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TILINGS OF CONVEX POLYGONS

by Richard KENYON [(1)]

1. Introduction.

In 1903, Dehn showed that an $a \times b$ rectangle is tileable by a finite number of squares only if a/b is rational. A similar result for equilateral triangles was obtained by Tutte [6] in 1948. These results together suggest the following generalization: a convex polygon can be tiled with *rational* polygons if and only if it is itself rational. We prove this result for three different definitions of *rational polygon* (Theorems 2 and 3, below). Whereas both Dehn's and Tutte's proofs used networks, our proof is group theoretic in nature.

For K a subfield of $\mathbb{R}$, let $T_1 = T_1(K)$ be the set of polygons P such that the ratio of the length of any two edges of P is in K. Let $T_2 = T_2(K)$ be the set of polygons which are real homotheties of polygons which satisfy the two properties: the x-coordinates of the vertices are in K, and the lengths of any vertical edges are in K. Let $T_3 = T_3(K)$ be the set of polygons obtained from polygons with vertices in $K^2 \subset \mathbb{R}^2$ by real homothety and translations.

For an arbitrary polygon P we compute two quadratic forms $q_1(P)$ and $q_2(P)$, where $q_1(P)$ depends on the rationality relations of the edge lengths and $q_2(P)$ depends on the rationality relations of the vertex coordinates. We show

THEOREM 1. — *For $i = 1, 2$, a polygon P can be tiled with polygons in T_i only if $q_i(P)$ is positive semidefinite.*

For convex polygons we can say much more:

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THEOREM 2. — *For $i = 1, 2$ a convex polygon P is tileable with polygons in T_i if and only if $P \in T_i$.*

THEOREM 3. — *A convex polygon P is tileable with polygons in T_3 if and only if $P \in T_3$.*

A variant of this argument gives necessity in the following theorem:

THEOREM 4. — *A convex polygon P in $\mathbb{C}$ with angles which are multiples of π/n and an edge from 0 to 1 is tileable with triangles having angles multiples of π/n if and only if the vertices of P are in $\mathbb{Q}[\mathrm{e}^{2\pi i/n}]$.*

The construction of the tiling in the above theorem uses the geometry of the field $\mathbb{Q}[\cos(2\pi/n)]$.

When n is even, the necessary condition for a tiling to exist in Theorem 4 is also a consequence of a theorem of Laczkovich (Theorem 2 of [3]), whose methods are very different.

Note also that the convexity condition in Theorems 2, 3, and 4 is essential; non-convex counterexamples can easily be found.

2. The group of polygonal paths.

Let $\mathcal{P}$ be the set of polygonal paths in $\mathbb{R}^2$ starting at the origin and parametrized by arc length. We define a product $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$ as follows. For $f, g \in P$, define fg to be the concatenation of f and g, obtained by translating g to the endpoint of f. In other words, letting ℓ be the length of f and ℓ' the length of g,

$$fg(t) = \begin{cases} f(t) & \text{if } t < \ell, \\ g(t-\ell) + f(\ell) & \text{if } \ell \leq t \leq \ell + \ell'. \end{cases}$$

We also define an involution $f \mapsto f^{-1}$ on $\mathcal{P}$, where if ℓ is the length of a path f then ℓ is also the length of f^{-1}, and

$$f^{-1}(t) = f(\ell - t) - f(\ell).$$

That is, f^{-1} is the path f traversed in the reverse direction, and translated so that it starts at the origin.

Let e denote the unique path of length 0. We define an equivalence relation on $\mathcal{P}$ which makes it into a group with the above operations: if a, b, and g are paths and a path f can be written as a product $f = agg^{-1}b$, then $f \sim ab$. Then $\sim$ is the equivalence relation generated by this relation (its transitive closure). In particular, $f \sim e$ if f can be reduced to the constant path by successively removing "backtrackings", that is, subpaths of f of the form gg^{-1}.

Let G be the set of equivalence classes of $\mathcal{P}$. Then G is a group: this can be proved in the same manner as in [4], or by noting that G is isomorphic to the (nonabelian) free product

$$G = \prod_{\mathbb{R}P^1} \mathbb{R}$$

of copies of $\mathbb{R}$, one for each direction in $\mathbb{R}P^1$. An element of G can be represented uniquely as a path with no backtrackings, that is, no subpath is of the form ff^{-1}. Call such a path *minimal.* When there is no risk of confusion we will consider elements of G to be minimal paths.

