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KARL-HERMANN NEEB

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ON THE COMPLEX GEOMETRY OF INVARIANT DOMAINS IN COMPLEXIFIED SYMMETRIC SPACES

par Karl-Hermann NEEB

Introduction.

Let (G,τ) be a compact symmetric group, i.e., G is a compact Lie group and τ is an involutive automorphism of G. Let G^τ denote the group of τ-fixed points. Then $M := G/G^\tau$ is a compact symmetric space, and it is easy to construct complexifications of such spaces. The most direct way is to consider the universal complexification $G_{\mathbb{C}}$ of G which is a complex reductive Lie group endowed with a holomorphic involutive automorphism τ obtained by extending τ using the universal property of $G_{\mathbb{C}}$. Now $M_{\mathbb{C}} := G_{\mathbb{C}}/(G_{\mathbb{C}})^\tau$ is a complex symmetric space and we have a natural embedding $M \hookrightarrow M_{\mathbb{C}}$. On the other hand the complex symmetric space $M_{\mathbb{C}}$ has an interesting fiber bundle structure which can be described as follows. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ denote the τ-eigenspace decomposition of the Lie algebra $\mathfrak{g}$, where $\mathfrak{h}$ is the 1-eigenspace and $\mathfrak{q}$ is the -1-eigenspace. Then the corresponding decomposition of $\mathfrak{g}_{\mathbb{C}}$ is given by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \mathfrak{q}_{\mathbb{C}}$, and we have an exponential map $\operatorname{Exp} : \mathfrak{q}_{\mathbb{C}} \to M_{\mathbb{C}}$. Now the mapping

$$G \times i\mathfrak{q} \to M_{\mathbb{C}}, \quad (g, X) \mapsto g \operatorname{Exp} X$$

turns out to be surjective, and it factors to a diffeomorphism

$$G \times_{G^\tau} i\mathfrak{q} \cong M_{\mathbb{C}}$$

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which exhibits a fiber bundle structure of $M_{\mathbb{C}}$ with base space M and fiber $i\mathfrak{q}$. Furthermore, if $\mathfrak{a} \subseteq i\mathfrak{q}$ is a maximal abelian subspace, then $i\mathfrak{q} = \mathrm{Ad}(G^\tau).\mathfrak{a}$, and one derives that $M_{\mathbb{C}} = G.\mathrm{Exp}(\mathfrak{a})$, i.e., that each G-orbit in $M_{\mathbb{C}}$ meets the subspace $\mathrm{Exp}(\mathfrak{a})$. A typical example of this fiber bundle structure is the polar decomposition of the complexification $K_{\mathbb{C}}$ of a compact Lie group K, viewed as a symmetric space of the group $G = K \times K$. In this case it is given by the polar map $K \times i\mathfrak{k} \to K_{\mathbb{C}}, (k, X) \mapsto k \,\mathrm{Exp}\, X$.

From a complex geometric point of view there are several natural questions about these spaces. Since $M_{\mathbb{C}} = G.\mathrm{Exp}(i\mathfrak{q}) = G.\mathrm{Exp}(\mathfrak{a})$, a G-invariant function on $M_{\mathbb{C}}$ is determined by its values on $\mathrm{Exp}(i\mathfrak{q})$, resp. $\mathrm{Exp}(\mathfrak{a})$, and likewise a G-invariant domain $D \subseteq M_{\mathbb{C}}$ is determined by a corresponding G^τ-invariant domain $D_{\mathfrak{q}} \subseteq i\mathfrak{q}$ with $D = G.\mathrm{Exp}(D_{\mathfrak{q}})$. In [AL92] Azad and Loeb give a characterization of the G-invariant plurisubharmonic functions on D (under the assumption that $D_{\mathfrak{q}} \cap \mathfrak{a}$ is convex) as those corresponding to Weyl group invariant convex functions on $D_{\mathfrak{q}} \cap \mathfrak{a}$. Moreover, they obtain a description of the envelopes of holomorphy of invariant domains which previously has been derived by Lasalle (*cf.* [Las78]).

The purpose of this paper is to extend these results concerning complexifications of compact symmetric spaces to an appropriate class of non-compact symmetric spaces naturally arising in representation theory and in the geometry of causal symmetric spaces (*cf.* [HÓ96]). Unfortunately it turns out that for a non-compact symmetric space one can no longer work with the full complexification because in general it does not permit a description as a domain of the type $G.\mathrm{Exp}(i\mathfrak{q})$. Nevertheless for a large class of spaces such descriptions exist for interesting G-invariant domains whose boundary contains the symmetric space G/G^τ.

Let us briefly describe our setup. Let $(\mathfrak{g}, \tau)$ be a real symmetric Lie algebra and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ the corresponding eigenspace decomposition for τ. We call an element $X \in \mathfrak{g}$ elliptic if the operator $\mathrm{ad}\, X$ is semisimple with purely imaginary spectrum. The existence of "enough" elliptic elements in $\mathfrak{q}$ is important in many contexts. If $\mathfrak{g}$ is a compact Lie algebra, then $\mathfrak{q}$ consists entirely of elliptic elements. If $(\mathfrak{g}, \tau)$ is a compactly causal symmetric Lie algebra in the sense of [HÓ96], then $\mathfrak{q}$ contains open convex cones which are invariant under the group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ of inner automorphisms of $\mathfrak{g}$ generated by $e^{\mathrm{ad}\,\mathfrak{h}}$ and which are elliptic in the sense that they consist of elliptic elements. In recent years this class of reductive symmetric Lie algebras and the associated symmetric spaces have become a topic of very active research spreading in more and more areas. For a survey of the state of the art we refer to [HÓ96] and the literature cited there. In [KN96] and [KNÓ97] we have investigated a generalization of this class of reductive symmetric Lie algebras which has quite similar structural properties and whose members are not necessarily reductive. The simplest type (called the group type) is the one with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$, where τ is the flip involution and $\mathfrak{h}$ is a Lie algebra containing open invariant elliptic convex cones. The domains associated to symmetric spaces of group type and the G-invariant complex analysis (invariant Stein domains and plurisubharmonic functions) resp. the consequences for the structure of representations of G in invariant Hilbert spaces of holomorphic functions have been investigated in [Ne96b] resp. [Ne97].

In this paper we address the case of symmetric spaces which are not necessarily of group type. As the detailed structural analysis in [KN96] shows, after some inessential reductions one sees that the class of symmetric spaces to which one might hope to extend the aforementioned complex geometric results corresponds to the class of symmetric Lie algebras $(\mathfrak{g}, \tau)$ which has the property that $i\mathfrak{g}$ contains an open hyperbolic invariant convex cone W, which, in addition, is invariant under $-\tau$. Then $W \cap i\mathfrak{q}$ is an $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant open hyperbolic cone in $i\mathfrak{q}$, and if W is maximal, then the maximal domains to which our methods apply are those of the form $\Xi(W \cap i\mathfrak{q}) := G.\operatorname{Exp}(W \cap i\mathfrak{q})$ (explained below). These are complex domains whose boundaries contain a real symmetric space of G, but if $\mathfrak{g}$ is not compact, then these domains are not complex symmetric spaces of $G_{\mathbb{C}}$.

Formally the domain $\Xi(W \cap i\mathfrak{q})$ can be defined as $G \times_H (W \cap i\mathfrak{q})$, where $H \subseteq G^\tau$ is an open subgroup. All these domains turn out to carry natural complex structures, and G-acts by holomorphic automorphisms (*cf.* [KNÓ97]). If $W \subseteq i\mathfrak{q}$ is a maximal hyperbolic cone, then we are interested in a description of the G-invariant Stein domains in $\Xi(W)$, the envelopes of holomorphy of any G-invariant domain, and a description of the G-invariant plurisubharmonic functions on invariant domains in $\Xi(W)$. Further we want to apply these complex geometric results to obtain some information on the structure of the representations of G in invariant Hilbert spaces of holomorphic functions on invariant domains.

An invariant domain $D \subseteq \Xi(W)$ can be described as $\Xi(D_{\mathfrak{q}}) := G \times_H D_{\mathfrak{q}}$, where $D_{\mathfrak{q}} \subseteq W$ is an H-invariant domain. Since the description of $\Xi(D_{\mathfrak{q}})$ does not exhibit its complex structure, it is in many cases preferable to use the following different realization. Let $\widehat{W} \subseteq i\mathfrak{g}$ denote an invariant open hyperbolic cone with $\widehat{W} \cap i\mathfrak{q} = W$ and $S := \Gamma_G(\widehat{W}) := G^c \operatorname{Exp}(\widehat{W})$ the associated complex Ol'shanskiĭ semigroup (*cf.* [Ne99, Ch. XI]). Then the antiinvolution $X \mapsto -\tau(X)$ integrates to a holomorphic involution $S \to S$, $s \mapsto s^\sharp$ satisfying

$$(s^\sharp)^\sharp = s \quad \text{and} \quad (st)^\sharp = t^\sharp s^\sharp$$

for $s, t \in S$. For $H = G^\tau$ we then have

$$\Xi(W) \cong Q(W) := \{ss^\sharp : s \in S\}$$

which yields a description as a connected component of the set $S^\sharp := \{s \in S : s^\sharp = s\}$ and hence implies that $Q(W)$ is a Stein manifold because this is true for S (*cf.* [Ne98]). Note that the semigroup $\overline{S} := \Gamma_G(\widehat{\overline{W}})$ acts on $Q(W)$ by $s.x := sxs^\sharp$. In this sense we have $Q(W) = G^c.\operatorname{Exp}(W)$, where we consider $\operatorname{Exp}(W)$ as $\{s \in \operatorname{Exp}(\widehat{W}) : s^\sharp = s\}$.

In the group case where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ and τ is the flip involution we have $G \cong H \times H$, $S_W := \Gamma_G(\widehat{W})$ is a complex Ol'shanskiĭ semigroup, and $S \cong S_W \times \overline{S_W}$, where $\overline{S_W}$ denote the complex manifold S_W endowed with the opposite complex structure. For $s = (s_1, s_2) \in S$ we have $\tau(s_1, s_2) = (\tau.s_2, \tau.s_1)$ and thus $s^\sharp = (s_2^*, s_1^*)$. So $s^\sharp = s$ means that $s = (s_1, s_1^*)$. We conclude that $Q(W) \cong S_W$, where the embedding $S_W \to S$ is given by $s \mapsto (s, s^*)$. In this realization of $Q(W)$ the action of S is given by $(s, t).x = sxt^*$.

The main results of this paper are the following:

(1) An invariant domain $\Xi(D_{\mathfrak{q}})$ is a Stein manifold if and only if $D_{\mathfrak{q}}$ is convex.

(2) The envelope of holomorphy of a domain $\Xi(D_{\mathfrak{q}})$ is given by the smallest Stein domain containing it, i.e., $\Xi(\operatorname{conv} D_{\mathfrak{q}})$.

(3) An invariant real C^2-function ϕ on an invariant domain $\Xi(D_{\mathfrak{q}})$ is plurisubharmonic if and only if the corresponding function $\psi : X \mapsto \phi(\operatorname{Exp} X)$ on $D_{\mathfrak{q}}$ is locally convex.

(4) The representation of G in any irreducible invariant Hilbert subspace of $\operatorname{Hol}\big(\Xi(D_{\mathfrak{q}})\big)$ is a unitary highest weight representation with spherical lowest K-type. In view of general desintegration results of B. Krötz ([Kr97]), this provides a suitable direct integral decomposition for any Hilbert subspace of $\operatorname{Hol}\big(\Xi(D_{\mathfrak{q}})\big)$ as a direct integral of irreducible ones.

1. Invariant locally convex functions.

In this section we collect all results on the Lie algebra level that we will need later on – in particular in Section 3. The main results of this section concern the characterization of the convexity of an invariant function on an invariant domain $D_{\mathfrak{q}} \subseteq \mathfrak{q}$ by properties of its restriction to a maximal abelian subspace $\mathfrak{a}$.

To simplify our notation, the class of Lie algebras considered in this section is a class which is dual to the one needed later on, but it will be no problem to translate the results from a symmetric Lie algebra to its dual.

DEFINITION 1.1. — (a) *A symmetric Lie algebra* $(\mathfrak{g}, \tau)$ *is a pair consisting of a finite dimensional Lie algebra* $\mathfrak{g}$ *and an involutive automorphism* τ *of* $\mathfrak{g}$. *We put* $\mathfrak{h} := \{X \in \mathfrak{g} : \tau.X = X\}$ *and* $\mathfrak{q} := \{X \in \mathfrak{g} : \tau.X = -X\}$, *and note that* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. *The subalgebra* $\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ *is called the dual symmetric Lie algebra. The corresponding involution is given by restricting the complex linear extension of* τ *on* $\mathfrak{g}_{\mathbb{C}}$ *to* $\mathfrak{g}^c$.

(b) *Let* $\mathfrak{g}$ *denote a finite dimensional real Lie algebra. An element* $X \in \mathfrak{g}$ *is called hyperbolic, resp. elliptic, if* $\operatorname{ad} X$ *is diagonalizable over* $\mathbb{R}$, *resp. semisimple with purely imaginary spectrum. A convex subset* $C \subseteq \mathfrak{g}$ *is called hyperbolic, resp. elliptic, if all its relative interior points are hyperbolic, resp. elliptic. A symmetric Lie algebra* $(\mathfrak{g}, \tau)$ *is said to be admissible if* $\mathfrak{q}$ *contains an open convex hyperbolic* $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$*-invariant subset not containing any affine subspace.*

An abelian subspace $\mathfrak{a} \subseteq \mathfrak{q}$ *is called abelian maximal hyperbolic if* $\mathfrak{a}$ *is abelian and consists of hyperbolic elements and is maximal w.r.t. this property. A subspace* $\mathfrak{l} \subseteq \mathfrak{q}$ *is called a Lie triple system if* $[\mathfrak{l}, [\mathfrak{l}, \mathfrak{l}]] \subseteq \mathfrak{l}$. *This means that the space* $\mathfrak{l}_L := \mathfrak{l} \oplus [\mathfrak{l}, \mathfrak{l}]$ *is a subalgebra of* $\mathfrak{g}$. *Recall that all abelian maximal hyperbolic subspaces as well as all maximal hyperbolic Lie triple systems in* $\mathfrak{q}$ *are conjugate under* $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$ *(cf. [KN96], Cor. II.9, Th. III.3).*

(c) *Let* $\mathfrak{a} \subseteq \mathfrak{q}$ *be an abelian maximal hyperbolic subspace. For every subalgebra* $\mathfrak{b} \subseteq \mathfrak{g}$ *we set* $\mathfrak{b}^0 := \mathfrak{z}_{\mathfrak{b}}(\mathfrak{a})$. *For* $\alpha \in \mathfrak{a}^*$ *we define*

$$\mathfrak{g}^\alpha := \{X \in \mathfrak{g} : (\forall Y \in \mathfrak{a})[Y, X] = \alpha(Y)X\}$$

and write $\Delta := \{\alpha \in \mathfrak{a}^* \setminus \{0\} : \mathfrak{g}^\alpha \neq \{0\}\}$ *for the set of roots. Then we get the root space decomposition* $\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha$.

We call a root $\alpha \in \Delta$ semisimple, resp. solvable, if $\mathfrak{s}^\alpha := \mathfrak{g}^\alpha \cap \mathfrak{s} \neq \{0\}$, resp. $\mathfrak{g}^\alpha \subseteq \mathfrak{r}$. The set of all semisimple, resp. solvable, roots is denoted by Δ_s, resp. Δ_r. Note that $\Delta = \Delta_r \dot\cup \Delta_s$ (cf. [KN96], Lemma IV.5(i)).

Let $\mathfrak{p} \subseteq \mathfrak{q}$ be a maximal hyperbolic Lie triple system containing $\mathfrak{a}$ and $\mathfrak{p}_L = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ the corresponding Lie subalgebra of $\mathfrak{g}$. A root $\alpha \in \Delta$ is called compact if $\mathfrak{p}_L^\alpha \neq \{0\}$ and non-compact otherwise. We write Δ_k, Δ_n, resp. Δ_p, for the set of all compact, non-compact, resp. non-compact semisimple roots. Note that Δ_k is independent of the choice of $\mathfrak{p} \supseteq \mathfrak{a}$ (cf. [KN96], Def. V.1) and that $\Delta = \Delta_k \dot\cup \Delta_n$ holds by definition.

