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KLAS DIEDERICH
EMMANUEL MAZZILLI

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EXTENSION AND RESTRICTION OF HOLOMORPHIC FUNCTIONS

by K. DIEDERICH and E. MAZZILLI

1. Introduction.

In [20] (see also [18]) T. Ohsawa and K. Takegoshi showed the following result:

THEOREM 1.1. — *Let $D \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex domain, $\psi : D \to \mathbf{R} \cup \{-\infty\}$ a plurisubharmonic function and $A \subset \mathbb{C}^n$ a complex linear hyperplane. Put $D' := D \cap A$. Then there is a constant C depending only on the diameter of D such that for any function $f \in \mathcal{O}(D')$ with*

$$\int_{D'} |f|^2 e^{-\psi}\, d\lambda_{n-1} < \infty$$

there is a function $F \in \mathcal{O}(D)$ satisfying $F|D' = f$ and

$$\int_D |F|^2 e^{-\psi}\, d\lambda_n \leqslant C \int_{D'} |f|^2 e^{-\psi}\, d\lambda_{n-1}.$$

Meanwhile, this fact has proved to be an extremely useful tool in complex analysis and complex algebraic geometry (see, for instance, the recent articles [5], [6], [7] and [21]).

There are many ways of generalizing the question of extending holomorphic functions from submanifolds. Here, we will be interested in the following two possibilities:

1) Replacing the L^2-norm by L^p-norms for $2 \leqslant p \leqslant \infty$.

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2) Introducing plurisubharmonic weight functions into these norms and investigating their dependence on the geometry of ∂D.

Both questions will also be asked for complex submanifolds A of higher codimension.

Many positive results concerning these questions can be found in the literature. We mention, for instance, the articles [13], [1], [4], [2], the book [14] and the articles [19], [12], [10], [9], [11]. A long time before this development H. Skoda showed in [22], how deep theorems on division and extension can be derived by refinements of the L^2-theory for the $\overline{\partial}$-equations.

In the case of question 1) the situation can be drastically different from the spirit of the result of T. Ohsawa/K. Takegoshi, as has already been shown in the article [17] by the second author (see also his thesis [15] and [16]). In fact, the proofs of the present article use some of the construction principles of [15] in an essential way. We will show, for instance, that for any $2 < p \leqslant \infty$ and any given $\varepsilon > 0$ there is, for any sufficiently large n, a pseudoconvex domain $D \subset\subset \mathbb{C}^n$ with smooth polynomial boundary and an affine linear subspace $A \subset \mathbb{C}^n$ of sufficiently large dimension and codimension, such that there is no extension operator from $H^p(D') := L^p(D') \cap \mathcal{O}(D')$ to $H^{2+\varepsilon}(D)$.

The main motivation for asking question 2) is the well-known observation (see also the above-mentioned literature), that in certain situations there is a "regularity gain" for the holomorphic extension of holomorphic functions from submanifolds. In fact, it has been tried to characterize this gain in terms of the geometry of $\partial D'$ and this should, in principle, be possible. However, in a certain sense it does not only depend on the Levi form of ∂D. Namely, in this article we give an example of a pair (D, A) where A has codimension 1 and the Levi form of ∂D is strongly positive definite in all directions transverse to A at all points of $\partial D'$, but, nevertheless, there is essentially no gain for the extension of holomorphic L^2-functions from D' to D.

An important question asked by several authors is, whether for any bounded pseudoconvex domain $D \subset\subset \mathbb{C}^n$ with $\mathcal{C}^\infty$-smooth boundary and any affine hyperplane A with $D' := D \cap A$ there is a plurisubharmonic function φ on D' such that one has for the restriction map $r : \mathcal{O}(D) \to \mathcal{O}(D')$

$$r\big(H^2(D)\big) = H^2(D', \varphi)\,. \tag{1.1}$$

(For notations see the next section.)

We show in this paper, that this is in general not the case. In fact, we will construct a pair (D, A) and functions $f \in H^2(D)$ and $g \in \mathcal{O}(D')$ with $|g| \leqslant |r(f)|$ everywhere on D', such that, nevertheless, the function g cannot be extended to a function in $H^2(D)$. This shows, surprisingly enough, that the L^2-holomorphic extendability of holomorphic functions $f \in \mathcal{O}(D')$ to D is not just a question of the size of f.

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2. Notations and results.

For an open set $U \subset \mathbb{C}^k$ and a measurable function $\varphi : U \to \mathbf{R} \cup \{-\infty\}$ we define the Hilbert space

$$L^2(U, \varphi) := \left\{ g : U \to \mathbb{C} | \int_U |g|^2 e^{-\varphi} \, d\lambda < \infty \right\}$$

and put $H^2(U, \varphi) := L^2(U, \varphi) \cap \mathcal{O}(U)$. Furthermore, for any real number $1 \leqslant q < \infty$, we use in this article the notation $H^q(U) := L^q(U) \cap \mathcal{O}(U)$ and, similarly, for $q = \infty$, $H^q(D) := L^\infty(D) \cap \mathcal{O}(D)$. (This should not be confused with Hardy spaces).

Finally, we denote for any integer $1 \leqslant p \leqslant n-1$ by A_p the complex linear hyperplane

$$A_p := \{ z \in \mathbb{C}^n | z_1 = \cdots = z_p = 0 \} \subset \mathbb{C}^n$$

of codimension p.

Our main results are:

THEOREM 2.1. — *Let $n \geqslant 3$ and $2 \leqslant q \leqslant \infty$. Then there is for any given $\varepsilon > 0$ a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ with smooth real-analytic boundary with $D' := A_1 \cap D \neq \varnothing$ and A_1 intersects ∂D transversally, such that there is a function $f \in H^q(D')$ which does not have a holomorphic extension in the Banach space $H^{\frac{6q}{q+4}+\varepsilon}(D)$ (resp. in $H^{6+\varepsilon}(D)$ for $q = \infty$).*

Remark 2.2. — Notice, that, according to [8], the domain D of the theorem necessarily is of finite type.