3. The $\mathbb{R}$-form of a path.

For a path $P \in \mathcal{P}$ define

$$A_{\mathbb{R}}(P) = \int_P x\,\mathrm{d}y.$$

Then $A_{\mathbb{R}}(P)$ depends only on the equivalence class of P in G. For a closed path P, $A_{\mathbb{R}}(P)$ is just the (signed) area enclosed by P.

Let P be a minimal path with n edges $v_1, v_2, \ldots, v_n$ (which are vectors in $\mathbb{R}^2$). Define $X(P)$ to be the set of paths with edges $w_1, \ldots, w_n$ in that order where for each i, the edge w_i is parallel to v_i. We allow w_i to be the zero vector. The *signed length* of w_i is defined to be $|w_i|$ if w_i points in the same direction as v_i, and $-|w_i|$ if it points in the opposite direction. The set of paths $X(P)$ is then parametrized by $(z_1, \ldots, z_n) \in \mathbb{R}^n$, the vector of signed lengths of $w_1, \ldots, w_n$.

On $X(P)$ the function $A_{\mathbb{R}}$ is a homogeneous quadratic function of the signed lengths $z_1, \ldots, z_n$. This follows from the fact that along each edge the contribution to $\int x\,\mathrm{d}y$ is just $\frac{1}{2}(x_1+x_2)(y_2-y_1)$, where $(x_1, y_1), (x_2, y_2)$ are the coordinates of the two endpoints of the edge: these coordinates are homogeneous linear functions of the z_i. Denote by $q(P)$ this quadratic form (the restriction of $A_{\mathbb{R}}$ to $X(P)$); $q(P)$ is a form on $\mathbb{R}^n$ which depends only on the edge directions of P (and their order).

Let $X_0(P)$ be the subset of $X(P)$ consisting of those paths which are closed. Then $X_0(P)$ corresponds to a codimension-2 subspace of $\mathbb{R}^n$, and can be parametrized by the first $(n-2)$ consecutive edge (signed) lengths $(z_1, \dots, z_{n-2})$ (the last two signed lengths z_{n-1}, z_n are uniquely determined by these if the path is to be closed, since v_{n-1} and v_n are independent vectors in $\mathbb{R}^2$). Let $q_0(P)$ be the restriction of $q(P)$ to $X_0(P)$.

LEMMA 5 (see Thurston [5]). — *If P is convex and has n sides then the signature of $q_0(P)$ is $(1, n-3)$.*

Proof (*see also* [1]). — The proof is by induction on n. This is clear if P is a triangle ($n = 3$): the area of a triangle with fixed angles is a positive constant times z_1^2. It is also clear in case P is a parallelogram: for then the area is proportional to $z_1 z_2$, the product of the side lengths of two consecutive sides. Note that the form $z_1 z_2$ has signature $(1,1)$:

$$z_1 z_2 = \tfrac{1}{4}\left((z_1 + z_2)^2 - (z_1 - z_2)^2\right).$$

In case $n \geq 4$ and P is not a parallelogram, there are three consecutive sides a, b, c such that the sum of the two exterior angles is less than π, that is, the extensions of the edges a and c past b intersect. In that case, let P' be the polygon with $n-1$ edges obtained by extending sides a and c to their point of intersection. If a, b, c are edges z_i, z_{i+1}, z_{i+2} of P, then we see that

$$\begin{aligned} q_0(P)(z_1, \dots, z_{n-2}) = q_0(P')(z_1, \dots, z_i + \alpha z_{i+1}, z_{i+2} + \beta z_{i+1}, \\ z_{i+3}, \dots, z_{n-2}) - \gamma z_{i+1}^2, \end{aligned}$$

where the constants $\gamma > 0$, α, β depend only on the angles between a, b and c.

The induction hypothesis is that $q_0(P')$ (which doesn't involve z_{i+1}) has signature $(1, n-4)$, and therefore $q_0(P)$ has signature $(1, n-3)$. $\square$

4. Rational motions and the K-form.

Let K be a subfield of $\mathbb{R}$. In what follows we consider $\mathbb{R}$ to be a vector space over K.

The group G of minimal polygonal paths has a very large group of endomorphisms. We shall be concerned with two subgroups H_1, H_2 of endomorphisms, both isomorphic to $\mathrm{Hom}_K(\mathbb{R}, \mathbb{R})$, but which act on G in different ways.