(d) *The Weyl group $\mathcal{W}$ of $(\mathfrak{g}, \tau)$ w.r.t. $\mathfrak{a}$ is defined by $\mathcal{W} := N_{\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})}(\mathfrak{a})/Z_{\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})}(\mathfrak{a})$. We call $X_0 \in \mathfrak{a}$ regular if $\alpha(X_0) \neq 0$ holds for all $\alpha \in \Delta$. A positive system is a subset $\Delta^+ \subseteq \Delta$ for which there exists a regular element $X_0 \in \mathfrak{a}$ with $\Delta^+ = \{\alpha \in \Delta : \alpha(X_0) > 0\}$. A positive system is called $\mathfrak{p}$-adapted if the set $\Delta_n^+ := \Delta_n \cap \Delta^+$ of positive non-compact roots is invariant under the Weyl group.*

(e) *The symmetric Lie algebra $(\mathfrak{g}, \tau)$ is called quasihermitian if $\mathfrak{z}_\mathfrak{q}(\mathfrak{z}(\mathfrak{p})) = \mathfrak{p}$. In this case $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$ and there exists a $\mathfrak{p}$-adapted positive system Δ^+ (cf. [KN96], Prop. V.10).*

(f) *Let V be a finite dimensional real vector space and V^* its dual. For a subset $E \subseteq V$ the dual cone is defined by $E^\star := \{\omega \in V^* : (\forall x \in E)\omega(x) \geq 0\}$. A cone $C \subseteq V$ is called generating if $V = C - C$ and pointed if $C \cap -C = \{0\}$.*

We associate to a positive system of non-compact roots Δ_n^+ the convex cones

$$C_{\min} := \mathrm{cone}\left(\{[X_\alpha, \tau(X_\alpha)] : X_\alpha \in \mathfrak{g}^\alpha, \alpha \in \Delta_n^+\}\right),$$

and $C_{\max} := (\Delta_n^+)^\star = \{X \in \mathfrak{a} : (\forall \alpha \in \Delta_n^+)\alpha(X) \geq 0\}$. □

In the following $(\mathfrak{g}^c, \tau)$ will always be a symmetric Lie algebra containing an invariant open elliptic invariant cone C which, in addition, is invariant under $-\tau$. Then $W := \mathfrak{q} \cap iC$ is an invariant open hyperbolic cone in $\mathfrak{q}$. In this section we will study invariant functions on domains $D_\mathfrak{q} \subseteq W$ which are invariant under the group $\mathrm{Inn}_\mathfrak{q}(\mathfrak{h})$.

To explain which symmetric Lie algebras $(\mathfrak{g}, \tau)$ arise in this context, and how they can be characterized intrinsically, we recall from [KN96], Cor. IX.10, that the existence of the invariant hyperbolic cone $W \subseteq \mathfrak{q}$ implies that $(\mathfrak{g}, \tau)$ is quasihermitian, and that there exists a $\mathfrak{p}$-adapted positive system Δ^+ such that $C_{\min} \subseteq W \cap \mathfrak{a} \subseteq C_{\max}$. Further there exists a maximal open convex hyperbolic cone $W_{\max} \subseteq \mathfrak{q}$ containing W.

On the other hand it follows from [KN96], Th. VIII.1, Cor. IX.10, that the existence of C can be characterized intrinsically by the condition that there exists a $\mathfrak{p}$-adapted positive system Δ^+ such that $C_{\min} \subseteq W \cap \mathfrak{a} \subseteq C_{\max}$, and, in addition, the subalgebra $\mathfrak{h}^0 = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$ of $\mathfrak{h}$ is compactly embedded in $\mathfrak{g}$. If these conditions are satisfied, then there exists a maximal open convex hyperbolic cone $W_{\max} \subseteq \mathfrak{q}$ containing W which arises from a maximal open elliptic cone in $\mathfrak{g}^c$. Hence we may w.l.o.g. assume that $W = W_{\max}^0$.

We briefly record the implications of this situation for the corresponding root decompositions of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}$. If $\mathfrak{t}_{\mathfrak{h}} \subseteq \mathfrak{h}^0$ is a Cartan subalgebra, then $\mathfrak{t}^c := i\mathfrak{a} + \mathfrak{t}_{\mathfrak{h}}$ is a compactly embedded Cartan subalgebra of the dual Lie algebra $\mathfrak{g}^c$, and if $\mathfrak{p}$ is a maximal hyperbolic Lie triple system in $\mathfrak{q}$ containing $\mathfrak{a}$, then $\mathfrak{k}^c := \mathfrak{h}^0 + [\mathfrak{p}, \mathfrak{p}] + i\mathfrak{p} \subseteq \mathfrak{g}^c$ is the uniquely determined maximal compactly embedded subalgebra of $\mathfrak{g}^c$ containing $\mathfrak{t}^c$ ([KN96], Th. VIII.1). Moreover $i\mathfrak{z}(\mathfrak{p}) \subseteq \mathfrak{z}(\mathfrak{k}^c)$ and $\mathfrak{z}_{\mathfrak{g}^c}(i\mathfrak{z}(\mathfrak{p})) = \mathfrak{k}^c$, so that $(\mathfrak{g}^c, \tau)$ is a quasihermitian Lie algebra. Furthermore there exists a $\mathfrak{k}$-adapted positive system $\widehat{\Delta}^+ \subseteq \widehat{\Delta} := \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}^c_{\mathbb{C}})$ which is compatible with Δ^+ in the sense that

$$\widehat{\Delta}^+_n|_{\mathfrak{a}} = \Delta^+_n, \quad -\tau(\Delta^+_n) = \Delta^+_n, \quad \text{and} \quad \widehat{\Delta}_k = \{\alpha \in \widehat{\Delta} : \alpha|_{\mathfrak{a}} \in \Delta_k \cup \{0\}\}.$$

In this section G^c denotes a connected group with Lie algebra $\mathfrak{g}^c$ and $H := (G^c)^\tau_0 = \langle \exp_{G^c} \mathfrak{h} \rangle$.

The restriction theorem.

For $f \in C^\infty(W)^H$ the restriction of f to $C^0_{\max}$ is a $\mathcal{W}$-invariant smooth function. The main objective of this subsection is to prove the following theorem which is a converse to this observation.

THEOREM 1.1 (Restriction Theorem). — *For each invariant open subset $\Omega \subseteq W$ the restriction map*

$$C^\infty(\Omega)^H \to C^\infty(\Omega \cap \mathfrak{a})^{\mathcal{W}}$$

is a bijection. □

We recall the set $\widehat{\Delta}^+_r$ of positive roots of solvable type and the set $\widehat{\Delta}^+_{n,s}$ of non-compact positive roots of semisimple type of $\mathfrak{g}_{\mathbb{C}}$. For $\alpha \in \widehat{\Delta}$ we set $[\alpha] := \{\alpha, -\alpha\}$ and $(\mathfrak{g}^c)^{[\alpha]} := \mathfrak{g}^c \cap (\mathfrak{g}_{\mathbb{C}}^{\alpha} + \mathfrak{g}_{\mathbb{C}}^{-\alpha})$. We set $\mathfrak{p}_r^c := \bigoplus_{\alpha\in\widehat{\Delta}_r^+} (\mathfrak{g}^c)^{[\alpha]}$ and $\mathfrak{p}_s^c := \bigoplus_{\alpha\in\widehat{\Delta}_{n,s}^+} (\mathfrak{g}^c)^{[\alpha]}$.

LEMMA 1.2. — *Let G^c be any connected group with Lie algebra $\mathfrak{g}^c$ such that the involution τ on $\mathfrak{g}^c$ integrates to an involution of G^c. Then the mapping*

$$(\mathfrak{h} \cap \mathfrak{p}_r^c) \times (\mathfrak{h} \cap \mathfrak{p}_s^c) \times (K^c)^\tau \to (G^c)^\tau, \quad (X, Y, k) \mapsto \exp X \exp Y k$$

is a diffeomorphism.

Proof. — From $\tau(\widehat{\Delta}_n) = \widehat{\Delta}_n$ we conclude that the subspace $\sum_{\alpha\in\widehat{\Delta}_n} (\mathfrak{g}^c)^{[\alpha]}$ is invariant under τ. Furthermore the uniqueness of the $\mathfrak{t}^c$-invariant Levi decomposition of $\mathfrak{g}^c$ ([Ne99], Prop. VII.2.5) and the invariance of $\mathfrak{t}^c$ imply that τ preserves the Levi decomposition $\mathfrak{g}^c = \mathfrak{r}^c + \mathfrak{s}^c$, hence that $\mathfrak{p}_r^c$ and $\mathfrak{p}_s^c$ are τ-invariant. Next we recall that the map

$$\mathfrak{p}_r^c \times \mathfrak{p}_s^c \times K^c \to G^c, \quad (X, Y, k) \mapsto \exp X \exp Y k$$

is a diffeomorphism ([Ne96a], Lemma III.2) to see that

$$(G^c)^\tau = \exp(\mathfrak{h} \cap \mathfrak{p}_r^c) \exp(\mathfrak{h} \cap \mathfrak{p}_s^c)(K^c)^\tau,$$

that the product mapping is bijective, and that it has a regular differential. This completes the proof. □

We will use the decomposition of the group stated in the preceding lemma to obtain a corresponding decomposition of the open cone W. We set $C_{\max,\mathfrak{p}} := \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k}_{\mathfrak{h}}^c).C_{\max} \subseteq \mathfrak{p}$. Since $(\mathfrak{k}^c, \tau)$ is a compact symmetric Lie algebra with $\mathfrak{k}^c \cap i\mathfrak{q} = i\mathfrak{p}$, it is clear that $\mathfrak{p}$ is invariant under the Lie algebra $\mathfrak{k}_{\mathfrak{h}}^c := \mathfrak{k}^c \cap \mathfrak{h}$, hence that $C_{\max,\mathfrak{p}}$ is a closed generating $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k}_{\mathfrak{h}}^c)$-invariant cone in $\mathfrak{p}$.

LEMMA 1.3. — *The mapping*

$$\phi : (\mathfrak{h} \cap \mathfrak{p}_r^c) \times (\mathfrak{h} \cap \mathfrak{p}_s^c) \times C_{\max,\mathfrak{p}}^0 \to W, \quad (X, Y, Z) \mapsto e^{\mathrm{ad}\, X} e^{\mathrm{ad}\, Y}.Z$$

is a diffeomorphism.

Proof. — The surjectivity of ϕ follows from Lemma 1.2 and the fact that each orbit of $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ in W intersects $C_{\max}^0$ ([KN96], Th. III.3(ii)). The injectivity of ϕ and the fact that it has a regular differential now follows from Lemma III.3 in [Ne96a] which states that the map

$$\phi^c : \mathfrak{p}_r^c \times \mathfrak{p}_s^c \times \widehat{C}_{\max,\mathfrak{p}}^0 \to \widehat{W}, \quad (X, Y, Z) \mapsto e^{\mathrm{ad}\, X} e^{\mathrm{ad}\, Y}.Z$$

is a diffeomorphism. □

Proof (of Theorem 1.1). — First we assume that $\Omega = W$. Let ϕ be the mapping of Lemma 1.3. Then Lemma 1.3 shows that the smooth H-invariant functions on W are in one-to-one correspondence with the smooth $\mathrm{Inn}_{\mathfrak{p}}(\mathfrak{k}_{\mathfrak{h}}^c)$-invariant functions on $C^0_{\max,\mathrm{p}}$. Hence it suffices to prove the assertion in the case where $\mathfrak{q} = \mathfrak{p}$ is a hyperbolic Lie triple system, i.e., the case of non-compact Riemannian symmetric spaces. In this case the assertion follows from [Hel84], Th. 5.8, which implies that for every smooth $\mathcal{W}$-invariant function on $C^0_{\max}$ with compact support its unique extension to $C_{\max,\mathrm{p}}$ is smooth. Note that Helgason assumes that the Riemannian space under consideration is semisimple but that his argument works equally well for the reductive case.

If $\Omega \subseteq W$ is an H-invariant open subset and $f \in C^\infty(\Omega \cap \mathfrak{a})^{\mathcal{W}}$, then we have to show that the unique extension to an H-invariant function $\tilde{f}$ on Ω is smooth. Multiplying f with a $\mathcal{W}$-invariant smooth function with compact support, we may w.l.o.g. assume that f has compact support in $\Omega \cap \mathfrak{a}$, hence that $f \in C^\infty(C^0_{\max})^{\mathcal{W}}$. Then the smoothness of $\tilde{f}$ follows from the first part of the proof. □

In the remainder of this subsection we turn to continuous functions.

LEMMA 1.4. — *Let $\mathfrak{a}^+ \subseteq \mathfrak{a}$ be a fundamental domain for the action of the Weyl group $\mathcal{W}$. For an invariant subset $\Omega \subseteq W$ the following are equivalent:*

(1) *Ω is open.*

(2) *$\Omega \cap \mathfrak{a}$ is open.*

(3) *$\Omega \cap \mathfrak{a}^+$ is open in $\mathfrak{a}^+$.*

Proof. — In view of Lemma 1.3, Ω is open if and only if $\Omega \cap C^0_{\max,\mathrm{p}}$ is open. Hence we may w.l.o.g. assume that $\mathfrak{q} = \mathfrak{p}$ is a hyperbolic Lie triple system.

Then each orbit $\mathcal{O}_X = \mathrm{Ad}(H).X$ intersects $\mathfrak{a}^+$ in a unique element $q(X)$. Let $q : \mathfrak{p} \to \mathfrak{a}^+$ denote the corresponding map. Since the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial, we only have to prove the converse. Assume that $\Omega \cap \mathfrak{a}^+$ is open. Then $\Omega = q^{-1}(\Omega \cap \mathfrak{a}^+)$. Hence it remains to show that q is continuous. Let $X_n \to X$ and assume that $q(X_n) \to Y$. It suffices to show that $q(X) = Y$. We pick $\gamma_n \in \mathrm{Inn}_{\mathfrak{p}}(\mathfrak{k}_{\mathfrak{h}}^c)$ with $q(X_n) = \gamma_n.X_n$.

Since $\mathrm{Inn}_{\mathfrak{p}}(\mathfrak{k}_{\mathfrak{h}}^c)$ is compact, we may w.l.o.g. assume that the sequence γ_n converges to $\gamma \in \mathrm{Inn}_{\mathfrak{p}}(\mathfrak{k}_{\mathfrak{h}}^c)$. Then $q(X_n) = \gamma_n.X_n \to \gamma.X \in \mathfrak{a}^+$ proves that $q(X) = \gamma.X = Y$. □

PROPOSITION 1.5. — *Let $\Omega \subseteq W$ be an open invariant subset and $\mathfrak{a}^+ \subseteq \mathfrak{a}$ a fundamental domain for the $\mathcal{W}$-action. Then the restriction maps*

$$C(\Omega)^H \to C(\Omega \cap \mathfrak{a})^{\mathcal{W}} \to C(\Omega \cap \mathfrak{a}^+)$$

are bijections.

Proof. — It only remains to show that for a continuous function ϕ on $\Omega \cap \mathfrak{a}^+$ the H-invariant extension $\widetilde{\phi}$ on Ω is continuous. But his is a direct consequence of Lemma 1.4. □

Invariant locally convex sets and functions.

To each $X \in C_{\max}^0 \subseteq \mathfrak{a}$ we associate the cone

$$C_X := \mathrm{cone}\{\alpha(X)[Z, \tau(Z)] : \alpha \in \Delta, Z \in \mathfrak{g}^\alpha\} = \overline{C_{\min} + C_{X,k}}$$

with $C_{X,k} := \mathrm{cone}\{\alpha(X)[Z, \tau(Z)] : \alpha \in \Delta_k, Z \in \mathfrak{g}^\alpha\}$. We also recall that

$$\mathfrak{a}^+ = \{X \in \mathfrak{a} : (\forall \alpha \in \Delta_k^+)\alpha(X) \geq 0\}$$

is a fundamental domain for the action of the Weyl group $\mathcal{W}$ on $\mathfrak{a}$ (cf. [KN96], Prop. V.2) and that for $X \in (\mathfrak{a}^+)^0$ we have $C_{X,k} = -\,\mathrm{cone}(\check{\Delta}_k^+)$. Note that $(\mathfrak{a}^+)^\star = \mathrm{cone}(\Delta_k^+)$.

We omit the proofs of Lemma 1.6 and Lemma 1.7 which consist in simply rewriting the proofs of Lemma III.15 and Lemma III.16 in [Ne96a] for the more general setting of symmetric spaces.

