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ON THE HAAGERUP INEQUALITY AND GROUPS ACTING ON $\tilde{A}_n$-BUILDINGS(*)

by Alain VALETTE

1. Introduction.

Let $\mathbf{F_n}$ be the free group on n generators; for $\gamma \in \mathbf{F_n}$, denote by $|\gamma|$ the word length of γ with respect to a free generating subset; for f a function with finite support on $\mathbf{F_n}$, denote by $\lambda(f)$ the operator of left convolution by f on the Hilbert space $\ell^2(\mathbf{F_n})$. In Lemma 1.5 of [Haa79], U. Haagerup proved the following remarkable inequality on the operator norm $\|\lambda(f)\|$:

$$\|\lambda(f)\| \leqslant 2\sqrt{\sum_{\gamma\in\mathbf{F_n}} |f(\gamma)|^2(1+|\gamma|)^4}.$$

In other words, the convolution norm of f, which is in general quite hard to compute (see e.g. [AO76]), can be estimated by a weighted ℓ^2-norm - or Sobolev norm - which is much easier to calculate.

Haagerup's inequality was studied in a systematic way by P. Jolissaint (see [Jol90], [Jol89]) in the setup of a group Γ endowed with a length function L. A *length function* is a function $L : \Gamma \to \mathbf{R}^+$ such that $L(1) = 0$, $L(\gamma^{-1}) = L(\gamma), L(\gamma_1\gamma_2) \leqslant L(\gamma_1) + L(\gamma_2)$ for every $\gamma_1, \gamma_2, \gamma \in \Gamma$, and for every $R > 0$ the set $\{\gamma \in \Gamma : L(\gamma) \leqslant R\}$ is finite (i.e. L is a proper function).

Apart from length functions given by word length with respect to a finite generating subset in a finitely generated Γ, examples of length

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functions are obtained by letting Γ act properly isometrically on a metric space (X, d) with base-point x_o, and setting

$$L(\gamma) = d(\gamma x_o, x_o)$$

for $\gamma \in \Gamma$ (actually this last example is general).

Denote by $\mathbf{C}\Gamma$ the group algebra of Γ, i.e. the space of complex-valued finitely supported functions on Γ, endowed with the convolution product; for $f \in \mathbf{C}\Gamma$ and $s > 0$, define the weighted ℓ^2-norm of f as:

$$\|f\|_{s,L} = \sqrt{\sum_{\gamma \in \Gamma} |f(\gamma)|^2 (1 + L(\gamma))^{2s}}.$$

DEFINITION 1. — *We say that* Γ *satisfies the Haagerup inequality, or has the (RD)-property with respect to* L*, if there exists constants* $C, s > 0$ *such that, for any* $f \in \mathbf{C}\Gamma$*, one has*

$$\|\lambda(f)\| \leqslant C \|f\|_{s,L}.$$

The papers [Jol90] and [dlH88] give the main known examples of (RD)-groups; among finitely generated groups with a word length, these are groups with polynomial growth and hyperbolic groups "à la Gromov".

The main feature of (RD)-groups (which explains the acronym RD) appears in [Jol89]: for an (RD)-group Γ, the space of rapidly decreasing functions on Γ (i.e. functions ϕ on Γ such that $\|\phi\|_{s,L} < \infty$ for every $s > 0$) is a dense subalgebra of the reduced C*-algebra $C^*_r(\Gamma)$, such that the inclusion induces isomorphisms in topological K-theory. (This fact played a crucial role in the Connes-Moscovici proof [CM90] of Novikov's conjecture for hyperbolic groups.) Applications of Haagerup's inequality to harmonic analysis were given in [Haa79] and [JV91]. More recently came other applications to spectra of Markov operators [dlHRV93].

Jolissaint gave a purely algebraic obstruction to property (RD): if Γ contains a subgroup which is solvable with exponential growth, then there is *no* length function on Γ for which Haagerup's inequality holds (combine 1.1.7, 2.1.1 and 3.1.8 in [Jol90]); this applies in particular to $SL_n(\mathbf{Z})$, with $n \geqslant 3$ ([Jol90], 3.1.9); more generally, this holds true for any non-uniform lattice in a simple real Lie group with real rank at least 2 (private communication of E. Leuzinger and C. Pittet). In contrast, a *uniform* lattice in such a Lie group (or in a simple p-adic group with split rank at least 2) has no solvable subgroup with exponential growth (see [GW71]). Thus the question was raised in the problem section of [FFR95] whether such a uniform lattice has property (RD); that was the motivation for the present paper. While this article was under completion, we received a very interesting preprint by J. Ramagge, G. Robertson and T. Steger [RRS] providing a proof of property (RD) for $\tilde{A}_2$-groups - these groups will be defined below.