An element $f \in H_1$ acts on a path P in the following way. If P has signed edge lengths $z_1, z_2, \dots, z_n$, then $f(P)$ is the path with the same edge directions as P but with signed lengths $f(z_1), f(z_2), \dots, f(z_n)$. It is not hard to see that f is an endomorphism of G: indeed, f is an endomorphism of $\mathbb{R}$ and G is a free product of copies of $\mathbb{R}$ (one for each direction); the action of f described above is just the diagonal action on each free factor.

An element $g \in H_2$ acts on G in the following way. For every real α and $x \neq 0$, the image of the segment from $(0,0)$ to $(x, \alpha x)$ is the segment from $(0,0)$ to $(g(x), \alpha g(x))$, and the image of the segment from $(0,0)$ to $(0,\alpha)$ is the segment from $(0,0)$ to $(0, g(\alpha))$. This map preserves directions and is an endomorphism in each direction, and so extends to an endomorphism of G. Note that a vertical segment gets special treatment.

Let $P \in G$ be a minimal path with edges $v_1, \dots, v_n$ having signed lengths $z_1, \dots, z_n$. For $i = 1, 2$, and $f \in H_i$ we have $f(P) \in X(P)$. In fact the map $f \mapsto (f(z_1), \dots, f(z_n))$ is an $\mathbb{R}$-*linear map* from H_1 to $X(P) = \mathbb{R}^n$. For H_2, if we let $\hat{e}_j = \hat{x}$ when v_j is not vertical and $\hat{e}_j = \hat{y}$ if v_j is vertical then the map $f \mapsto (f(v_1) \cdot \hat{e}_1, \dots, f(v_n) \cdot \hat{e}_n)$ is an $\mathbb{R}$-*linear map* of H_2 into $\mathbb{R}^n$.

For $i = 1, 2$ define $q_i(P)$ to be the pullback of $q(P)$ to H_i.

Let V_1 be the K-vector space spanned by the edge lengths of P. Let V_2 be the K-vector space spanned by the differences in x-coordinates of vertices of P and the differences in y-coordinates of vertical edges of P. The forms $q_i(P)$ may be thought of as forms on the finite dimensional $\mathbb{R}$-vector space $\mathrm{Hom}_K(V_i, \mathbb{R}) \subset H_i$, since they are zero on the orthogonal space to this subspace. However later on we will be adding forms having different V_i's and so we will retain the larger space of definition.

Example. — Let P be the closed path in Figure 1, starting at the lower left vertex and going counterclockwise.

If we change the edge lengths to z_1, z_2, z_3, z_4, z_5 then we find

$$q(P) = z_1 z_2 + \frac{z_3}{\sqrt{2}} \frac{(2z_1 - z_3/\sqrt{2})}{2} - z_5\Big(z_1 - \frac{z_3}{\sqrt{2}} - z_4\Big).$$

For a closed path we must have $z_1 = z_3/\sqrt{2} + z_4$ and $z_5 = z_2 + z_3/\sqrt{2}$, and so

$$q_0(P) = z_1 z_2 + \frac{z_3}{2\sqrt{2}}\Big(2z_1 - \frac{z_3}{\sqrt{2}}\Big),$$

which has signature $(1, 2)$ as predicted by Lemma 5.

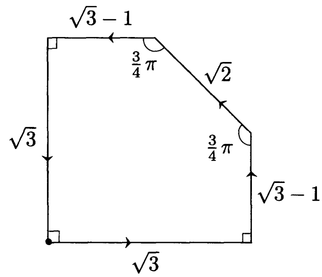

Figure 1. The polygon P of the example.

Let $K = \mathbb{Q}$; then $V_1 = \operatorname{span}_{\mathbb{Q}}\{1, \sqrt{2}, \sqrt{3}\}$. An element of $\operatorname{Hom}_{\mathbb{Q}}(V_1, \mathbb{R})$ is determined by the images of $1, \sqrt{2}$ and $\sqrt{3}$, which can be arbitrary. Let $f_1 \in H_1$ map 1 to x, $\sqrt{2}$ to y, and $\sqrt{3}$ to z; then

$$f_1\big(\sqrt{3}, \sqrt{3}-1, \sqrt{2}, \sqrt{3}-1, \sqrt{3}\big) = (z, z-x, y, z-x, z).$$

This yields

$$q_1(P) = z(z-x) + \frac{y}{\sqrt{2}}\frac{(2z - y/\sqrt{2})}{2} - z\Big(x - \frac{y}{\sqrt{2}}\Big).$$

Note that $f_1(P)$ is not necessarily a closed path.