When affine subspaces A of arbitrarily high dimension and codimension are admitted, the regularity bound in terms of H^q can get arbitrarily close to 2, as the following result shows:

THEOREM 2.3. — *Choose arbitrary integers $p \geqslant 1$ and $2 \leqslant q \leqslant \infty$. Then there is for any $\varepsilon > 0$ a pseudoconvex domain $D \subset\subset \mathbb{C}^{2p+1}$ with smooth polynomial boundary and a function $f \in H^q(D')$, $D' := D \cap A_p$, such that f does not have a holomorphic extension in $H^{\frac{2q(p+2)}{qp+4}+\varepsilon}(D)$ (resp. $H^{\frac{2(p+2)}{p}+\varepsilon}(D)$ for $q = \infty$).*

An immediate consequence of this is:

COROLLARY 2.4. — *For each $\varepsilon > 0$ there is a positive integer p and a pseudoconvex domain $D \subset\subset \mathbb{C}^{2p+1}$ with smooth polynomial boundary, such that A_p intersects ∂D transversally at all points of $\partial D \cap A_p \neq \varnothing$ and such that there is a holomorphic function $f \in H^\infty(D')$, $D' := D \cap A_p$, not admitting an extension in $H^{2+\varepsilon}(D)$.*

We, now, come to the question of the possible regularity gain, when extending holomorphic functions from $D' = D \cap A_1$ to D, when $\mathcal{L}_{\partial D}(z;t) > 0$ at all points $z \in \partial D'$ and for all vectors $t \in T_z^{10}\partial D$ transverse to A_1. If, in this case, the boundary ∂D is extendable in a pseudoconvex way of order 2 along $\partial D'$, which one might expect at first (for the definition see [12] and [10]), then there is always an extension with a "good" gain with respect to L^2-norms with weights ([10]). However, the following result shows, that the situation can be quite bad:

THEOREM 2.5. — *Let $\chi > 0$ be a real number. Then there is a pseudoconvex domain $D \subset\subset \mathbb{C}^3$ with $\mathcal{C}^\infty$-smooth boundary, such that $z_0 := (0,0,1) \in \partial D$, $T_{z_0}^{1,0}\partial D = \{t = (t_1, t_2, 0) \in \mathbb{C}^3\}$ and $\mathcal{L}_{\partial D}(z_0;t) > 0$ for all $t \in T_{z_0}^{1,0}\partial D$ with $t_1 \neq 0$, and a function $f \in H^2(D', -\chi \log d(z, \partial D))$, such that, for any strictly positive numbers $r, \varepsilon \in \mathbf{R}$ the function f does not admit an extension to $H^2(D \cap B(z_0;r), -(\chi - \varepsilon)\log d(z, \partial D))$ (here $d(z, \partial D)$ denotes the euclidean distance between z and ∂D).*

Let, again, $D \subset\subset \mathbb{C}^n$ be a pseudoconvex domain such that $D' := D \cap A_1 \neq \varnothing$. As in the introduction we denote by $r : \mathcal{O}(D) \to \mathcal{O}(D')$ the restriction map. We want to ask, whether the space $r(H^2(D))$ always can be characterized as $H^2(D', \varphi)$ with a suitable plurisubharmonic function φ on D'. Unfortunately, this is far from being true, because we have

THEOREM 2.6. — *There is a pseudoconvex domain $D \subset\subset \mathbb{C}^3$ with smooth polynomial boundary and A_1 intersecting ∂D everywhere transversally, such that the following holds:*

There is a function $f \in H^2(D)$ and a function $h \in \mathcal{O}(D')$ with $|h(z)| \leqslant |f(z)|$ for all $z \in D'$, such that $h \notin r(H^2(D))$.

An immediate consequence of this obviously is

COROLLARY 2.7. — *Let D be as in Theorem 2.6. Then there is no positive measure $d\nu$ on $D' = D \cap A_1$, such that*

$$r\big(H^2(D)\big) = H^2(D', d\nu).$$

Remark 2.8. — If $D \subset\subset \mathbb{C}^n$ is pseudoconvex with smooth $\mathcal{C}^2$ boundary and such that $D' := D \cap A_1 \neq \varnothing$ and A_1 intersects ∂D everywhere transversally, then there always exist plurisubharmonic functions φ on D', such that $r(H^2(D)) \subset H^2(D', \varphi)$. Namely, for instance, the function $\varphi := -2 \log d(z, \partial D)$ has this property.

3. Cauchy type estimates.

The domains the existence of which is claimed in our theorems are almost always biholomorphic images of pseudoellipsoids. It is, therefore, useful for us to introduce the following notations fixed throughout this article. For any $N \in \mathbb{N}$, $N \geqslant 2$, and any $n \geqslant 2$ we denote by D_N the following pseudoellipsoid:

(3.1)
$$D_N := \left\{ z \in \mathbb{C}^n : \rho_N(z) := |z_1|^{2^{N+1}} + |z_2|^2 + \sum_{j=3}^{n} |z_j|^{2^{N+1}} - 1 < 0 \right\}.$$

For any $\varepsilon' > 0$ small enough the circle

$$S(\varepsilon') := \left\{ z = \left(\varepsilon'^{\frac{1}{2^{N+1}}} e^{i\theta}, 0, \ldots, 0, 1 - \varepsilon' \right) : \theta \in [0, 2\pi] \right\} \tag{3.2}$$

is then contained in D_N and is close to the boundary point $z_0 := (0, \ldots, 0, 1)$.

Furthermore, if $p \geqslant 1$ is an integer we write

(3.3)
$$D_{N,p} := \left\{ z \in \mathbb{C}^{2p+1} : \rho_{N,p}(z) := \sum_{j=1}^{p} |z_j|^{2^{N+1}} + \sum_{j=p+1}^{2p+1} |z_j|^2 - 1 < 0 \right\}.$$

In this case, for $\epsilon' > 0$ small enough, the torus

(3.4)
$$S_p(\varepsilon') := \left\{ z \in \mathbb{C}^{2p+1} \, : \, z = \left(\varepsilon'^{\frac{1}{2^{N+1}}} e^{i\theta_1}, \dots, \varepsilon'^{\frac{1}{2^{N+1}}} e^{i\theta_p}, 0, \dots, 0, 1 - p\varepsilon' \right) \right\}$$

of real dimension p is contained in $D_{N,p}$ close to the boundary point $z_0 = (0, \dots, 0, 1)$.

As an important technical tool for the proofs we will need Cauchy type estimates for the values of functions in $H^q(D_N)$ resp. $H^q(D_{N,p})$ on $S(\varepsilon')$ resp. $S_p(\varepsilon')$. Here comes the first:

LEMMA 3.1. — *For any $N \geqslant 2$ there is a constant $C_N > 0$ such that one has for all $2 \leqslant q < \infty$ and all functions $f \in H^q(D_N)$ the estimate*

$$|f(z)| \leqslant C_N \|f\|_{L^q(D_N)} d(z, \partial D_N)^{-\frac{3}{q} - \frac{n-2}{q 2^N}}$$

for all $z \in S(\varepsilon')$, $\varepsilon' > 0$ sufficiently small.