LEMMA 1.6. — *Let $\Omega^+ \subseteq \mathfrak{a}^+$ be a relatively open convex subset and $C^+ \subseteq \Omega^+$ relatively closed and convex. Suppose that for each $X \in C^+$ there exists an open neighborhood U of X in $\mathfrak{a}^+$ with $(X + C_X) \cap U \subseteq C^+$. Then $(X + C_X) \cap \Omega^+ \subseteq C^+$ holds for all $X \in C^+$.* □

LEMMA 1.7. — *Let $\Omega \subseteq \mathfrak{a}^+$ be a relatively open convex subset and $F \subseteq \Omega$ relatively closed. Suppose that $(X + C_X) \cap \Omega \subseteq F$ holds for all $X \in F$. Then*

$$(F - \mathrm{cone}(\check{\Delta}_k^+)) \cap \Omega \subseteq F \quad \textit{and} \quad F = \mathrm{conv}(\mathcal{W}.F) \cap \Omega.$$

□

DEFINITION 1.8. — *Let V be a real vector space and $\Omega \subseteq V$ be a subset.*

(a) *A function $\psi : \Omega \to \mathbb{R}$ is said to be locally convex if each $x \in \Omega$ has a convex neighborhood on which ϕ is a convex function.*

(b) *Suppose, in addition, that Ω is open. Then $\phi \in C^2(\Omega)$ is called stably locally convex if all the bilinear forms $d^2\phi(x) : V \times V \to \mathbb{R}$, $x \in \Omega$, are positive definite. If, in addition, Ω is convex, then a stably locally convex function ϕ on Ω is called stably convex. Note that stable local convexity implies local convexity because a two times differentiable function with positive semidefinite second derivative is convex.* □

THEOREM 1.9. — *Let $\Omega \subseteq W$ be an open H-invariant subset such that $\Omega^+ := \Omega \cap \mathfrak{a}^+$ is convex. Let further $\widetilde{C} \subseteq \Omega$ be a relatively closed invariant subset such that $C^+ := \widetilde{C} \cap \mathfrak{a}^+$ is connected. Then the following are equivalent:*

(1) *$\widetilde{C}$ is locally convex.*

(2) *C^+ is convex and for each $X \in C := \widetilde{C} \cap \mathfrak{a}$ there exists an open neighborhood U of X in $\mathfrak{a}$ with $(X + C_X) \cap U \subseteq C$.*

(3) *There exists an invariant convex subset $\widetilde{D} \subseteq W$ such that $\widetilde{D} \cap \Omega = \widetilde{C}$.*

Proof. — (1) $\Rightarrow$ (2): If $\widetilde{C}$ is locally convex, then $C^+ = \widetilde{C} \cap \mathfrak{a}^+$ inherits this property. Hence C^+ is a closed connected locally convex subset of the convex set $\Omega \cap \mathfrak{a}^+$. Therefore [Ne96a], Prop. I.5, implies that C^+ is convex.

Let $X \in C$ and $V \subseteq \Omega$ be an open convex neighborhood of X in $\mathfrak{q}$ such that $V \cap \widetilde{C}$ is convex and note that this implies that $V \cap C = V \cap \widetilde{C} \cap \mathfrak{a}$ is also convex. Pick $\alpha \in \Delta^+$ with $\alpha(X) \neq 0$ and $Z \in \mathfrak{g}^\alpha \setminus \{0\}$. Then [KN96, Lemma VI.1] shows that if $p_\mathfrak{a} : \mathfrak{q} \to \mathfrak{a}$ is the projection along $\mathfrak{q} \cap [\mathfrak{a}, \mathfrak{g}]$, then

$$\begin{aligned} p_\mathfrak{a}\big(e^{\mathrm{ad}(Z+\tau(Z))}.X\big) &= \cosh\big(\mathrm{ad}(Z+\tau(Z))\big).X \\ &= X + \begin{cases} \alpha(X)[Z,\tau(Z)] & \text{for } \alpha([Z,\tau(Z)]) = 0 \\ \alpha(X)\frac{\cosh(\sqrt{2\alpha([Z,\tau(Z)])})-1}{\alpha([Z,\tau(Z)])}[Z,\tau(Z)] & \text{for } \alpha([Z,\tau(Z)]) > 0 \\ \alpha(X)\frac{\cos(\sqrt{2\alpha([\tau(Z),Z])})-1}{\alpha([Z,\tau(Z)])}[Z,\tau(Z)] & \text{for } \alpha([Z,\tau(Z)]) < 0. \end{cases} \end{aligned}$$

If t is so small that $e^{\pm t\,\mathrm{ad}(Z+\tau(Z))}.X \in V$, then

$$\cosh\big(t\,\mathrm{ad}(Z+\tau(Z))\big).X = \frac{1}{2}\big(e^{t\,\mathrm{ad}(Z+\tau(Z))}.X + e^{-t\,\mathrm{ad}(Z+\tau(Z))}.X\big)$$
$$\in \frac{1}{2}(V \cap \widetilde{C} + V \cap \widetilde{C}) \subseteq V \cap \widetilde{C}.$$

Thus we find $t > 0$ with $Y + t\alpha(X)[Z, \tau(Z)] \in C \cap V$. Applying [Ne96a], Lemma I.8, Prop. I.9, shows that $\big((C \cap V) + C_X\big) \cap V \subseteq C$. In particular we obtain $(X + C_X) \cap V \subseteq C$ which proves (2).

(2) $\Rightarrow$ (3): The same arguments as in the proof of the corresponding implication in [Ne96a], Th. III.17, show that the set

$$D := \mathrm{conv}\,\big(\mathcal{W}.(C^+ + (C_{\min} \cap \mathfrak{a}^+))\big)$$

satisfies $D \cap \Omega^+ = C^+$, hence that $\widetilde{D} := \mathrm{Ad}(H).D$ satisfies $\widetilde{D} \cap \Omega = \widetilde{C}$. It remains to see that $\widetilde{D}$ is convex. In view of [KN96], Th. X.2(i), it suffices to show that $D + C_{\min} \subseteq D$. Using [Ne99, Prop. V.2.9], we see that

$$D = \bigcap_{\gamma \in \mathcal{W}} \gamma.\big(C^+ + (C_{\min} \cap \mathfrak{a}^+) - \mathrm{cone}(\check{\Delta}_k^+)\big) = \bigcap_{\gamma \in \mathcal{W}} \gamma.\big(C^+ + C_{\min} - \mathrm{cone}(\check{\Delta}_k^+)\big).$$

Since each of the convex sets $\gamma.\big(C^+ + C_{\min} - \mathrm{cone}(\check{\Delta}_k^+)\big)$ is invariant under addition of elements of $C_{\min}$, the same holds for D, i.e., $D + C_{\min} \subseteq D$. Thus $\widetilde{D}$ is convex and $\widetilde{C} = \Omega \cap \widetilde{D}$.

(3) $\Rightarrow$ (1): This is trivial. □

THEOREM 1.10. — *Let $\Omega \subseteq W$ be an open invariant subset such that $\Omega \cap \mathfrak{a}^+$ is convex, $\psi : \Omega \to \mathbb{R}$ an invariant function, and $\psi_{\mathfrak{a}} := \psi|_{\Omega \cap \mathfrak{a}}$. Then the following are equivalent:*

(1) *ψ is locally convex.*

(2) *$\psi_{\mathfrak{a}}$ is locally convex and $d\psi_{\mathfrak{a}}(X)(C_X) \subseteq \mathbb{R}^-$ for all $X \in \Omega \cap \mathfrak{a}$.*

(3) *There exists a convex invariant function $\widetilde{\psi}$ on $\mathrm{conv}(\Omega)$ such that $\widetilde{\psi}|_{\Omega} = \psi$.*

Proof. — (1) $\Rightarrow$ (2): If ψ is locally convex, then $\psi_{\mathfrak{a}}$ is locally convex.

Let $\alpha \in \Delta$ with $\alpha(X) \neq 0$, $Z \in \mathfrak{g}^\alpha \setminus \{0\}$, and put $\gamma(t) = e^{t\,\mathrm{ad}(Z+\tau.Z)}.X$. We have

$$\gamma'(0) = [Z + \tau.Z, X] = -\alpha(X)(Z - \tau.Z) \neq 0,$$

and

$$2\gamma''(0) = \mathrm{ad}(Z + \tau.Z)^2(X) = -\alpha(X)[Z + \tau.Z, Z - \tau.Z] = 2\alpha(X)[Z, \tau.Z].$$

Hence [Ne96a], Prop. I.17, shows that $d\psi_{\mathfrak{a}}(X)(C_X) \subseteq \mathbb{R}^-$ because the function $\psi_{\mathfrak{a}}$ is constant on the curve γ.

(2) $\Rightarrow$ (3): We consider the symmetric Lie algebra $\mathfrak{g}^\sharp := \mathfrak{g} \times \mathbb{R}$ with $\mathfrak{q}^\sharp := \mathfrak{q} \times \mathbb{R}$, $\mathfrak{a}^\sharp := \mathfrak{a} \times \mathbb{R}$, and the corresponding invariant cone $W^\sharp = W \times \mathbb{R}$. Then $\widetilde{C} := \operatorname{epi}(\psi) \subseteq \Omega \times \mathbb{R}$ is an invariant subset of $W^\sharp$, and we put $C := \widetilde{C} \cap \mathfrak{a}^\sharp = \operatorname{epi}(\psi_{\mathfrak{a}})$. Since $\psi_{\mathfrak{a}}$ is locally convex, it is continuous, and therefore C is a relatively closed locally convex subset of $(\Omega \times \mathbb{R}) \cap \mathfrak{a}^\sharp$.

To see that $C^+ := C \cap \mathfrak{a}^+$ is convex, let $\psi'_{\mathfrak{a}} := \psi_{\mathfrak{a}}|_{\Omega\cap\mathfrak{a}^+}$. Since $\psi'_{\mathfrak{a}}$ is a locally convex function on a convex set, it is convex ([Ne96a], Cor. I.6), and therefore its epigraph $C^+ = \operatorname{epi}(\psi'_{\mathfrak{a}})$ is convex.

Let $X \in C$ and U be a convex $\mathcal{W}^X$-invariant neighborhood of X in C such that $Y \in U$ and $\alpha(X) \neq 0$ imply $\alpha(Y) \neq 0$, hence that $C_X \subseteq C_Y$. This implies that $d\psi_{\mathfrak{a}}(Y)(C_X) \subseteq d\psi_{\mathfrak{a}}(Y)(C_Y) \subseteq \mathbb{R}^-$ holds for all $Y \in U$. Since $\psi_{\mathfrak{a}}|_U$ is locally convex and hence convex ([Ne96a], Cor. I.6), we find for each $(Y, t) \in U$ and $Z \in C_X$ a $\delta > 0$ with $(Y, t) + \delta(Z, 0) \in \operatorname{epi}(\psi_{\mathfrak{a}}|_U)$. Thus [Ne96a], Lemma I.8, entails that

$$\big(\operatorname{epi}(\psi_{\mathfrak{a}}|_U) + (C_X \times \{0\})\big) \cap (U \times \mathbb{R}) \subseteq \operatorname{epi}(\psi_{\mathfrak{a}}).$$

This means that $\operatorname{epi}(\psi_{\mathfrak{a}}) \subseteq \mathfrak{a}^\sharp$ satisfies the assumptions of Theorem 1.9(ii). So there exists an invariant convex subset $\widetilde{D} \subseteq W$ with $\widetilde{D} \cap (\Omega \times \mathbb{R}) = \operatorname{epi}(\psi)$. We define a function $\widetilde{\psi} : \operatorname{conv}(\Omega) \to \mathbb{R}$ by

$$\widetilde{\psi}(X) := \inf\{t \in \mathbb{R} : (X, t) \in \widetilde{D}\}.$$

To see that this function is well defined, we first note that by the convexity of $\widetilde{D}$, for each $X \in \operatorname{conv}(\Omega)$ there exists $t \in \mathbb{R}$ with $(X, t) \in \widetilde{D}$. Moreover, if $\widetilde{\psi}(X) = -\infty$, then $\{X\} \times \mathbb{R} \subseteq \widetilde{D}$ and therefore $\widetilde{D} + (\{0\} \times \mathbb{R}) = \widetilde{D}$ (cf. [Ne99], Prop. V.1.5(vi)), a contradiction. So we have constructed a convex H-invariant function on $\operatorname{conv}(\Omega)$ with $\widetilde{\psi}|_\Omega = \psi$.

(3) $\Rightarrow$ (1): This is trivial. □

Remark 1.11. — To obtain the equivalence between (1) and (2) in Theorem 1.10, we do not need that the set $\Omega \cap \mathfrak{a}^+$ is convex. In fact, this equivalence is a purely local assertion. Hence one can shrink Ω in such a way that $\Omega \cap \mathfrak{a}^+$ becomes convex and then apply Theorem 1.10. □

Before we come to the characterization of the stably convex invariant functions, we first need at least some stably convex invariant function. We will see below that the assumptions made on $\mathfrak{g}$ in the following lemma are even necessary for the existence of such functions.

For a convex subset C of a vector space V we write $\lim(C) := \{v \in V : v + C \subseteq C\}$ for the *recession cone* of C. We put $H(C) := \lim(C) \cap -\lim(C) = \{v \in V : v + C = C\}$.

LEMMA 1.12. — *Suppose, in addition, that $\mathfrak{g}$ has cone potential, i.e., $0 \neq Z \in \mathfrak{g}^\alpha$ implies that $[Z, \tau.Z] \neq 0$, that $C_{\min}$ is pointed, and let $X \in W$. Then there exists an open convex H-invariant subset $\Omega \subseteq W$ containing X with $H(\Omega) = \{0\}$ and a stably convex invariant function on Ω.*

Proof. — First we note that we may w.l.o.g. assume that $X \in C_{\max}^0$ ([KN96], Th. III.3(i)).

Let K be a compact convex $\mathcal{W}$-invariant neighborhood of X in $C_{\max}^0$ and put $C := K^0 + C_{\min}$. Then C is an open convex $\mathcal{W}$-invariant subset of $\mathfrak{a}$ and [KN96], Th. X.2(i), implies that $\Omega := \mathrm{Ad}(H).C \subseteq W$ is an open convex invariant subset of $\mathfrak{q}$. We have $H(C) = H(\overline{C}) = H(K + C_{\min}) = H(C_{\min}) = \{0\}$, and therefore $p_{\mathfrak{a}}(H(\Omega)) \subseteq H(C) = \{0\}$. Thus $H(\Omega)$ is an H-invariant subspace of $[\mathfrak{a}, \mathfrak{h}] \subseteq \mathfrak{q}$, and therefore trivial because $\mathfrak{g}$ has cone potential ([KN96], Prop. VII.2(iv)).

To construct a stably convex invariant function on Ω, we first use [KN96], Th. X.7(i), to find an open $G^\tau \cup \{-\tau\}$-invariant hyperbolic convex invariant subset $\hat{\Omega}$ of $\hat{\mathfrak{q}} = i\mathfrak{g}^c$ with $\hat{\Omega} \cap \mathfrak{q} = \Omega$. Factoring the largest ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$, we may, in addition, assume that the symmetric Lie algebra $(\mathfrak{g}, \tau)$ is effective. Then the proof of [KN96], Th. X.7(ii), shows that $\hat{\Omega}$ contains no affine lines, so that we can use [Ne96a], Lemma III.11, to find a stably convex G^c-invariant function $\hat{\psi}$ on $\hat{\Omega}$, in such a way that the restriction $\psi := \hat{\psi}|_\Omega$ is a stably convex H-invariant function on Ω. $\square$

For the following theorem we recall that a symmetric Lie algebra $(\mathfrak{g}, \tau)$ is said to be *admissible* if $\mathfrak{q}$ contains H-invariant generating hyperbolic subsets not containing affine lines.

THEOREM 1.13. — *Let $\Omega \subseteq W$ be an open invariant subset, $\psi : \Omega \to \mathbb{R}$ an invariant smooth function, and $\psi_{\mathfrak{a}} := \psi|_{\Omega \cap \mathfrak{a}}$. Then the following are equivalent:*

(1) ψ is stably locally convex.

(2) $\psi_{\mathfrak{a}}$ is stably locally convex, $d\psi_{\mathfrak{a}}(X) \in -\operatorname{int} C_X^\star$ for all $X \in \Omega \cap \mathfrak{a}$, and $(\mathfrak{g}, \tau)$ is admissible.

(3) $\psi_{\mathfrak{a}}$ is stably locally convex, $d\psi_{\mathfrak{a}}(X) \in -\operatorname{int} C_X^\star$ for all $X \in \Omega \cap \mathfrak{a}$,

and $(\mathfrak{g}, \tau)$ *has cone potential.*

Proof. — (1) $\Rightarrow$ (2): We proceed as in the proof of Theorem 1.10. If $\psi_{\mathfrak{a}}$ is stably locally convex, then [Ne96a], Prop. I.17, yields

$$d\psi_{\mathfrak{a}}(X)\big(\alpha(X)[Z, \tau.Z]\big) < 0$$

for $X \in \Omega \cap \mathfrak{a}$, $\alpha(X) \neq 0$, and $0 \neq Z \in \mathfrak{g}^{\alpha}$. Thus $d\psi_{\mathfrak{a}}(X) \in -\operatorname{int} C_X^{\star}$. We conclude in particular that $d\psi_{\mathfrak{a}}(X) \in -\operatorname{int} C_{\min}^{\star}$, hence that $C_{\min}$ is pointed. Moreover, the preceding argument implies that $[Z, \tau.Z] \neq 0$ for $\alpha \in \Delta_n^+$ and $0 \neq Z$, i.e., that $(\mathfrak{g}, \tau)$ has cone potential. Therefore Lemma 1.12 proves the existence of a closed convex hyperbolic invariant subset $C \subseteq \mathfrak{q}$ with $H(C) = \{0\}$, hence that $(\mathfrak{g}, \tau)$ is admissible.

