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RECOVERING THE TOTAL SINGULARITY OF A CONORMAL POTENTIAL FROM BACKSCATTERING DATA

by Mark S. JOSHI

1. Introduction.

In this paper, we show that in certain cases the total singularity of a potential with a singularity conormal to a hypersurface bounding a smooth convex domain is determined by backscattering data. This extends previous work of Greenleaf and Uhlmann, [2], who showed that the location of a conormal singularity and its principal symbol could be recovered from backscattering data for any submanifold. Our approach is to refine the proof of Greenleaf and Uhlmann by using the more exact calculus of paired Lagrangian distributions developed in [8] and [9] to compute the effects of lower order terms by regarding them as a perturbation – *cf.* [10].

Greenleaf and Uhlmann proceeded by constructing an approximate solution to

$$(\Box + q)u(x,t) = 0 \ \text{ on } \mathbb{R}^{n+1} \tag{1.1}$$

$$u(x,t) = \delta(t - x.\omega), t << 0, \tag{1.2}$$

by using an iterative scheme $u_0 = \delta(t - x.\omega)$, $u_j = -\Box^{-1}(q(x)u_{j-1})$. In fact, their proof only required u_0, u_1; we will however use the entire iterative scheme – how far we go depends the strength of the perturbation

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considered. The scattering matrix and the backscattering matrix can then be obtained by applying elliptic Fourier integral operators to u. A part of their proof was that this implies that away from a bad set that the scattering kernel, $\alpha(s,\theta,\omega)$, is microlocally a Lagrangian distribution associated to a Lagrangian submanifold – $\hat{\Lambda}_-$ – the reflected Lagrangian, in the area which occurs in backscattering. They showed that the principal symbol on $\hat{\Lambda}_-$ restricted to $\theta = -\omega$ determines the principal symbol of q. Here, we show that for q_1, q_2 of a particular form, the principal symbol of the difference of the scattering matrices on $\theta = -\omega$ determines the principal symbol of the difference of the potentials. This in particular means that if the difference of the backscattering kernels is smooth then $w_1 - w_2$ is smooth.

We need a condition on the potential to ensure that the scattering kernel is defined, see [15] or [2]. Let

$$\gamma_n = \begin{cases} -\frac{1}{2} & n = 3,4 \\ \frac{2-n}{n} & n \geq 5. \end{cases}$$

The scattering kernel is then defined for potentials conormal to a hypersurface with Lagrangian orders less than

$$\gamma_n - 1/2 + n/4.$$

It may be possible to make the Lax-Phillips theory work for these potentials with weaker assumptions on their regularity as the singularities are of a very precise type, in which case our arguments would continue to work but we leave this to the interested reader.

THEOREM 1.1. — *Let $\Omega \subset \mathbb{R}^n$ be a normally accessible, precompact domain with smooth boundary, $\partial\Omega$, in $\mathbb{R}^n$ with $n \geq 3$. Suppose $q_i = v + w_i$, $i = 1,2$ with $v \in C_c^\infty(\mathbb{R}^n)$, $w_i \in I^\mu_{phg}(\mathbb{R}^n, N^*(\partial\Omega))$, $\operatorname{supp}(w_i) \subset \bar{\Omega}$ with*

$$\mu + 1/2 - n/4 < \gamma_n$$

and let $\beta_i(s,\omega)$ be the associated back-scattering kernels then if $\beta_1(s,\omega) - \beta_2(s,\omega)$ is smooth then $w_1 - w_2 \in C^\infty$.

We note that as γ_n is negative the potential is always representable by an L^p_{loc} function with $p > 1$. For most of the arguments in this paper, we only need the potential to be of order less than $n/4 - 1/2$ but the extra smoothness is required to ensure the scattering theory for the potential is defined. We recall that a normally accessible domain is a domain such that the outer normal line at each point of the boundary does not reintersect the domain. This is clearly weaker than weak convexity.

We note an alternative formulation of our hypotheses avoiding microlocal analysis. Let y be a function which is zero on $\partial\Omega$ with non-zero derivative there and positive on Ω. Let χ be the function which is one on Ω and zero off Ω. Our condition on w_i is equivalent to saying that

$$w_i = y^a \chi g_i$$

with g_i a smooth function and $a > -1 - \gamma_n$. Theorem 1.1 says that if the difference of the backscattering matrices is smooth then g_1, g_2 have the same Taylor series at every point of $\partial\Omega$.

In Section 2, we review the theory of paired Lagrangian distributions. In Section 3, we prove estimates for products of Lagrangian and paired Lagrangian distributions. In Section 4, we examine the mapping properties of paired Lagrangian distributions under the forward fundamental solution of the wave operator. We then put all this together in Section 5 to prove the main result. We refer the reader to [2] for discussion of other related papers.

2. Paired Lagrangian distributions.

In this section, we review the theory of paired Lagrangian distributions. Recall that a Lagrangian distribution is a distribution that lies in a fixed Sobolev (or Besov) class under repeated application of first order pseudo-differential operators which are characteristic on a given conic Lagrangian submanifold. Paired Lagrangian distributions are similarly associated to a cleanly intersecting pair of submanifolds. Paired Lagrangian distributions were introduced by Melrose and Uhlmann in [14] to give a symbolic construction of parametrices of operators of real principal type. However that calculus is too narrow for our purposes. Greenleaf and Uhlmann worked in a much wider calculus consisting of distributions of fixed Sobolev order under repeated application of first order pseudo-differential operators which are characteristic on both Lagrangian submanifolds. (See [3], [4].) This is a very wide class as it contains distributions which have wavefront contained in the intersection of the two submanifolds and so behaviour off the intersection is not controlled by behaviour at the intersection. Here, we work with this class as well as with two narrower classes. The first narrower class is the calculus developed in [8] and [9] which is general enough to contain most of the distributions we are studying, whilst being narrow enough to retain tight control of them. The second is a compromise class which is polyhomogeneous on one of the Lagrangian submanifolds up to the intersection but not both. There are many different order conventions for paired Lagrangian distributions; we define ours so that microlocally away from the intersection the order will be the same as that as a Lagrangian distribution in [6] which is consistent with the order conventions in [8] and [9]. (This is not consistent with [2]– we shall discuss the relationship below.) For brevity, we shall assume that all Lagrangian submanifolds mentioned in the following are embedded, conic, Lagrangian submanifolds of the cotangent bundle of $\mathbb{R}^r$ (or a smooth manifold X of dimension r).