We have $V_2 = \operatorname{span}_{\mathbb{Q}}\{1, \sqrt{3}\}$. Let $f_2 \in H_2$ map 1 to x and $\sqrt{3}$ to z; then

$$f_2\big(\sqrt{3}, \sqrt{3}-1, 1, \sqrt{3}-1, \sqrt{3}\big) = (z, z-x, x, z-x, z)$$

which gives

$$q_2(P) = y(y-x) + \frac{yx}{\sqrt{2}} - \frac{x^2}{4}.$$

In this case $f_2(P)$ is always a closed path.

Note that $q_1(P)$ has signature $(1,2)$ and $q_2(P)$ has signature $(1,1)$.

5. Rational decompositions.

LEMMA 6. — *For $i = 1,2$, if $P \in T_i$ then $q_0(P) = q_i(P)$ and has signature* $(1,0)$.

Proof. — If $P \in T_1$, then V_1, the K-vector space spanned by the edge lengths of P, is one-dimensional. If $P \in T_2$, then V_2, the K-vector space spanned by the lengths of the projections of the edges to the x-axis and lengths of vertical edges, is one-dimensional over K. If $f \in H_i$ and $P \in T_i$ then this implies that $f(P)$ is a homothety of P. Thus $f(P)$ is closed. Since the area is positive, the signature of $q_0(P)$ is $(1,0)$. □

LEMMA 7. — *If a polygon P is tiled with tiles $\{P_j\}$, with $P_j \in T_1$ for all j, then $q_1(P) = \sum_j q_1(P_j)$. If P is tiled with tiles $P_j \in T_2$, then $q_2(P) = \sum_j q_2(P_j)$.*

Proof. — A tiling of P with tiles $\{P_j\}$ allows us to write the polygonal path P, considered as an element of G, as a product of lassos, that is, paths of the form $x_j P_j x_j^{-1}$ (see [2]). So $P = \prod_j x_j P_j x_j^{-1}$. Since $f \in H_i$ is an endomorphism of G we have

$$f(P) = \prod_j f(x_j) f(P_j) f(x_j)^{-1}.$$

But now since each $P_j \in T_i$, Lemma 6 above shows that the paths $f(P_j)$ are *closed.* Thus for each lasso,

$$A_{\mathbb{R}}\big(f(x_j)f(P_j)f(x_j)^{-1}\big) = A_{\mathbb{R}}\big(f(P_j)\big).$$

This implies that $A_{\mathbb{R}}(f(P)) = \sum_j A_{\mathbb{R}}(f(P_j))$ for all f, from which the result follows. □

Proof of Theorem 1. — The theorem follows from Lemmas 6 and 7: the $q_i(P_j)$ have signature $(1,0)$, and a sum of positive semidefinite forms is positive semidefinite. □

Proof of Theorem 2. — By Lemma 5, $q_0(P)$ has signature $(1, n-3)$. Since $q_i(P)$ is positive semidefinite and is the (pull-back of a) restriction of $q_0(P)$ to a subspace of $\mathbb{R}^{n-2}$, this subspace must be one-dimensional. This subspace also contains the homotheties, and so it must consist solely of homotheties. In particular any $f \in H_i$ acts by homothety on P. This implies that $P \in T_i$. □

Note that this proof says something about the case when P is not convex. Namely, if the signature of $q_0(P)$ is (r,s) (see [1] for how to compute the signature) and P is tileable with tiles in T_i then $q_i(P)$ has signature at most $(r,0)$; the image of H_i in $X(P)$ has dimension at most r. Thus in the case $i=1$, the K-vector space generated by the edge lengths of P

has dimension at most r. In case $i = 2$, the K-vector space generated by the differences in x-coordinates of the vertices of P and the lengths of the vertical edges of P has dimension at most r.

Proof of Theorem 3. — If P is tiled with polygons in T_3, consider rotations of P by angles 0, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$. In each case the tiles are in T_2 and so the rotated P is in T_2 by Theorem 2.

Let (a, b) and $(c, d) \in \mathbb{R}^2$ be vectors representing any two non-parallel edges of P. We conclude from the first rotation that $c/a \in K$ or $a = 0$; we conclude from the second rotation that $(c - d)/(a - b) \in K$ or $a = b$; and we conclude from the third that $d/b \in K$ or $b = 0$. These three equations (including the degenerate cases) imply that all finite ratios of a, b, c, d are in K. Since this is true for any two edges of P, we conclude that $P \in T_3$. $\square$

6. Tilings with triangles having angles $k\pi/n$.

Before we begin the proof of Theorem 4, we illustrate with some examples.

