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LIE-RINEHART ALGEBRAS, GERSTENHABER ALGEBRAS AND BATALIN–VILKOVISKY ALGEBRAS

by Johannes HUEBSCHMANN

1. Introduction.

It has been known for a while that, given a vector space $\mathfrak{g}$ over a field k or, more generally, a projective module $\mathfrak{g}$ over a commutative ring R with 1, Lie brackets on $\mathfrak{g}$ are in bijective correspondence with differentials on the graded exterior coalgebra $\Lambda'_R\mathfrak{g}$ yielding a differential graded coalgebra structure. The homology and cohomology of $\mathfrak{g}$ with coefficients in any $\mathfrak{g}$-module M may then be computed from suitable complexes involving $\Lambda'_R\mathfrak{g}$ and M and a suitable additional structure, that of a twisting cochain; in particular, the homology of $\Lambda'_R\mathfrak{g}$ is that of $\mathfrak{g}$ with trivial coefficients. We refrain from spelling out details since we shall not need them; see e.g. [11] for the notion of twisting cochain and that of twisted object. For a general *Lie-Rinehart algebra* (A, L) (a definition will be reproduced in Section 1 below), a description of its homology and cohomology is no longer available in terms of twisting cochains; in fact, when the action of L on A is non-trivial there is no way to endow $\Lambda'_A L$ with a differential graded coalgebra structure corresponding to the Lie-Rinehart structure. Rather, given A, a commutative algebra over the ground ring R, and an

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A-module L, *Lie-Rinehart structures* on (A, L) correspond bijectively to *Gerstenhaber algebra* structures on the graded exterior A-algebra $\Lambda_A L$ over L. In the Gerstenhaber algebra picture, there is no differential in sight and the question arises as to what the link between Gerstenhaber structures and homology and cohomology of Lie-Rinehart algebras might be. The purpose of the present note is to show that an answer and some new insight may be obtained by means of the notion of *Batalin-Vilkovisky algebra*. This will also shed considerable light on these algebras themselves. Batalin-Vilkovisky algebras have recently become important in string theory and elsewhere [1], [2], [3], [7], [14], [16].

Let (A, L) be a Lie-Rinehart algebra. Given A, we shall also refer to L as an (R, A)-*Lie algebra*, cf. [15]. Our first result, Theorem 1, will say that *there is a bijective correspondence between right* (A, L)-*connections on* A (in a sense made precise in Section 1 below) *and* R-*linear operators generating the Gerstenhaber bracket on* $\Lambda_A L$; moreover, *under this correspondence, flat right* (A, L)-*connections on* A, *that is, right* (A, L)-*module structures on* A, *correspond to operators of square zero, that is, to differentials.*

To explain our next result, write $U(A, L)$ for the universal algebra for (A, L) and recall the complex $(K(A, L), d)$ generalizing the usual resolution used in Cartan-Chevalley-Eilenberg Lie-algebra cohomology [15], [9]. We refer to $(K(A, L), d)$ as the *Rinehart complex* for (A, L); it is an acyclic relatively free chain complex in the category of left $U(A, L)$-modules. When L is projective as an A-module, $(K(A, L), d)$ is in fact a projective resolution of A in the category of left (A, L)-modules. Whether or not L is projective as an A-module, given an exact generator ∂ for the Gerstenhaber algebra $\Lambda_A L$, let A_∂ denote A together with the corresponding right (A, L)-module structure (given by our first result). Our second result, Theorem 2 below, will say that, *as a chain complex, the Batalin-Vilkovisky algebra* $(\Lambda_A L, \partial)$ *coincides with* $(A_\partial \underset{U(A,L)}{\otimes} K(A, L), d)$. Thus, when L is projective as an A-module, the Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ computes the homology

$$\mathrm{H}_*(L, A_\partial) \quad (= \mathrm{Tor}_*^{U(A,L)}(A_\partial, A))$$

of L with coefficients in the right (A, L)-module A_∂; we mention in passing that in general $(\Lambda_A L, \partial)$ will compute a certain relative homology in the sense of relative homological algebra [8].

Thereafter we consider the special case where, as an A-module, L is projective of finite constant rank (say) n, so that $\Lambda_A^n L$ is the highest non-zero exterior power of L in the category of A-modules. In an earlier paper [10], we introduced the concept of *dualizing module* for such a Lie-Rinehart algebra. Our next result, Theorem 3, will exploit this notion and will say that *there is a bijective correspondence between (A, L)-connections on $\Lambda_A^n L$ and right (A, L)-connections on A*; moreover, under this correspondence, *left (A, L)-module structures on $\Lambda_A^n L$, that is, flat connections, correspond to right (A, L)-module structures on A*. Theorem 1 and Theorem 3 together imply at once, *cf.* the corollary in Section 2, that *there is a bijective correspondence between (A, L)-connections on $\Lambda_A^n L$ and linear operators generating the Gerstenhaber bracket on $\Lambda_A L$*; further, *under this correspondence, flat connections correspond to operators of square zero, that is, to differentials.* This generalizes certain observations made by Koszul [13] and Xu [18], *cf.* the remark in Section 2.