Proof. — We use the Henkin-Ramirez integral formulas with weight factors as established by B. Berndtsson and M. Andersson in [3]. (The use of the usual Cauchy estimates on suitable polydiscs also would be possible.) They give for any function $f \in H^q(D_N)$ a representation of the form

$$f(z) = \int_{\zeta \in D_N} f(\zeta) \mathcal{P}(\zeta, z) \tag{3.5}$$

for all $z \in D_N$. Here

$$\mathcal{P}(\zeta, z) = \left(\frac{\rho_N(\zeta)}{\langle \partial \rho_N(\zeta), z - \zeta \rangle + \rho_N(\zeta)} \right)^{n+1} \left(\partial \bar{\partial} \log \left(-\frac{1}{\rho_N(\zeta)} \right)^n \right). \tag{3.6}$$

From the Hölder inequalities we, hence, get

$$|f(z)| \leqslant \left(\int_{D_N} |f(\zeta)|^q \, d\lambda(\zeta) \right)^{\frac{1}{q}} \left(\int_{D_N} |\mathcal{P}(\zeta, z)|^{\frac{q}{q-1}} \, d\lambda(\zeta) \right)^{\frac{q-1}{q}}. \tag{3.7}$$

We consider for a sufficiently small radius $r > 0$ the ball $B := B(z_0, r)$ and choose a constant $C > 0$ such that $|\frac{\partial \rho_N}{\partial \zeta_n}| > C$ for all $\zeta \in B$. If $\varepsilon' > 0$ is small enough, we have $S\left(\varepsilon'\right) \subset B(z_0, \frac{r}{2}\Big)$. Hence it follows from (3.5) and (3.6), that there is a constant $K > 0$, such that

$$\int_{D_N} |\mathcal{P}(\zeta, z)|^{\frac{q}{q-1}}\, d\lambda(\zeta) \leqslant K \int_{D_N \cap B} |\mathcal{P}(\zeta, z)|^{\frac{q}{q-1}}\, d\lambda(\zeta)$$

for all $z \in S(\varepsilon')$. On the other hand, the classical way of estimating Cauchy-Fantappié kernels of the form (3.6) gives the abbreviation

$$\begin{aligned}\mathcal{D}(\zeta, z) :=& \rho_N(z) - \rho_N(\zeta) + |z_1 - \zeta_1|^{2^{N+1}} \\ &+ \sum_{j=3}^{n-1} |\zeta_j|^{2^{N+1}} + |\zeta_2|^2 + |\operatorname{Im}\langle \partial\rho_N(\zeta), z - \zeta\rangle|\end{aligned}$$

the inequality

$$|\mathcal{P}(\zeta, z)| \leqslant C_N \frac{\prod_{j=1, j\neq 2}^{n-1} |\zeta_j|^{2^{N+1}-2}}{[\mathcal{D}(\zeta, z)]^{n+1}}. \tag{3.8}$$

With the triangle inequality $|\zeta_1| \leqslant |\zeta_1 - z_1| + |z_1|$ one obtains for all $z \in S(\varepsilon')$ the estimate

$$|\mathcal{P}(\zeta, z)|^{\frac{q}{q-1}} \leqslant C_N [\mathcal{D}(\zeta, z)]^{\alpha} \tag{3.9}$$

with $\alpha := -\frac{q}{q-1}\left(3 + \frac{n-2}{2^N}\right)$. For integrating this inequality we make the following Henkin type change of variables:

$$\begin{cases} u_1 = \zeta_1 - z_1 \\ \vdots \\ u_{n-1} = \zeta_{n-1} - z_{n-1} \\ u_n = \rho_N(\zeta) + i \operatorname{Im}\langle \partial\rho_N(\zeta), z - \zeta\rangle. \end{cases} \tag{3.10}$$

With this the result follows easily. □

We now formulate the necessary Cauchy estimate for the domains $D_{N,p}$:

LEMMA 3.2. — *Let $N \geqslant 2$ and $p \geqslant 1$ be integers. Then there is a constant $C_{N,p} > 0$, such that one has for all $2 \leqslant q < \infty$ and all functions $f \in H^q(D_{N,p})$ the estimate*

$$|f(z)| \leqslant C_{N,p} \|f\|_{L^q(D_{N,p})} d(z, \partial D_{N,p})^{-\frac{p+2}{q} - \frac{p}{q2^N}}$$

for all $z \in S_p(\varepsilon')$ and all $\varepsilon' > 0$ sufficiently small.

Proof. — The proof of this lemma is exactly parallel to the proof of Lemma 3.1. For brevity we now use the definition

$$\mathcal{D}(z,\zeta) := -\rho(z) - \rho(\zeta) + \sum_{j=1}^{p} |\zeta_j - z_j|^{2^{N+1}} + \sum_{j=p+1}^{2p} |\zeta_j|^2 + |\mathrm{Im}\langle \partial\rho(\zeta), z-\zeta\rangle|.$$

Estimate (3.8) becomes

$$|\mathcal{P}(\zeta,z)| \leqslant C_{N,p} \frac{\prod_{j=1}^{p} |\zeta_j|^{2^{N+1}-2}}{[\mathcal{D}(\zeta,z)]^{2p+2}}$$

where ρ always means $\rho_{N,p}$. Together with the estimates $|\zeta_j| \leqslant |\zeta_j - z_j| + |z_j|$ we get for all $z \in S_p(\varepsilon')$

$$|\mathcal{P}(\zeta,z)|^{\frac{q}{q-1}} \leqslant C_{N,p}[\mathcal{D}(\zeta,z)]^{\alpha}$$

with $\alpha = -\frac{q}{q-1}\left(p + 2 + \frac{p}{2^N}\right)$. By integration using again the change of variables (3.10), we obtain the desired estimate. □

For the proof of Theorem 2.5 we will need the following third version of Cauchy estimates:

LEMMA 3.3. — *Put for any integer $k \geqslant 2$*

$$\hat{D}_k := \left\{ z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^{2k} + |z_3|^2 - 1 < 0 \right\}$$

and $z_0 := (0,0,1)$. Let $\chi > 0$ be a real number. Then there is a constant $\hat{C}_q > 0$ such that for all sufficiently small numbers $r, \varepsilon' > 0$ and all functions $h \in H^2(\hat{D}_k \cap B(z_0,r), -\chi \log d(z, \partial\hat{D}_k))$ the estimate

$$|h(z)| \leqslant C_k \|h\|_{L^2\left(\hat{D}_k \cap B(z_0,r), -\chi \log d\left(z,\partial\hat{D}_k\right)\right)} \varepsilon'^{-\frac{1}{2}\left(3+\frac{1}{k}+\chi\right)}$$

holds for all z on the circle $\hat{S}(\varepsilon') := \{z = (0, \varepsilon'^{\frac{1}{2k}} e^{i\theta}, 1 - (2^{2k}+1)\varepsilon') : \theta \in [0,2\pi]\}$.