The theorem in case $n = 3$ implies that a hexagon with angles $\frac{2}{3}\pi$ can be tiled with equilateral triangles if and only if all edge length ratios are rational. To see this, note that a real element of $\mathbb{Q}[\mathrm{e}^{2\pi i/3}]$ is rational, since the only Galois automorphism of $\mathbb{Q}[\mathrm{e}^{2\pi i/3}]$ is complex conjugation, which preserves reals. Similarly if z is an element with argument $k\frac{1}{3}\pi$, then $z\,\mathrm{e}^{-k\pi i/3} \in \mathbb{Q}[\mathrm{e}^{2\pi i/3}]$ is real; hence z has rational modulus.

As another example, in the case $n = 4$, an octagon with angles $\frac{3}{4}\pi$ can be tiled with isosceles right triangles if and only if ratios of adjacent edges are rational multiples of $\sqrt{2}$: an element $z \in \mathbb{Q}[i]$ with argument $k\frac{1}{4}\pi$ has rational modulus if k is even, and modulus in $\sqrt{2}\mathbb{Q}$ if k is odd (since then $z/(1 + i)$ has argument $k'\frac{1}{2}\pi$ for some k').

Proof of Theorem 4, necessity. — (As the proof of sufficiency is slightly longer, the reader will permit us to relegate it to the next section.)

Let $L = \mathbb{Q}[\mathrm{e}^{2\pi i/n}]$.

Let us first show that a triangle with angles which are multiples of π/n is a homothetic copy of a triangle with vertices in L. Let t be triangle with vertices $0, 1, z \in \mathbb{C}$ and angles α at 0 and β at 1. Then taking $a = \mathrm{e}^{i\alpha}$

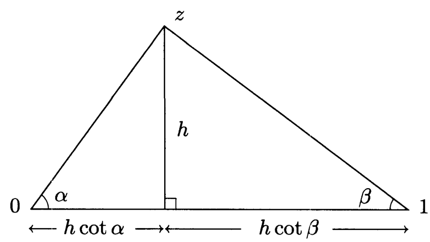

Figure 2. ASA-formula for the third vertex of a triangle

and $b = \mathrm{e}^{i\beta}$, we have $z = h\cot(\alpha) + ih$, where

$$h = \frac{1}{\cot(\alpha) + \cot(\beta)} = \frac{1}{i\left(\frac{a^2+1}{a^2-1} + \frac{b^2+1}{b^2-1}\right)}$$

(see Figure 2). Thus

$$z = \frac{\frac{a^2+1}{a^2-1} + 1}{\frac{a^2+1}{a^2-1} + \frac{b^2+1}{b^2-1}}.$$

In particular if α, β are multiples of π/n then a, b are powers of $\mathrm{e}^{i\pi/n}$, so a^2, b^2 are powers of $\mathrm{e}^{2\pi i/n}$ and so $z \in L = \mathbb{Q}[\mathrm{e}^{2\pi i/n}]$.

Note also if t is the above triangle then the triangle $(1 + \mathrm{e}^{2\pi i/n})t$ has vertices in L and an edge vector of argument π/n.

In a tiling of P with triangles having angles multiples of π/n, edges of tiles have arguments (with respect to the x-axis) which are multiples of π/n. This implies that each triangle is real homothetic either to one of the form $\mathrm{e}^{2\pi ik/n}t$ or to one of the form $(1 + \mathrm{e}^{2\pi i/n})\,\mathrm{e}^{2\pi ik/n}t$, which both have vertices in L.

Let K be the real subfield of L, *i.e.* $K = \mathbb{Q}[\cos(2\pi/n)]$. The real parts of all points in L are in K. The imaginary parts of points in L are in $\sin(2\pi/n)K$, since $\dfrac{\sin(2k\pi/n)}{\sin(2\pi/n)} \in K$ for all integers k.

So, given a tiling of P, apply the linear map

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sin(2\pi/n) \end{pmatrix}$$

to all the tiles; their images are in $T_3(K)$. Since AP is convex, Theorem 2 implies $AP \in T_3(K)$ and so P has vertices in $K + i\sin(2\pi/n)K \subset L$ (note that $i\sin(2\pi/n) \in L$). $\square$

7. Construction of a tiling.

Proof of Theorem 4, sufficiency.

Fix $n \geq 3$. Suppose P satisfies the hypotheses of Theorem 4 and has vertices in L. We must construct a tiling of P.