We recall Hörmander's constructive definition of a Lagrangian distribution that u is Lagrangian with respect to Λ of order m if locally u is of the form

$$\int e^{i\phi(x,\theta)}a(x,\theta)d\theta,$$

where ϕ is a generating function for Λ and a is a symbol of order $m - N/2 + r/4$ where N is the number of θ variables. A Fourier integral operator is an operator whose Schwartz kernel is a Lagrangian distribution.

The marked Lagrangian distributions of Melrose are a variant of these distributions which are allowed more singular behaviour on an isotropic submanifold. These are defined with reference to a model,

$$\tilde{\Lambda}_0 = N^*(x=0), \Sigma = N^*(x=0) \cap \{\xi' = 0\}$$

for some splitting of the coordinates $x = (x', x'')$ and with ξ the dual coordinates. Any Lagrangian submanifold containing an isotropic submanifold can be reduced to this form by a homogeneous symplectomorphism.

Definition 2.1. — *The distribution u is in $I^{m,s}_{ma}(\tilde{\Lambda}_0, \Sigma)$ if u can be written up to smooth terms as*

$$\int e^{ix.\xi}a(\xi)d\xi$$

where

$$|D^\alpha_\xi a(\xi)| \leq C_\alpha \langle\xi\rangle^{m-|\alpha''|}\langle|\xi''|^{1/2} + |\xi'|\rangle^{-|\alpha'|}\left(\frac{\langle\xi\rangle}{\langle|\xi''|^{1/2} + |\xi'|\rangle}\right)^{2s}.$$

In general, if E is a submanifold of the Lagrangian Λ, we say $u \in I^{m,s}_{ma}(\Lambda, E)$ if u is in $I^m(\Lambda)$ microlocally away from E and if for any zeroth order Fourier integral operator, F, associated to the graph of a local symplectomorphism which maps (Λ, E) to $(\tilde{\Lambda}_0, \Sigma)$, we have $Fu \in I^{m,s}_{ma}(\tilde{\Lambda}_0, \Sigma.)$

These distributions map in a natural way under pseudo-differential operators.

PROPOSITION 2.1. — *If P is a pseudo-differential operator of order m then P induces a map*

$$P : I^{p,l}_{ma}(\Lambda, E) \to I^{p+m,l}_{ma}(\Lambda, E).$$

For further details see [13]. An important use of marked Lagrangian distributions is that they can be used to decompose paired Lagrangian distributions. We take such a decomposition as our definition.

DEFINITION 2.2. — *If (Λ_0, Λ_1) are a cleanly intersecting pair of Lagrangian submanifolds. We define $I^{m,p}(\Lambda_0, \Lambda_1)$ to be equal to*

$$I^{m,\frac{m-p+d/2}{2}}_{ma}(\Lambda_0, \Lambda_0 \cap \Lambda_1) + I^{p,\frac{p-m+d/2}{2}}_{ma}(\Lambda_1, \Lambda_0 \cap \Lambda_1)$$

where d in the codimension of the intersection in each of the two.

The mapping properties of this class under pseudo-differential operators is immediate from the marked case. We note that away from $\Lambda_0 \cap \Lambda_1$, these are just ordinary Lagrangian distributions. In the special case that $\Lambda_j = N^*(X_j)$ with X_j submanifolds, $X_0 \subset X_1$, of codimension $d_1 + d_2$ and d_2, then there is an alternative definition. Away from the intersection, the distributions are Lagrangian of the correct orders and in local coordinates (x', x'', x''') such that

$$X_0 = \{x'' = 0, x''' = 0\}, X_1 = \{x''' = 0\},$$

the distribution near the intersection can be written as

$$\int e^{i(x''.\xi''+x'''.\xi''')} a(x', \xi'', \xi''') d\xi'' d\xi''',$$

where a obeys symbol estimates

$$|D^{\alpha}_{x'} D^{\beta}_{\xi''} D^{\gamma}_{\xi'''} a(x', \xi'', \xi''')| \le C_{\alpha,\beta,\gamma} \langle \xi'', \xi''' \rangle^{\mu-|\gamma|} \langle \xi'' \rangle^{\mu'-|\beta|}$$

which are uniform for x' in compact sets. Here

$$\mu + \mu' + \frac{d_1 + d_2}{2} = m + r/4,$$

$$\mu + \frac{d_1}{2} = p + r/4.$$

We note that Greenleaf and Uhlmann used μ, μ' as their orders when the Lagrangian submanifolds were conormal bundles.

In order to define the concept of a polyhomogeneous, paired, Lagrangian distribution, we need the concept of a radial operator. We recall from [7], DEFINITION 2.3. — *A radial operator for a Lagrangian submanifold Λ is a properly supported, classical pseudo-differential operator, R, of first order such that*

1. *the principal symbol of R vanishes on Λ,*
2. *the subprincipal symbol of R equals $n/4$ on Λ,*
3. *the bicharacteristic field of R, H_R, on Λ is equal to the radial vector field multiplied by i^{-1}.*

THEOREM 2.1. — *An element u of $I^{\Re\mu}(\Lambda)$ is polyhomogeneous of order μ if and only for some (and hence any) radial operator R associated to Λ,*

$$\left(\prod_{j=0}^{N-1}(R+\mu-j)\right)u \in I^{\Re\mu-N}(\Lambda), \quad \textit{for all } N. \tag{2.1}$$

We make the convention that

$$\prod_{j=0}^{-1}(R+\mu-j) = \mathrm{Id}\,.$$

DEFINITION 2.4. — *The pair of operators, (R_0, R_1) is said to be radial for the pair of Lagrangian submanifolds (Λ_0, Λ_1), if R_j is radial for Λ_j and is characteristic on $\Lambda_0 \cup \Lambda_1$.*

We then recall from [9],

DEFINITION 2.5. — *If (Λ_0, Λ_1) is a cleanly intersecting pair of Lagrangian submanifolds and (R_0, R_1) is an associated pair of radial operators then $u \in I^{m,p}_{phg}(\Lambda_0, \Lambda_1)$ if and only if*

$$\prod_{j=0}^{N-1}(R_0+m-j)\prod_{k=0}^{M-1}(R_1+p-k)u \in I^{m-N,p-M}(\Lambda_0,\Lambda_1).$$

This definition was shown to be independent of the choice of radial operators there. A full symbol calculus can be developed for this class but we shall not make use of it here. We recall that

$$I^{p}_{phg}(\Lambda_0) \subset I^{p,p-1/2}_{phg}(\Lambda_0,\Lambda_1).$$

We also introduce a new class where polyhomogeneity is required on only one of the two Lagrangian submanifolds.

