ANNALES DE L'INSTITUT FOURIER

BRUCE L. REINHART

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Annales de l'institut Fourier, tome 14, nº 1 (1964), p. 49-52
<http://www.numdam.org/item?id=AIF_1964__14_1_49_0>

Ann. Inst. Fourier, Grenoble
14, 1 (1964), 49-52.

COBORDISM AND FOLIATIONS

by Bruce L. REINHART [1]
(RIAS and University of Maryland.)

Let V be a connected, oriented differential manifold, on manifold, on which is given an oriented foliation of codimension 1. Let $M_1, \ldots, M_k$ be compact leaves. If we cut V open along $M_1 \cup \ldots \cup M_k$, we get a certain number of manifolds with boundary, hence a certain number of cobordism relations. These are of three types:

1. The usual oriented cobordism of Thom.

2. A vector cobordism, in which we assume the existence of a non-singular vector field normal to the boundary.

3. A foliated cobordism, in which we assume the existence of a foliation of which the components of the boundary are leaves. In the present paper, we shall apply these three types types of cobordism to the study of the leaves of the foliation on V.

LEMMA. — *If* $M_1, \ldots, M_k$ *are disjoint compact oriented submanifolds of dimension n in* V^{n+1} *and if k is greater then the first Betti number of* V, *then there exists a cobordism relation*

$$\sum_{i=1}^{k} \varepsilon_i M_i = 0 \quad \varepsilon_i = 0, \pm 1 \sum \varepsilon_i^2 > 0.$$

[1] This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research, Office of Aerospace Research under Contract. No. AF 49 (638)-1242, in part by the Office of Naval under Research Contract Nonr-3692 (00), and in part by the National Aeronautics and Space Administration under Contract NASw-718. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Proof. — Let $M = M_1 \cup \ldots \cup M_k$ and consider the exact sequence for cohomology with compact supports

$$\ldots \to H^{n-1}(V) \xrightarrow{i^*} H^{n-1}(M) \xrightarrow{\delta} H^n(V - M) \xrightarrow{j^*} H^n(V) \to 0.$$

Since the first group has rank $b_1(V)$ and the second is free abelian of rank k, i^* cannot be onto, and in fact the quotient by its image contains elements of infinite order. Hence, the image of δ has rank $r > 0$, so that $(V - M)$ has $r + 1$ components, $r + 1 > 1$. Each component V_α is a manifold with boundary, and

$$\delta_i : H^{n-1}(M_i) \to H^n(V_\alpha - M_i),$$

is an isomorphism if $M_i \subset V_\alpha$. Hence the coefficients of δ with respect to the natural bases in its domain and range are $0, \pm 1$.

Since M is a boundary if and only if its Pontryagin and Stiefel-Whitney numbers vanish [3, 4], we have

$$\Sigma \varepsilon_i W(M_i) = 0, \qquad \Sigma \varepsilon_i P(M_i) = 0,$$

for each such number W or P.

Since the foliation is oriented, there is a complementary vector field which is interior normal on M_i if $\varepsilon_i = 1$ and exterior normal if $\varepsilon_i = -1$. Thus, the following definition is relevant to our problem [2] :

M_1 is vector cobordant to M_2 if and only if there is a manifold N and a vector field F on N such that

a) $\partial N = M_1 \cup M_2'$ where M_2' is M_2 with the orientation reversed

b) F is nonsingular.

c) F is interior normal on M_1 and exterior normal on M_2.

The set of cobordism classes with the operation disjoint union is a commutative monoid with a neutral element (the empty manifold), of which every element is regular :

$$xTy = xTz \qquad \text{implies} \qquad y = z.$$

In fact, this monoid is a group Ω_0^n for each $n > 0$. This group admits a canonical homomorphism onto the Thom group Ω^n. The kernel is a cyclic group, and Ω_0^n is isomorphic to the direct sum of Ω^n and this kernel. Finally, two manifolds belong to the same vector cobordism class if and only if either $n \neq 4k + 1$

and the manifolds have the same Euler, Pontryagin, and Stiefel-Whitney numbers, or $n = 4k + 1$ and the manifolds bound an oriented manifold with even Euler number [2]. Hence, we have the following result.

THEOREM. — *If* $M_1, \ldots, M_k$ *are compact leaves of an oriented foliation of codimenison* 1 *on an oriented* V^{n+1}, *and* $k > b_1(V)$, *then there are relations*

$$\Sigma\varepsilon_i X(M_i) = 0 \quad \Sigma\varepsilon_i P(M_i) = 0 \quad \Sigma\varepsilon_i W(M_i) = 0$$

where X, P, and W are Euler, Pontryagin, and Stiefel-Whitney numbers respectively, and

$$\varepsilon_i = 0, \ \pm 1 \qquad \text{but not all} \qquad \varepsilon_i = 0.$$

Furthermore, if $n = 4k + 1$, each component of

$$V - (M_1 \cup \ldots \cup M_k)$$

has even Euler number.

Finally, we may introduce foliated cobordism by requiring that N admit a foliation complementary of F of which M_1 and M_2 are leaves. Again the cobordism classes from a commutative monoid with neutral element, of which every element is regular. We shall see that in general, inverses do not exist. However, we can still embed this monoid in a group, which admits a canonical homomorphism onto Ω_0^n. There are at present no general results on foliated cobordism. In dimension 2, the foliation described by Reeb [1] shows that the torus is a boundary. On the other hand, the global stability theorem of Reeb [1] shows that if M_1 is the sphere S_2, then so is M_2 and in fact $N = S^2 \times [0, 1]$. Hence, S^2 admits no inverse. Since Ω_0^n is free cyclic, one may conjecture that the foliated cobordism monoid is the direct sum of two copies of the non-negative integers. By use of the global stability theorem and the Euler number, one may show that neither the four sphere nor the real projective plane admit an inverse.

Throughout this paper, orientation conditions have been assumed. We may discard the assymptions that V and the foliation are oriented, provided we assume that the transverse line element field is oriented. The changes that follows have been discussed elsewhere [2].

As Thom has pointed out (oral communication) a natural generalization of cobordism is obtained by supposing that M_i and N are foliated of the same codimension k. The case $k = 0$ is the usual cobordism and the case $k = \dim M_i$ is vector cobordism. Nothing is presently known about the intermediate cases.

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