Finally, we combine the present results with those obtained in [10] establishing homological duality for Lie-Rinehart algebras, in the following way: Let ∇ be a flat (A, L)-connection on $\Lambda_A^n L$, let A_∇ be the corresponding right (A, L)-module (Theorem 3), and let ∂ be the corresponding exact generator for the Gerstenhaber algebra $\Lambda_A L$ (Theorem 1). By Theorem 2, as a chain complex, the Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ coincides with $(A_\nabla \underset{U(A,L)}{\otimes} K(A, L), d)$ and the latter computes the homology of L with coefficients in the right (A, L)-module A_∇. Theorem 4 below will say that this homology and hence that of the Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ *is naturally isomorphic to the cohomology*

$$\mathrm{H}^*(L, \Lambda_A^n L_\nabla) \quad (= \mathrm{Ext}^*_{U(A,L)}(A, \Lambda_A^n L_\nabla))$$

of L with coefficients in $\Lambda_A^n L_\nabla$, the A-module $\Lambda_A^n L$ with left (A, L)-module structure given by the connection ∇.

In a sense, the approach in the present paper relies on a careful study of the interplay between left- and right modules and of the resulting homological algebra for a general Lie-Rinehart algebra or, more generally, of the interplay between left and right connections. Once the appropriate notions and language have been found, the theory will somewhat take care of itself.

To some extent, our terminology is the same as that in [12]. I am indebted to Jim Stasheff for a number of comments on an earlier draft which helped improve the exposition and for having drawn my attention to Xu's paper [18]; in fact, the latter prompted me to write the present paper, and in a final section we explain briefly how it is related to that of Xu [18].
We mention at this stage that, though our approach includes results of

Koszul's [13] and Xu's [18], it goes beyond that. By means of right module structures and right connections, we place the relationship between Lie-Rinehart algebras, Gerstenhaber algebras, and Batalin-Vilkovisky algebras into the framework of standard homological algebra, following Hochschild and Rinehart [15]; this is completely formal and has nothing to do with the question whether or not one is working over a smooth manifold (as Koszul and Xu do) or over more general spaces including singularities. Apparently at the time, 1963, when Gerstenhaber's and Rinehart's papers [5] and [15] appeared, the link between them which we work out here was completely missed. By extending our results to appropriate strong homotopy notions, we intend to *tame* elsewhere the bracket zoo that arose recently in topological field theory, *cf.* e.g. [16].

1. Lie-Rinehart-, Gerstenhaber-, BV-algebras.

In this section, we will explain the significance of the generator of a Gerstenhaber structure and that of an exact structure. For ease of exposition we first recall the definitions.

Let R be a commutative ring with 1 and A a commutative R-algebra. An (R,A)-*Lie algebra* [15] is a Lie algebra L over R which acts (by derivations) on A (from the left) and is also an A-module satisfying suitable compatibility conditions which generalize the usual properties of the Lie algebra of vector fields on a smooth manifold viewed as a module over its ring of functions; these conditions read

$$\begin{aligned}
[\alpha, a\beta] &= \alpha(a)\beta + a[\alpha,\beta],\\
(a\alpha)(b) &= a(\alpha(b)),
\end{aligned}$$

where $a,b \in A$ and $\alpha,\beta \in L$. When the emphasis is on the pair (A,L) with the mutual structure of interaction between A and L, we refer to the pair (A,L) as a *Lie-Rinehart* algebra; in [6] the terminology *Palais pair* is used. Some relevant history may be found in the introduction and in Section 1 of [9].

We now suppose given an arbitrary Lie-Rinehart algebra (A,L). Consider the graded exterior A-algebra $\Lambda_A L$ over L where L is taken concentrated in degree 1. Write typical elements of $\Lambda_A L$ in the form

$$\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n, \quad \alpha_1, \alpha_2, \ldots, \alpha_n \in L.$$

The Lie bracket $[\cdot,\cdot]$ of L induces an R-linear bracket on $\Lambda_A L$ which endows the latter with a *Gerstenhaber algebra* structure. Abusing notation somewhat, we continue to denote the resulting bracket by

$$[\cdot,\cdot]\colon \Lambda_A L \underset{R}{\otimes} \Lambda_A L \to \Lambda_A L.$$

To get an explicit formula for it, let $u = \alpha_1 \wedge \ldots \wedge \alpha_\ell \in \Lambda_A^\ell L$ and $v = \alpha_{\ell+1} \wedge \ldots \wedge \alpha_n \in \Lambda_A^{n-\ell} L$, where $\alpha_1, \ldots, \alpha_n \in L$; then

$$(1.1)\quad [u,v] = (-1)^{|u|} \sum_{j\le \ell<k} (-1)^{(j+k)} [\alpha_j,\alpha_k] \wedge \alpha_1 \wedge \ldots \widehat{\alpha_j} \ldots \widehat{\alpha_k} \ldots \wedge \alpha_n.$$