Proof. — For $\varepsilon' > 0$ sufficiently small and any point $z = (z_1, z_2, z_3) \in \hat{S}(\varepsilon')$, we consider the polydisc

$$\mathcal{P}_z := \left\{ \zeta : |\zeta_1| \leqslant \varepsilon'^{1/2}, |\zeta_2 - z_2| \leqslant \varepsilon'^{\frac{1}{2k}}, |\zeta_3 - z_3| \leqslant \varepsilon' \right\}. \tag{3.11}$$

Then $\mathcal{P}_z \subset \hat{D}_k \cap B(z_0; r)$ and from the Cauchy mean value inequality we get

$$|h(z)| \leqslant \frac{1}{\operatorname{vol}(\mathcal{P}_z)} \int_{\mathcal{P}_z} |h(\zeta)|\, d\lambda_\zeta \tag{3.12}$$

for all $z \in \hat{S}(\varepsilon')$ and for all $h \in \mathcal{O}(\hat{D}_k)$. If we take $h \in H^2(\hat{D}_k \cap B(z_0, r), -\chi \log d(z, \partial \hat{D}_k))$, we get by applying the Hölder inequality to (3.12)

$$|h(z)| \leqslant C_k \frac{\varepsilon'^{-\frac{\chi}{2}}}{\operatorname{vol}(\mathcal{P}_z)} \|h\|_{L^2\left(\hat{D}_k \cap B(z_0, r), -\chi \log d\left(z, \partial \hat{D}_k\right)\right)}$$

which implies the lemma. □

4. The relevant examples.

In this section we construct the examples of domains D and holomorphic functions h on $D' = D \cap A_p$ needed for the proofs of the theorems. At the same time we will investigate in a series of lemmas the growth behavior of possible holomorphic extensions of the functions h to D. In fact, more precisely, the domains D of the theorems will be certain biholomorphic images of pseudoellipsoids. In this section we will, as in the previous section, only consider the relevant pseudoellipsoids $\tilde{D}$. They, then, will have to be intersected by suitable complex manifolds $V \subset \mathbb{C}^n$ and we will construct the required holomorphic functions on $\tilde{D}' := D \cap V$ and investigate the growth behavior of their possible holomorphic extensions on the circles as considered in Lemmas 3.1, 3.2 and 3.3.

At first we show

LEMMA 4.1. — *Let $N \geqslant 2$ be an integer and define the domain D_N as in (3.1). Put*

$$V_N := \left\{ z \in \mathbb{C}^n : z_1^N + z_2 = 0 \right\}.$$

Furthermore, let $2 \leqslant q \leqslant \infty$ be any real number and

$$f_N(z) := \frac{z_1^{N-1}}{(1-z_n)^{\frac{N-1}{2N}+\frac{2}{q}}}.$$

Then V_N intersects ∂D_N everywhere transversally and $f_N \in H^q(D'_N)$ where $D'_N := D_N \cap V_N$. Furthermore, there is no holomorphic extension $F \in \mathcal{O}(D_N)$ of f_N, such that, for some $\varepsilon > 0$ there exists a constant $C > 0$, such that one has for all $\varepsilon' > 0$ sufficiently small and all $z \in S(\varepsilon')$ the inequality

$$|F(z)| \leqslant Cd(z, \partial D_N)^{-\frac{N-1}{2N}-\frac{2}{q}+\frac{N-1}{2^{N+1}}+\varepsilon}. \tag{4.1}$$

Proof. — We leave the verification of the fact, that V_N and ∂D_N intersect everywhere transversally, to the reader. For estimating the function f_N we use, that according to (3.1)

$$1 - |z_n|^{2^{N+1}} = -\rho(z) + |z_1|^{2^{N+1}} + |z_2|^2 + \sum_{j=3}^{n-1} |z_j|^{2^{N+1}}. \tag{4.2}$$

Hence, on D_N, with $x_n = \operatorname{Re} z_n$ and $y_n = \operatorname{Im} z_n$ the estimate holds

$$1 - x_n \geqslant C\Big[-\rho(z) + |z_1|^{2^{N+1}} + |z_2|^2 + |y_n|\Big]. \tag{4.3}$$

Restricting this to $D'_N := D_N \cap V_N$ gives

$$1 - x_n \geqslant C\Big[-\rho(z) + |z_1|^{2N} + |y_n|\Big] \tag{4.4}$$

with, possibly, a new constant $C > 0$. If we plug these estimates into the definition of the function f_N, we get for f_N on D'_N

$$|f_N(z)| \leqslant C\frac{|z_1|^{N-1}}{|z_1|^{N-1}} \frac{1}{\Big(-\rho(z) + |z_1|^{2N} + |y_n|\Big)^{\frac{2}{q}}} = \frac{C}{\Big(-\rho(z) + |z_1|^{2N} + |y_n|\Big)^{\frac{2}{q}}}. \tag{4.5}$$

So, in order to show, that $f_N \in H^q(D'_N)$, we have to verify the integrability of

$$\frac{C}{\Big(-\rho(z) + |z_1|^{2N} + |y_n|\Big)^2} \tag{4.6}$$

over the domain D'_N. This follows, if one observes, that after integrating over the variable z_1 at first, it remains to see, that the integral of $\frac{C}{(-\rho(z)+|y_n|)^{2-\frac{1}{2N}}}$ with respect to the real coordinates $-\rho(z)$ and y_n is finite, which, obviously is the case. After that the integration over the remaining variables is trivial.

