\Delta^+_\omega(f;R,T)(t) \to 0, R \to -\infty, T \to \infty$$

for each $t \in (-\pi,\pi)$.

Therefore to estimate the quantity $|\sigma_\omega^+(f;R,T)(t)|$ it suffices to construct a sequence $\mathcal{R}(=\mathcal{R}(f,\omega)) = \{R_N\}_{-\infty}^{\infty}; R_N \to \pm\infty,$ as $N \to \pm\infty$ so that

$$\sigma_\omega^+(f;R_N) \to 0, N \to \pm\infty \tag{7.11}$$

for any $t \in (-\pi,\pi)$.

g. Denote $G = \{\zeta \in \mathbb{C}^+; \Im\zeta < \log^2|\Re\zeta| + 2\}$. We need the relation

$$\int\int_G (\log|\zeta|)^{-2}|\Theta(\zeta)|^2 dm_\zeta < \text{Const}\,\|\Theta\|_{H^2(\mathbb{C}^+)}, \tag{7.12}$$

here, as usually, dm_ζ denotes the planar Lebesgue measure.

In order to obtain this relation we recall that a measure μ in $\mathbb{C}^+$ is a *Carleson measure* if, for all $x \in \mathbb{R}$ and $h > 0$

$$\mu(\{\zeta = \xi + i\eta;\ x \leqslant \xi \leqslant x+h, 2 \leqslant \eta \leqslant 2+h\}) \leqslant Ch$$

with a constant C independent of x and h. (For definition and properties of Carleson measures see e.g. [25], [26].) For such measures the Carleson theorem says

$$\int\int_{\mathbb{C}^+} |\Theta(z)|^2 d\mu(z) \leqslant \text{Const}\,\|\Theta\|^2_{H^2(\mathbb{C}^+)}, \quad \Theta \in H^2(\mathbb{C}^+).$$

To obtain (7.12) it now suffices to mention that the measure $d\mu_z = (\log(1+|z|))^{-2}\chi_G(z)dm_z$, (here χ_G is the characteristic function of G) is a Carleson measure in $\mathbb{C}^+$.

Now denote $G_n = \{\zeta \in G; n < \Re\zeta \leqslant n+1\}$ and rewrite (7.12) with $\Theta = \Theta_+^0$ in the form

$$\sum_n [(1+|n|^\alpha)\log^2|n|]^{-1} \int\int_{G_n} (1+|\zeta|^\alpha|)\Theta_+^0(\zeta)|^2 dm_\zeta < Const\,\|\Theta_+^0\|_{H^2(\mathbb{C}^+)},$$

which, in particularly, yields

$$\liminf_{n\to\pm\infty} |n|^{1-2\alpha} \int\int_{G_n} (1+|\Re\zeta|^\alpha)|\Theta_+^0(\zeta)|^2 dm_\zeta = 0.$$

For each $\epsilon > 0$ denote

$$E_{\epsilon,n} = \Big\{\xi \in [n,n+1); |n|^\alpha \int_2^{\log^2|\xi|} |\Theta_+^0(\xi+i\eta)|^2 d\eta > \epsilon\Big\}.$$

The Chebyshev inequality yields

$$\liminf_{n\to\pm\infty} |n|^{1-2\alpha}\mathrm{mes}E_{\epsilon,n} = 0$$

and, thus, for each $\epsilon > 0$, there exists a sequence $\{r_k(\epsilon)\}_{-\infty}^{\infty}; r_k \to \pm\infty, k \to \pm\infty$ such that

$$\int_{I^+_{r_k(\epsilon)}} (1+|\zeta|^\alpha)|\Theta^0_+(\zeta)|^2 d|\zeta| < \epsilon.$$

In order to obtain a sequence satisfying (7.11) it now suffices to use the diagonalization procedure.

h. One can consider the quantity $\sigma^-_\omega(f; R, T)$ in a similar way by taking into account that the half-plane $\mathbb{C}^-$ contains the zeroes of the functions S and S_ω. Therefore, to estimate the function $S_\omega(\lambda)$ from below we also need to estimate the Blaschke product with respect to the zeroes Λ_ω. This may be done in the same way as in the previous section.

In addition it can easily be seen that the sequence $\mathcal{R} = \{R_N\}$ may be chosen so that $\mathrm{dist}(\mathcal{R}, \Lambda_\omega \cup \Lambda) > 0$ (thus satisfying the hypothesis of Theorem 6) and *simultaneously*

$$\sigma^+_\omega(f; R_{-M}, R_N) \to 0, M, N \to \infty,$$

$$\sigma^-_\omega(f; R_{-M}, R_N) \to 0, M, N \to \infty.$$

This is the desired sequence.

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