In this paper, we first generalize Definition 1 as follows:

DEFINITION 2. — *Let E be a linear subspace of $\mathbf{C}\Gamma$; we say that E satisfies the Haagerup inequality if there exists constants $C, s > 0$ such that, for any $f \in E$, one has*

$$\|\lambda(f)\| \leqslant C\|f\|_{s,L}.$$

(Somewhat pedantically: $\mathbf{C}\Gamma$ satisfies the Haagerup inequality, according to Definition 2, if and only if Γ satisfies the Haagerup inequality, according to Definition 1.) The purpose of this generalization is twofold. First, even if Γ does not have property (RD), it may happen that some interesting subspaces of $\mathbf{C}\Gamma$ satisfy the Haagerup inequality (as an illustration, see [Jol96] for the case of a free product $\Gamma = G * \mathbf{Z}$, with G arbitrary). Second, it may be easier to prove Haagerup's inequality for a subspace, as we will show.

Our main results are as follows:

(1) A *-subspace E of $\mathbf{C}\Gamma$ satisfies the Haagerup inequality if and only if there exists constants $C, s > 0$ such that, for any self-adjoint $f \in E$ and any $k \in \mathbf{N}$:

$$f^{(2k)}(1) \leqslant C^{2k}\|f\|_{s,L}^{2k},$$

where $f^{(j)}$ is the j-th convolution power of f in $\mathbf{C}\Gamma$.

(2) We get the following new characterization of property (RD): Γ has property (RD) if and only if there exists constants $C, s > 0$ such that, for any symmetric, finitely supported probability measure μ on Γ and any $k \in \mathbf{N}$:

$$\mu^{(2k)}(1) \leqslant C^{2k}\|\mu\|_{s,L}^{2k}.$$

Noticing that $\mu^{(2k)}(1) = \sup\{\mu^{(2k)}(x) : x \in \Gamma\}$ measures the decay of the random walk on Γ associated with μ, one sees that this is close to results linking decay of random walks with growth properties of Γ, as they appear e.g. in Chapters VI and VII of [VSCC92].

(3) Denote by $\mathrm{Rad}_L(\Gamma)$ the space of *radial functions*, i.e. the space of functions in $\mathbf{C}\Gamma$ that depend only on L. If L is a word length function on a finitely generated group Γ, we are able to relate growth and amenability as follows. Suppose that $\mathrm{Rad}_L(\Gamma)$ satisfies Haagerup's inequality; we prove that Γ is non-amenable if and only if Γ has superpolynomial growth. (This was known to Jolissaint [Jol90], under the stronger assumption that Γ has property (RD)).

(4) Assume that L is integer valued (e.g. L is a word length). It turns out that Haagerup's inequality for $\mathrm{Rad}_L(\Gamma)$ has a purely combinatorial interpretation. Define a *strict N-loop with length* $2k$ in Γ as a sequence $(v_0 = 1, v_1, \ldots, v_{2k-1}, v_{2k} = 1)$ such that $L(v_{i-1}^{-1}v_i) = N$ for $i = 1, \ldots, 2k$; the *sphere* S_N of radius N is the level set $S_N = L^{-1}(N)$. We show that $\mathrm{Rad}_L(\Gamma)$ satisfies Haagerup's inequality if and only if there exists constants $C, s > 0$ such that for any $k, N \in \mathbf{N}$:

$$\mathrm{card}\{\textit{strict } N\textit{-loops with length } 2k \textit{ in } \Gamma\} \leqslant C^{2k}(1+N)^{2ks}(\mathrm{card}\, S_N)^k.$$