Cut P with a horizontal line through each vertex of P; this dissects P into trapezoids (and triangles) whose angles are multiples of π/n, and whose vertices are in L. (Note that the intersection of two line segments with vertices in L is again in L.) We will show that each such trapezoid is tileable.

By shaving triangles off the corners of the trapezoid we can reduce ourselves to the case where the trapezoid is a parallelogram with one angle $2\pi/n$: first, take rays spaced evenly every π/n radians from a vertex of the trapezoid. One of these π/n-sectors contains the opposite vertex.

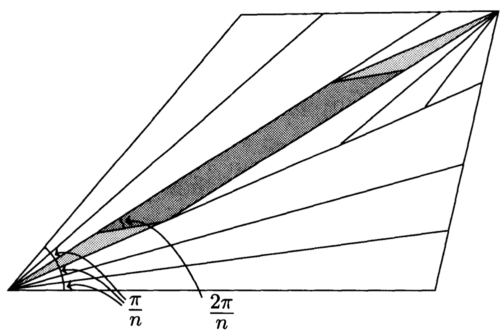

Figure 3. Shaving off triangles from a trapezoid to leave a parallelogram with a $2\pi/n$-angle

In this sector, do the same starting from the opposite vertex. The remaining non-triangle is a parallelogram with angle π/n (shaded in Fig. 3); shave off two $(\pi/n, \pi/n, \pi - 2\pi/n)$ triangles (lightly shaded in the figure) to leave a parallelogram with a $2\pi/n$ angle.

The lengths of the two sides of the parallelogram are either both in $K = \mathbb{Q}[\cos(2\pi/n)]$ or both in $\sin(2\pi/n)K$ (depending on whether the argument of one of its edges is $2\pi k/n$ or $(2k+1)\pi/n$), so the *ratio* of these lengths is in K. We show that, for any positive $u \in K$, a parallelogram P_u with an angle $2\pi/n$ and side length ratio u is tileable. This will complete the proof.

Let

$$S = \{u \mid P_u \text{ is tileable}\}.$$

Note that if $x, y \in S$ then

$$x + y \in S, \tag{1}$$

$$x^{-1} \in S, \tag{2}$$

$$\frac{px}{q} \in S \text{ for all } p/q \in \mathbb{Q}_+, \tag{3}$$

(in fact (3) follows from (1) and (2)).

From the law of sines applied in Fig. 4, we have that

$$\frac{\sin((k+2)\pi/n)}{\sin(k\pi/n)} \in S \text{ for integers } k \in [1, n-3], \tag{4}$$

$$1 \in S. \tag{5}$$

We will show that, starting from (4) and (5) and applying rules (1)–(3) that

$$S = \{u \in K \mid u > 0\}.$$

- If $n = 3$ then $K = \mathbb{Q}$ and we're done by (5) and (3).
- If $n \geq 4$ then the element $z_0 = \dfrac{\sin(3\pi/n)}{\sin \pi/n}$ is in S by (4). We have

$$z_0 = 4\cos^2\left(\frac{\pi}{n}\right) - 1 = 2\cos\left(\frac{2\pi}{n}\right) + 1,$$

which generates K as a field.

Let $\{\sigma_1, \ldots, \sigma_m\}$ be all the embeddings of the totally real field K into $\mathbb{R}$ (here $m = \phi(n)$). We suppose that σ_1 is the identity embedding. Let σ be the embedding of K into $\mathbb{R}^m$, which is the product of the σ_j. This embedding σ has the property that, if C is the convex hull of a set of points $X \subset \sigma(K)$, then $C \cap K$ is exactly the set of convex rational linear combinations of elements of X.

By the *ith coordinate* of $x \in K$ we mean $\sigma_i(x)$, which we also denote x_i.

By the rules (3) and (1), $\sigma(S)$ is the intersection of $\sigma(K)$ with a convex cone in $\mathbb{R}^m$. Furthermore by (2), $\sigma(S)$ is invariant under the map

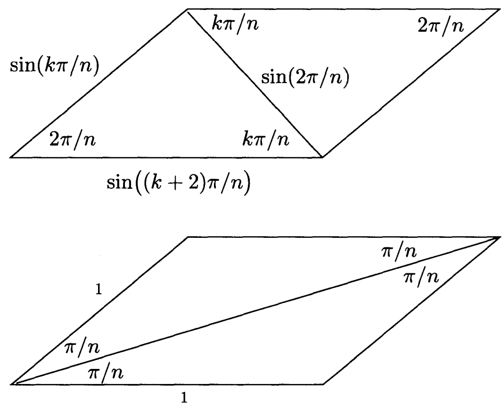

Figure 4. The edge ratios of some basic parallelograms

$\varphi(x_1, \ldots, x_m) = \left(\frac{1}{x_1}, \ldots, \frac{1}{x_m}\right)$. This map φ leaves the open orthants of $\mathbb{R}^m$ invariant.