For the remaining part of the lemma we argue by contradiction. Namely, let us suppose, that there is a constant $C>0$ and a holomorphic extension $F \in \mathcal{O}(D_N)$ of $f_N|D'_N$ such that (4.1) holds on $S(\varepsilon')$ for a certain $\varepsilon>0$. Since the function $z_1^N+z_2$ vanishes to first order on V_N, we can factorize F as

$$F(z) = \frac{1}{(1-z_n)^{\frac{N-1}{2N}+\frac{2}{q}}}\left(z_1^{N-1}+\left(z_1^N+z_2\right)a_1(z)\right) \tag{4.7}$$

with a holomorphic function a_1 on D_N. Plugging this into (4.1) for any point $z \in S(\varepsilon')$ gives the estimate

$$\left|\frac{1}{(1-z_n)^{\frac{N-1}{2N}+\frac{2}{q}}}\left(z_1^{N-1}+z_1^N a_1(z)\right)\right| \leqslant C(-\rho_N(z))^{-\frac{N-1}{2N}-\frac{2}{q}+\frac{N-1}{2^{N+1}}+\varepsilon} \tag{4.8}$$

for all $\varepsilon'>0$ sufficiently small and all $z \in S(\varepsilon')$ with a (possibly different) constant $C>0$. This gives for the same ε' and z the inequality

$$\left|\frac{1}{z_1}+a_1(z)\right| \leqslant C(\varepsilon')^{\varepsilon-\frac{1}{2^{N+1}}}. \tag{4.9}$$

Since the holomorphic disc $\Delta(\varepsilon') := \{z = (z_1, 0, \dots, 0, 1-\varepsilon') : |z_1| \leqslant \varepsilon'^{\frac{1}{2^{N+1}}}\} \subset D_N$, we get from Cauchy's theorem

$$I := \int_{S(\varepsilon')} a_1(z)\, dz_1 = 0. \tag{4.10}$$

On the other hand, we have

$$I = \int_{S(\varepsilon')} -\frac{1}{z_1}\, dz_1 + \int_{S(\varepsilon')} \left(\frac{1}{z_1}+a_1(z)\right) dz_1.$$

The first term being $-2\pi i$ and the second term according to (4.9) $O(\varepsilon'^{\varepsilon})$, this is a contradiction to (4.10), if $\varepsilon'>0$ is taken small enough. $\square$

Next we consider the domains $D_{N,p}$ as defined in (3.3). We show

LEMMA 4.2. — *Let $N \geqslant 2$ and $p \geqslant 1$ be integers and $\varepsilon' > 0$. Define the pseudoellipsoid $D_{N,p} \subset\subset \mathbb{C}^{2p+1}$ as in (3.3) and the torus $S_p(\varepsilon') \subset D_{N,p}$ as in (3.4). Put*

$$V_{N,p} := \left\{ z \in \mathbb{C}^{2p+1} : z_1^N + z_{p+1} = \cdots = z_p^N + z_{2p} = 0 \right\}$$

$D'_{N,p} := D_{N,p} \cap V_{N,p}$ and for $2 \leqslant q < \infty$

$$f_{N,p}(z) := \frac{z_1^{N-1} \cdot \cdots \cdot z_p^{N-1}}{(1 - z_{2p+1})^{\frac{p(N-1)}{2N} + \frac{2}{q}}}.$$

Then $f_{N,p} \in H^q(D'_{N,q})$ and $f_{N,p}$ does not have an extension $F \in \mathcal{O}(D_{N,p})$, such that for an $\varepsilon > 0$ and a suitable constant $C > 0$ the estimate

$$|F(z)| \leqslant Cd(z, \partial D_{N,p})^{-\frac{p(N-1)}{2N} - \frac{2}{q} + \frac{p(N-1)}{2^{N+1}} + \varepsilon} \tag{4.11}$$

holds on $S_{N,p}(\varepsilon')$ for all $\varepsilon' > 0$ sufficiently small.

Proof. — The proof is parallel to the proof of Lemma 4.1. The transversality of $V_{N,p}$ and $\partial D_{N,p}$ can again easily be verified. Instead of the estimate (4.3) we, now, have

$$1 - x_{N,p} \geqslant C \left[-\rho_{N,p}(z) + \sum_{j=1}^{p} |z_j|^{2^{N+1}} + \sum_{j=p+1}^{2p} |z_j|^2 + |y_{2p+1}| \right]. \tag{4.12}$$

Restricting this to $D'_{N,p}$ gives

$$1 - x_{2p+1} \geqslant C \left[-\rho_{N,p}(z) + \sum_{j=1}^{p} |z_j|^{2N} + |y_{2p+1}| \right] \tag{4.13}$$

such that $f_{N,p}|D'_{N,p}$ satisfies the estimate

$$|f_{N,p}(z)| \leqslant \frac{C}{\left(-\rho_{N,p}(z) + \sum_{j=1}^{p} |z_j|^{2N} + |y_{2p+1}| \right)^{\frac{2}{q}}}. \tag{4.14}$$

Hence, $f_{N,p} \in H^q(D'_{N,p})$ as above.

We, now, suppose, that there is a holomorphic extension F of $f_{N,p}|D'_{N,p}$ to $D_{N,p}$ satisfying for some $\varepsilon > 0$ the inequality (4.11) on the torus $S_p(\varepsilon')$ for all $\varepsilon' > 0$ sufficiently small. We have the representation

$$F(z) = \frac{1}{(1 - z_{2p+1})^{\frac{p(N-1)}{2N} + \frac{2}{q}}} \left(z_1^{N-1} \cdots z_p^{N-1} + \sum_{k=1}^{p} \left(z_k^N + z_{p+k}\right) a_k(z) \right)$$

with holomorphic functions a_k on $D_{N,p}$. By putting this into (4.11) the following estimate on $S_p(\varepsilon')$ follows after a few calculations

$$\left| \frac{1}{z_1 \cdots z_p} + \sum_{k=1}^{p} \frac{a_k(z)}{\prod_{\substack{j=1 \\ j \neq k}}^{p} z_j^N} \right| \leqslant C(\varepsilon')^{\varepsilon - \frac{p}{2^{N+1}}}. \tag{4.15}$$

We, next, observe, that for $\varepsilon' > 0$ small enough and $\eta := (\varepsilon')^{2^{N+1}}$ the whole solid torus

$$T_p(\varepsilon') := \left\{ z = (\eta\zeta_1, \ldots, \eta\zeta_p, 0, \ldots, 0, 1 - p\varepsilon') \in \mathbb{C}^{2p+1} : |\zeta_j| \leqslant 1 \; \forall j = 1, \ldots, p \right\}$$

is contained in $D_{N,p}$. Hence, since in the denominator of the kth term of the sum the factor z_k is missing, we get from Cauchy's theorem

$$I := \int_{S_p(\varepsilon')} \sum_{k=1}^{p} \frac{a_k(z)}{\prod_{\substack{j=1 \\ j \neq k}}^{p} z_j^N} \, dz_1 \wedge \ldots \wedge dz_p = 0. \tag{4.16}$$