(2) $\Rightarrow$ (3) follows from [KN96], Th. VI.6.

(3) $\Rightarrow$ (1): Let $X \in C$. Since $C_{\min} \subseteq C_X$, the assumption $d\psi_{\mathfrak{a}}(X) \in -\operatorname{int}(C_X)^{\star} \subseteq -\operatorname{int} C_{\min}^{\star}$ entails in particular that $\operatorname{int} C_{\min}^{\star} \neq \emptyset$, i.e., that the cone $C_{\min}$ is pointed. Hence the assumptions of Lemma 1.12 are satisfied, so that, after shrinking Ω if necessary, we find a stably convex invariant function f on Ω.

We choose an open convex neighborhood U of X which is invariant under the group $\mathcal{W}^X := \{\gamma \in \mathcal{W} : \gamma.X = X\}$, such that $Y \in U$ and $\alpha(X) \neq 0$ implies $\alpha(Y) \neq 0$, and $\overline{U} \subseteq \Omega$ is compact. Then the restriction of $\psi_{\mathfrak{a}}$ to U is a convex $\mathcal{W}^X$-invariant function ([Ne96a], Cor. I.6). Since U is relatively compact in $\Omega \cap \mathfrak{a}$ and $\psi_{\mathfrak{a}}$ is stably convex on $\overline{U}$, there exists $\varepsilon > 0$ such that $\phi_{\mathfrak{a}} := \psi_{\mathfrak{a}} - \varepsilon f$ is convex on U and $d\phi_{\mathfrak{a}}(Y) \in -\operatorname{int} C_X^{\star}$ holds for all $Y \in U$.

In view of our condition on U, we have $C_X \subseteq C_Y$ for all $Y \in U$. Suppose that $\alpha(X) = 0$. Then the reflection s_α is contained in $\mathcal{W}^X$ and since U is invariant under this reflection, it contains with each element Y the whole line segment $[Y, s_\alpha(Y)]$ with

$$s_\alpha(Y) = Y - \alpha(Y)\check{\alpha} = Y + \alpha(Y)[X_\alpha, \tau.X_\alpha],$$

where $0 \neq X_\alpha \in \mathfrak{g}^\alpha$. Now the invariance of the convex function $\phi_{\mathfrak{a}}|_{[Y, s_\alpha(Y)]}$ under this reflection implies that

$$d\phi_{\mathfrak{a}}(Y)\big(\alpha(Y)[X_\alpha, \tau.X_\alpha]\big) \leq 0.$$

So we even have $d\phi_{\mathfrak{a}}(Y) \in -C_Y^{\star}$ for all $Y \in U$. For $\gamma.Y \in \mathcal{W}.U$ the invariance of $\phi_{\mathfrak{a}}$ entails

$$d\phi_{\mathfrak{a}}(\gamma.Y) = d\phi_{\mathfrak{a}}(Y) \circ \gamma \in -\gamma^*.(C_Y)^{\star} = -C_{\gamma.Y}^{\star}.$$

Thus $d\phi_{\mathfrak{a}}(Y) \in -C_Y^\star$ for all $Y \in \mathcal{W}.U$.

Now Theorem 1.10 and Remark 1.11 imply that the function $\phi := \psi - \varepsilon f$ on $\widetilde{U} := \mathrm{Ad}(H).U \subseteq W$ is locally convex. We conclude in particular that $d^2\phi(Y) = d^2\psi(Y) - \varepsilon d^2 f(Y)$ is positive semidefinite for all $Y \in \widetilde{U}$, whence $d^2\psi(Y)$ is positive definite for all $Y \in \widetilde{U}$. Since X was arbitrary in $\Omega \cap \mathfrak{a}$, it follows that ψ is a stably convex function on Ω. □

Remark 1.14. — (a) The proof of Theorem 1.13 shows that, if the open convex subset $U \subseteq \mathfrak{a}$ containing X satisfies $\alpha(Y) \neq 0$ for $Y \in U$ and $\alpha(X) \neq 0$, and $\psi_{\mathfrak{a}}$ is a convex $\mathcal{W}$-invariant function on $\mathcal{W}.U$, then $d\psi_{\mathfrak{a}}(Y) \in -C_X^\star$ for all $Y \in U$ implies $d\psi_{\mathfrak{a}}(Y) \in -C_Y^\star$ for all $Y \in \mathcal{W}.U$.

(b) If, in particular, $\Omega \cap \mathfrak{a}$ is convex, the condition $d\psi_{\mathfrak{a}}(X) \in -C_X^\star$ follows from $d\psi_{\mathfrak{a}}(X) \in -C_{\min}^\star$. Therefore we obtain the simpler criterion in this case saying that ψ is (stably) locally convex if and only if $\psi_{\mathfrak{a}}$ is (stably) locally convex and $d\psi_{\mathfrak{a}}(X) \in -C_{\min}^\star$ ($d\psi_{\mathfrak{a}}(X) \in -\operatorname{int} C_{\min}^\star$) for all $X \in \Omega \cap \mathfrak{a}$. □

Extending invariant convex sets and functions.

In this subsection we deal with the problem of extending convex H-invariant subsets of $\mathfrak{q}$ to G^c-invariant convex subsets of $i\mathfrak{g}^c = \mathfrak{q} + i\mathfrak{h}$ and of extending H-invariant convex functions to G^c-invariant convex functions. Our main result is Proposition 1.19 which will be important later on to make the results for the group case (see [Ne96a]) available in our present setting.

LEMMA 1.15. — *For each $X \in \mathfrak{a}$ we have $(\widehat{\mathcal{W}}.X) \cap \mathfrak{a} = \mathcal{W}.X$.*

Proof. — Let $p : \hat{\mathfrak{a}} \to \mathfrak{a}$ denote the projection given by $p_{\mathfrak{a}}(X) = \frac{1}{2}(X - \tau.X)$. Using [KN96], Prop. X.5(iii), we see that $p_{\mathfrak{a}}(\operatorname{conv} \widehat{\mathcal{W}}.X) = \operatorname{conv}(\mathcal{W}.X)$. This shows that if $\gamma \in \widehat{\mathcal{W}}$ satisfies $Y := \gamma.X \in \mathfrak{a}$, then $Y \in \operatorname{conv}(\mathcal{W}.X)$. If $\|\cdot\|$ is a $\widehat{\mathcal{W}}$-invariant euclidean norm on $\hat{\mathfrak{a}}$, then we conclude that $\|Y\| = \|X\|$, hence that Y is an exposed point of the polyhedron $\operatorname{conv}(\mathcal{W}.X)$, and therefore

$$Y \in \operatorname{Ex}(\operatorname{conv} \mathcal{W}.X) = \mathcal{W}.X,$$

where $\operatorname{Ex}(C)$ denotes the set of exposed points of the closed convex set C. □

LEMMA 1.16. — *For each $\mathcal{W}$-invariant open subset $\Omega \subseteq \mathfrak{a}$ there exists a $\widehat{\mathcal{W}}$-invariant open subset $\widehat{\Omega} \subseteq \hat{\mathfrak{a}}$ with $\widehat{\Omega} \cap \mathfrak{a} = \Omega$.*

Proof. — For each $X \in \Omega$ we choose an open neighborhood $U_X \subseteq \hat{\mathfrak{a}}$ which is invariant under the finite group $\widehat{\mathcal{W}}^X$, satisfies $\gamma.U_X \cap \mathfrak{a} = \emptyset$ for $\gamma.X \notin \mathfrak{a}$, and $\gamma.U_X \cap \mathfrak{a} \subseteq \Omega$ whenever $\gamma.X \in \mathfrak{a}$ (*cf.* Lemma 1.15). We define

$$\widehat{\Omega} := \bigcup_{X \in \Omega} \widehat{\mathcal{W}}.U_X$$

and note that this is an open $\widehat{\mathcal{W}}$-invariant subset of $\hat{\mathfrak{a}}$ containing Ω. If for $\gamma \in \widehat{\mathcal{W}}$ and $X \in \Omega$ the set $\gamma.U_X$ intersects $\mathfrak{a}$, then $\gamma.X \in \mathfrak{a}$, and therefore the construction of U_X implies that $(\gamma.U_X) \cap \mathfrak{a} \subseteq \Omega$. We conclude that $\Omega \subseteq \widehat{\Omega} \cap \mathfrak{a} \subseteq \Omega$, and this proves the lemma. □

In the following a *domain* always means an open connected set and an *extension* of an H-invariant domain $D_\mathfrak{q} \subseteq \mathfrak{q}$ is a G-invariant domain $\widehat{D}_\mathfrak{q} \subseteq \mathfrak{g}$ with $\widehat{D}_\mathfrak{q} \cap \mathfrak{q} = D_\mathfrak{q}$.

PROPOSITION 1.17. — *For each H-invariant domain $D_\mathfrak{q} \subseteq W$ there exists a G^c-invariant domain $\widehat{D}_\mathfrak{q} \subseteq i\mathfrak{g}^c$ with $\widehat{D}_\mathfrak{q} \cap \mathfrak{q} = D_\mathfrak{q}$. If, in addition, $D_\mathfrak{q}$ is convex, then $\widehat{D}_\mathfrak{q}$ can be chosen to be convex.*

Proof. — Using Lemma 1.16, we find a $\widehat{\mathcal{W}}$-invariant domain $\widehat{\Omega} \subseteq \hat{\mathfrak{a}}$ with $\widehat{\Omega} \cap \mathfrak{a} = D_\mathfrak{q} \cap \mathfrak{a}$. Since $\widehat{C}^0_{\max}$ is invariant under $\widehat{\mathcal{W}}$ and $D_\mathfrak{q} \cap \mathfrak{a} \subseteq C^0_{\max} \subseteq \widehat{C}^0_{\max}$, we may w.l.o.g. assume that $\widehat{\Omega} \subseteq \widehat{C}^0_{\max}$. Then Lemma 1.4 shows that $\widehat{D}_\mathfrak{q} := G^c.\widehat{\Omega} \subseteq \widehat{W}$ is a G^c-invariant open subset.

Further $\widehat{D}_\mathfrak{q} \cap \mathfrak{q}$ is H-invariant, and therefore

$$\widehat{D}_\mathfrak{q} \cap \mathfrak{q} = H.(\widehat{D}_\mathfrak{q} \cap \mathfrak{a}) = H.(\widehat{\Omega} \cap \mathfrak{a}) = H.(D_\mathfrak{q} \cap \mathfrak{a}) = D_\mathfrak{q}.$$

This proves the existence of an extension $\widehat{D}_\mathfrak{q}$ of $D_\mathfrak{q}$.

Suppose, in addition, that $D_\mathfrak{q}$ is convex. We recall from [KN96], Th. X.7, that each closed convex hyperbolic subset $C \subseteq W_{\max}$ has a maximal extension

$$\widehat{C} := \{X \in \overline{\widehat{W}} : p_\mathfrak{a}(G^c.X) \subseteq C \cap \mathfrak{a}\}$$

which, in addition, is $-\tau$-invariant and satisfies $p_\mathfrak{q}(\widehat{C}) = \widehat{C} \cap \mathfrak{q} = C$. Applying this extension to the closure of an invariant convex domain $D_\mathfrak{q} \subseteq W$, it follows from the $-\tau$-invariance of the extension that the interior $\widehat{D}_\mathfrak{q}$ of the extension of the closure extends the domain $D_\mathfrak{q}$. □

COROLLARY 1.18. — *Every H-invariant domain $D_{\mathfrak{q}} \subseteq W$ is also invariant under the group $(G^c)^\tau$.*

Proof. — Writing $D_{\mathfrak{q}} = \widehat{D}_{\mathfrak{q}} \cap \mathfrak{q}$ as in Proposition 1.17, we observe that the group $(G^c)^\tau$ preserves $\mathfrak{q}$ and $\widehat{D}_{\mathfrak{q}}$, hence also $D_{\mathfrak{q}}$. □

PROPOSITION 1.19. — *If $D_{\mathfrak{q}} \subseteq W$ is an invariant convex domain and $\psi : D_{\mathfrak{q}} \to \mathbb{R}$ is an invariant convex function, then there exists a convex extension $\widehat{D}_{\mathfrak{q}}$ of $D_{\mathfrak{q}}$ and a convex G^c-invariant convex function $\widehat{\psi} : \widehat{D}_{\mathfrak{q}} \to \mathbb{R}$ with $\psi = \widehat{\psi}|_{D_{\mathfrak{q}}}$.*

Proof. — The epigraph $B := \operatorname{epi}(\psi) \subseteq D_{\mathfrak{q}}^\sharp := D_{\mathfrak{q}} \times \mathbb{R} \subseteq \mathfrak{q}^\sharp := \mathfrak{q} \times \mathbb{R}$ is an H-invariant domain, hence extends to an invariant convex domain $\widehat{B} \subseteq \mathfrak{q}^\sharp$ (Proposition 1.17). Now $\{0\} \times \mathbb{R} \not\subseteq H(B)$ implies that $\{0\} \times \mathbb{R} \not\subseteq H(\widehat{B})$, and therefore that whenever $(x, t_0) \in \widehat{B}$ for some t_0, the number

$$\widehat{\psi}(x) := \inf\{t : (x,t) \in \widehat{B}\}$$

is well defined and defines an invariant convex function on the domain $\widehat{D}_{\mathfrak{q}} := p_{\widehat{\mathfrak{q}}}(\widehat{B})$ which in turn extends $D_{\mathfrak{q}}$. Since $\widehat{B}$ extends B, it follows that $\widehat{\psi}|_{D_{\mathfrak{q}}} = \psi$. □

2. Calculations in low dimensional cases.

In this section we collect the calculations in low dimensional symmetric Lie algebras that will be needed to obtain the characterizations of invariant (strictly) plurisubharmonic functions on domains of the type $Q(C)$. We keep the notation and the assumptions of Section 2. The most relevant cases are $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ with $\mathfrak{h} = \mathfrak{so}(1,1)$ and $\mathfrak{g}^c = \mathfrak{su}(1,1)$, $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ with $\mathfrak{h} = \mathfrak{so}(2,\mathbb{R})$ and $\mathfrak{g}^c = \mathfrak{su}(2)$, and where $\mathfrak{g}^c$ is the four dimensional oscillator algebra.

The solvable type.

In this subsection $\mathfrak{g}^c$ denotes a solvable Lie algebra with compactly embedded Cartan subalgebra $\mathfrak{t} = i\hat{\mathfrak{a}}$. We assume that $\Delta^+ = \{\alpha\}$, so that there exist only two root spaces. Then $\mathfrak{g}^c = \mathfrak{u} \rtimes \mathfrak{l}$, where $\mathfrak{l} \subseteq \mathfrak{t}$

complements the center $\mathfrak{z}(\mathfrak{g})$ and $\mathfrak{u} = \mathfrak{z}(\mathfrak{g}^c) + \mathfrak{g}^{[\alpha]}$ is the nilradical. Note that $\mathfrak{u}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{\alpha} + \mathfrak{g}_{\mathbb{C}}^{-\alpha} + \mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$ so that $[\mathfrak{u}_{\mathbb{C}}, \mathfrak{u}_{\mathbb{C}}] \subseteq [\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{-\alpha}] \subseteq \mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$. We extend the root α to a functional on $\mathfrak{g}_{\mathbb{C}}$ that vanishes on $\mathfrak{u}_{\mathbb{C}}$. If $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^{\alpha} = 1$, $[\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{-\alpha}] \neq \{0\}$ and $\dim \mathfrak{t} = 2$, then $\mathfrak{g}$ is called the *four dimensional oscillator algebra*. It has a basis (P, Q, Z, H) where the non-zero brackets are given by

$$[P, Q] = Z, \quad [H, P] = Q, \quad \text{and} \quad [H, Q] = -P.$$

Let $G_{\mathbb{C}}$ denote the simply connected group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and $G \subseteq G_{\mathbb{C}}$ the simply connected subgroup with Lie algebra $\mathfrak{g}$. Then $G_{\mathbb{C}} = G \exp(i\mathfrak{g}^c)$, where the map

$$G^c \times i\mathfrak{g}^c \to G_{\mathbb{C}}, \quad (g, X) \mapsto g \exp(X)$$

is a diffeomorphism (*cf.* [Ne98], Sect. 3). For $s = g\exp(X)$, $X \in i\mathfrak{g}$, we have

$$s^* = \exp(X) g^{-1} = g^{-1} \exp\big(\operatorname{Ad}(g).X\big),$$

so that $s^* = s$ entails $g^2 = \mathbf{1}$, hence $g = \mathbf{1}$ since G^c is simply connected. For $s = s^* = \exp X$ we define $\log s := X$.