DEFINITION 2.6. — *If* (Λ_0, Λ_1) *is a cleanly intersecting pair of Lagrangian submanifolds and* R_0 *is a radial operator for* Λ_0 *which is characteristic on* Λ_1 *then* $u \in I^{m,p}_{pphg}(\Lambda_0, \Lambda_1)$ *if and only if*

$$\prod_{j=0}^{N-1} (R_0 + m - j)u \in I^{m-N,p}(\Lambda_0, \Lambda_1).$$

The proof that this class is independent of the choice of R_0 is just a special case of that for the I_{phg} classes. It is important to realize that this class is smaller than the class of paired Lagrangian distributions which are polyhomogeneous on Λ_0 off Λ_1.

3. Product estimates.

The most important part of our constructions involve studying the action of multiplying a paired Lagrangian distribution by a singular function. All the cases we need will involve distributions conormal to transverse hypersurfaces and their intersections. Since locally, transverse hypersurfaces can be reduced to the vanishing of coordinate functions, we study a model problem, let

$$\begin{aligned} X_1 &= \{x_1 = 0\}, \\ X_2^{\pm} &= \{x_2 = 0, \pm x_1 \geq 0\}, \\ X_3 &= X_1 \cap X_2^{\pm} = \{x_1 = 0, x_2 = 0\}. \end{aligned}$$

We work in a manifold of dimension r (later in the paper we will take $r = 2n$) and take a splitting of coordinates

$$x = (x_1, x_2, x'').$$

Suppose $w \in I^{\mu}_{phg}(N^*X_1)$ and $\mathrm{supp}(w) \subset \{x_1 \geq 0\}$. We want to understand the action of multiplying by w, M_w, on $I(N^*(X_2^{\pm}), N^*(X_3))$ and on $I(N^*(X_1), N^*(X_3))$. We first look at the wider non-polyhomogeneous classes and then deduce results for the polyhomogeneous ones.

Now w can be written as

$$\int e^{ix_1 \cdot \xi_1} b(x_2, x'', \xi_1) d\xi_1$$

with b a symbol of order $\tilde{\mu} = \mu + n/4 - 1/2$ with asymptotic expansion $\sum_j b_j(x_2, x'', \xi_1)$. The support condition on w means that

$$b_j(x_2, x'', \xi_1) = (\xi_1 - i0)^{\tilde{\mu}-j} b_j(x_1, x'', 1)$$

– see [6] Vol. 3, Chap. 18, for a discussion of this. This means that w has an expansion

$$\sum \chi_+^{-\tilde{\mu}-1+j}(x_1)c_j(x_2,x'')$$

with c_j smooth, in the sense that w can be made arbitrarily smooth by subtracting a finite number of these terms. Applying the Borel lemma, this means that w is of the form $\chi_+^a(x_1)f(x)$ with f a smooth function, where $a = -\mu - n/4 - 1/2$. So to study mapping properties it is enough to study multiplication by smooth functions and by $\chi_+^a(x_1)$. Multiplication by a smooth function is a special case of applying a zeroth order, classical, pseudo-differential operator so it will leave the paired Lagrangian spaces invariant. We denote M_a the operation of multiplying by χ_+^a.

We start with the action of M_a on distributions conormal to a transverse hypersurface.

PROPOSITION 3.1. — *For any a, M_a induces a map*

$$I^p_{phg}(N^*(X_2)) \to I_{phg}^{-a-1+1/2-r/4,p-a-1/2}(N^*(X_1),N^*(X_2^+)) + I_{phg}^{p-a-1/2,p}(N^*(X_3),N^*(X_2^+)).$$

Proof. — For the non-polyhomogeneous spaces this is Lemma 1.1 of [2]. We obtain $N^*(X_2^+)$ because of the support of $H(x_1)$.

To see the polyhomogeneity, note that the operators

$$R_1 = x_1\frac{\partial}{\partial x_1} + 1/2 - r/4,$$
$$R_2 = x_2\frac{\partial}{\partial x_2} + 1/2 - r/4,$$
$$R_3 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 1 - r/4$$

are radial for X_1, X_2, and X_3 respectively. As M_a commutes with R_2, the improvement under application of radial operators on $N^*(X_2)$ is immediate.

Applying $R_1 - a - 1 + 1/2 - r/4$ to $M_a u$ will yield

$$M_{a+1}u + M_{a+1}\frac{\partial u}{\partial x_1}$$

and, of course, $\dfrac{\partial u}{\partial x_1} \in I^p_{phg}(N^*(X_1))$. So the order on $(N^*(X_1), N^*(X_3))$ is lowered.

To get the improvement under R_3, just write it as $R_1 + R_2$ plus a constant and then combine the arguments above. □

PROPOSITION 3.2. — *If $a > -1$ and $u \in I^{p,m}(N^*(X_1), N^*(X_3))$ with*

$$p - 1/2 + r/4 + a < 0,$$

then

$$M_a u \in I^{p-a,m-a}(N^*(X_1), N^*(X_3)) + I_{phg}^{-a-1+1/2-r/4}(N^*(X_1))$$

if a is not an integer, and

$$M_a u \in I^{p-a+,m-a+}(N^*(X_1), N^*(X_3)) + I_{phg}^{-a-1+1/2-r/4}(N^*(X_1)),$$

otherwise.