On the other hand, we can split

$$\begin{aligned} I = &- \int_{S_p(\varepsilon')} \frac{dz_1 \wedge \ldots \wedge dz_p}{z_1 \cdots z_p} \\ &+ \int_{S(\varepsilon')} \left(\frac{1}{z_1 \cdots z_p} + \sum_{k=1}^{p} \frac{a_k(z)}{\prod_{\substack{j=1 \\ j \neq k}}^{p} z_j^N} \right) dz_1 \wedge \ldots \wedge dz_p. \end{aligned} \tag{4.17}$$

The first integral is $-(2\pi i)^p$, whereas, according to estimate (4.15), the second integral is $O((\varepsilon')^{\varepsilon})$. Choosing $\varepsilon' > 0$ sufficiently small, for $\varepsilon > 0$ fixed, this becomes a contradiction. $\square$

For the proof of Theorem 2.5 we need

LEMMA 4.3. — *Let for $k \geqslant 2$ the domain $\hat{D}_k$ and the circle $\hat{S}(\varepsilon')$ be defined as in Lemma 3.3, put $z_0 := (0,0,1) \in \partial\hat{D}_k$, let $\chi > 0$ be given and define*

$$\tag{4.18} \begin{cases} V := \{z \in \mathbb{C}^3 : z_1 + z_2^2 = 0\} \\ \hat{D}'_k := \hat{D}_k \cap V \\ f(z) := \frac{z_2}{(1-z_3)^{\frac{3}{2}+\frac{\chi}{2}} \log(z_3-1)}. \end{cases}$$

Then $f \in H^2(\hat{D}'_k, -\chi \log d(z, \partial\hat{D}_k))$ and, for any $r > 0$ and $\varepsilon > 0$ small enough, the function $f|\hat{D}'_k$ does not allow an extension $h \in \mathcal{O}(\hat{D}_k \cap B(z_0, r))$ satisfying the estimate

$$\tag{4.19} |h(z)| \leqslant C d\Big(z, \partial\hat{D}_k\Big)^{-\frac{3}{2}-\frac{\chi}{2}+\frac{1}{2k}+\varepsilon}$$

for all $z \in \hat{S}(\varepsilon')$ and $\varepsilon' > 0$ sufficiently small.

Proof. — We have the inequality

$$\tag{4.20} 1 - x_3 \geqslant C\Big[-\rho(z) + |z_1|^2 + |z_2|^{2k} + |y_3|\Big]$$

on $\hat{D}_k$, and, hence,

$$\tag{4.21} 1 - x_3 \geqslant C\Big[-\rho(z) + |z_2|^4 + |y_3|\Big]$$

on $\hat{D}'_k$. From it it follows by the same arguments as used in the proof of Lemma 4.1, that $f \in H^2(\hat{D}'_k)$. Let us, now, assume, that h is a holomorphic extension of $f|\hat{D}'_k$ satisfying the estimate (4.19). Then we can factorize h as

$$h(z) = \frac{1}{(1-z_3)^{\frac{3}{2}+\frac{\chi}{2}} \log(z_3 - 1)} \big(z_2 + (z_2^2 + z_1) a_1(z)\big)$$

with a holomorphic function a_1 on $\hat{D}_k \cap B(z_0, r)$. A small calculation using (4.19) and (4.20) gives

$$\tag{4.22} \left|\frac{1}{z_2} + a_1(z)\right| \leqslant C\varepsilon'^{\varepsilon - \frac{1}{2k}} \log\varepsilon'.$$

On the other hand, we have

$$0 = \int_{\hat{S}(\varepsilon')} a_1(z)\, dz_2 = -\int_{\hat{S}(\varepsilon')} \frac{1}{z_2}\, dz_2 + \int_{\hat{S}(\varepsilon')} \left(\frac{1}{z_2} + a_1(z)\right) dz_2.$$

The first integral on the right is $-2\pi i$, whereas the second integral is according to (4.22) just $O(\varepsilon'^{\varepsilon}\log)$. This is a contradiction, if ε' is chosen small enough. □

Almost a special case of Lemma 4.2 is the following statement needed for the proof of Theorem 2.6.

LEMMA 4.4. — *Let $q > 1$ be an integer and $D := \{z \in \mathbb{C}^3 : \rho(z) := |z_1|^{2q} + |z_2|^2 + |z_3|^3 - 1 < 0\}$. Put*

$$V := \{z \in \mathbb{C}^3 : z_1^2 + z_2 = 0\}$$

and

$$f(z) := \frac{1}{(1-z_3)^{\frac{3}{2}}} \quad \text{and} \quad g(z) := \frac{z_1}{(1-z_3)^{\frac{1}{4}}}.$$

Then V and ∂D intersect transversally, $f \in H^2(D)$, $g \in H^\infty(D')$, and the function $h := g \cdot f|D' \in \mathcal{O}(D')$ does not allow an extension $F \in \mathcal{O}(D)$, such that for a suitable constant $C > 0$

$$|F(z)| \leqslant Cd(z, \partial D)^{-\frac{7}{4}+\frac{1}{2q}+\varepsilon} \tag{4.23}$$

for all $z \in S(\varepsilon') := \{z = (\varepsilon'^{\frac{1}{2q}}e^{i\theta}, 0, 1-\varepsilon') : \theta \in [0, 2\pi]\}$ for all $\varepsilon' > 0$ sufficiently small.

Proof. — The proof of this lemma is just a simplified version of the proof of Lemma 4.2 with $p = 1$ and q instead of 2^{N+1}. We, therefore, omit it. □

5. Proofs of the theorems.

We, now, come to the proofs of the theorems of section 2. Most of them follow easily by combining lemmas from sections 3 and 4.