Our first goal is to construct an element in S which is negative in every coordinate except the first. We accomplish this as follows.

By Lemma 8 below, there is an element $z \in S$ with $\sigma_2(z) < 0$. For a sufficiently small positive rational α, the element $z + \alpha z_0 \in S$ satisfies $\sigma_2(z + \alpha z_0) < 0$ and no two coordinates of $z + \alpha z_0$ are equal (no two coordinates of z_0 are equal since z_0 generates K). Replacing z by $z + \alpha z_0$ if necessary we may assume that no two coordinates of z are equal. Now by Lemma 9 below, S contains the intersection of $\sigma(K)$ with a neighborhood of the point

$$e_z \stackrel{\text{def}}{=} \left(\frac{z_1}{|z_1|}, \ldots, \frac{z_m}{|z_m|}\right) \in \mathbb{R}^m.$$

Suppose there is a coordinate $i > 1$ such that $z_i > 0$. Choose i minimal, that is, such that $z_i > 0$ and $z_j < 0$ for $1 < j < i$). By Lemma 8, there is a $z' \in S$ with $\sigma_i(z') < 0$ (and as before we can assume no two coordinates of z' are equal). As before, S contains a neighborhood of

$$e_{z'} = \left(\frac{z'_1}{|z'_1|}, \ldots, \frac{z'_m}{|z'_m|}\right).$$

There is a point z'', which is the sum of a point close to e_z and a point close to $e_{z'}$, in which after the first, all coordinates up to *and including* the ith coordinate of z'' are negative. Now $z'' \in S$ by (1), and again we can assume that z'' has no two coordinates equal. By Lemma 9 again, S contains a neighborhood of $e_{z''}$. Continue in this manner until an element $x \in S$ is constructed for which all coordinates after the first are negative and no two coordinates are equal.

Now S contains a neighborhood (in $\sigma(K)$) of

$$e_x = \left(\frac{x_1}{|x_1|}, \dots, \frac{x_m}{|x_m|}\right)$$

and also contains the point $\sigma(1) = (1, 1, \dots, 1)$. The convex hull of this neighborhood and the point $(1, \dots, 1)$ contains a neighborhood of the point $(1, 0, 0, \dots, 0)$. So S contains a conical neighborhood of the ray $\{(t, 0, \dots, 0)\}_{t>0}$. Inversion maps this neighborhood to a conical neighborhood of the plane $\{x_1 = 0\}$ intersected with the half-space $\{x_1 > 0\}$. The convex hull of this set is all of $\sigma(K_+)$. □

Lemma 8. — *For each embedding of K not equal to the identity, the image of one of the elements*

$$\frac{\sin((k+2)\pi/n)}{\sin(k\pi/n)}$$

for $k \in [1, n-3]$ is negative.

Proof of Lemma 8. — The Galois conjugates of $\dfrac{\sin((k+2)\pi/n)}{\sin(k\pi/n)}$ are the numbers $\left\{\dfrac{\sin((k+2)p\pi/n)}{\sin(kp\pi/n)}\right\}$, where p is relatively prime to n.

Fix p relatively prime to n. Note that

$$\frac{\sin\left(\frac{(k+2)p\pi}{n}\right)}{\sin\left(\frac{kp\pi}{n}\right)} = \frac{\sin\left(\frac{(k+2)(n-p)\pi}{n}\right)}{\sin\left(\frac{k(n-p)\pi}{n}\right)},$$

so we can assume that $p < \frac{1}{2}n$.