Note that the term $I_{phg}^{-a-1+1/2-r/4}(N^*(X_1))$ comes from multiplying smooth functions by $\chi_+^a(x_1)$.

Proof. — Up to smooth terms,

$$u = \int e^{i(x_1\xi_1 + x_2\xi_2)} b(x'', \xi_1, \xi_2) d\xi_1 d\xi_2$$

where b is such that

$$\left| D_{x''}^{\beta} D_{\xi}^{\alpha} b(x'', \xi_1, \xi_2) \right| \leq C \langle \xi_1, \xi_2 \rangle^{\mu - \alpha_1} \langle \xi_2 \rangle^{\mu' - \alpha_2}$$

and

$$m = \mu + 1/2 - r/4,$$
$$p = \mu + \mu' + 1 - r/4.$$

Multiplying by x_1 is equivalent to differentiating b with respect to ξ_1 so multiplying by x_1^k has this mapping property. This means that writing $a = a - k + k$, we can assume that a is non-positive. The smooth error yields an element of $I_{phg}^{-a-1+1/2-r/4}(N^*(X_1))$ as noted above.

Before proceeding to the estimate, we note that as the model is invariant under changes of coordinates of the form

$$x_1 \mapsto x_1,$$
$$x_2 \mapsto x_1 + s x_2,$$

it is enough to estimate the convolution in the domain where ξ_1, the dual to x_1, is elliptic.

As only behaviour near $x_1 = 0$ is important we can cut off near it, and so the total symbol of $M_a u$ is estimated, up to constants, by the convolution,

$$\int \langle \eta \rangle^{-a-1} |b(x'', \xi_1 - \eta, \xi_2)| d\eta.$$

The symbol estimates on b imply that this is bounded, up to constants by

$$\langle\xi_2\rangle^{\mu'}\int\langle\eta\rangle^{-a-1}\langle\xi_1-\eta_1,\xi_2\rangle^{\mu}d\eta.$$

As $\mu<0$, this is thus less than

$$C\langle\xi_2\rangle^{\mu'}\int\langle\xi_1-\eta_1\rangle^{\mu}\langle\eta\rangle^{-a-1}d\eta.$$

We consider first the integral over the domain where $|\eta|\leq\frac{1}{2}|\xi_1|$. This is less than

$$C\langle\xi_2\rangle^{\mu'}\langle\xi_1\rangle^{\mu}\int_{|\eta|\leq\frac{1}{2}|\xi_1|}\langle\eta\rangle^{-a-1}d\eta.$$

For $a<0$, this is bounded by

$$C\langle\xi_2\rangle^{\mu'}\langle\xi_1\rangle^{\mu-a}$$

and for $a=0$ by

$$C\langle\xi_2\rangle^{\mu'}\langle\xi_1\rangle^{\mu}\log(\langle\xi_1\rangle)$$

which is what we want.

To estimate in the domain where $\frac{1}{2}|\xi_1|\leq|\eta|\leq 2|\xi_1|$, we cut use a smooth cutoff to obtain a term supported in $\frac{1}{3}|\xi_1|\leq|\eta|\leq 3|\xi_1|$, which we then estimate in the same away but interchanging the roles of $-a-1$ and μ, integrating by parts if necessary to ensure that $\mu>-1$. (The smooth cutoff ensures the absence of boundary terms when integrating by parts.)

In the domain, $|\eta|\geq 2|\xi_1|$ we make the change of variables $\eta=|\xi_1|\eta'$ which yields

$$C\langle\xi_2\rangle^{\mu'}\int_{|\eta'|\geq 2}\langle\xi_1-\eta'|\xi_1|\rangle^{\mu}|\xi_1|\langle|\xi_1|\eta'\rangle^{-a-1}d\eta'.$$

This can be written as

$$C\langle\xi_2\rangle^{\mu'}\langle\xi_1\rangle^{\mu-a}\int\left(\frac{\langle\xi_1-\tilde{\eta}|\xi_1|\rangle}{\langle\xi_1\rangle}\right)^{\mu}\left(\frac{\langle|\xi_1|\eta'\rangle}{\langle\xi_1\rangle}\right)^{-a-1}d\eta'.$$

The integrand is uniformly integrable provided $\mu-a-1<0$ that is provided $p-1/2+r/4+a<0$ which was our hypothesis. The derivative estimates following by applying the derivatives to b and then applying the same estimate. □

If we make an assumption on the polyhomogeneity of u up to $N^*(X_1)$, we get a corresponding result about the polyhomogeneity of $M_a u$.

COROLLARY 3.1. — *If in addition $u \in I^{p,m}_{phg}(N^*(X_1), N^*(X_3))$ then*

$$M_a u \in I^{m-a,p-a}_{phg}(N^*(X_1), N^*(X_3)) + I^{-a-1+1/2-r/4}_{phg}(N^*(X_1)).$$

Proof. — This follows immediately from noting that

$$R_3 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + 1 - r/4$$

is radial for $N^*(X_3)$, and

$$R_1 = x_1 \frac{\partial}{\partial x_1} + 1/2 - r/4$$

is radial for $N^*(X_1)$, as

$$(R_j - a)M_a = M_a R_j.$$

□

We denote by $\tilde{I}^{m,p}(\Lambda_0, \Lambda_1, \Lambda_2)$ elements of $I^{m,p}(\Lambda_0, \Lambda_1)$ which have wavefront set disjoint from Λ_2.

PROPOSITION 3.3. — *If $m - p - 1/2 \leq -1$ and if $a > -1$ and non-integral, then M_a maps*

$$\tilde{I}^{p,m}(N^*(X_2^+), N^*(X_3), N^*(X_1))$$

into

$$I^{p,m-a}(N^*(X_2), N^*(X_3)) + I^{m-p-1/2-a+1/2-r/4,m-a}(N^*(X_1), N^*(X_2^+)) + I^{-1-a+1/2-r/4}_{phg}(N^*(X_1))$$

and if a is an integer into

$$I^{p,m-a-}(N^*(X_2), N^*(X_3)) + I^{m-p-1/2-a+1/2-r/4-,m-a-}(N^*(X_1), N^*(X_2^+)) + I^{-1-a+1/2-r/4}_{phg}(N^*(X_1)).$$

Proof. — We can assume u is supported near $x = 0$ as the mapping property on elements of $I^p(N^*(X_2))$ supported away from $x_1 = 0$ is trivial.