This last result allows us to make the link with J. Swiatkowski's paper [Swi], to which the present paper is an appendix. Indeed, let Γ be an $\tilde{A}_n$-group, i.e. a group acting simply transitively on the vertices of a thick euclidean building Δ of type $\tilde{A}_n$; $\tilde{A}_2$-groups have been studied for some years now, first from a combinatorial point of view [CMSZ93], then from the point of view of harmonic analysis [CMS93]; for $n \geqslant 3$ the existence of $\tilde{A}_n$-groups has been established by D. Cartwright and T. Steger [CS]. Let $v_0 \in \Delta$ be a base-vertex; consider the length function $L(\gamma) = d_\Delta(\gamma v_o, v_o)$, where d_Δ is the combinatorial distance on the 1-skeleton of Δ. Swiatkowski's loop inequality (Theorem 0.6.(a) in [Swi]) is nothing but our combinatorial criterion, equivalent to the Haagerup inequality for $\mathrm{Rad}_L(\Gamma)$. Moreover, the fact that Γ has exponential growth (proved in Proposition 1.9 of [Swi]) gives a direct, combinatorial proof of the non-amenability of Γ.

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2. Property (RD) for a subspace of $\mathbf{C}\Gamma$.

We shall consider two involutions on $\mathbf{C}\Gamma$:

$f \to f^*$ where $f^*(\gamma) = \overline{f(\gamma^{-1})}$;

$f \to \check{f}$ where $\check{f}(\gamma) = f(\gamma^{-1})$.

We say that f is *self-adjoint* if $f = f^*$, and *symmetric* if $f = \check{f}$. A linear subspace E of $\mathbf{C}\Gamma$ is a **-subspace* if $E^* = E$.

Proposition 1. — *For a *-subspace E of* $\mathbf{C}\Gamma$, *the following conditions are equivalent:*

(i) *E satisfies the Haagerup inequality;*

(ii) *there exists constants $C_1, s > 0$ such that for any self-adjoint $f \in E$, one has*

$$\|\lambda(f)\| \leqslant C_1 \|f\|_{s,L};$$

(iii) *there exists constants $C_2, s > 0$ such that for any $k \in \mathbf{N}$ and any self-adjoint $f \in E$, one has*

$$f^{(2k)}(1) \leqslant C_2^{2k} \|f\|_{s,L}^{2k}.$$

Proof. — (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i) This follows easily from the fact that the involution $f \mapsto f^*$ on $\mathbf{C}\Gamma$ is an isometry both for the norm $\|\lambda(f)\|$ and $\|f\|_{s,L}$.

(ii) $\Rightarrow$ (iii) Notice that $g^* \star g(1) = \|g\|_2^2$ for any $g \in \mathbf{C}\Gamma$. Then, for a self-adjoint $f \in E$:

$$f^{(2k)}(1) = \|f^{(k)}\|_2^2 \leqslant \|\lambda(f^{(k)})\|^2 = \|\lambda(f)\|^{2k} \leqslant C_1^{2k} \|f\|_{s,L}^{2k}.$$

(iii) $\Rightarrow$ (ii) It follows from the spectral theorem (see e.g. [Kes59], lemma 2.2) that, for any self-adjoint $g \in \mathbf{C}\Gamma$:

$$\lim_{k\to\infty} \left(g^{(2k)}(1)\right)^{\frac{1}{2k}} = \|\lambda(g)\|.$$

This concludes the proof of Proposition 1.

PROPOSITION 2. — *Let E be a *-subspace of $\mathbf{C}\Gamma$ which is stable under the map $f \to |f|$. The subspace E has property (RD) if and only if there exists constants $C, s > 0$ such that for any symmetric non-negative $f \in E$ and any $k \in \mathbf{N}$:*

$$f^{(2k)}(1) \leqslant C^{2k}\|f\|_{s,L}^{2k}.$$

Proof. — The direct implication follows from Proposition 1. For the converse, notice that for g a self-adjoint element in E, we have $|g^{(2k)}| \leqslant |g|^{(2k)}$ pointwise, and $|g|$ is non-negative and symmetric in E. Then

$$g^{(2k)}(1) = |g^{(2k)}(1)| \leqslant |g|^{(2k)}(1) \leqslant C^{2k}\||g|\|_{s,L}^{2k} = C^{2k}\|g\|_{s,L}^{2k},$$

so that the result follows from (iii) $\Rightarrow$ (i) in Proposition 1.

We single out as a corollary what Proposition 2 says for $E = \mathbf{C}\Gamma$.