For example when A is the ring of functions on a smooth manifold and L the Lie algebra of vector fields, $\Lambda_A L$ is the algebra of multivector fields and the bracket is the *Schouten* bracket. In general, a *Gerstenhaber algebra* is a graded commutative algebra $\mathcal{A}$ together with a Lie bracket from $\mathcal{A} \underset{R}{\otimes} \mathcal{A}$ to $\mathcal{A}$ of degree -1 (in an appropriate sense: it is a graded Lie bracket in the usual sense when the degrees of the elements of $\mathcal{A}$ are lowered by 1); see [6] where these objects are called G-algebras, or [12], [14], [16], [18]. We recall from Theorem 5 of [6] that the assignment to $\mathcal{A}$ of the pair (A_0, A_1) consisting of the homogeneous degree zero and degree one components A_0 and A_1, respectively, yields a functor from Gerstenhaber algebras to Lie-Rinehart algebras, and that this functor has a left adjoint which assigns the graded exterior A-algebra $\Lambda_A L$ over L to the Lie-Rinehart algebra (A, L), together with the bracket operation (1.1) on $\Lambda_A L$. Thus, given any Gerstenhaber algebra $\mathcal{A}$, there is a canonical morphism of Gerstenhaber algebras from $\Lambda_{A_0} A_1$ to $\mathcal{A}$. In particular, given the underlying A-module of L which we write L_0 (for the moment), there is a bijective correspondence between (R, A)-Lie algebra structures on L_0 and Gerstenhaber algebra structures on $\Lambda_A L_0$.

For a general Gerstenhaber algebra $\mathcal{A}$ over R, with bracket operation written $[\cdot,\cdot]$, an R-linear operator D on $\mathcal{A}$ of degree -1 is said to *generate* the Gerstenhaber bracket provided, for every homogeneous $a, b \in \mathcal{A}$,

$$(1.2)\qquad [a,b] = (-1)^{|a|}\,(D(ab) - (Da)b - (-1)^{|a|} a(Db));$$

the operator D is then called a *generator*. For the special case where $\mathcal{A} = \Lambda_A L$, A and L being the algebra of smooth functions and Lie algebra of smooth vector fields on a smooth manifold, respectively, this terminology goes back at least to Koszul [13]. A general Gerstenhaber algebra $\mathcal{A}$ is said to be *exact* if it is generated by an operator D of square zero, and the generator is then likewise said to be *exact*; it is then manifestly a differential on $\mathcal{A}$. An exact generator will henceforth be written ∂. For an exact structure, up to a sign, the bracket operation in $\mathcal{A}$, viewed as an element of the usual Hom-complex $\mathrm{Hom}(\mathcal{A} \underset{R}{\otimes} \mathcal{A}, \mathcal{A})$ with its differential induced by that on $\mathcal{A}$ and that on the tensor square, is the boundary of the multiplication map of $\mathcal{A}$, viewed as a chain in $\mathrm{Hom}(\mathcal{A} \underset{R}{\otimes} \mathcal{A}, \mathcal{A})$. A Gerstenhaber algebra with an exact generator is called a *Batalin-Vilkovisky* algebra [12], [14], [16], [18]; in the literature, the terminology *exact* Gerstenhaber algebra occurs as well.

Let M be an A-module. Recall that a left L-module structure $L \underset{R}{\otimes} M \to M$ on M, written $(\alpha, x) \mapsto \alpha(x)$, is called a *left* (A, L)-module structure provided

$$\alpha(ax) = \alpha(a)x + a\alpha(x), \tag{1.3.1}$$

$$(a\alpha)(x) = a(\alpha(x)), \tag{1.3.2}$$

where $a \in A$, $x \in M$, $\alpha \in L$. More generally, such an assignment $L \underset{R}{\otimes} M \to M$, not necessarily a left L-module structure but still satisfying (1.3.1) and (1.3.2), is referred to as an (A, L)-*connection*, cf. [9]; in this language, a left (A, L)-module structure is called a *flat* (A, L)-*connection*. See [9] (2.16) for historical remarks on these algebraic notions of connection etc. Likewise, let N be an A-module, and let there be given an assignment $N \underset{R}{\otimes} L \to N$, written $(x, \alpha) \mapsto x \circ \alpha$ or, somewhat simpler, $(x, \alpha) \mapsto x\alpha$ (when there is no risk of confusion); it is called a *right* (A, L)-module structure provided it is a right L-module structure and, moreover, satisfies

$$(ax)\alpha = a(x\alpha) - (\alpha(a))x, \tag{1.4.1}$$

$$x(a\alpha) = a(x\alpha) - (\alpha(a))x, \tag{1.4.2}$$

where $a \in A$, $x \in N$, $\alpha \in L$; we will refer to such an assignment as a *right* (A, L)-*connection* provided it only satisfies (1.4.1) and (1.4.2) without necessarily being a right L-module structure. Again, a right (A, L)-module structure is also said to be a *flat right* (A, L)-*connection*. The formula (1.4.2) might look somewhat puzzling at first glance; it is *not* precisely the expected replica of (1.3.2). In view of the associativity law, there is only one consistent way to interpret the expression $x(a\alpha)$, though, that is, we must have

$$x(a\alpha) = (xa)\alpha = (ax)\alpha$$

and the value of $(ax)\alpha$ is forced by (1.4.1).

We can now spell out our first result.