So, if

$$u \in \tilde{I}^{p,m}(N^*(X_2^+), N^*(X_3), N^*(X_1))$$

then up to smooth terms u can be written

$$\int e^{i(x_1\xi_1 + x_2\xi_2)} b(x'', \xi_1, \xi_2) d\xi_1 d\xi_2$$

and b a generalized symbol, precisely

$$|D^\beta_{x''}D^\alpha_\xi b(x'',\xi_1,\xi_2)| \leq C_{\alpha,\beta}\langle\xi_1,\xi_2\rangle^{\mu-\alpha_2}\langle\xi_1\rangle^{\mu'-\alpha_1}$$

where

$$p = \mu + \frac{1}{2} - \frac{r}{4},$$
$$m = \mu + \mu' + 1 - \frac{r}{4}.$$

We can also take b supported in the set where $|\xi_2|$ is elliptic.

Multiplying u by x_1^k with k a positive integer is equivalent to differentiating the symbol b with respect to ξ_1 k times. So we can immediately reduce to the case when $-1 < a \leq 0$.

By hypothesis, we have $\mu' < 0$. If we then estimate the convolution as in Proposition 3.2, we obtain that if c is the partial Fourier transform of $M_a u$ that

$$|D^\alpha_{\xi_1,\xi_2}D^\beta_{x''}c(\xi_1,\xi_2,x'')| \leq C\langle\xi_1\rangle^{\mu-\alpha_1}\langle\xi_2\rangle^{\mu'-a-\alpha_2}$$

for $a < 0$. Cutting up into domains according to the ellipticity of ξ_1,ξ_2 the result follows.

For $a = 0$ we gain a log term, as before, to give the ϵ error. □

COROLLARY 3.2. — *If $m - p - 1/2 \leq -1$ and if $a > -1$ and non-integral, then M_a maps*

$$\tilde{I}^{p,m}_{pphg}(N^*(X_3), N^*(X_2), N^*(X_1))$$

into

$$I^{p,m-a}_{pphg}(N^*(X_3), N^*(X_2^+)) + I^{m-p-1/2-a+1/2-r/4,m-a}_{pphg}(N^*(X_3), N^*(X_1)) + I^{-1-a+1/2-r/4}_{phg}(N^*(X_1)),$$

and if a is an integer into

$$I^{p,m-a-}(N^*(X_2), N^*(X_3)) + I^{m-p-1/2-a+1/2-r/4-,m-a-}(N^*(X_1), N^*(X_2^+)) + I^{-1-a+1/2-r/4}_{phg}(N^*(X_1)).$$

Proof. — This follows by commuting the radial operator R_3 with M_a and noting that the condition $m-p-1/2 \leq -1$ is preserved upon lowering m. □

Another case we need to study is applying M_a to the class

$$I^{m,p}_{phg}(N^*(X_3), N^*(X_2^-)).$$

In this case, the wavefront set of the product will be contained in $N^*(X_3) \cup N^*(X_1)$. We want to show that near $N^*(X_2) \cap N^*(X_3)$ $M_a u$ is in $I_{phg}(N^*(X_3))$ - it is not obvious that the distribution is even Lagrangian there.

PROPOSITION 3.4. — *Let $u \in I_{phg}^{m,p}(N^*(X_3), N^*(X_2^-))$, $m - p - 1/2 < -1$ and micro-supported near $N^*(X_2^-)$ then*
$M_a u \in I_{pphg}^{m-p-1/2-a+1/2-r/4, m-a}(N^*(X_1), N^*(X_3)) + I_{phg}^{-1-a+1/2-r/4}(N^*(X_1))$.

Proof. — Away from $N^*(X_2) \cap N^*(X_3)$ the proof is as in the case for $N^*(X_2^+)$. Now as $u \in I_{phg}^{m,p}(N^*(X_3), N^*(X_2^-))$, we can decompose $u = u_1 + u_2$ with

$$u_2 = \int e^{ix_2\xi_2}\phi(|\xi_2|^{\frac{1}{2}}x_1)\tilde{u}(x_1, \xi_2, x'')d\xi_2,$$

where $\phi(s)$ is smooth and identically one in $s > -1/2$ and identically zero in $s < -1$. This gives

$$u_1 \in I_{ma}^{p, \frac{m-p+\frac{1}{2}}{2}}(N^*(X_2^-), N^*(X_2^-) \cap N^*(X_3)),$$

$$u_2 \in I_{ma}^{m, \frac{p-m+\frac{1}{2}}{2}}(N^*(X_3), N^*(X_2^-) \cap N^*(X_3)),$$

– see [9]. We have that $M_a u = M_a u_2$. Now u_2 can be written

$$\int e^{i(x_1\xi_1 + x_2\xi_2)}a(x'', \xi_1, \xi_2)d\xi_1 d\xi_2$$

with a of type $(1/2, 0)$ distribution conormal to X_3 of order $\dfrac{m+p+1/2}{2}$. So arguing as in the proof of Proposition 3.2, we deduce that $M_a u$ is of type $(1/2, 0)$ of order $\dfrac{m+p+1/2}{2} - a$ with respect to $N^*(X_3)$ near $N^*(X_2) \cap N^*(X_3)$.

But, we also know that $M_a u$ will have an expansion in homogeneous terms near the intersection as commuting $\prod_{j=0}^{N-1}(R + m - s - j)$ through M_a we have that

$$\prod_{j=0}^{N-1}(R + m - s - j)M_a u \in I_{\frac{1}{2},0}^{\frac{m+p+1/2}{2} - s}(N^*(X_3)),$$

micro-locally. The proof that this implies polyhomogeneity of order m is the same as that in [7] and the result follows. $\square$

4. Action of parametrices.

We will need to study the action of applying the fundamental solution of the wave operator to various classes of paired Lagrangian distributions. Following Greenleaf and Uhlmann, we prove the requisite results for general operators of real principal type as this is no harder. We first recall the action of pseudo-differential operators as this allows us to reduce to first order and will be needed in any case.