COROLLARY 1. — *Γ has property (RD) if and only if there exists constants $C, s > 0$ such that, for any symmetric, finitely supported probability measure μ on Γ and any $k \in \mathbf{N}$:*

$$\mu^{(2k)}(1) \leqslant C^{2k}\|\mu\|_{s,L}^{2k}.$$

(By homogeneity, the condition in the corollary is clearly equivalent to the one in Proposition 2.) On purpose, we expressed the corollary by appealing to probability measures μ on Γ; indeed, $\mu^{(2k)}(1)$ is just the probability of return to 1, in $2k$ steps, of the random walk on Γ with probability transitions $p(x,y) = \mu(y^{-1}x)$. There are numerous results on the decay of $\mu^{(2k)}(1)$ as $k \to \infty$; see especially Chapters VI and VII of [VSCC92] for the relation between decay of random walks and growth of the group.

For the rest of the paper, we assume that *the length function L is integer-valued* (this will be the case if L comes from a proper isometric action of Γ on a graph). We denote by χ_N the characteristic function of the sphere S_N.

PROPOSITION 3. — *Let E be a *-subspace of $\mathbf{C}\Gamma$.*

(a) *If there exists constants $C, s > 0$ such that, for any $N, k \in \mathbf{N}$ and any self-adjoint $f \in E$:*

$$(*) \qquad (f\chi_N)^{(2k)}(1) \leqslant C^{2k}\|f\chi_N\|_{s,L}^{2k}$$

(where $f\chi_N$ denotes the pointwise product), then E satisfies the Haagerup inequality.

(b) *Assume moreover that E is stable under $f \to |f|$. Then E satisfies the Haagerup inequality provided (*) holds for any non-negative symmetric $f \in E$.*

Proof. — (a) The following computation is inspired by the proof of Lemma 1.5 in [Haa79]. First, as in the proof of Proposition 1 above, we have for any self-adjoint $f \in E$ and any $N \in \mathbf{N}$:

$$\|\lambda(f\chi_N)\| \leqslant C\|f\chi_N\|_{s,L}.$$

But $f = \sum_{N=0}^{\infty} f\chi_N$, hence

$$\|\lambda(f)\| \leqslant \sum_{N=0}^{\infty} \|\lambda(f\chi_N)\| \leqslant C\sum_{N=0}^{\infty} \|f\chi_N\|_{s,L} = C\sum_{N=0}^{\infty} \|f\chi_N\|_{s,L}(1+N)(1+N)^{-1}$$

$$\leqslant C\left(\sum_{N=0}^{\infty} \|f\chi_N\|_{s,L}^2(1+N)^2\right)^{\frac{1}{2}} \left(\sum_{N=0}^{\infty}(1+N)^{-2}\right)^{\frac{1}{2}} \text{(by Cauchy–Schwarz)}$$

$$= C\sqrt{\frac{\pi^2}{6}}\left(\sum_{N=0}^{\infty} \|f\chi_N\|_{s+1,L}^2\right)^{\frac{1}{2}} = C\sqrt{\frac{\pi^2}{6}}\|f\|_{s+1,L}.$$

One concludes as in the proof of Proposition 1, (ii) $\Rightarrow$ (i).

(b) This follows immediately from (a) and the proof of Proposition 2.

Taking $E = \mathbf{C}\Gamma$, one immediately sees that Corollary 1 may be improved:

COROLLARY 2. — *The group Γ has property (RD) if and only if there exists constants $C, s > 0$ such that, for any $k, N \in \mathbf{N}$ and any symmetric probability measure μ supported in S_N:*

$$\mu^{(2k)}(1) \leqslant C^{2k}\|\mu\|_{s,L}^{2k}.$$

3. Radial functions.

We restrict attention to the subspace $E = \mathrm{Rad}_L(\Gamma)$ of radial functions in $\mathbf{C}\Gamma$; note that this is exactly the linear span of the χ_N's. It turns out that property (RD) for $\mathrm{Rad}_L(\Gamma)$ has a purely combinatorial meaning.