For $s = g\exp(X)$ we further have $s^\sharp = \exp(X)^\sharp \tau(g)^{-1}$ so that $s^\sharp = s$ means $\tau(g) = g^{-1}$ and $X = -\operatorname{Ad}(g)^{-1}.\tau(X)$.

PROPOSITION 2.1. — *Let $s = \exp(Z_\alpha + Z_{-\alpha})\exp(Z)$ with $Z \in \mathfrak{l}_{\mathbb{C}}$, $\alpha(Z) \neq 0$, $Z_{\pm\alpha} \in \mathfrak{g}_{\mathbb{C}}^{\pm\alpha}$, and suppose that $s^* = s$ as well as $s^\sharp = s$. Then there exists $h \in H$ such that*

$$hsh^{-1} = \exp\left(Z + \frac{1}{2}\coth\left(\frac{\alpha(Z)}{2}\right)[Z_\alpha, Z_{-\alpha}]\right).$$

Proof. — With the same argument as in [Ne98], Prop. 3.1, we conclude that $\exp(Z)^* = \exp Z$, hence that $Z \in \mathfrak{a}$ and therefore that $\alpha(Z) \in \mathbb{R}$.

We have $s^\sharp = \tau(s)^{-1}$, so that $\tau.\mathfrak{g}_{\mathbb{C}}^{\alpha} = \mathfrak{g}_{\mathbb{C}}^{-\alpha}$ and

$$\begin{aligned}\exp(-\tau.Z)\exp(-\tau.Z_\alpha - \tau.Z_{-\alpha}) &= s^\sharp = s = \exp(Z_\alpha + Z_{-\alpha})\exp(Z)\\ &= \exp(Z)\exp(e^{-\alpha(Z)}Z_\alpha + e^{\alpha(Z)}Z_{-\alpha})\end{aligned}$$

imply that $\tau.Z_\alpha = -e^{\alpha(Z)}Z_{-\alpha}$. Similarly $s^* = s$ entails that $\overline{Z_\alpha} = -e^{\alpha(Z)}Z_{-\alpha}$.

In [Ne98], Prop. 3.1, we have seen that there exists $g \in G^c$ such that

$$gsg^{-1} = \exp\left(Z + \frac{1}{2}\coth\left(\frac{\alpha(Z)}{2}\right)[Z_\alpha, Z_{-\alpha}]\right).$$

Then $Z_{-\alpha} = -e^{-\alpha(Z)}\tau.Z_\alpha = -e^{-\alpha(Z)}\overline{Z_\alpha}$ implies that

$$[Z_\alpha, Z_{-\alpha}] = -e^{-\alpha(Z)}[Z_\alpha, \tau.Z_{-\alpha}] = -e^{-\alpha(Z)}[Z_\alpha, \overline{Z_{-\alpha}}].$$

Therefore $[Z_\alpha, Z_{-\alpha}] \in i\mathfrak{g}^c \cap \mathfrak{a}_{\mathbb{C}} = \mathfrak{a}$. Writing $s = \exp X$ with $X \in \mathfrak{q}$, this means that

$$\mathrm{Ad}(g).X \in \mathfrak{a} = i\mathfrak{t}.$$

Now $\mathcal{W}_{\mathfrak{k}} = \{\mathbf{1}\} = \mathcal{W}$ implies that $\mathrm{Ad}(G).X \cap \mathfrak{a} = \{X\} = \mathrm{Ad}(H).X$, so that we may w.l.o.g. assume that $g \in H$. This completes the proof. □

PROPOSITION 2.2. — *Let $X \in \mathfrak{a}$ with $\alpha(X) \neq 0$ and $X_\alpha \in \mathfrak{g}^\alpha$.*

(i) *If $s = \exp(X_\alpha - \tau.X_\alpha)\exp(X)\exp(X_\alpha - \tau.X_\alpha)$, then there exists $h \in H$ with*

$$hsh^{-1} = \exp\left(X - 2\coth\left(\frac{\alpha(X)}{2}\right)[X_\alpha, \tau.X_\alpha]\right).$$

(ii) *If $X_\alpha \in \mathfrak{g}^\alpha$, $\gamma(z) := \exp(zX_\alpha)\exp(X)\exp(-z\tau.X_\alpha)$, $z \in \mathbb{C}$, then there exists $g \in G^c$ with*

$$g\gamma(z)g^\sharp = \exp\left(X + 2\left(\frac{(\mathrm{Re}\, z)^2}{e^{\alpha(X)} - 1} - \frac{(\mathrm{Im}\, z)^2}{e^{\alpha(X)} + 1}\right)[\tau.X_\alpha, X_\alpha]\right).$$

Proof. — (i) From $X_\alpha \in \mathfrak{g}^\alpha$ we deduce that $\tau(X_\alpha) = \hat{\tau}(X_\alpha) = \overline{X_\alpha}$. Therefore [Ne98], Cor. 3.2, implies the existence of a $g \in G^c$ with $gsg^{-1} = \exp Y$ for

$$Y = \exp\left(X - 2\coth\left(\frac{\alpha(X)}{2}\right)[X_\alpha, \tau.X_\alpha]\right).$$

If $s = \exp Y'$, then this means that $Y = \mathrm{Ad}(g).Y'$. Now the same argument as in Lemma II.1 shows that we may w.l.o.g. assume that $g \in H$.

(ii) We may w.l.o.g. assume that $\alpha(X) > 0$ holds for $\Delta^+ = \{\alpha\}$. Let

$$\gamma(z) := \exp(zX_\alpha)\exp(X)\exp(-z\tau.X_\alpha)$$

and note that if z is small enough, then $\gamma(z) \in Q(C_{\max})$.

We make the following ansatz:

$$g = \exp(\mu X_\alpha + \overline{\mu}\overline{X_\alpha})\exp Z = \exp(\mu X_\alpha + \overline{\mu}\tau.X_\alpha)\exp Z$$

with $\mu \in \mathbb{C}$ and $Z \in i\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g}^c)$. More explicitly we put

$$\mu := e^{-\alpha(X)}\left(\frac{\mathrm{Re}\, z}{1 - e^{-\alpha(X)}} - i\frac{\mathrm{Im}\, z}{1 + e^{-\alpha(X)}}\right) = \frac{\mathrm{Re}\, z}{e^{\alpha(X)} - 1} - i\frac{\mathrm{Im}\, z}{e^{\alpha(X)} + 1}.$$

This means that $\overline{\mu} = (\mu + z)e^{-\alpha(X)}$. On the other hand we have

$$\exp(\mu X_\alpha + \overline{\mu}\tau.X_\alpha)\exp(zX_\alpha) = \exp(\overline{\mu}z[\tau.X_\alpha, X_\alpha])\exp\big((z+\mu)X_\alpha + \overline{\mu}\tau.X_\alpha\big)$$

which we use to see that

$$\begin{aligned}
g\gamma(z)g^\sharp &= \exp(\mu X_\alpha + \overline{\mu X_\alpha})\exp Z\exp(zX_\alpha)\exp(X)\\
&\qquad \exp(-z\tau.X_\alpha)\exp Z\exp(-\mu\tau.X_\alpha - \overline{\mu}X_\alpha)\\
&= \exp(\mu X_\alpha + \overline{\mu}\tau.X_\alpha)\exp Z\exp(zX_\alpha)\exp(X)\\
&\qquad \big(\exp(\mu X_\alpha + \overline{\mu}\tau.X_\alpha)\exp Z\exp(zX_\alpha)\big)^\sharp\\
&= \exp(2Z + 2\overline{\mu}z[\tau.X_\alpha, X_\alpha])\exp\big((z+\mu)X_\alpha + \overline{\mu}\tau.X_\alpha\big)\\
&\qquad \exp(X)\exp\big(-(z+\mu)\tau.X_\alpha - \overline{\mu}X_\alpha\big)\\
&= \exp(2Z + 2\overline{\mu}z[\tau.X_\alpha, X_\alpha])\exp\big((z+\mu)X_\alpha + \overline{\mu}\tau.X_\alpha\big)\\
&\qquad \exp\big(-(z+\mu)e^{-\alpha(X)}\tau.X_\alpha - \overline{\mu}e^{\alpha(X)}X_\alpha\big)\exp(X)\\
&= \exp(2Z + X + 2\overline{\mu}z[\tau.X_\alpha, X_\alpha]).
\end{aligned}$$

Next we choose $Z \in i\mathfrak{z}(\mathfrak{g})$ in such a way that $2Z + X + 2\overline{\mu}z[\tau.X_\alpha, X_\alpha] \in \mathfrak{a}$, which makes this expression equal to $X + 2\operatorname{Re}(\overline{\mu}z)[\tau.X_\alpha, X_\alpha]$. Now the assertion follows from

$$\operatorname{Re}(\overline{\mu}z) = \operatorname{Re}(z)\operatorname{Re}(\mu) + \operatorname{Im}(z)\operatorname{Im}(\mu) = \Big(\frac{(\operatorname{Re} z)^2}{e^{\alpha(X)} - 1} - \frac{(\operatorname{Im} z)^2}{e^{\alpha(X)} + 1}\Big).$$

□

The non-compact simple case.

Example 2.3. — We consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{h} = \mathfrak{so}(1,1)$ and $\mathfrak{g}^c = \mathfrak{su}(1,1)$, so that $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(2,\mathbb{C})$. For $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we then have

$$\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} = F\begin{pmatrix} a & b \\ c & d \end{pmatrix}F = \begin{pmatrix} d & c \\ b & a \end{pmatrix}. \tag{2.1}$$

We conclude that for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ we have

$$g^\sharp = \tau(g)^{-1} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}^{-1} = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}$$

and therefore

$$Q = \{s \in SL(2,\mathbb{C}) : s^\sharp = s\} = \Big\{\begin{pmatrix} a & b \\ -b & d \end{pmatrix} : a, b, d \in \mathbb{C}, ad + b^2 = 1\Big\}.$$

We consider the action of $SU(1,1)$ on Q given by $g.s := gsg^\sharp$. In view of (2.1), we have $(g.s)F = gs\tau(g)^{-1}F = g(sF)g^{-1}$, so that the map

$$Q \to \mathfrak{sl}(2,\mathbb{C}), \quad s = \begin{pmatrix} a & b \\ -b & d \end{pmatrix} \mapsto sF = \begin{pmatrix} b & a \\ d & -b \end{pmatrix}$$

is equivariant with respect to the adjoint action of $SU(1,1)$ on $\mathfrak{sl}(2,\mathbb{C})$. The subset $QF \subseteq \mathfrak{sl}(2,\mathbb{C})$ is a complex quadric, the coadjoint orbit of the diagonal matrix H.

We observe that the coadjoint action of $SU(1,1)$ on $\mathfrak{sl}(2,\mathbb{C})$ preserves the two subspaces $\mathfrak{g}^c$ and $i\mathfrak{g}^c$, where

$$i\mathfrak{g}^c = \{X \in \mathfrak{sl}(2,\mathbb{C}) : X^o = X\},$$

with $X^o = BX^*B$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. More explicitly this means that for $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ we have

$$X^o = B\begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & -\overline{a} \end{pmatrix}B = \begin{pmatrix} \overline{a} & -\overline{c} \\ -\overline{b} & -\overline{a} \end{pmatrix}.$$

Therefore the projecton

$$q : \mathfrak{sl}(2,\mathbb{C}) \to i\mathfrak{su}(1,1), \quad X \mapsto \tfrac{1}{2}(X + X^o)$$

is given by

$$q\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} \operatorname{Re} a & \frac{1}{2}(b-\overline{c}) \\ \frac{1}{2}(c-\overline{b}) & -\operatorname{Re} a \end{pmatrix}, \text{ and } \det q(X) = -(\operatorname{Re} a)^2 + \frac{1}{4}|b-\overline{c}|^2.$$

The function $X \mapsto \det q(X)$ on QF is invariant under the action of $SU(1,1)$.

We choose $\mathfrak{a} = \mathbb{R}H$ with $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\Delta^+ = \Delta_n^+ = \{\alpha\}$ with $\alpha(H) = 2$, i.e., $H = \check{\alpha}$. Then $C_{\max} = \mathbb{R}^+ H \subseteq \mathfrak{a}$. Let $\mu > 0$ and consider the holomorphic curve

$$\gamma : \mathbb{C} \to Q, \quad z \mapsto \begin{pmatrix} z & e^{\mu}(1-z^2) \\ e^{-\mu} & -z \end{pmatrix}F = \begin{pmatrix} e^{\mu}(1-z^2) & z \\ -z & e^{-\mu} \end{pmatrix}.$$

Then

$$\det q\big(\gamma(z)F\big) = -(\operatorname{Re} z)^2 + \frac{1}{4}|e^{\mu}(1-z^2) - e^{-\mu}|^2$$

and if $\gamma(z)$ is $SU(1,1)$-conjugate to $\exp\big(\nu(z)H\big)$, then

$$\begin{aligned}
\big(\sinh\mu(z)\big)^2 &= -(\operatorname{Re} z)^2 + \frac{1}{4}|e^{\mu}(1-z^2) - e^{-\mu}|^2 \\
&= -(\operatorname{Re} z)^2 + \Big|\sinh\mu - e^{\mu}\frac{z^2}{2}\Big|^2 \\
&= -(\operatorname{Re} z)^2 + (\sinh\mu)^2 + e^{2\mu}\frac{|z|^4}{4} - 2\operatorname{Re}\Big(\sinh\mu e^{\mu}\frac{\overline{z}^2}{2}\Big) \\
&= -(\operatorname{Re} z)^2 + (\sinh\mu)^2 + e^{2\mu}\frac{|z|^4}{4} - \frac{1}{2}(e^{2\mu}-1)\operatorname{Re}(\overline{z}^2) \\
&= -(\operatorname{Re} z)^2 + (\sinh\mu)^2 + e^{2\mu}\frac{|z|^4}{4} - \frac{1}{2}(e^{2\mu}-1)\big((\operatorname{Re} z)^2 \\
&\qquad - (\operatorname{Im} z)^2\big) \\
&= (\sinh\mu)^2 + e^{2\mu}\Big(\frac{|z|^4}{4} - \frac{1}{2}\big((\operatorname{Re} z)^2 - (\operatorname{Im} z)^2\big)\Big) - (\operatorname{Re} z)^2 \\
&\qquad + \frac{1}{2}\big((\operatorname{Re} z)^2 - (\operatorname{Im} z)^2\big) \\
&= (\sinh\mu)^2 + e^{2\mu}\Big(\frac{|z|^4}{4} - \frac{1}{2}(\operatorname{Re} z)^2 + \frac{1}{2}(\operatorname{Im} z)^2\Big) - \frac{1}{2}|z|^2.
\end{aligned}$$

$\square$

PROPOSITION 2.4. — *Let $(\mathfrak{g},\tau)$ be a reductive symmetric Lie algebra with $\mathfrak{g}' \cong \mathfrak{sl}(2,\mathbb{R})$, $X_\alpha \in \mathfrak{g}^\alpha$ with $[X_\alpha, \tau.X_\alpha] = \check{\alpha}$, $X \in C^0_{\max}$, and*

$$\gamma : \mathbb{C} \to G_{\mathbb{C}}, \quad z \mapsto \exp(ze^{\alpha(X)}X_\alpha)\exp(X)\exp(ze^{\alpha(X)}X_\alpha^\sharp).$$

If $z \in \mathbb{C}$ is sufficiently small, then there exists a $g \in G^c$ with

$$g\gamma(z)g^\sharp = \exp(X + \nu(z)[X_\alpha, \tau.X_\alpha])$$

and

$$\cosh\big(\alpha(X) + 2\nu(z)\big) = \cosh\alpha(X) + e^{\alpha(X)}\Big(\frac{|z|^4}{2} - (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2\Big) - |z|^2.$$

Proof. — Splitting off the center, we may w.l.o.g. assume that $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ because we may write $X = X_0 + \frac{\alpha(X)}{2}[X_\alpha, \tau.X_\alpha]$ with $X_0 \in \mathfrak{z}(\mathfrak{g})$. Furthermore $[X_\alpha, \tau.X_\alpha] = \check{\alpha}$ shows that we further may assume that

$$X_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \check{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With $\mu = \alpha(X) > 0$ we then obtain

$$\gamma(z) = \begin{pmatrix} 1 & ze^{\mu} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{\mu} & 0 \\ 0 & e^{-\mu} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -ze^{\mu} & 1 \end{pmatrix} = \begin{pmatrix} e^{\mu}(1-z^2) & z \\ -z & e^{-\mu} \end{pmatrix}.$$

Now the assertion follows from the calculations made above and the relation

$$\sinh(x)^2 = \frac{\cosh(2x) - 1}{2}.$$

$\square$

The compact simple case.