PROPOSITION 4.1. — *Let (Λ_0, Λ_1) be a cleanly intersecting pair of Lagrangian submanifolds and let P be a properly supported, classical, elliptic pseudo-differential operator of order w. Then P induces maps*

$$I^{m,p}(\Lambda_0, \Lambda_1) \to I^{m-w,p-w}(\Lambda_0, \Lambda_1),$$
$$I^{m,p}_{phg}(\Lambda_0, \Lambda_1) \to I^{m-w,p-w}_{phg}(\Lambda_0, \Lambda_1),$$
$$I^{m,p}_{pphg}(\Lambda_0, \Lambda_1) \to I^{m-w,p-w}_{pphg}(\Lambda_0, \Lambda_1).$$

Proof. — The first two maps are discussed in [9], [13]. To see the third, note that if Q is a parametrix for P and R is radial for Λ_0 then $QRP - w$ is radial for Λ_0. The result then follows by commuting P through the radial operators. (The same argument would also work for the second case.) □

PROPOSITION 4.2. — *Suppose P is an operator of real principal type of order m and Q is a forward parametrix for P. If Λ_0 is a Lagrangian submanifold and Λ_1 is the flow-out of $\Lambda_0 \cap \operatorname{char}(P)$ then provided the pair (Λ_0, Λ_1) is clean, we have*

$$Q : I^{p,l}(\Lambda_0, \Lambda_1) \to I^{p+m,l+m-1}(\Lambda_0, \Lambda_1),$$
$$Q : I^{p,l}_{phg}(\Lambda_0, \Lambda_1) \to I^{p+m,l+m-1}_{phg}(\Lambda_0, \Lambda_1),$$
$$Q : I^{p,l}_{pphg}(\Lambda_0, \Lambda_1) \to I^{p+m,l+m-1}_{pphg}(\Lambda_0, \Lambda_1),$$
$$Q : I^{l,p}_{pphg}(\Lambda_1, \Lambda_0) \to I^{l+m-1,p+m}_{pphg}(\Lambda_1, \Lambda_0).$$

Proof. — The first mapping is part of Proposition 2.2 in [2]. To see the others, we first apply an elliptic operator of order $1-m$ to reduce to the case where $m = 1$. As in [1], [2], [9], one can then conjugate by zeroth order, elliptic Fourier integral operators to reduce the operator P micro-locally to $\dfrac{\partial}{\partial x_1}$, Λ_0 to $N^*(x'' = 0)$, and Λ_1 to $N^*(x'' = 0, x_2 = 0, x_1 \geq 0)$.

In this case, the operator Q has kernel $H(x_1-y_1)\delta(x_2,x'')$. Computing the commutator of Q with $x_1\frac{\partial}{\partial x_1}$ and $x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}$, we see that Q commutes with the radial operators of Λ_0,Λ_1 up to a shift in index of one. The result then follows in the remaining cases by commuting Q through the radial operators and applying the estimate from the first case. □

5. Recovering the total singularity.

In this section, we use the calculi developed in previous sections to prove Theorem 1.1. We proceed by taking two potentials q_1, q_2 of the form $v+w_j$ with w_j polyhomogeneous conormal to the boundary of the domain Ω and supported inside $\overline{\Omega}$. We know from [2] that the principal symbols of w_1 and w_2 must be the same if the backscattering matrices are equal up to smoothing. What we shall show is that if $w_j \in I^{\alpha}_{phg}(N^*(\partial\Omega))$,

$$w_1 - w_2 \in I^{\alpha-N}_{phg}(N^*(\partial\Omega))$$

and the principal symbol of the difference of the backscattering matrices is zero then

$$w_1 - w_2 \in I^{\alpha-N-1}_{phg}(N^*(\partial\Omega)).$$

The main theorem will then follow by induction.

In any local coordinates such that $\Omega = \{x_1 \geq 0\}$, w_j is of the form $H(x_1)x_1^a g_j(x)$ with g_j smooth and $w_1 - w_2$ is of the form $H(x_1)x_1^{a+N}h(x)$, h smooth. We have that $a = -\alpha - 1/2 - n/4 > -1$.

We define the various submanifolds which arise both in our proof and in [2]. Let

$$\begin{aligned} S_1 &= \{(x,t,\omega) \in \mathbb{R}^n \times \mathbb{R} \times S^{n-1} \mid x \in \partial\Omega\}, \\ S_+ &= \{(x,t,\omega) \in \mathbb{R}^n \times \mathbb{R} \times S^{n-1} \mid t = x.\omega\}, \\ S_2 &= S_+ \cap S_1. \end{aligned}$$

We also let Λ_- be the flow-out in forward time of the intersection of the characteristic variety of $\Box$ with $N^*(S_2)$ with respect to $\Box$ – this was shown to be the conormal bundle of a smooth hypersurface, S_- away from a degenerate set, L. The set S_- expresses the set where singularities, which have come straight in, bounce straight back off S_1 and it is these singularities with which we are concerned. The degenerate set L being the places where the wave is tangential to S_1. In our case, the degenerate set L splits S_- into a number of components which are classified by how many times the wave has passed into and out of Ω before bouncing off $\partial\Omega$. We will concern ourselves purely with the component where the wave bounces before passing across Ω. In particular, we shall work away from the set where the wave reaches $\partial\Omega$ after passing over and out of Ω. (This could not happen if we assumed the domain was convex rather than normally accessible.) We denote the subset of S_- thus obtained as $S_-^{'}$. Our condition of normal accessibility means that the wave can always come in straight along the normal to $\partial\Omega$ and bounce back without crossing Ω - so the set we shall need in backscattering will be $S_-^{'}$.

Greenleaf and Uhlmann let $u_0 = \delta(t - x.\omega)$, $u_1 = -\Box^{-1}(qu_0)$ and showed that, away from the bad set L, u_1 was paired Lagrangian with respect to $N^*(S_2)$ and the disjoint spaces $N^*(S_1), N^*(S_+), N^*(S_-.)$ They also showed that $u - u_0 - u_1$ was lower order on $N^*(S_-)$ which means that the principal symbol of u on $N^*(S_-)$ is equal to that of u_1.