THEOREM 1. — *There is a bijective correspondence between right (A, L)-connections on A and R-linear operators D generating the Gerstenhaber bracket on $\Lambda_A L$. Under this correspondence, flat right (A, L)-connections on A, that is, right (A, L)-module structures on A, correspond to operators of square zero. More precisely: given an R-linear operator D generating the Gerstenhaber bracket on $\Lambda_A L$, the formula*

$$a \circ \alpha = a(D\alpha) - \alpha(a), \quad a \in A, \ \alpha \in L, \tag{1.5}$$

defines a right (A, L)-connection on A. Conversely, given a right (A, L)-connection $(a, \alpha) \mapsto a \circ \alpha$ on A ($a \in A$, $\alpha \in L$), the operator D on $\Lambda_A L$ defined by means of

$$\begin{aligned} D(\alpha_1 \wedge \ldots \wedge \alpha_n) = &\sum_{i=1}^{n} (-1)^{(i-1)} (1 \circ \alpha_i)(\alpha_1 \wedge \ldots \widehat{\alpha_i} \ldots \wedge \alpha_n) \\ &+ \sum_{j<k} (-1)^{(j+k)} [\alpha_j, \alpha_k] \wedge \alpha_1 \wedge \ldots \widehat{\alpha_j} \ldots \widehat{\alpha_k} \ldots \wedge \alpha_n, \end{aligned} \tag{1.6}$$

yields an R-linear operator D generating the Gerstenhaber bracket on $\Lambda_A L$.

Before proving this theorem we recall that any (R, A)-Lie algebra admits a universal algebra $U(A, L)$ [15] (the algebra of differential operators when A is the ring of smooth functions and L the Lie algebra of smooth vector fields on a smooth manifold). We note that left- and right (A, L)-modules plainly correspond to left- and right $U(A, L)$-modules, and vice versa; see e.g. [9] for details. More generally, left- and right (A, L)-connections may be shown to correspond bijectively to certain left- and right $U(A, E)$-module structures, for suitable (R, A)-Lie algebras E mapping surjectively onto L. The action of L on A which is part of the Lie-Rinehart algebra structure endows A with a left (A, L)- and hence with a left $U(A, L)$-module structure.

Proof of the first half of Theorem 1. — Suppose given an R-linear operator D generating the Gerstenhaber bracket on $\Lambda_A L$. A straightforward verification shows that (1.5) then yields a right (A, L)-connection on A. Indeed, let $x \in A$, $a \in A$ and $\alpha \in L$. Then

$$\begin{aligned} (ax) \circ \alpha &= (ax)(D\alpha) - \alpha(ax) \\ &= a(x(D\alpha)) - \alpha(a)x - a(\alpha(x)) \\ &= a(x(D\alpha) - \alpha(x)) - (\alpha(a))x \\ &= a(x \circ \alpha) - \alpha(a)x \end{aligned}$$

whence (1.3.1) holds; here $\alpha(a)$ etc. refers to the result of the action on $a \in A$ by $\alpha \in L$ via the left L-structure on A (which is part of the Lie-Rinehart structure of (A, L)). Moreover, from (1.1) we deduce

$$D(a\alpha) = a(D\alpha) + [a, \alpha] = a(D\alpha) - \alpha(a).$$

Hence

$$\begin{aligned} x \circ (a\alpha) &= x(D(a\alpha)) - (a\alpha)(x) \\ &= x(a(D\alpha) - \alpha(a)) - a(\alpha(x)) \\ &= a(x(D\alpha) - \alpha(x)) - x(\alpha(a)) \\ &= a(x \circ \alpha) - (\alpha(a))x \end{aligned}$$

whence (1.3.2) holds as well. Thus (1.5) yields a right (A, L)-connection on A. Another straightforward calculation shows that the vanishing of DD is equivalent to this right connection being flat, that is, to being a right (A, L)-module structure on A.

Rather than giving a direct proof for the converse, we will place the argument to be offered in its proper context, in the following way.

Recall briefly the Rinehart complex for (A, L): Consider the graded left $U(A, L)$-module $U(A, L) \underset{A}{\otimes} \Lambda_A L$ where A acts on $U(A, L)$ from the right by means of the canonical map from A to $U(A, L)$. For $u \in U(A, L)$ and $\alpha_1, \ldots, \alpha_n \in L$, let

$$\begin{aligned} d\Big(u \underset{A}{\otimes} (\alpha_1 \wedge \ldots \wedge \alpha_n)\Big) &= \sum_{i=1}^{n} (-1)^{(i-1)} u\alpha_i \underset{A}{\otimes} (\alpha_1 \wedge \ldots \widehat{\alpha_i} \ldots \wedge \alpha_n) \\ &+ \sum_{j<k} (-1)^{(j+k)} u \underset{A}{\otimes} ([\alpha_j, \alpha_k] \wedge \alpha_1 \wedge \ldots \widehat{\alpha_j} \ldots \widehat{\alpha_k} \ldots \wedge \alpha_n). \end{aligned} \tag{1.7}$$

Rinehart [15] has proved that this yields an R-linear operator

$$d \colon U(A, L) \underset{A}{\otimes} \Lambda_A L \longrightarrow U(A, L) \underset{A}{\otimes} \Lambda_A L.$$