PROPOSITION 4. — $\mathrm{Rad}_L(\Gamma)$ *satisfies the Haagerup inequality if and only if there exists constants* $C, s > 0$ *such that, for any* $k, N \in \mathbf{N}$*:*

$$\mathrm{card}\{\textit{strict } N\textit{-loops with length } 2k \textit{ in } \Gamma\} \leqslant C^{2k}(1+N)^{2ks}(\mathrm{card}\, S_N)^k.$$

Proof. — It follows from Propositions 2 and 3(b) that $\mathrm{Rad}_L(\Gamma)$ satisfies the Haagerup inequality if and only if there exists $C, s > 0$ such that, for any $k, N \in \mathbf{N}$:

$$\chi_N^{(2k)}(1) \leqslant C^{2k}\|\chi_N\|_{s,L}^{2k}.$$

Now

$$\|\chi_N\|_{s,L} = \sqrt{\sum_{\gamma : L(\gamma)=N} (1+L(\gamma))^{2s}} = (1+N)^s(\mathrm{card}\, S_N)^{\frac{1}{2}}$$

and

$$\chi_N^{(2k)}(1) = \sum_{(s_1,s_2,\ldots,s_{2k}):s_1 s_2\ldots s_{2k}=1} \chi_N(s_1)\chi_N(s_2)\ldots\chi_N(s_{2k}).$$

With $v_0 = 1 = v_{2k}$ and $v_{i-1}^{-1}v_i = s_i$ for $i = 1, \ldots, 2k$, this yields:

$$\begin{aligned}\chi_N^{(2k)}(1) &= \sum_{(v_0,v_1,\ldots,v_{2k}):v_0=v_{2k}=1} \chi_N(v_0^{-1}v_1)\chi_N(v_1^{-1}v_2)\ldots\chi_N(v_{2k-1}^{-1}v_{2k}) \\ &= \mathrm{card}\{\textit{strict } N - \textit{loops of length } 2k\}\end{aligned}$$

since $(v_0, v_1, \ldots, v_{2k})$ contributes a non-zero term to the summation if and only if $L(v_0^{-1}v_1) = L(v_1^{-1}v_2) = \ldots = L(v_{2k-1}^{-1}v_{2k}) = N$. This concludes the proof.

An *N-loop with length 2k* in Γ is a sequence $(v_0 = 1, v_1, \ldots, v_{2k-1}, v_{2k} = 1)$ such that $L(v_{i-1}^{-1}v_i) \leqslant n$ for $i = 1, \ldots, 2k$. Consider also the ball with radius N in Γ:

$$B_N = \{\gamma \in \Gamma : L(\gamma) \leqslant N\}.$$

LEMMA 1. — *Assume that* $\mathrm{Rad}_L(\Gamma)$ *satisfies the Haagerup inequality. Then there exists constants* $C, s > 0$ *such that, for any* $k, N \in \mathbf{N}$:

$$\mathrm{card}\{N - \textit{loops with length } 2k \textit{ in } \Gamma\} \leqslant C^{2k}(1+N)^{2ks}(\mathrm{card}\, B_N)^k.$$

Proof. — Denote by η_N the characteristic function of B_N. Since $\mathrm{Rad}_L(\Gamma)$ satisfies the Haagerup inequality, we find by Proposition 1 constants $C, s > o$ such that, for any $k, N \in \mathbf{N}$:

$$\eta_N^{(2k)}(1) \leqslant C^{2k}\|\eta_N\|_{s,L}^{2k}.$$

But

$$\|\eta_N\|_{s,L}^{2k} = \left(\sum_{\gamma\in B_N}(1+L(\gamma))^{2s}\right)^k \leqslant (1+N)^{2ks}(\mathrm{card}\, B_N)^k.$$

On the other hand, the same calculation as in the proof of Proposition 4 yields:

$$\eta_N^{(2k)}(1) = \mathrm{card}\{N - \textit{loops with length } 2k \textit{ in } \Gamma\}.$$

This concludes the proof of Lemma 1.

Suppose now that Γ is a finitely generated group, and that L is a word length function with respect to some finite, symmetric, generating subset. Lemma 1 exhibits a link between the Haagerup inequality and growth properties of Γ, i.e. the behaviour of the growth function $N \to \mathrm{card}\, B_N$. It turns out that amenability also plays a subtle role, as the following two propositions illustrate.