Let $u^{(j)}$ denote the solution of

$$(\Box + q_j)u^{(j)}(x,t) = 0 \text{ on } \mathbb{R}^{n+1} \tag{5.1}$$

$$u^{(j)}(x,t) = \delta(t - x.\omega),\ t << 0. \tag{5.2}$$

We also let $u_0^{(j)} = \delta(t - x.\omega)$, and $u_{l+1}^{(j)} = -\Box^{-1}(q_j u_l^{(j)})$. These being the iterative schemes for each potential which provide successively better approximations to $u^{(j)}$. Our approach is to show that the principal symbol of $u^{(1)} - u^{(2)}$ on $N^*(S_-)$ is equal to that of $u_1^{(1)} - u_1^{(2)}$. Note that the first term in the analogous iteration for the potential associated to $q_1 - q_2$ is equal to $u_1^{(1)} - u_1^{(2)}$.

LEMMA 5.1. — *Away from L, and close to $N^*(S_-^{'})$, the distribution $u_l^{(j)}$ is in the class*

$$I_{pphg}(N^*(S_+), N^*(S_2)) + I_{pphg}(N^*(S_1), N^*(S_2)) + I_{phg}(N^*(S_-^{'}), N^*(S_2))$$

and the Lagrangian orders off $N^*(S_1)$ converge to $-\infty$ as l tends to ∞.

Proof. — Fix j and let $u_l = u_l^{(j)}$ and $q = q_j$. Away from the set L, the geometry is that Λ_- is the conormal bundle of the hypersurface S_-. The hypersurfaces S_1, S_-, S_+ are pairwise transverse with intersection S_2 so we can apply the results of Section 3.

Let $p = 1/2 - n/2$ then $u_0 \in I_{phg}^p(N^*(S_+))$. So by Proposition 3.1, we have that

$$qu_0 \in I_{phg}^{-a-1+1/2-n/2,p-a-1/2}(N^*(S_1), N^*(S_2)) + I_{phg}^{-a-1+1/2-n/2,p-a-1/2}(N^*(S_+), N^*(S_2)) + I_{phg}^p(S_+)$$

(with an extra ϵ error if a is an integer – we will suppress this in the sequel as it makes no material difference). Note the final term comes from multiplying u_0 by the smooth part of q. We also remark that one could do better here by using the support of u_0 but that this will not help later on so we do not bother to do so.

We then want to apply $\Box^{-1}$, using Proposition 4.2, and the mapping properties of paired Lagrangian distributions under elliptic pseudo-differential operators, the order will decrease by two on the non-characteristic Lagrangian submanifolds and by one on the characteristic submanifold $N^*(S_+)$; it will also generate the new Lagrangian Λ_- where it is of one order lower than on $N^*(S_2)$. So we have that

$$\begin{aligned} u_1 \in\ & I_{phg}^{-a-3+1/2-n/2,p-a-1/2-2}(N^*(S_1), N^*(S_2)) \\ &+ I_{phg}^{-a-2+1/2-n/2,p-a-1/2-2}(N^*(S_+), N^*(S_2)) \\ &+ I_{phg}^{p-a-1/2-1,p-a-1/2-2}(N^*(S_-), N^*(S_2)) + I_{phg}^{p-1}((S_+)). \end{aligned}$$

We now have to repeatedly do this and see that the order decreases. First note that the final term here can be discarded, as it will just feed back in as something of one lower order at each stage in the iteration. On applying q to the sum of spaces, the smooth term will leave them all invariant and the singular term can be computed by the results of Section 3. But applying the individual results for each these spaces, one sees that wu_1 will be in the analogous *pphg* spaces with orders shifted by $-a-1$ on $N^*(S_1)$ and $N^*(S_2)$. By Proposition 3.4 the term on $N^*(S_-^{'})$ will be killed - here we are using the support condition on w_i and the convexity condition on Ω. (There will also be a new term conormal to $N^*(S_1)$. However this is irrelevant as multiplying such an object by M_a will yield something conormal to S_1 also and the space $N^*(S_1)$ is invariant under the action of $\Box^{-1}$, as this is equivalent to applying an elliptic pseudo-differential operator.)

We have from Proposition 4.2, that applying $\Box^{-1}$ leaves all these spaces invariant (except for reintroducing $N^*(S_-^{'})$) and reduces the order by 1 on the characteristic Lagrangians and 2 on the non-characteristic Lagrangians. So the action of applying $\Box^{-1}M_q$ will leave the spaces invariant and always reduce the order by at least $1+a>0$ off $N^*(S_1)$. The result now follows inductively. $\Box$

LEMMA 5.2. — *For any $N \geq 1$, the principal symbol of $\sum_{l=0}^{N} u_l^{(1)} - u_l^{(2)}$ on $N^*(S_-^{'})$ is equal to that of $u_1^{(1)} - u_1^{(2)}$ modulo terms of order $-1-a$ lower.*

Note that it is essential here that $q_1 - q_2$ is supported inside $\bar{\Omega}$. Note that if $a \geq 0$ the fact the symbol is determined modulo terms of order $-1-a$ lower is irrelevant but that for $-1 < a < 0$ this is a real effect.

Proof. — First, we discard the zeroth terms as they are equal. Now $u_1^{(1)} - u_1^{(2)}$ is equal to

$$\Box^{-1}((q_1 - q_2)\delta(t - x.\omega)).$$

Thus as above, we obtain

$$\begin{aligned} u_1^{(1)} - u_1^{(2)} \in\ & I_{phg}^{-a-N-3+1/2-n/2,p-a-N-1/2-2}(N^*(S_1), N^*(S_2)) \\ &+ I_{phg}^{-a-N-2+1/2-n/2,p-a-N-1/2-2}(N^*(S_+), N^*(S_2)) \\ &+ I_{phg}^{p-a-N-1/2-1,p-a-1/2-2}(N^*(S_-^{'}), N^*(S_2)), \end{aligned}$$

without the additional term in $I_{phg}(N^*(S_+))$ as we have no smooth error. So we need to show that $u_j^{(1)} - u_j^{(2)}$ is of order lower than $p-a-N-1/2-1$ on $N^*(S_1)$ for any j bigger than 1. To see this we observe that if q_2 is written as $q_1 + (q_2 - q_1)$ and the expression for $u_j^{(2)}$ is then expanded, we obtain $u_j^{(1)}$ plus a sum of terms of the form

$$\prod_{l=1}^{j}(\Box^{-1}M_{r_l})\delta(t - x.\omega),$$

where r_l is either q_1 or $q_2 - q_1$ and at least one r_1 will be $q_2 - q_1$. These terms will necessarily be of lower order than $u_1^{(1)} - u_1^{(2)}$ on $N^*(S_-^{'})$ as applying $q_2 - q_1$ reduces order by $-a-N$ and applying $\Box^{-1}M_{q_1}$ lowers order by $-1-a$. The result then follows. □