We note that the non-trivial fact to be verified here is that, for every $u \in U(A, L)$, $a \in A$, and $\alpha_1, \ldots, \alpha_n \in L$,

$$\begin{aligned} d((ua) \otimes (\alpha_1 \wedge \ldots \wedge \alpha_n)) &= d(u \otimes ((a\alpha_1) \wedge \ldots \wedge \alpha_n)) \\ &= \ldots = d(u \otimes (\alpha_1 \wedge \ldots \wedge (a\alpha_n))). \end{aligned} \tag{1.8}$$

The resulting graded object

$$K(A, L) = (U(A, L) \underset{A}{\otimes} \Lambda_A(sL), d)$$

is the *Rinehart complex* for (A, L). Rinehart [15] has also proved that $dd = 0$, that is, d is an $U(A, L)$-linear differential whence $K(A, L)$ is indeed a chain complex, *cf.* also [9]. When L is projective as an A-module, $K(A, L)$ is in fact a projective resolution of A in the category of left $U(A, L)$-modules.

Proof of the second half of Theorem 1. — Suppose given a right (A, L)-connection on A, written $(a, \alpha) \mapsto a \circ \alpha$. Formally the same argument as that given by Rinehart in [15] establishing (1.8) shows that the operator D on $\Lambda_A L$ given by (1.6) is well defined, that is to say, for every $a \in A$ and $\alpha_1, \dots, \alpha_n \in L$,

$$(1.9) \qquad D((a\alpha_1) \wedge \dots \wedge \alpha_n) = \dots = D(\alpha_1 \wedge \dots \wedge (a\alpha_n));$$

this relies on the defining properties (1.4.1) and (1.4.2) of a right (A, L)-connection. Alternatively, when the right (A, L)-connection on A is flat, that is, a right (A, L)-module and hence $U(A, L)$-module structure on A, comparing (1.6) with (1.7), we see that, ignoring the algebra structure of $\Lambda_A L$, the operator (1.6) then coincides with that of the chain complex $(A \underset{U(A,L)}{\otimes} K(A, L), d)$ which arises when the tensor product of A with $K(A, L)$ over $U(A, L)$ is taken, with reference to the right (A, L)-module structure on A. For a general right (A, L)-connection on A, this kind of argument can still be used, with L being replaced by a suitable (R, A)-Lie algebra E mapping surjectively onto L, so that the right (A, L)-connection on A comes from a genuine right (A, E)-module structure on A.

To complete the proof, we consider the exterior algebra $\Lambda_R L$; the formula (1.6) yields an operator on $\Lambda_R L$, too, which we denote by D as well, with an abuse of notation. We then write this operator as a sum

$$D = D^\wedge + D^{[\cdot,\cdot]} \colon \Lambda_R L \to \Lambda_R L$$

where, given $\alpha_1, \dots, \alpha_n \in L$,

$$D^\wedge(\alpha_1 \wedge \dots \wedge \alpha_n) = \sum_{i=1}^{n} (-1)^{(i-1)} (1 \circ \alpha_i)(\alpha_1 \wedge \dots \widehat{\alpha_i} \dots \wedge \alpha_n)$$

$$D^{[\cdot,\cdot]} = \sum_{j<k} (-1)^{(j+k)} [\alpha_j, \alpha_k] \wedge \alpha_1 \wedge \dots \widehat{\alpha_j} \dots \widehat{\alpha_k} \dots \wedge \alpha_n.$$

We note that neither $D^\wedge$ nor $D^{[\cdot,\cdot]}$ descends from $\Lambda_R L$ to an operator on $\Lambda_A L$ but their sum does. Likewise we write

$$[\cdot,\cdot] \colon \Lambda_R L \underset{R}{\otimes} \Lambda_R L \to \Lambda_R L$$

for the corresponding bracket. A straightforward calculation shows that, given $\alpha_1, \dots, \alpha_n \in L$ and letting $u = \alpha_1 \wedge \dots \wedge \alpha_\ell \in \Lambda_R^\ell L$ and $v = \alpha_{\ell+1} \wedge \dots \wedge \alpha_n \in \Lambda_R^{n-\ell} L$, we get

$$D^\wedge(uv) = (D^\wedge u)v + (-1)^{|u|} u(D^\wedge v) \in \Lambda_R L,$$

$$D^{[\cdot,\cdot]}(uv) = (D^{[\cdot,\cdot]} u)v + (-1)^{|u|} u(D^{[\cdot,\cdot]} v) + (-1)^{|u|} [u, v] \in \Lambda_R L.$$

Since the sum $D^\wedge + D^{[\cdot,\cdot]}$ descends to an operator on $\Lambda_A L$ we conclude that, on $\Lambda_A L$,

$$D(uv) = (Du)v + (-1)^{|u|} u(Dv) + (-1)^{|u|} [u, v]$$

or, equivalently,

$$[u,v] = (-1)^{|u|}(D(uv) - (Du)v - (-1)^{|u|}u(Dv))$$

as asserted. □

Remark 1. — Given a right (A,L)-connection $(a,\alpha) \mapsto a \circ \alpha$ on A, the square DD of the operator D given by (1.6) will in general be non-zero – in fact the failure from being zero is precisely measured by the appropriate notion of curvature for this right (A,L)-connection. We refrain from spelling out details here.