Proof of Theorem 2.1. — Let $2 \leqslant q < \infty$ and $\varepsilon > 0$ be given. We choose a pseudoellipsoid D_N as in (3.1) (specifying the number N a little bit later) and define $S(\varepsilon')$ for $\varepsilon' > 0$ by (3.2). Furthermore, V_N and f_N are taken as in Lemma 4.1 and $D'_N = D_N \cap V_N$. Hence, $f_N \in H^q(D'_N)$. Let us, now, suppose, that f_N has an extension $F \in H^{\frac{6q}{q+4}+\varepsilon}(D_N)$. Then we get from Lemma 3.1, that there is a constant $C_N > 0$, such that F satisfies the estimate

$$|F(z)| \leqslant C_N \|F\|_{L^q(D_N)} d(z, \partial D_N)^{-\frac{3}{\tilde{q}} - \frac{n-2}{\tilde{q}2^N}} \tag{5.1}$$

for all $z \in S(\varepsilon')$ and all $\varepsilon' > 0$ sufficiently small, with $\tilde{q} := \dfrac{6q}{q+4} + \varepsilon$. On the other hand we know from Lemma 4.1, that there cannot be an $\tilde{\varepsilon} > 0$ and a constant $C > 0$, for which the extension F would satisfy (4.1) for all $z \in S(\varepsilon')$ with $\varepsilon' > 0$ sufficiently small. Hence, it follows, that the extension F is impossible, if $\tilde{q}$ and N are such, that the inequality

$$d(z, \partial D_N)^{-\frac{3}{\tilde{q}} - \frac{n-2}{\tilde{q}2^N}} \lesssim d(z, \partial D_N)^{-\frac{N-1}{2N} - \frac{2}{q} + \frac{N-1}{2^{N+1}} + \tilde{\varepsilon}} \tag{5.2}$$

is satisfied for all $z \in S(\varepsilon')$ with $\varepsilon' > 0$ sufficiently small. A small calculation shows, however, that (5.2) holds, if

$$\tilde{q} > \frac{3 + \frac{n-2}{2^N}}{\frac{1}{2} + \frac{2}{q} - \left(\frac{1}{2N} + \frac{N-1}{2^{N+1}}\right)}. \tag{5.3}$$

But

$$\tilde{q} = \frac{6q}{q+4} + \varepsilon = \frac{3}{\frac{1}{2} + \frac{2}{q}} + \varepsilon. \tag{5.4}$$

Therefore, (5.3) is satisfied, if only N has been chosen sufficiently large. This shows, that the extension $F \in H^{\tilde{q}}(D_N)$ is impossible.

In order to deduce Theorem 2.1 from this, we just have to observe, that

$$\begin{cases} z_1^* := z_1^N + z_2 \\ z_2^* := z_1 \\ z_j^* := z_j \ \forall j = 3, \ldots, n \end{cases} \tag{5.5}$$

is an algebraic biholomorphism of $\mathbb{C}^n$ onto itself and turns the situation constructed here into the desired one.

We, still, should mention, that we excluded for pratical reasons the case $q = \infty$ in these arguments. However, if they are properly changed, they carry over also to this case. □

We, next, come to the

Proof of Theorem 2.3. — The argument is completely parallel to the proof of Theorem 2.1. Instead of the pseudoellipsoids D_N we, now, use the pseudoellipsoids $D_{N,p}$ as defined in (3.3) with the corresponding tori $S_p(\varepsilon')$ from (3.4). We assume again for practical reasons $2 \leqslant q < \infty$, use the varieties $V_{N,p}$ and functions $f_{N,p}$ from Lemma 4.2 (specifying again N later), such that $f_{N,p} \in H^q(D'_{N,p})$. We assume that the function $f_{N,p}|D'_{N,p}$ has an extension $F \in H^{\tilde{q}}(D_{N,p})$ with $\tilde{q} := \dfrac{2q(p+2)}{qp+4} + \varepsilon$ and want to show, that this leads to a contradiction in case N is chosen suitably large. Comparing Lemmas 3.2 and 4.2, we see, that this contradiction is reached, if we can show, that for such N the inequality

$$\tilde{q} > \frac{p+2+\frac{p}{2N}}{\frac{p}{2}+\frac{2}{q}-p\left(\frac{1}{2N}+\frac{N-1}{2^{N+1}}\right)} \tag{5.6}$$

holds (this follows in total analogy to (5.3)). Since, however,

$$\tilde{q} = \frac{p+2}{\frac{p}{2}+\frac{2}{q}} + \varepsilon \tag{5.7}$$

(5.6) can, indeed, be realized by choosing N sufficiently large.

In order to finish the proof of Theorem 2.3, we just have to observe, that $V_{N,p}$ again can be linearized by an algebraic biholomorphism of $\mathbb{C}^{2p+1}$.

□

Corollary 2.4 is a direct consequence of Theorem 2.3. We, therefore, can now come to the

Proof of Theorem 2.5. — We define

$$\begin{cases} D := \left\{ z \in \mathbb{C}^3 : \rho(z) := |z_1|^2 + e \cdot e^{-\frac{1}{|z_2|^2}} + |z_3|^2 - 1 < 0 \right\} \\ V : \{ z \in \mathbb{C}^3 : z_1 + z_2^2 = 0 \} \\ D' := D \cap V. \end{cases} \tag{5.8}$$

The domain D is pseudoconvex with C^∞-smooth boundary, $z_0 := (0,0,1) \in \partial D$, $V \ni z_0$ intersects ∂D transversally everywhere, $\mathcal{L}_{\partial D}(z_0, t) > 0$ for all $t = (t_1, t_2, 0)$ with $t_1 \neq 0$, and, from the inequality

$$1 - x_3 \geqslant C\left[-\rho(z) + |z_1|^2 + |y_3|\right] \tag{5.9}$$

on D, it follows in complete analogy to the proof of Lemma 3.1, that the function

$$f(z) := \frac{z^2}{(1-z_3)^{\frac{3}{2}+\frac{\chi}{2}} \log(z_3 - 1)}$$

(see also Lemma 4.3) satisfies $f \in H^2(D', -\chi \log d(z, \partial D))$. We assume, that for the given $\varepsilon > 0$ and some $r > 0$ this function has an extension $h \in H^2(D \cap B(z_0, r), -(\chi - \varepsilon) \log d(z, \partial D))$ and observe at first, that for all $k \geqslant 2$ the domain $\hat{D}_k$ as defined in Lemma 3.3 lies in D with $z_0 = (0,0,1) \in \partial D \cap \partial \hat{D}_k$. Hence, $h \in H^2(\hat{D}_k \cap B(z_0, r), -(\chi - \varepsilon) \log d(z, \partial \hat{D}_k))$. We, now use the same strategy as in the previous proofs, this time by comparing Lemmas 3.3 and 4.3. We obtain, that, for the given $\varepsilon > 0$, the extension h is impossible, namely cannot be in $H^2(\hat{D}_k \cap B(z_0, r), -(\chi - \varepsilon) \log d(z, \partial \hat{D}_k))$, if k can be chosen such that

$$-\frac{3}{2} - \frac{1}{2k} - \frac{\chi}{2} + \frac{\varepsilon}{2} > -\frac{3}{2} - \frac{\chi}{2} + \frac{1}{2k}.$$