PROPOSITION 5. — *Suppose that* Γ *is not amenable. The following statements are equivalent:*

(i) $\mathrm{Rad}_L(\Gamma)$ *satisfies the Haagerup inequality;*

(ii) *There exists constants* $C, s > 0$ *such that, for any* $k, N \in \mathbf{N}$:

$$\mathrm{card}\{N - \textit{loops with length } 2k \textit{ in } \Gamma\} \leqslant C^{2k}(1+N)^{2ks}(\mathrm{card}\, B_N)^k.$$

Proof. — (i) $\Rightarrow$ (ii) This is just Lemma 1 (which does not depend on amenability).

(ii) $\Rightarrow$ (i) We assume that (ii) holds. Since Γ is non-amenable, by Folner's property there exists $\epsilon > 0$ such that $\operatorname{card} S_N \geqslant \epsilon.\operatorname{card} B_N$ for any $N \in \mathbf{N}$. Then, for $k, N \in \mathbf{N}$:

$$\begin{aligned}\operatorname{card}\{\textit{strict } N - \textit{loops of length } 2k\} &\leqslant \operatorname{card}\{N - \textit{loops of length } 2k\} \\ &\leqslant C^{2k}(1+N)^{2ks}(\operatorname{card} B_N)^k \\ &\leqslant \left(\frac{C}{\sqrt{\epsilon}}\right)^{2k}(1+N)^{2ks}(\operatorname{card} S_N)^k.\end{aligned}$$

It follows from Proposition 4 that $\operatorname{Rad}_L(\Gamma)$ satisfies the Haagerup inequality.

The following proposition extends Jolissaint's result that an amenable group with property (RD) (with respect to a word length function) necessarily has polynomial growth; see Corollary 3.1.8 in [Jol90]. Following [VSCC92], we say that a finitely generated group is *superpolynomial* if its growth function grows faster than any polynomial.

PROPOSITION 6. — *Assume that, for some word length function L, the space $\operatorname{Rad}_L(\Gamma)$ satisfies the Haagerup inequality. The following are then equivalent:*

(i) Γ *is not amenable;*

(ii) Γ *has exponential growth;*

(iii) Γ *is superpolynomial.*

Proof. — (i) $\Rightarrow$ (ii) It is a general fact that any non-amenable group has exponential growth.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) Assume that Γ is superpolynomial. Let $C, s > 0$ be such that $\|\lambda(f)\| \leqslant C\|f\|_{s,L}$ for any $f \in \operatorname{Rad}_L(\Gamma)$. Take $f = \dfrac{\eta_N}{\operatorname{card} B_N}$, the uniform probability measure on B_N. Then:

$$\|\lambda\left(\frac{\eta_N}{\operatorname{card} B_N}\right)\| \leqslant \frac{C}{\operatorname{card} B_N}\sqrt{\sum_{\gamma\in B_N}(1+L(\gamma))^{2s}} \leqslant \frac{C(1+N)^s}{(\operatorname{card} B_N)^{\frac{1}{2}}}.$$

Since Γ is superpolynomial, we have $\|\lambda\left(\dfrac{\eta_N}{\operatorname{card} B_N}\right)\| < 1$ for N big enough. By Kesten's well-known characterization of amenability [Kes59], the group Γ has to be non-amenable.

It is often useful to have criteria for non-amenability that do not depend on the presence inside the group of a free group on two generators. Proposition 6 provides such a criterion. It will be used in the next section to deduce that $\tilde{A}_n$-groups are non-amenable. It would be interesting to use this criterion to prove non-amenability for other finitely generated groups.

4. From Jolissaint to Tits: groups acting on buildings.

Here we make the connection with the companion paper by J. Swiatkowski [Swi]. It is noticed in [CMSZ93] that an irreducible euclidean building with a vertex-transitive group of automorphisms is necessarily of type $\tilde{A}_n$. So let Δ be a locally finite, thick euclidean building of type $\tilde{A}_n$. Following Definition 0.1.2 in [Swi], we say that Δ is *uniformly thick* if there exists $q \in \mathbf{N}$ such that any codimension 1 face in Δ is contained in $q+1$ chambers. We thank the referee for suggesting the next lemma, that improves a previous version.

LEMMA 2. — *For $n \geqslant 2$, a thick building of type $\tilde{A}_n$ is uniformly thick.*

Proof. — For $n \geqslant 3$, this follows from Tits'result [Tit86] that a thick building of type $\tilde{A}_n$ is "classical", i.e. comes from a (not necessarily commutative) field K endowed with a discrete valuation v (see §2 of Chapter 9 in [Ron89] for a construction of the building $\tilde{A}_n(K,v)$). For $n=2$, the lemma follows from the fact that the link of a vertex in an $\tilde{A}_2$-building is a generalized 3-gon (see §2 in Chapter 3 in [Ron89]), and all vertices in a thick generalized 3-gon have the same valency (Proposition (3.3) in [Ron89]).