Remark 2. — For an ordinary Lie algebra $\mathfrak{g}$ over the ground ring R, viewed as an (R,R)-Lie algebra with $A = R$, the statement of Theorem 1 comes down to the observation, *cf.* [12], that the complex $(\Lambda_R \mathfrak{g}, \partial)$ which computes the homology of $\mathfrak{g}$ with trivial coefficients may be viewed as that underlying the Batalin-Vilkovisky algebra $(\Lambda_R \mathfrak{g}, [\cdot,\cdot], \partial)$ with bracket $[\cdot,\cdot]$ defined by (1.1).

Given a generating operator ∂ of square zero for the Gerstenhaber bracket, we will accordingly write A_∂ for A together with the right (A,L)-module structure given by (1.5). An exact Gerstenhaber algebra or, equivalently, Batalin-Vilkovisky algebra structure on $\Lambda_A L$ has an underlying chain complex structure, and we now spell out explicitly an observation which was already implicit in the proof of Theorem 1.

THEOREM 2. — *Given an exact generator ∂ for the Gerstenhaber algebra $\Lambda_A L$, the Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ coincides as a chain complex with $(A_\partial \underset{U(A,L)}{\otimes} K(A,L), d)$. In particular, when L is projective as an A-module, the Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ computes*

$$\mathrm{H}_*(L, A_\partial) \quad (= \mathrm{Tor}_*^{U(A,L)}(A_\partial, A)),$$

the homology of L with coefficients in A_∂. □

In general, that is, when L is not necessarily projective, $(\Lambda_A L, \partial)$ computes a certain relative Tor-functor; see [8] for details on relative homological algebra.

2. Duality.

We now suppose that, as an A-module, L is projective of finite constant rank (say) n, so that $\Lambda_A^n L$ is the highest non-zero exterior power of L in the category of A-modules. In an earlier paper [10], we introduced the concept of *dualizing module* C_L for such a Lie-Rinehart algebra. Recall that, by definition, $C_L = \mathrm{H}^n(L, U(A,L))$, with its induced right (A,L)-module structure (coming from the obvious right $U(A,L)$-module structure on $U(A,L)$). Moreover, the negative of the Lie-derivative endows $\mathrm{Hom}(\Lambda_A^n L, A)$ with a right (A,L)-module structure and, by Theorem 2.8 of [10], C_L and $\mathrm{Hom}(\Lambda_A^n L, A)$ are isomorphic as right (A,L)-modules.

THEOREM 3. — *There is a bijective correspondence between (A,L)-connections on $\Lambda_A^n L$ and right (A,L)-connections on A. Under this correspondence, left (A,L)-module structures on $\Lambda_A^n L$ (i.e. flat connections) correspond to right (A,L)-module structures on A. More precisely: Given an (A,L)-connection ∇ on $\Lambda_A^n L$, the negative of the (generalized) Lie-derivative on $A \cong \mathrm{Hom}_A(\Lambda_A^n L, M)$ with reference to the connection ∇ on $M = \Lambda_A^n L$, that is, the formula*

$$(\phi\alpha)x = \phi(\alpha x) - \nabla_\alpha(\phi(x)), \tag{2.1}$$

where $x \in \Lambda_A^n L$, $\alpha \in L$, $\phi \in \mathrm{Hom}_A(\Lambda_A^n L, \Lambda_A^n L) \cong A$, yields a right (A,L)-connection on A, written $(a,\alpha) \mapsto a\alpha$ $(a \in A,\ \alpha \in L)$. Conversely, given a right (A,L)-connection on A (written $(a,\alpha) \mapsto a\alpha$), on $\Lambda_A^n L \cong \mathrm{Hom}_A(C_L, A)$, the assignment

$$(\nabla_\alpha\psi)x = \psi(x\alpha) - (\psi x)\alpha, \quad x \in C_L,\ \alpha \in L,\ \psi \in \mathrm{Hom}_A(C_L, A), \tag{2.2}$$

yields an (A,L)-connection ∇ (written $L \underset{R}{\otimes} \Lambda_A^n L \to \Lambda_A^n L$, $(\alpha,\psi) \mapsto \nabla_\alpha\psi$).

Proof. — This is straightforward and left to the reader. □

Combining Theorem 1 with Theorem 3, we obtain the following:

COROLLARY. — *There is a bijective correspondence between (A,L)-connections on $\Lambda_A^n L$ and linear operators D generating the Gerstenhaber bracket on $\Lambda_A L$. Under this correspondence, flat connections correspond to operators of square zero, that is, to differentials. The relationship is made explicit by means of* (1.5), (1.6), (2.1) *and* (2.2). □

Remark. — The statement of the corollary, without the explicit relationship given by (1.5), (1.6), (2.1), and (2.2), was proved by Koszul [13] (Section 2) for the special case where A is the ring of smooth functions and L the $(\mathbb{R}, A)$-Lie algebra of smooth vector fields on a smooth manifold, and by Xu [18] (3.8) for the special case where A is the ring of smooth functions and L the $(\mathbb{R}, A)$-Lie algebra of sections of a general Lie algebroid on a smooth manifold. However, our approach in terms of right (A,L)-module structures on A (coming into play in Theorem 1 and Theorem 3) seems to be new and is far more general.