Example 2.5. — We consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{h} = \mathfrak{so}(2)$ and $\mathfrak{g}^c = \mathfrak{su}(2)$, so that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$. Then $\tau(X) = -X^\top = FXF$ with $F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $g^\sharp = g^\top = Fg^{-1}F$ holds for $g \in SL(2,\mathbb{C})$. Hence

$$Q = \{s \in SL(2,\mathbb{C}) : s^\top = s\} = \Big\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} : a, b, d \in \mathbb{C}, ad - b^2 = 1\Big\}.$$

We consider the action of $G^c = SU(2)$ on Q given by $g.s := gsg^\sharp = gsg^\top$. From $\tau(g) = FgF$ we get $(g.s)F = gs\tau(g)^{-1}F = g(sF)g^{-1}$, so that the map

$$Q \to \mathfrak{sl}(2,\mathbb{C}), \quad s = \begin{pmatrix} a & b \\ -b & d \end{pmatrix} \mapsto sF = \begin{pmatrix} -b & a \\ -d & b \end{pmatrix}$$

is equivariant with respect to the adjoint action of $SU(2)$ on $\mathfrak{sl}(2,\mathbb{C})$. The subset $QF \subseteq \mathfrak{sl}(2,\mathbb{C})$ is a complex quadric, the coadjoint orbit of the diagonal matrix H.

As in Example 2.3, we see that if

$$q : \mathfrak{sl}(2,\mathbb{C}) \to i\mathfrak{su}(2),\ X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \frac{1}{2}(X + X^*) = \begin{pmatrix} \mathrm{Re}\, a & \frac{1}{2}(b+\overline{c}) \\ \frac{1}{2}(c+\overline{b}) & -\mathrm{Re}\, a \end{pmatrix}$$

is the projection, then $X \mapsto \det q(X) = -(\mathrm{Re}\, a)^2 - \frac{1}{4}|b+\overline{c}|^2$ is a function on QF which is invariant under the action of the group $SU(2)$.

We choose

$$\mathfrak{a} = \mathbb{R}H \quad \text{with} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\Delta^+ = \Delta_k^+ = \{\alpha\}$ with $\alpha(H) = 2$. Then $C_{\max} = \mathfrak{a}$. Let $\mu \in \mathbb{R}$ and define the holomorphic curve

$$\gamma : \mathbb{C} \to Q, \quad z \mapsto \begin{pmatrix} z & e^{\mu}(1+z^2) \\ -e^{-\mu} & -z \end{pmatrix} F = \begin{pmatrix} e^{\mu}(1+z^2) & z \\ z & e^{-\mu} \end{pmatrix}.$$

Then

$$\det q\big(\gamma(z)F\big) = -(\operatorname{Re} z)^2 - \frac{1}{4}|e^{\mu}(1+z^2) - e^{-\mu}|^2.$$

If $\gamma(z)F$ is $SU(2)$-conjugate to $\exp\big(\nu(z)H\big)$, then $\mu(z)$ satisfies the relation

$$\begin{aligned}
\big(\sinh\nu(z)\big)^2 &= (\operatorname{Re} z)^2 + \frac{1}{4}|e^{\mu}(1+z^2) - e^{-\mu}|^2 \\
&= (\operatorname{Re} z)^2 + \left|\sinh\mu + e^{\mu}\frac{z^2}{2}\right|^2 \\
&= (\operatorname{Re} z)^2 + (\sinh\mu)^2 + e^{2\mu}\frac{|z|^4}{4} + 2\operatorname{Re}\Big(\sinh\mu e^{\mu}\frac{\overline{z}^2}{2}\Big) \\
&= (\operatorname{Re} z)^2 + (\sinh\mu)^2 + e^{2\mu}\frac{|z|^4}{4} + \frac{1}{2}(e^{2\mu}-1)\operatorname{Re}(\overline{z}^2) \\
&= (\operatorname{Re} z)^2 + (\sinh\mu)^2 + e^{2\mu}\frac{|z|^4}{4} + \frac{1}{2}(e^{2\mu}-1)\big((\operatorname{Re} z)^2 \\
&\qquad - (\operatorname{Im} z)^2\big) \\
&= (\sinh\mu)^2 + e^{2\mu}\Big(\frac{|z|^4}{4} + \frac{1}{2}\big((\operatorname{Re} z)^2 - (\operatorname{Im} z)^2\big)\Big) + (\operatorname{Re} z)^2 \\
&\qquad - \frac{1}{2}\big((\operatorname{Re} z)^2 - (\operatorname{Im} z)^2\big) \\
&= (\sinh\mu)^2 + e^{2\mu}\Big(\frac{|z|^4}{4} - \frac{1}{2}(\operatorname{Re} z)^2 + \frac{1}{2}(\operatorname{Im} z)^2\Big) + \frac{1}{2}|z|^2.
\end{aligned}$$

□

PROPOSITION 2.6. — *If $(\mathfrak{g}, \tau)$ is a reductive symmetric Lie algebra with $\mathfrak{g}' \cong \mathfrak{sl}(2,\mathbb{R})$, $X_\alpha \in \mathfrak{g}^\alpha$ with $[X_\alpha, \tau.X_\alpha] = -\check{\alpha}$, $X \in \mathfrak{a}$, and*

$$\gamma(z) = \exp(ze^{\alpha(X)}X_\alpha)\exp(X)\exp(ze^{\alpha(X)}X_\alpha^\sharp),$$

then there exists a $g \in G^c$ with $g\gamma(z)g^\sharp = \exp\big(X + \nu(z)[X_\alpha, \tau.X_\alpha]\big)$ and

$$\cosh\big(\alpha(X) - 2\nu(z)\big) = \cosh\alpha(X) + e^{\alpha(X)}\Big(\frac{|z|^4}{2} - (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2\Big) + |z|^2.$$

Proof. — As in the proof of Proposition 2.4, we may w.l.o.g. assume that $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$. Furthermore $[X_\alpha, \tau.X_\alpha] = -\check{\alpha}$ shows that $\mathfrak{h} \cong \mathfrak{so}(2)$, and so we may assume that

$$X_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \check{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now

$$\gamma(z) = \begin{pmatrix} 1 & ze^{\mu} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{\mu} & 0 \\ 0 & e^{-\mu} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ ze^{\mu} & 1 \end{pmatrix} = \begin{pmatrix} e^{\mu}(1+z^2) & z \\ z & e^{-\mu} \end{pmatrix}.$$

As in Proposition 2.4, the assertion now follows from the calculations in Example 2.5. □

3. Envelopes of holomorphy of invariant domains.

For the following section we slightly change the global notation from the preceding section in that we interchange the symmetric Lie algebra $(\mathfrak{g}, \tau)$ and its dual $(\mathfrak{g}^c, \tau)$. In these terms our setup is the following. The Lie algebra $\mathfrak{g}$ is assumed to contain an open invariant elliptic cone $\widehat{W}$ which, in addition, is $-\tau$-invariant. We may w.l.o.g. assume that $\widehat{W}$ is maximal and put $W := \widehat{W} \cap \mathfrak{q}$.

On the global level (G, τ) denotes a connected symmetric Lie group with symmetric Lie algebra $(\mathfrak{g}, \tau)$, and $H \subseteq G^\tau$ an open subgroup of the group of τ-fixed points. Invariance of a domain in $\mathfrak{q}$ or of a function on such a domain will always refer to the group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ acting on $\mathfrak{q}$. We recall from Corollary 1.18 that this leads to the same notion of invariance for each open subgroup $H \subseteq G^\tau$.

Now we explain the construction of the domains we are interested in. Let $D_{\mathfrak{q}} \subseteq iW$ be an invariant domain. Then we define

$$\Xi(D_{\mathfrak{q}}) := G \times_H D_{\mathfrak{q}},$$

which is the orbit space for the action of H on $G \times D_{\mathfrak{q}}$ given by $h.(g, X) := (gh^{-1}, \mathrm{Ad}(h).X)$. We write $[g, X] := H.(g, X)$ for the elements of $\Xi(D_{\mathfrak{q}})$. To endow this domain with a natural complex structure, we will use the complex Ol'shanskiĭ semigroup $S := \Gamma_G(\widehat{W})$. As already mentioned in the introduction, the antiinvolution $X \mapsto -\tau(X)$ integrates to a holomorphic involutive antiautomorphism $S \to S, s \mapsto s^\sharp$. We consider the domain $S^\sharp := \{s \in S : s^\sharp = s\}$ of $\sharp$-fixed points in S. The involution $\sharp$ extends to the closed Ol'shanskiĭ semigroup $\overline{S} := \Gamma_G(\overline{\widehat{W}})$ which acts on $S^\sharp$ by $s.x := sxs^\sharp$. Thus we obtain in particular an action of G by $g.x = gxg^\sharp = gx\tau(g)^{-1}$. It is clear that $\mathrm{Exp}(iW) \subseteq S^\sharp$ because W is the set of $-\tau$-fixed points in the larger cone $\widehat{W}$. We have seen in [KNÓ97], Th. IV.3, that the connected component $S_0^\sharp$ of $S^\sharp$ containing $\mathrm{Exp}(iW)$ coincides with $G.\mathrm{Exp}(iW)$. Motived by this equality we define for an invariant domain $D_{\mathfrak{q}} \subseteq iW$:

$$Q(D_{\mathfrak{q}}) := G.\mathrm{Exp}(2D_{\mathfrak{q}}) \subseteq S^\sharp.$$

The following theorem shows that for $H = G^\tau$ the domains $\Xi(D_{\mathfrak{q}})$ and $Q(D_{\mathfrak{q}})$ can be identified, and furthermore that $Q(D_{\mathfrak{q}})$ is a submanifold of the complex manifold $S^\sharp$.

For the following theorem we recall from Proposition 1.17 that extensions of invariant open domains $D_{\mathfrak{q}} \subseteq iW$ always exist.

THEOREM 3.1. — *If $D_{\mathfrak{q}} \subseteq iW$ is an invariant domain and $\widehat{D}_{\mathfrak{q}} \subseteq i\mathfrak{g}$ an invariant extension of $D_{\mathfrak{q}}$, then the following assertions hold:*

(i) *The set $Q(\widehat{D}_{\mathfrak{q}}) := \{ss^\sharp : s \in \Gamma_G(\widehat{D}_{\mathfrak{q}})\}$ coincides with the connected component of the set $\Gamma_G(2\widehat{D}_{\mathfrak{q}}) \cap S^\sharp$ containing $\mathrm{Exp}(2D_{\mathfrak{q}})$ which in turn coincides with $Q(D_{\mathfrak{q}})$.*

(ii) *The set $Q(D_{\mathfrak{q}})$ is an open submanifold of $S^\sharp_0$, and the mapping*

$$q_{D_{\mathfrak{q}}} : \Xi(D_{\mathfrak{q}}) \to Q(D_{\mathfrak{q}}), \quad [g, X] \mapsto g\exp(2X)g^\sharp$$

is a covering whose fiber is given by the group G^τ/H acting on $\Xi(D_{\mathfrak{q}})$ as a group of deck transformations by $gH.[z, X] = [zg^{-1}, \mathrm{Ad}(g).X]$. It follows in particular that for $H = G^\tau$ we obtain a diffeomorphism.

Proof. — (i) The chain of inclusions

(3.1)
$$Q(D_{\mathfrak{q}}) := \{g\exp(2X)g^\sharp : g \in G, X \in D_{\mathfrak{q}}\} \subseteq Q(\widehat{D}_{\mathfrak{q}}) \subseteq \{s \in \Gamma_G(2\widehat{D}_{\mathfrak{q}}) : s^\sharp = s\}_0$$

is clear. Thus we have to show that the right hand side of (3.1) is contained in the left hand side. Take an arbitary element from the right hand side and write $s = g\exp(X)$, where $g \in G$, $X \in 2\widehat{D}_{\mathfrak{q}}$. Then $s^\sharp = s$ entails that

$$g^\sharp \exp(\mathrm{Ad}(g^\sharp)^{-1}.X^\sharp) = \exp(X^\sharp)g^\sharp = s^\sharp = s = g\exp(X),$$

and so $g = g^\sharp$ by the uniqueness of the polar decomposition in S. Since the projection of the right hand side of (3.1) onto G obtained by the polar map is connected, we see that $g \in \{g_1 \in G : g_1^\sharp = g_1\}_0$. Therefore [Lo69], Prop. 4.4, p.182, implies the existence of an element $g_1 \in G$ such that $g = g_1 g_1^\sharp$, and thus

$$s = g_1 g_1^\sharp \exp(X) = g_1 \exp(\mathrm{Ad}(g_1^\sharp).X)g_1^\sharp = g_1\exp(Y)g_1^\sharp \tag{3.2}$$

holds with $Y := \mathrm{Ad}(g_1^\sharp).X \in 2\widehat{D}_{\mathfrak{q}}$. So $s^\sharp = s$ together with (3.2) yields

$$\exp(Y) = \exp(Y)^\sharp = \exp(-\tau(Y)),$$

whence $Y = -\tau(Y)$ and thus $Y \in \mathfrak{q} \cap 2\widehat{D}_{\mathfrak{q}} = 2D_{\mathfrak{q}}$. Now the assertion follows from (3.2).

(ii) This follows by restriction from [KNÓ97], Th. IV.3(ii), Prop. IV.4(iii). $\square$

Remark 3.2. — Let $q : S = \Gamma_G(\widehat{W}) \to G$ denote the projection defined by the polar decomposition, i.e., $q(g\,\mathrm{Exp}\,X) = g$ for $g \in G$ and

$X \in \widehat{W}$. If we consider the action of G on S given by $g.x := gxg^\sharp$, then the invariance of $\widehat{W}$ shows that the map q is G-equivariant. Further $q(S^\sharp) \subseteq G^\sharp$, and therefore $q(S_0^\sharp) \subseteq G_0^\sharp = G$. From that it is clear that q may be viewed as the projection map of a G-fiber bundle, and that the same holds for the restriction to the symmetric space $G_0^\sharp \cong G/G^\tau$. We see in particular that for each G-invariant domain $D \subseteq S_0^\sharp$ we have

$$D = G.\big(D \cap q^{-1}(\mathbf{1})\big) = G.\big(D \cap \operatorname{Exp}(iW)\big).$$

□

If $D = \Xi(D_{\mathfrak{q}})$ is a G-invariant domain, then we will show in this section that each holomorphic function on D extends to the domain $\Xi(\operatorname{conv} D_{\mathfrak{q}})$ (Theorem 3.4), and from that in particular derive that D is Stein if and only if $D_{\mathfrak{q}}$ is convex (Theorem 3.5).

The following theorem will turn out to be an important tool to reduce problems on the domains $Q(D_{\mathfrak{q}})$ to the group case dealt with in [Ne98]. It generalizes Lemma V.5 in [KNÓ97].

THEOREM 3.3. — *Let $\widehat{D}_{\mathfrak{q}} \subseteq \widehat{W}$ be an extension of the H-invariant domain $D_{\mathfrak{q}} \subseteq W$ and $\gamma : \Gamma_G(\widehat{D}_{\mathfrak{q}}) \to Q(D_{\mathfrak{q}})$ denote the holomorphic map given by $\gamma(s) = ss^\sharp$. Then $\gamma^* : f \mapsto f \circ \gamma$ induces a bijection*

$$\operatorname{Hol}\big(Q(D_{\mathfrak{q}})\big) \to \operatorname{Hol}\big(\Gamma_G(\widehat{D}_{\mathfrak{q}})\big)^H$$

whose inverse is given by the push forward mapping

$$\gamma_* : \operatorname{Hol}\big(\Gamma_G(\widehat{D}_{\mathfrak{q}})\big)^H \to \operatorname{Hol}\big(Q(D_{\mathfrak{q}})\big), \quad f \mapsto \big(\gamma(g \operatorname{Exp} X) \mapsto f(g \operatorname{Exp} X)\big)$$

for $g \in G$, $X \in D_{\mathfrak{q}}$.

Proof. — First we note that the proof of [KNÓ97], Lemma V.5, shows that for each $f \in \operatorname{Hol}\big(\Gamma_G(\widehat{D}_{\mathfrak{q}})\big)^H$ the function $\gamma_* f$ on $Q(D_{\mathfrak{q}})$ is well defined (because of the H-right-invariance), and furthermore that $\gamma_* f$ is a holomorphic function.