As the order of the distributions have been shown to go down off $N^*(S_1)$, we can asymptotically sum $u_l^{(j)}$ in

$$I(N^*(S_1), N^*(S_2)) + I(N^*(S_-^{'}), N^*(S_2)) + I(N^*(S_+), N^*(S_2))$$

to obtain $v^{(j)}$. By construction, $(\Box + q_j)v^{(j)}$ is Lagrangian with respect to $N^*(S_1)$ near $S_-^{'}$. So

$$(\Box + q_j)(u^{(j)} - v^{(j)})$$

is zero in the far past and near $S_-^{'}$ is Lagrangian to $N^*(S_1)$. As the space of distributions Lagrangian to $N^*(S_1)$ is an algebra and invariant under application of $\Box^{-1}$, we deduce that near $S_-^{'}$, for all k

$$\left(I + (\Box^{-1}M_{q_j})^k\right)(u^{(j)} - v^{(j)})$$

is Lagrangian to $N^*(S_1)$ near $S_-^{'}$ (*cf.* proof of Theorem 3.1 in [2]). As applying $\Box^{-1}M_{q_j}$ smooths Sobolev order off $N^*(S_1)$, this implies that

$u^{(j)}, v^{(j)}$ are equal up to smoothing near S_-' off $N^*(S_1)$. Thus $u^{(j)}$ will be paired Lagrangian with respect to $(N^*(S_2), N^*(S_-))$ and $N^*(S_+)$ off $N^*(S_1)$ near S_-'. We also have from Lemma 5.2 that the principal symbol of $u^{(1)} - u^{(2)}$ on $N^*(S_-)$ is equal to that of $u_1^{(1)} - u_2^{(1)}$.

Now to pass from $u^{(j)}$ to the scattering matrix α_j, we subtract $\delta(t - x.\omega)$, apply an elliptic Fourier integral, F, associated to the canonical graph

$$N^*(x.\theta = s) \subset T^*(\mathbb{R}_t \times S_\theta^{n-1} \times \mathbb{R}_s \times S_\omega^{n-1} \times \mathbb{R}_x^n \times \mathbb{R}_t \times S_\omega^{n-1})$$

(see [2]) and then restrict via pulling back by the map

$$\rho(s, \theta, \omega) = (s + t_0, \theta, t_0, \omega)$$

for some t_0 large.

The main theorem now follows by a repetition of the arguments of Greenleaf and Uhlmann, [2], applied to the difference of the scattering matrices rather than a single one. We sketch the argument but do not give the details for brevity. The principal symbol of $u^{(1)} - u^{(2)}$ on $N^*(S_-')$ is equal to that of $u_1^{(1)} - u_1^{(2)}$ and this will be proportional to that of $g_1 - g_2$. Applying F will map $N^*(S_-')$ to a Lagrangian $\hat{\Lambda}_-'$ and will change the principal symbol by a known elliptic factor which is thus irrelevant. We deduce that after the restriction, the principal symbol on $\hat{\Lambda}_-$, the restricted Lagrangian, will be proportional to that $g_1 - g_2$ and this then means that the principal symbol of the difference of the backscattering matrices then determines the principal symbol of $g_1 - g_2$ and the result follows.

BIBLIOGRAPHY

[1] J.J. DUISTERMAAT and L. HÖRMANDER, Fourier Integral Operators II, Acta Mathematicae, 128 (1972), 183–269.

[2] A. GREENLEAF and G. UHLMANN, Recovering Singularities of a Potential from Singularities of Scattering Data, Commun. Math. Phys., 157 (1993), 549–572.

[3] A. GREENLEAF and G. UHLMANN, Estimates for singular Radon transforms and pseudo-differential operators with singular symbols, J. Funct. Anal., 89 (1990), 202–232.

[4] V. GUILLEMIN, G. UHLMANN, Oscillatory integrals with Singular Symbols, Duke Math. J., 48 (1981), 251–267.

[5] L. HÖRMANDER, Fourier Integral Operators I, Acta Mathematicae, 127 (1971), 79–183.

[6] L. HÖRMANDER, Analysis of Linear Partial Differential Operators, Vol. 1 to 4, Springer Verlag, Berlin, 1985.

[7] M.S. Joshi, An Intrinsic Characterisation of Paired Lagrangian Distributions, Proc. Amer. Math. Soc., 125 (1997), N° 5, 1537–1543.

[8] M.S. Joshi, A Precise Calculus of Paired Lagrangian Distributions, M.I.T. thesis, 1994.

[9] M.S. Joshi, A Symbolic Contruction of the Forward Fundamental Solution of the Wave Operator, preprint.

[10] M.S. Joshi and A. Sa Barreto, Recovering Asymptotics of Short Range Potentials, to appear in Commun. in Math. Phys.

[11] P. Lax, R. Phillips, Scattering Theory, Revised Edition. New York, London: Academic Press, 1989.

[12] R.B. Melrose, Differential Analysis on Manifolds with Corners, forthcoming.

[13] R.B. Melrose, Marked Lagrangian Distributions, manuscript.

[14] R.B. Melrose and G. Uhlmann, Lagrangian Intersection and the Cauchy Problem, Comm. on Pure and Applied Math., 32 (1979), 482–512.

[15] R. Philips, Scattering Theory for the Wave Equation with Short Range Potential, Indiana Univ. Math. Journal, 31 (1982), 609–639.

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