This inequality, however, can easily be satisfied by choosing $k > \dfrac{4}{\varepsilon}$. Of course, the variety V can again be linearized and turned into A_1 by a global algebraic change of coordinates, thus generating the situation desired in Theorem 2.5. □

It only remains to give the

Proof of Theorem 2.6. — We define for $q := 2^N$, $N \geqslant 2$, the domain D, the variety V, the circle $S(\varepsilon')$ and the functions f, g, h as in Lemma 4.4. Furthermore, $D' := D \cap V$. After multiplying g with a suitable constant $c > 0$ we may assume that $|g| < 1$ on D', such that $|h| < |f|$ on D'. Since $f \in H^2(D)$, we can apply to it Lemma 3.2 for $q = 2$ (namely, $D = D_{N,p}$, if $p = 1$). If we assume, that $h|D'$ admits an extensions $F \in H^2(D)$, we, therefore, get the estimate

$$|F(z)| \leqslant C \|F\|_{L^2(D)} d(z, \partial D)^{-\frac{3}{2} - \frac{1}{2q}} \tag{5.10}$$

for all $z \in S(\varepsilon')$ and all $\varepsilon' > 0$ sufficiently small. On the other hand, it follows from Lemma 4.4, that no estimate of the following form can hold with some $\varepsilon > 0$

$$|F(z)| \leqslant \tilde{C} d(z, \partial D)^{-\frac{7}{4} + \frac{1}{2q} + \varepsilon} \tag{5.11}$$

for all $z \in S(\varepsilon')$ and all $\varepsilon' > 0$ sufficiently small. Comparing these inequalities it follows, that the extension $F \in H^2(D)$ cannot exist, if only $q > 4$, in other words, $N > 2$. Finally, V can again be turned into A_1 by a global algebraic biholomorphism. This proves the theorem. □

6. An open question.

Let $D \subset\subset \mathbb{C}^n$ be a $\mathcal{C}^\infty$-smooth pseudoconvex domain, $z_0 \in \partial D$. Let, furthermore, $H \subset \mathbb{C}^n$ be a complex hyperplane which is transverse to $T^{1,0}_{z_0}\partial D$. We put $H' := H \cap T^{1,0}_{z_0}\partial D$, hence, $\dim H' = n-1$.

DEFINITION 6.1. — *The domain D is called transversally strictly pseudoconvex with respect to H at z_0, if*

$$\mathcal{L}_{\partial D}(z_0; t) > 0 \ \ \forall t \in T^{1,0}_{z_0}\partial D \setminus H'.$$

As briefly mentioned already in section 2 before Theorem 2.5, the following fact holds true:

Suppose, V is a closed complex submanifold of a neighborhood U of z_0, $\dim V = n-1$, $z_0 \in V$, and such that $T_{z_0}V = H$. Put $D' := D \cap H$. If, in this situation, ∂D is extendable of order 2 in a pseudoconvex way along $\partial D'$ near z_0 (see [10] together with [19]), then we have

THEOREM 6.2. — *Let $\chi > 1$ be a real number. Then there is a (small) radius $r > 0$ and a constant $C > 0$, such that any function $f \in H^2(D', -\chi \log d(z, \partial D))$ has an extension $F \in H^2(D, -(\chi - 1)\log d(z, \partial D))$ with $\|F\|_2 \leqslant C\|f\|_1$, where $\|\cdot\|_1$ is the L^2-norm on D' with weight $-\chi \log d(z, \partial D)$ and $\|\cdot\|_2$ is the L^2-norm on D with weight $-(\chi - 1)\log d(z, \partial D)$.*

It is not known, whether pseudoconvex extendability of order 2 is also necessary for having such a gain in the extension from D' to D. Therefore, we give the

DEFINITION 6.3. — *Let $D \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with $\mathcal{C}^\infty$-smooth boundary, $z_0 \in \partial D$, V a closed complex submanifold of a neighborhood U of z_0, $z_0 \in V$ and V intersecting ∂D at z_0 transversally. Suppose, furthermore, that D is transversally strictly pseudoconvex with respect to $H := T_{z_0}\partial D$. Then we call V of good type, if there is a constant $C > 0$ and a (small) radius $r > 0$, such that any function $f \in H^2(D', -\chi \log d(z, \partial D))$ has an extension $F \in H^2(D, -(\chi - 1)\log d(z, \partial D))$ with $\|F\|_2 \leqslant C\|f\|_1$, where $\|\cdot\|_1$ is the L^2-norm on D' with weight $-\chi \log d(z, \partial D)$ and $\|\cdot\|_2$ is* *the L^2-norm on D with weight $-(\chi-1)\log d(z,\partial D)$. Furthermore, we say in this situation that the triple (D, z_0, H) admits a hypersurface of good type.*

In Theorem 2.5 we have shown, that not every hypersurface V intersecting the smooth pseudoconvex domain $D \subset\subset \mathbb{C}^n$ at $z_0 \in \partial D \cap V$ transversally and such that D is transversally strictly pseudoconvex with respect to $H := T_{z_0}\partial D$, is necessarily of good type. In fact, we have shown, that in some sense it can be arbitrarily bad. On the other hand, it easily can be shown, that in the example of the domain D from (5.8) the hypersurface A_1 itself is of good type, allthough it has the same tangent space at z_0 as the hypersurface V of (5.8). – This suggests the following very important open

Question: Suppose, that $D \subset\subset \mathbb{C}^n$ is smooth, pseudoconvex, $z_0 \in \partial D$. Let $H \subset T^{1,0}_{z_0}\partial D$ be a linear subspace of dimension $n-1$, such that D is transversally strictly pseudoconvex with respect to H. Does in this situation the triple (D, z_0, H) always admit a hypersurface of good type?

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K. DIEDERICH,
Universität Wuppertal
Mathematik
Gausstr. 20
D-42095 Wuppertal (Allemagne).
&
E. MAZZILI,
Université de Lille
UFR de Mathématiques Pures et Appliquées
F-59655 Villeneuve d'Ascq Cedex (France).