Now $\gamma^*\gamma_* f$ and f are two holomorphic functions on $\Gamma_G(\widehat{D}_{\mathfrak{q}})$ that coincide on $G \cdot \operatorname{Exp}(D_{\mathfrak{q}})$, and the proof of [KNÓ97], Lemma V.5, shows that both functions coincide. This proves that γ^* is surjective, hence bijective with inverse γ_*, and this completes the proof. □

THEOREM 3.4. — *Let $D = \Xi(D_{\mathfrak{q}})$ be an invariant domain. Then each holomorphic function on D extends to the domain $\widehat{D} := \Xi(\operatorname{conv} D_{\mathfrak{q}})$.*

Proof. — First we assume that $G^\tau = H$, so that $\Xi(D_{\mathfrak q}) \cong Q(D_{\mathfrak q})$ by Theorem 3.1. Let $\widehat{D}_{\mathfrak q} \subseteq \widehat{W}$ be an extension of the domain $D_{\mathfrak q}$ and $f \in \mathrm{Hol}\big(\Xi(D_{\mathfrak q})\big)$. We consider the holomorphic map $\gamma : \Gamma_G(\widehat{D}_{\mathfrak q}) \to Q(D_{\mathfrak q}), s \mapsto ss^\sharp$, and write $f = \gamma_*(f \circ \gamma)$. Using [Ne98], Th. 7.9, we find a holomorphic extension h of $f \circ \gamma$ to the larger G-biinvariant domain $\Gamma_G(\operatorname{conv} \widehat{D}_{\mathfrak q})$. But this function is right-H-invariant and therefore $\gamma_* h$ defines a holomorphic function on the domain $Q\big(\operatorname{conv}(\widehat{D}_{\mathfrak q}) \cap \mathfrak q\big) \supseteq Q(\operatorname{conv} D_{\mathfrak q})$ which, in view of the construction of $\gamma_* h$, extends the function f.

Now we turn to the general case, where H might be different from G^τ. Let $p_G : \tilde G \to G$ denote the universal covering morphism of the group G and note that τ lifts to an involution $\tilde\tau$ on $\tilde G$. Furthermore the subgroup $\tilde H := \tilde G^{\tilde\tau}$ of $\tilde\tau$-fixed points is connected ([Lo69], Th. 3.4), hence coincides with the analytic subgroup of $\tilde G$ corresponding to the Lie subalgebra $\mathfrak h$ of $\mathfrak g$. According to [KNÓ97], Prop. IV.4, the domain $\tilde\Xi(W) := \tilde G \times_{\tilde H} W$ is simply connected because the symmetric space $\tilde G/\tilde H$ is simply connected. Now the map $\tilde\Xi(W) \to \Xi(W), [g, X] \mapsto [p_G(g), X]$ is a covering whose fiber is the discrete group $B := p_G^{-1}(H)/\tilde H$. It is clear that this covering map restricts to a covering $p_\Xi : \tilde\Xi(D_{\mathfrak q}) \to \Xi(D_{\mathfrak q})$. Now let $f \in \mathrm{Hol}\big(\Xi(D_{\mathfrak q})\big)$. Then the first part of the proof shows that the function $p_\Xi^* f = f \circ p_\Xi$ extends to the larger domain $\tilde\Xi(\operatorname{conv} D_{\mathfrak q})$. Further the B-invariance of $p_\Xi^* f$ and the uniqueness of extension imply that the extension is B-invariant, hence factors to a holomorphic extension of f to the domain $\Xi(\operatorname{conv} D_{\mathfrak q})$. This completes the proof of the general case. □

THEOREM 3.5. — *An invariant domain $D = \Xi(D_{\mathfrak q}) \subseteq \Xi(W)$ is Stein if and only if $D_{\mathfrak q}$ is convex.*

Proof. — If $D_{\mathfrak q}$ is not convex, then Theorem 3.4 shows that each holomorphic function on $\Xi(D_{\mathfrak q})$ extends to some bigger domain, and therefore D cannot be Stein.

If, conversely, $D_{\mathfrak q}$ is convex, and $\widehat{D}_{\mathfrak q} \subseteq \widehat{W}$ is a convex extension, then $Q(D_{\mathfrak q})$ coincides with the connected component of $\Gamma_G(2\widehat{D}_{\mathfrak q}) \cap S^\sharp$ containing $\operatorname{Exp}(2D_{\mathfrak q})$ (Theorem 3.1(i)). According to [Ne98], Th. 6.1, the complex manifold $\Gamma_G(\widehat{D}_{\mathfrak q})$ is Stein. As a connected component of the closed subset $\Gamma_G(2\widehat{D}_{\mathfrak q}) \cap S^\sharp$, the domain $Q(D_{\mathfrak q}) \subseteq \Gamma_G(2\widehat{D}_{\mathfrak q})$ is a closed complex submanifold of the Stein manifold $\Gamma_G(2\widehat{D}_{\mathfrak q})$. Hence $Q(D_{\mathfrak q})$ is a Stein manifold ([Hö73], Th. 5.1.5), and since $\Xi(D_{\mathfrak q})$ is a covering of $Q(D_{\mathfrak q})$ (Theorem 3.1(ii)), it is also a Stein manifold (*cf.* [MaMo60], Th. 4). Using the fact that this covering is finite (*cf.* [Lo69], Th. IV.3.4), this also follows more directly from the observation that the pullback of a strictly plurisubharmonic exhaustion function on $Q(D_{\mathfrak{q}})$ yields an exhaustion function on $\Xi(D_{\mathfrak{q}})$, so that [Hö73], Th. 5.2.10, applies. $\square$

4. Invariant plurisubharmonic functions.

We keep the setup of Section 3, i.e., $(\mathfrak{g}, \tau)$ is a symmetric Lie algebra, $\widehat{W} \subseteq \mathfrak{g}$ is a maximal open elliptic $-\tau$-invariant cone, and $W = \widehat{W} \cap \mathfrak{q}$. We consider domains of the type $\Xi(D_{\mathfrak{q}})$, where $D_{\mathfrak{q}} \subseteq W$ is an H-invariant domain.

Then the G-invariant functions on $\Xi(D_{\mathfrak{q}})$ are in natural one-to-one correspondence with H-invariant functions on $D_{\mathfrak{q}}$ and therefore with $\mathcal{W}$-invariant functions on $\mathfrak{a} \cap D_{\mathfrak{q}}$ (cf. Proposition 1.5). We will show that a G-invariant C^2-function ϕ on a G-invariant domain $D = \Xi(D_{\mathfrak{q}})$ is plurisubharmonic if and only if the corresponding H-invariant function ψ on $D_{\mathfrak{q}}$ is locally convex.

To see that the plurisubharmonicity of ϕ implies the local convexity of ψ, we will use the explicit calculations from Section 2 and then the results of Section 1 to pass from properties of $\psi_{\mathfrak{a}}$ on $\mathfrak{a} \cap D_{\mathfrak{q}}$ to properties of the function ψ on $D_{\mathfrak{q}}$. For the converse we can directly use the corresponding results in [Ne98] for the group case. To do this, we will first shrink $D_{\mathfrak{q}}$ such that $\mathfrak{a}^+ \cap D_{\mathfrak{q}}$ is convex, and then extend the function ψ to an invariant convex function on the domain $\operatorname{conv}(D_{\mathfrak{q}})$ (cf. Theorem 1.10). This provides a reduction to the case where $D_{\mathfrak{q}}$ is convex. Now we extend the H-invariant convex function ψ on $D_{\mathfrak{q}}$ to a convex G-invariant function $\hat{\psi}$ on $\widehat{D}_{\mathfrak{q}}$ (Proposition 1.19). Next we use [Ne98] to see that the corresponding function $\hat{\phi}$ on $\Gamma_G(\widehat{D}_{\mathfrak{q}})$ is plurisubharmonic and by restriction we then derive that ϕ is plurisubharmonic. We now turn to the details of the proof.

LEMMA 4.1. — *Let $h :]-\varepsilon, \varepsilon[\to \mathbb{R}$ be a C^2-function, $U \subseteq \mathbb{R}^2$ a 0-neighborhood, and $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ a smooth function with $f(0) = 0$, $df(0,0) = 0$, and $f(U) \subseteq]-\varepsilon, \varepsilon[$. Suppose that the function $h \circ f$ is subharmonic. Then $h'(0)(\Delta f)(0,0) \geq 0$ and strict positivity holds if $h \circ f$ is strictly plurisubharmonic in $(0,0)$.*

Proof. — Since h is a C^2-function, we have

$$\partial_x(h \circ f) = (h' \circ f)\partial_x f \quad \text{and} \quad \partial_x^2(h \circ f) = (h'' \circ f)(\partial_x f)^2 + (h' \circ f)(\partial_x^2 f)$$

and therefore

$$\Delta(h \circ f) = (h'' \circ f)\big((\partial_x f)^2 + (\partial_y f)^2\big) + (h' \circ f)\Delta f.$$

Now the assertion follows from $\Delta(h \circ f)(0,0) = h'(0)\Delta f(0,0)$. □

PROPOSITION 4.2. — *Let $D_{\mathfrak{q}} \subseteq W$ be an H-invariant domain, ϕ a G-invariant C^2-function on $\Xi(D_{\mathfrak{q}})$, and $\psi : D_{\mathfrak{q}} \to \mathbb{R}, X \mapsto \phi([\mathbf{1}, X])$. If ϕ is plurisubharmonic, then ψ is a locally convex function.*

Proof. — First we reduce the problem to the case where G is simply connected. We use the same setup as in the proof of Theorem 3.4. Let $p_G : \tilde{G} \to G$ denote the universal covering, group of G, $\tilde{H} := \tilde{G}^{\tilde{\tau}}$, and $\tilde{\Xi}(D_{\mathfrak{q}}) := \tilde{G} \times_{\tilde{H}} D_{\mathfrak{q}}$. Then we have a covering map $\tilde{\Xi}(D_{\mathfrak{q}}) \to \Xi(D_{\mathfrak{q}}), [g, X] \mapsto [p_G(g), X]$ whose fiber is the discrete group $B := p_G^{-1}(H)/\tilde{H}$. If ϕ is a G-invariant plurisubharmonic function on $\Xi(D_{\mathfrak{q}})$, then $\phi \circ p_\Xi$ is a $\tilde{G}$-invariant plurisubharmonic function on $\tilde{\Xi}(D_{\mathfrak{q}})$, and since every $\tilde{G}$-invariant function on $\tilde{\Xi}(D_{\mathfrak{q}})$ is constant on the fibers of the covering map, it is the pullback of a function on $\Xi(D_{\mathfrak{q}})$. Since both functions ϕ and $\phi \circ p_\Xi$ correspond to the same function ψ on $D_{\mathfrak{q}}$, and ϕ is plurisubharmonic if and only if this holds for $\phi \circ p_\Xi$, we may w.l.o.g. assume that G is simply connected and therefore that $H = G^\tau$.

Since the assertion of the proposition is a local one, we may shrink $D_{\mathfrak{q}}$ and therefore w.l.o.g. assume that $\mathfrak{a}^+ \cap D_{\mathfrak{q}}$ is convex. Let $\psi_{\mathfrak{a}} := \psi|_{\mathfrak{a} \cap D_{\mathfrak{q}}}$. In view of Theorem 1.10, we have to show that $\psi_{\mathfrak{a}}$ is a locally convex function with $d\psi_{\mathfrak{a}}(X)(C_X) \subseteq \mathbb{R}^-$ for all $X \in \mathfrak{a} \cap D_{\mathfrak{q}}$.

Since the mapping $\operatorname{Exp} : i\mathfrak{a} + (\mathfrak{a} \cap D_{\mathfrak{q}}) \to \Xi(D_{\mathfrak{q}})$ is holomorphic, the function ψ is an $i\mathfrak{a}$-invariant plurisubharmonic function on the tube domain $i\mathfrak{a} + (\mathfrak{a} \cap D_{\mathfrak{q}})$, hence is locally convex (*cf.* [AL92], p.369).

Let $X_0 \in \mathfrak{a} \cap D_{\mathfrak{q}}$, $\alpha \in \Delta$ and $0 \neq X_\alpha \in \mathfrak{g}^\alpha$. We have to show that

$$\alpha(X_0) d\psi_{\mathfrak{a}}(X_0)([X_\alpha, \tau.X_\alpha]) \leq 0 \tag{4.1}$$

(*cf.* Section 1). If $\alpha(X_0) = 0$, then this relation holds trivially, and if $\alpha(X_0) < 0$, then $D_{\mathfrak{q}} \subseteq W$ implies that $\alpha \in \Delta_k^+$. Let $s_\alpha : \mathfrak{a} \to \mathfrak{a}$ denote the corresponding reflection given by $s_\alpha(X) = X - \alpha(X)\check{\alpha}$. Then the invariance of ψ under H implies the invariance of $\psi_{\mathfrak{a}}$ under $\mathcal{W}$, hence that $d\psi_{\mathfrak{a}}(s_\alpha.X_0) \circ s_\alpha = d\psi_{\mathfrak{a}}(X_0)$. Therefore

$$\begin{aligned}
\alpha(X_0) d\psi_{\mathfrak{a}}(X_0)([X_\alpha, \tau.X_\alpha]) &= \alpha(X_0) d\psi_{\mathfrak{a}}(s_\alpha.X_0)(s_\alpha.[X_\alpha, \tau.X_\alpha]) \\
&= -\alpha(X_0) d\psi_{\mathfrak{a}}(s_\alpha.X_0)([X_\alpha, \tau.X_\alpha]) \\
&= \alpha(s_\alpha.X_0) d\psi_{\mathfrak{a}}(s_\alpha.X_0)([X_\alpha, \tau.X_\alpha]).
\end{aligned}$$

Since $\alpha(s_\alpha.X_0) > 0$, it therefore suffices to prove (4.1) under the assumption that $\alpha(X_0) > 0$.

We consider the subalgebra $\mathfrak{g}_1^c := \mathfrak{a} + \operatorname{span}\{X_\alpha \pm \tau.X_\alpha\}$ and note that it is one of the three types considered in Propositions 2.2/4/6. For $\alpha([X_\alpha, \tau.X_\alpha]) > 0$ it is of the non-compact reductive type, for $\alpha([X_\alpha, \tau.X_\alpha]) < 0$ it is compact, and otherwise it is solvable (*cf.* [KN96], Th. IV.1). Putting $W_1 := W \cap \mathfrak{q}_1$ and writing G_1 for the universal covering group of $\langle \exp \mathfrak{g}_1 \rangle \subseteq G$, we now obtain a holomorphic G_1-equivariant map $\Xi(W_1^0) \to \Xi(W)$ which is induced by the injection $\mathfrak{g}_1 \to \mathfrak{g}$ of symmetric Lie algebras.

Let $B_\varepsilon := \{z \in \mathbb{C} : |z| < \varepsilon\}$ and $\gamma : B_\varepsilon \to \Xi(W)$ a holomorphic map with $\gamma(0) = \exp X_0$ which factors over the mapping $\Xi(W_1) \to \Xi(W)$ to a map $\gamma_1 : B_\varepsilon \to \Xi(W_1)$ of the type considered in Section II. This means in particular that there exists for each $z \in B_\varepsilon$ an element $g_z \in G^c$ with $g_z.\gamma(z) \in \operatorname{Exp}(D_\mathfrak{q} \cap \mathfrak{a})$ (cf. [KN96, Th. III.3]). Then the function

$$B_\varepsilon \to \mathbb{R}, \quad z \mapsto \phi\big(\gamma(z)\big) = \phi\big(g_z.\gamma(z)\big) = \psi_\mathfrak{a}\big(\log(g_z.\gamma(z))\big)$$

is subharmonic.