Of course this lemma does not hold for $n=1$, since an $\tilde{A}_1$-building is just a tree. We shall use the fact that the lemma is (trivially!) true if this tree admits a vertex-transitive group of automorphisms.

Let Γ be an $\tilde{A}_n$-group, i.e. a group acting simply transitively on the vertices of a thick $\tilde{A}_n$-building Δ (examples of such groups appear in [CMSZ93], [CS]). Fix a base-vertex $v_o \in \Delta$; let S be the set of elements $\gamma \in \Gamma$ such that γv_o is a neighbour of v_o in the 1-skeleton $\Delta^{(1)}$ of Δ. Then S is a finite, symmetric, generating subset for Γ, and the Cayley graph of Γ with respect to S identifies with $\Delta^{(1)}$. Consider the length function $L(\gamma) = d_\Delta(\gamma v_o, v_o)$, where d_Δ is the combinatorial distance in $\Delta^{(1)}$; alternatively, L is the word length function with respect to S. From Swiatkowski's loop inequality (Theorem 0.6.(a) in [Swi]) together with our Proposition 4, we immediately get:

THEOREM 1. — *Let Γ be an $\tilde{A}_n$-group, with L as above. Then the space $\mathrm{Rad}_L(\Gamma)$ satisfies the Haagerup inequality.*

COROLLARY 3. — *An $\tilde{A}_n$-group is non-amenable.*

Proof. — From Claims 1 and 2 in the proof of Proposition 1.9 in [Swi], it follows that an $\tilde{A}_n$-group has exponential growth. Then combine Proposition 6 with Theorem 1.

Of course this corollary is known, and we indicate two other possible proofs.

First, for $n \geqslant 2$, one may prove the stronger statement that an $\tilde{A}_n$-group Γ has Kazhdan's property (T). For $n = 2$, this is done in [CMS93] when Γ acts in a type-rotating way and the building is locally Desarguesian (these assumptions were dropped in [Pan] and [Zuk96]); for $n \geqslant 3$, first use Tits'result [Tit86] (see also p.137 in [Ron89]) that a euclidean building with dimension at least 3 is "classical", i.e. comes from some simple algebraic group G with F-rank at least 2, defined over some non-archimedean local field F. So Γ is essentially a lattice in G, and one may prove as in [dlHV89] that G and Γ have property (T).

Alternatively, one may construct free subgroups inside Γ. For $n = 1$, this is a simple exercise. For $n = 2$, this is a recent result of W. Ballmann and M. Brin (Theorem E in [BB]). For $n \geqslant 3$, using the fact that Γ is essentially a lattice in G, one may appeal to the celebrated Tits alternative [Tit72].

As a final remark, we mention that Swiatkowski proves in Proposition 1.9 of [Swi] that, for any uniformly thick building of type $\tilde{A}_n$, one has

$$\mathrm{card}\, B_N(v_o) \leqslant C(1+N)^{\dim \Delta} \mathrm{card}\, S_N(v_o)$$

for any $N \in \mathbf{N}$. If Γ acts simply transitively on the vertices of Δ, then Γ is non-amenable (by Corollary 3), so that the above inequality may be improved to the strong isoperimetric inequality

$$\mathrm{card}\, B_N(v_o) \leqslant C' \,\mathrm{card}\, S_N(v_o)$$

for any $N \in \mathbf{N}$. Actually the latter inequality holds for any thick building Δ that admits a discrete group Γ acting properly co-compactly. Indeed, such a Γ is non-amenable (Theorem F in [BB]), so that $\operatorname{card} B_N \leqslant K \operatorname{card} S_N$ by Følner's property. But the assumptions are such that Γ is quasi-isometric to Δ; and it is known that satisfying a strong isoperimetric inequality is a quasi-isometry invariant among graphs (see Proposition 4.1 in [Pit] for a recent proof of this fact). If Δ comes from a simple algebraic group G defined over some non-archimedean local field of characteristic zero, then such co-compact lattices do exist [BH78].

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