Next we recall from Theorem 2.8 of [10] that (A, L) satisfies duality and inverse duality in dimension n; this means that there are natural isomorphisms

$$\phi: \mathrm{H}^k(L, M) \to \mathrm{H}_{n-k}\Big(L, C_L \underset{A}{\otimes} M\Big)$$

for all non-negative integers k and all left (A, L)-modules M and, furthermore, natural isomorphisms

$$\psi: \mathrm{H}_k(L, N) \to \mathrm{H}^{n-k}(L, \mathrm{Hom}_A(C_L, N))$$

for all non-negative integers k and all right (A, L)-modules N. We now combine this with the results of the present paper, in the following way: let ∇ be a flat (A, L)-connection on $\Lambda_A^n L$; from Theorem 3 we know that ∇ determines a unique right (A, L)-module A_∇ having A as underlying A-module. Further, by Theorem 1, this right (A, L)-module structure determines a corresponding exact generator ∂ for the Gerstenhaber algebra $\Lambda_A L$ and, by Theorem 2, as a chain complex, the resulting Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ coincides with the chain complex $(A_\partial \underset{U(A,L)}{\otimes} K(A, L), d)$ arising from the Rinehart resolution $K(A, L)$ of A in the category of left (A, L)-modules and computing

$$\mathrm{H}_*(L, A_\nabla) \quad (= \mathrm{Tor}_*^{U(A,L)}(A_\nabla, A)),$$

the homology of L with coefficients in the right (A, L)-module A_∇.

THEOREM 4. — *The homology of L with coefficients in the right (A, L)-module A_∇, that is, that of the Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$, is naturally isomorphic to the cohomology*

$$\mathrm{H}^*(L, \Lambda_A^n L_\nabla) \quad (= \mathrm{Ext}^*_{U(A,L)}(A, \Lambda_A^n L_\nabla))$$

of L with coefficients in $\Lambda_A^n L_\nabla$, the A-module $\Lambda_A^n L$, with left (A, L)-module structure given by the flat connection ∇. □

3. Some applications and concluding remarks.

(3.1) *Lie algebroids.* Let (A, L) be the Lie-Rinehart algebra arising from a Lie algebroid of dimension n over a smooth manifold, so that A is the algebra of smooth functions and the L underlying A-module the space of sections of the Lie algebroid. Then n is the rank of L, and $\Lambda_A^n L$, being the space of sections of a real line bundle, manifestly has a flat (A, L)-connection ∇. (Over the complex numbers, in particular in the holomorphic context, there will in general be obstructions to the existence of such a flat connection.) Hence A has a right (A, L)-module structure (Theorem 3), to be referred to as A_∇ which, by Theorem 1, determines a generator ∂ for the Gerstenhaber algebra $\Lambda_A L$; by Theorem 2, the homology of the resulting Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ is naturally isomorphic to the homology of L with coefficients in A_∇ and, by Theorem 4, to the cohomology of L with values in $\Lambda_A^n L_\nabla$, the A-module $\Lambda_A^n L$, with left (A, L)-module structure given by the connection ∇. The isomorphism between the homology of $(\Lambda_A L, \partial)$ and the cohomology of L with values in $\Lambda_A^n L_\nabla$ has been observed in [18] (4.6) for the special case where the line bundle underlying $\Lambda_A^n L$ is trivial in such a way that, as left (A, L)-modules, $\Lambda_A^n L$ and A are isomorphic or, more formally, when the modular class of the Lie-Rinehart algebra (A, L) is trivial, *cf.* [10] and also [4], [17]. Xu's isomorphism (4.6) is in fact a special case of the duality isomorphism

$$\mathrm{H}_*(L, A_\nabla) \cong \mathrm{H}^{n-*}(L, \Lambda_A^n L_\nabla) \tag{3.1.1}$$

established in our paper [10]. Notice that when L is the $(\mathbb{R}, A)$-Lie algebra of smooth vector fields, we can view the elements of $\Lambda_A L$ as de Rham currents and hence the Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ as a chain complex defining *de Rham homology*.

(3.2) *Poisson structures.* Let A be a Poisson algebra, with Poisson structure $\{\cdot,\cdot\}$, and let $D_{\{\cdot,\cdot\}}$ be its (R, A)-Lie algebra, coming from the Poisson structure, see [9] (3.8) for details and also [10]. Besides its obvious left $(A, D_{\{\cdot,\cdot\}})$-module structure, from the Poisson structure, the algebra A inherits also a right $(A, D_{\{\cdot,\cdot\}})$-module structure by means of the formula