Let $k = \left\lfloor \dfrac{n}{3p} \right\rfloor + 1$, that is, $k \geq 1$ is the smallest integer such that

$$\frac{\pi}{3} < \frac{kp\pi}{n} < \frac{2\pi}{3}. \tag{6}$$

Now the product

$$\frac{\sin\left(\frac{(k+2)p\pi}{n}\right)}{\sin\left(\frac{kp\pi}{n}\right)} \frac{\sin\left(\frac{(k+4)p\pi}{n}\right)}{\sin\left(\frac{(k+2)p\pi}{n}\right)} \cdots \frac{\sin\left(\frac{3kp\pi}{n}\right)}{\sin\left(\frac{(3k-2)p\pi}{n}\right)} = \frac{\sin\left(\frac{3kp\pi}{n}\right)}{\sin\left(\frac{kp\pi}{n}\right)}$$

is less than zero since (6) implies $\pi < \dfrac{3kp\pi}{n} < 2\pi$. Since the product is negative, some factor must be negative.

Furthermore,

$$3k = 3\left(\left\lfloor \frac{n}{3p} \right\rfloor + 1\right) \leq 3\left(\frac{n}{3p} + 1\right) = \frac{n}{p} + 3 < n - 2$$

if $n > 10$ since $p \geq 2$. So in this case all factors are in S by (4). This completes the proof if $n > 10$.

To treat the remaining cases $n \leq 10$, we can immediately eliminate $n = 8$ and $n = 10$ since for these $p \geq 3$ and the above inequality holds.

- For $n = 9$ there are two non-identity embeddings, with $p = 2$ and $p = 4$. The embedding $p = 4$ is eliminated as above, and for $p = 2$ we have

$$\frac{\sin\left(2 \cdot \frac{5\pi}{9}\right)}{\sin\left(2 \cdot \frac{3\pi}{9}\right)} < 0.$$

- For $n = 7$ there are two non-identity embeddings corresponding to $p = 2$ and $p = 3$. For $p = 2$ we have

$$\frac{\sin\left(2 \cdot \frac{4}{7}\pi\right)}{\sin\left(2 \cdot \frac{2}{7}\pi\right)} < 0,$$

and for $p = 3$ we have

$$\frac{\sin\left(3 \cdot \frac{3}{7}\pi\right)}{\sin\left(3 \cdot \frac{1}{7}\pi\right)} < 0.$$

- For $n = 6$ and $n \leq 4$ there is only one embedding, and
- for $n = 5$ the other embedding corresponds to $p = 2$, where we have

$$\frac{\sin\left(2 \cdot \frac{3}{5}\pi\right)}{\sin\left(2 \cdot \frac{1}{5}\pi\right)} < 0. \qquad \square$$

LEMMA 9. — *If $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ has nonzero coordinates and no two coordinates equal then any convex cone containing x and closed under φ contains a neighborhood of the axis of symmetry of the orthant containing x (i.e. a neighborhood of the element $e_x = \left(\dfrac{x_1}{|x_1|}, \ldots, \dfrac{x_m}{|x_m|}\right)$).*

Proof of Lemma 9. — Suppose without loss of generality that all coordinates of x are positive.

The proof is by induction on the dimension m. If $m = 2$, then the convex hull in projective space ($\mathbb{R}P^1$ in this case) of $x = (x_1, x_2)$ and $\varphi(x) = \left(\frac{1}{x_1}, \frac{1}{x_2}\right)$ is a segment containing $e_x = (1,1)$ in its interior.

Suppose the result is true for dimensions $< m$. Let S be a convex cone containing x and closed under inversion.

For $a, b \in \mathbb{R}_+$, the quantity $ax + b\varphi(x)$ has coordinates i and j equal if and only if

$$ax_i + \frac{b}{x_i} = ax_j + \frac{b}{x_j},$$

or in other words

$$a(x_i - x_j) = b\left(\frac{1}{x_j} - \frac{1}{x_i}\right)$$

which is true if and only if $\frac{b}{a} = x_i x_j$.

Suppose without loss of generality that $x_1 > x_2 > \cdots > x_m$. Then the point

$$y = (x_1 \ldots, x_m) + x_1 x_2 \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_m}\right)$$

has the first two coordinates equal and no two other coordinates are equal since $x_1 x_2 > x_i x_j$ for any other pair i, j.

We can now work in the subspace $\Delta_{12} = \{x \mid x_1 = x_2\}$. Inversion preserves this $(n-1)$-dimensional subspace, which we parametrize by the last $(n-1)$ coordinates. Since $y \in \Delta_{12}$, and y has no two coordinates equal, by induction $S \cap \Delta_{12}$ is a convex cone containing a neighborhood U of e_y ($= e_x$). The convex hull of $x \notin \Delta_{12}$ and U contains the intersection of a neighborhood of e_x with the half-space bounded by Δ_{12} and containing x. Since the derivative of φ at e_x is minus the identity, S contains a neighborhood of e_x. □

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