$$a(b(du)) = \{ab, u\}$$

which we refer to by the notation $A_{\{\cdot,\cdot\}}$; here $a, b, u \in A$ and du denotes the differential of u. By Theorem 1, this structure induces an exact generator for the Gerstenhaber algebra $\Lambda_A D_{\{\cdot,\cdot\}}$; we denote this generator by $\partial_{\{\cdot,\cdot\}}$. The resulting Batalin-Vilkovisky algebra $(\Lambda_A D_{\{\cdot,\cdot\}}, \partial_{\{\cdot,\cdot\}})$ computes the Poisson homology of A. This follows at once from Theorem 2, in view of the definition of Poisson homology, *cf.* [9]. In fact the chain complex which underlies $(\Lambda_A D_{\{\cdot,\cdot\}}, \partial_{\{\cdot,\cdot\}})$ coincides with the chain complex $(A_{\{\cdot,\cdot\}} \underset{U(A, D_{\{\cdot,\cdot\}})}{\otimes} K(A, D_{\{\cdot,\cdot\}}), d)$ and, by definition, the Poisson homology of A is the homology of this complex [9]. For the special case where, as an A-module, $D_{\{\cdot,\cdot\}}$ is finitely generated projective of constant rank n, the corresponding left $(A, D_{\{\cdot,\cdot\}})$-module structure on the top exterior power $\Lambda^n_A D_{\{\cdot,\cdot\}}$ has been introduced in (7.8) of our paper [10]. When A is the Poisson algebra of smooth functions on a smooth Poisson manifold and $D_{\{\cdot,\cdot\}}$ the corresponding $(\mathbb{R}, A)$-Lie algebra arising from the cotangent Lie algebroid, the chain complex $(A_{\{\cdot,\cdot\}} \underset{U(A,D_{\{\cdot,\cdot\}})}{\otimes} K(A, D_{\{\cdot,\cdot\}}), d)$ coincides with Koszul's complex defining Poisson homology [13]. In fact, Koszul has indeed shown that then the operator $\partial_{\{\cdot,\cdot\}}$ is a generator for the corresponding Gerstenhaber algebra. For this special case, a geometric description of the corresponding left $(A, D_{\{\cdot,\cdot\}})$-module structure on $\Lambda^n_A D_{\{\cdot,\cdot\}}$ or, rather, that of the corresponding flat connection on the underlying line bundle was given in [4]; an expression relating the latter with the generator of the corresponding Batalin-Vilkovisky algebra has been given in (18) of [18]. In (4.8) of that paper, a certain isomorphism between Poisson homology and cohomology is given which is again a special case of (3.1.1) above; see our paper [10] (7.4) for details.

Let $L = \mathrm{Der}(A)$, with its obvious (R, A)-Lie algebra structure. The pair $(L, D_{\{\cdot,\cdot\}})$ constitutes the structure of what we will call elsewhere an *(R, A)-Lie bialgebra*, in fact, a *triangular (R, A)-Lie bialgebra*. These are abstractions from the notion of a Lie bialgebroid and from that of a triangular one, respectively, *cf.* [12]. Here we only remark that the notion of (R, A)-Lie bialgebra is equivalent to that of strong differential Gerstenhaber algebra, that is, to a Gerstenhaber algebra endowed with a derivation of degree 1 and square zero which behaves as a derivation for the algebra and bracket structures; *cf.* [12] and [18].

(3.3) Finally we comment on question 2 and question 3 in Section 5 of [18]:

(3.3.1) Given a general Lie algebroid, for its Gerstenhaber algebra, there does not seem to exist a canonical generating operator corresponding to its modular class: in fact, write (A, L) for the Lie-Rinehart algebra arising in the usual fashion from the Lie algebroid under consideration. We know from Theorem 1 above that these generating operators correspond bijectively to right (A, L)-module structures (or connections) on A and, for a Poisson algebra A, the Poisson structure (somewhat miraculously) induces a right module structure for the corresponding Lie-Rinehart algebra as well and hence a canonical exact generator. Now, on the one hand, by Theorem 3, right (A, L)-module structures on A correspond bijectively to left (A, L)-module structures on the top exterior power $\Lambda^n_A L$ of L. On the other hand, the modular class is determined by the left (A, L)-module

$Q_L = \mathrm{Hom}_A(C_L, \omega_A)$ (see [10] for the right (A, L)-module ω_A), but there is no obvious way to relate these structures with requisite right (A, L)-module structures (or right connections) on A. The fact that there is a canonical such structure in the Poisson case seems to reflect certain compatibility properties between left and right structures in this case. Indeed, the Koszul operator $\partial_{\{\cdot,\cdot\}}$ on $\Lambda_A D_{\{\cdot,\cdot\}}$ looks canonical but depends tacitly also on the right module structure on A which, in turn, depends on the Poisson structure, too. Hence the answer to Xu's question 2 will presumably be No.

(3.3.2) The family of Poisson homologies parametrized by the first Poisson cohomology are precisely the homologies $\mathrm{H}_*^{\mathrm{Poisson}}(A, A_\nabla)$ where ∇ runs through connections on $\Lambda_A^n L$; these homologies are isomorphic to the cohomologies $\mathrm{H}^*_{\mathrm{Poisson}}(A, \Lambda_A^n L_\nabla)$, where $L = D_{\{\cdot,\cdot\}}$, the corresponding $(\mathbb{R}, A)$-Lie algebra having constant rank n as a projective A-module. This answers Xu's question 3.

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