To prove the required assertions on the function $\psi_\mathfrak{a}$, we now have to take a closer look at the different cases. In all cases we may write

$$\log\big(g_z.\gamma(z)\big) = X_0 + \nu(z)[X_\alpha, \tau.X_\alpha],$$

where in the solvable case

$$\nu(z) = -\frac{2(\operatorname{Re} z)^2}{e^{\alpha(X_0)} - 1} + \frac{2(\operatorname{Im} z)^2}{e^{\alpha(X_0)} + 1},$$

in the non-Riemannian reductive case

$$2\nu(z) = -\alpha(X_0) + \cosh^{-1}\Big(\cosh\big(\alpha(X_0)\big) + e^{\alpha(X_0)}\Big(\frac{|z|^4}{2} - (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2\Big) - |z|^2\Big),$$

where $\cosh^{-1} : [1, \infty[\to \mathbb{R}^+$ is the positive branch of the inverse function of cosh. In the Riemannian reductive case we have

$$2\nu(z) = \alpha(X_0) - \cosh^{-1}\Big(\cosh\big(\alpha(X_0)\big) + e^{\alpha(X_0)}\Big(\frac{|z|^4}{2} - (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2\Big) + |z|^2\Big).$$

From these formulas we see that in all cases we may write with $z = x + iy$:

$$\psi_\mathfrak{a}\big(\log(g_z.\gamma(z))\big) = (h \circ f)(x, y) \text{ with } f(x, y) = ax^2 + by^2 + cx^4 + dy^4 + ex^2y^2.$$

Then the assumption that ψ is a C^2-function implies that h is a C^2-function, and, since $f(0,0)=0$ and $df(0,0)=0$, Lemma 4.1 implies that

$$0 \leq h'(0)\Delta f(0,0) = 2h'(0)(a+b). \tag{4.2}$$

The function h is given by the following formulas: In the solvable case

$$h(t) = \psi_{\mathfrak{a}}(X_0 + t[X_\alpha, \tau.X_\alpha]) \quad \text{with} \quad h'(0) = d\psi_{\mathfrak{a}}(X_0)([X_\alpha, \tau.X_\alpha]),$$

in the non-Riemannian reductive case

$$h(t) = \psi_{\mathfrak{a}}\Big(X_0 - \Big(\frac{\alpha(X_0)}{2} - \frac{1}{2}\cosh^{-1}\big(\cosh\big(\alpha(X_0)\big) + t\big)\Big)[X_\alpha, \tau.X_\alpha]\Big),$$

with

$$\begin{aligned} h'(0) &= \frac{1}{2} d\psi_a(X_0)([X_\alpha, \tau.X_\alpha])(\cosh^{-1})'\Big(\cosh\big(\alpha(X_0)\big)\Big) \\ &= \frac{1}{2}\frac{d\psi_a(X_0)([X_\alpha, \tau.X_\alpha])}{\cosh'\big(\alpha(X_0)\big)} = \frac{1}{2}\frac{d\psi_a(X_0)([X_\alpha, \tau.X_\alpha])}{\sinh\big(\alpha(X_0)\big)}, \end{aligned}$$

and in the Riemannian reductive case:

$$h(t) = \psi_{\mathfrak{a}}\Big(X_0 + \Big(\frac{\alpha(X_0)}{2} - \frac{1}{2}\cosh^{-1}\big(\cosh\big(\alpha(X_0)\big) + t\big)\Big)[X_\alpha, \tau.X_\alpha]\Big),$$

with

$$h'(0) = -\frac{1}{2}\frac{d\psi_a(X_0)([X_\alpha, \tau.X_\alpha])}{\sinh\big(\alpha(X_0)\big)}.$$

The coefficients a and b are given by

$$a = -\frac{2}{e^{\alpha(X_0)-1}} \quad \text{and} \quad b = \frac{2}{e^{\alpha(X_0)}+1}$$

in the solvable case, so that

$$\frac{a+b}{2} = \frac{e^{\alpha(X_0)} - 1 - (e^{\alpha(X_0)}+1)}{e^{2\alpha(X_0)}-1} = \frac{-2}{e^{2\alpha(X_0)}-1} < 0.$$

In the non-Riemannian reductive case:

$$a = -\frac{e^{\alpha(X_0)}+1}{2} \quad \text{and} \quad b = \frac{e^{\alpha(X_0)}-1}{2}$$

with $a+b=-1<0$. In the Riemannian reductive case:

$$a = \frac{1-e^{\alpha(X_0)}}{2} \quad \text{and} \quad b = \frac{e^{\alpha(X_0)}+1}{2}$$

with $a+b=1>0$.

Considering all three cases, we see that in each case (4.2) implies that

$$d\psi(X)([X_\alpha, \tau.X_\alpha]) \leq 0,$$

and this completes the proof. $\square$

Problem 4. — Is it possible to drop the differentiability assumption in Proposition 4.2 resp. Lemma 4.1? This means that one has to prove the following assertion: Let $h :]-\varepsilon, \varepsilon[\to \mathbb{R}$ be a continuous function whose one-sided derivatives exist in each point, and f as in Lemma 4.1. Suppose that the function $h \circ f$ is subharmonic. Does this imply that

$$h'(0+)\Delta f(0,0) \geq 0 \ ?$$

According to Lemma 4.1, the assertion holds if h is a C^2-function. $\square$

THEOREM 4.3. — *Let $D_\mathfrak{q} \subseteq W$ be an H-invariant domain, ϕ a G-invariant C^2-function on $\Xi(D_\mathfrak{q})$, and $\psi : D_\mathfrak{q} \to \mathbb{R}, X \mapsto \phi([\mathbf{1}, X])$. Then ϕ is plurisubharmonic if and only if ψ is locally convex.*

Proof. — In view of Proposition 4.2, it remains to show that if ψ is locally convex, then ϕ is plurisubharmonic. By the same argument as in the proof of Proposition 4.2, we may w.l.o.g. assume that $H = G^\tau$. Since the assertion is a local one, we may w.l.o.g. assume that $\mathfrak{a} \cap D_\mathfrak{q}$ is a convex set. Then we use Theorem 1.10 to extend ψ to a convex function on a convex invariant subdomain of W, and therefore may assume that $D_\mathfrak{q}$ is convex. Next we use Proposition 1.19 to find a G-invariant convex extension $\widehat{\psi}$ of ψ to a convex invariant extension $\widehat{D}_\mathfrak{q}$ of the domain $D_\mathfrak{q}$. We define the function $\widehat{\phi} : \Gamma_G(2\widehat{D}_\mathfrak{q}) \to \mathbb{R}$ by

$$\widehat{\phi}(g \operatorname{Exp} X) := \widehat{\psi}\Big(\frac{1}{2}X\Big)$$

for $g \in G$, $X \in \widehat{D}_\mathfrak{q}$. Since the function $\widehat{\psi}$ is convex, [Ne98], Th. 4.10, implies that the G-biinvariant function $\widehat{\phi}$ is plurisubharmonic.

In view of $H = G^\tau$, we have a natural holomorphic embedding

$$\Xi(D_\mathfrak{q}) \to Q(D_\mathfrak{q}) \subseteq \Gamma_G(2\widehat{D}_\mathfrak{q}), \quad [g, X] \mapsto g \operatorname{Exp}(2X) g^\sharp$$

so that the function

$$\Xi(D_\mathfrak{q}) \to \mathbb{R}, \quad [g, X] \mapsto \widehat{\phi}(g \operatorname{Exp} 2X g^\sharp) = \widehat{\phi}(\operatorname{Exp} 2X) = \psi(X) = \phi([g, X])$$

is plurisubharmonic. This completes the proof. $\square$

Strictly plurisubharmonic functions.

THEOREM 4.4. — *Let $D_\mathfrak{q} \subseteq W$ be an H-invariant domain, ϕ a G-invariant C^2-function on $\Xi(D_\mathfrak{q})$, and $\psi : D_\mathfrak{q} \to \mathbb{R}, X \mapsto \phi(\operatorname{Exp} X)$. Then ϕ is strictly plurisubharmonic if and only if ψ is stably locally convex.*

Proof. — Again we use the same argument as in the proof of Proposition 4.2 to see that we may w.l.o.g. assume that $H = G^\tau$.

Suppose first that ϕ is strictly plurisubharmonic. Then the same arguments as in the proof of Proposition 4.2 show that $\psi_\mathfrak{a}$ is a stably locally convex function on $\mathfrak{a} \cap D_\mathfrak{q}$, and that for $0 \neq X_\alpha \in \mathfrak{g}^\alpha$ and $X \in \mathfrak{a} \cap D_\mathfrak{q}$ with $\alpha(X) \neq 0$ we have

$$\alpha(X) d\psi_\mathfrak{a}(X)([X_\alpha, \tau.X_\alpha]) > 0. \tag{4.3}$$

This implies in particular that $\mathfrak{g}$ has cone potential and that $C_{\min}$ is pointed with $d\psi_\mathfrak{a}(X) \in \operatorname{int} C_X^\star$ for each $X \in \mathfrak{a} \cap D_\mathfrak{q}$. Therefore Theorem 1.13 implies that ψ is a stably locally convex function on $D_\mathfrak{q}$.

Conversely, suppose that ψ is stably convex, i.e., $\psi_\mathfrak{a}$ is stably locally convex, $d\psi_\mathfrak{a}(X) \in -\operatorname{int}(C_X)^\star$ for all $X \in \mathfrak{a} \cap iD_\mathfrak{q}$, and $\mathfrak{g}$ is admissible (Theorem 1.13).

Let $X \in \mathfrak{a} \cap D_\mathfrak{q}$ and U be a relatively compact open convex set containing X which is invariant under the stabilizer $\mathcal{W}^X$ of X and satisfies $\overline{U} \subseteq \mathfrak{a} \cap D_\mathfrak{q}$. Further we require that $\alpha(Y) \neq 0$ for $Y \in U$ whenever $\alpha(X) \neq 0$. We embedd $\Xi(D_\mathfrak{q}) \hookrightarrow \Gamma_G(\widehat{W})$ and then use [Ne98], Lemma 4.12, to find a strictly plurisubharmonic G-biinvariant function on $\Gamma_G(\widehat{W})$ which restricts to a G-invariant strictly plurisubharmonic function ϕ^1 on $\Xi(D_\mathfrak{q})$. In view of the compactness of $\overline{U}$, there exists an $\varepsilon > 0$ such that $\psi_\mathfrak{a}^2 := \psi_\mathfrak{a} - \varepsilon\psi_\mathfrak{a}^1$ is convex and satisfies $d\psi_\mathfrak{a}^2(Y) \in -\operatorname{int}(C_X)^\star$ for all $Y \in U$. In view of Remark 1.14, this also implies that $d\psi_\mathfrak{a}^2(Y) \in -C_Y^\star$ for all $Y \in \mathcal{W}.U$. Now Theorem 1.10 shows that the function ψ^2 is locally convex on $\operatorname{Ad}(H).U$. Using Theorem 4.3, we conclude that the function ϕ^2 is plurisubharmonic on the G-invariant domain $G.\operatorname{Exp}(U) \subseteq \Xi(W)$. Now $\phi = \phi^2 + \varepsilon\phi^1$ and the fact that ϕ^1 is strictly plurisubharmonic imply that ϕ is strictly plurisubharmonic on $G.\operatorname{Exp} U$. Since X was arbitrary in $\mathfrak{a} \cap D_\mathfrak{q}$, the function ϕ is strictly plurisubharmonic on $\Xi(D_\mathfrak{q})$. $\square$

5. Representation on invariant Hilbert spaces.

Let $D = \Xi(D_{\mathfrak{q}}) \subseteq \Xi(W)$ be a G-invariant domain. We call a Hilbert subspace $\mathcal{H}$ of the space $\mathrm{Hol}(D)$ of holomorphic functions on D an *invariant Hilbert space* if the inclusion $\mathcal{H} \hookrightarrow \mathrm{Hol}(D)$ is continuous with respect to the natural Fréchet space structure on $\mathrm{Hol}(D)$ (the topology of uniform convergence on compact subsets), and G acts unitarily on $\mathcal{H}$ by

$$(g.f)(z) = f(g^{\sharp}.z).$$

It easily follows from the Closed Graph Theorem that the continuity of the point evaluations

$$\mathcal{H} \to \mathbb{C}, \quad f \mapsto f(z)$$

for all $z \in D$ is equivalent to the continuity of the embedding $\mathcal{H} \to \mathrm{Hol}(D)$. This means that $\mathcal{H}$ has a reproducing kernel which is given by $f(z) = \langle f, K_z \rangle$ for all $f \in \mathcal{H}$ and $K(z,w) = K_w(z)$. The invariance of $\mathcal{H}$ means that

$$K(g.z, w) = K(z, g^{-1}w)$$

for $z, w \in D$ and $g \in G$ (*cf.* [Ne99], Rem. II.4.5).

We recall that an invariant domain $D = \Xi(D_{\mathfrak{q}})$ is Stein if and only if $D_{\mathfrak{q}}$ is convex (Theorem 3.5), and furthermore, that the envelope of holomorphy of an invariant domain $D = \Xi(D_{\mathfrak{q}})$ is given by $\widehat{D} = \Xi(\operatorname{conv} D_{\mathfrak{q}})$, i.e., that each holomorphic function on D extends to the smallest Stein domain containing D (Theorem 3.4).

LEMMA 5.1. — *Let $\widehat{D} := \Xi(\operatorname{conv} D_{\mathfrak{q}})$ denote the envelope of holomorphy of D. Then each function in $\mathcal{H}$ extends holomorphically to $\widehat{D}$ and we thus obtain a realization of $\mathcal{H}$ as a biinvariant Hilbert subspace of* $\mathrm{Hol}(\widehat{D})$.

Proof. — This is the same argument as in [Ne97], Lemma III.5. □

According to the preceding lemma, it suffices to consider Hilbert spaces which live on domains of holomorphy in $\Xi(W)$, i.e., we may assume that $D_{\mathfrak{q}}$ is a convex subset of W.

LEMMA 5.2. — *If $\mathcal{H}_K \subseteq \mathrm{Hol}(D)$ is a non-zero invariant Hilbert space and $K : D \times D \to \mathbb{C}$ its reproducing kernel, then*

$$\phi : D \to \mathbb{R}, \quad z \mapsto \log K(z,z)$$

is a G-invariant smooth plurisubharmonic function.

Proof. — In view of [Ne98], Prop. 4.6, it remains to show that $K(z,z) \neq 0$ for all $z \in D$. Suppose that $K(z_0, z_0) = 0$, i.e., $K_{z_0} = 0$. This means that all functions in $\mathcal{H}_K$ vanish in z_0. Therefore the invariance implies that they also vanish on $G.z_0$.

Let $U \subseteq \mathfrak{g}_\mathbb{C}$ be a convex neighborhood of X for which there exists a holomorphic map $\gamma : U \to D$ extending the map $U \cap \mathfrak{g} \to D, X \mapsto \exp(X).z_0$. Then for each $f \in \mathrm{Hol}(D)$ the function $f \circ \gamma$ vanishes on $U \cap \mathfrak{g}$, hence on U, i.e., f vanishes on the open subset $\gamma(U)$ of D, and thus $f = 0$. Therefore $\mathcal{H}_K = \{0\}$, contradicting our assumption. □

LEMMA 5.3. — *Let $D_\mathfrak{q} \subseteq W$ be an H-invariant open subset and $D := Q(D_\mathfrak{q}) \subseteq \Gamma_G(\widehat{W})$. Further let $\phi : D \to \mathbb{R}$ be a G-invariant function and define $\psi : D_\mathfrak{q} \to \mathbb{R}$ by $\psi(X) := \phi(\exp X)$. Then the following assertions hold:*

(i) $D^* = D$.

(ii) $\phi(s^*) = \phi(s)$ *for all $s \in D$.*

Proof. — (i) For $s = g.\operatorname{Exp} X = g \operatorname{Exp} X g^\sharp \in D$ we have

$$s^* = g^{-\sharp} \operatorname{Exp} X g^{-1} = g^{-\sharp}.\operatorname{Exp} X \in D.$$

(ii) We have just seen that $D^* = D$, so that (ii) makes sense. For $s = g.\operatorname{Exp} X \in D$ with $X \in D_\mathfrak{q}$ we have

$$\phi(s^*) = \phi(g^{-\sharp}.\operatorname{Exp} X) = \phi(\operatorname{Exp} X) = \phi(s).$$

□

Invariant functions and semigroup actions.

Let $D_\mathfrak{q} \subseteq W$ be a convex invariant domain and $\widehat{D}_\mathfrak{q} \subseteq \mathfrak{g}$ an invariant extension which, in addition, is invariant under $-\tau$.

LEMMA 5.4. — *The semigroup $S_{D_\mathfrak{q}} := \Gamma_G(\lim D_\mathfrak{q})$ acts on $Q(D_\mathfrak{q})$ by $s.x := sxs^\sharp$ and this action lifts to an action on $\Xi(D_\mathfrak{q})$.*

Proof. — We first use [Ne97], Th. II.5, to see that

$$S_{D_\mathfrak{q}} \Gamma_G(\widehat{D}_\mathfrak{q}) S_{D_\mathfrak{q}} \subseteq \Gamma_G(\widehat{D}_\mathfrak{q}).$$

For $x \in \Gamma_G(\widehat{D}_{\mathfrak{q}})$ and $s \in S_{D_{\mathfrak{q}}}$ we therefore have $sx \in \Gamma_G(\widehat{D}_{\mathfrak{q}})$ and hence $sx(sx)^\sharp = sxx^\sharp s^\sharp \in Q(\widehat{D}_{\mathfrak{q}}) = Q(D_{\mathfrak{q}})$ (Theorem 3.1). This proves the assertion about the action of $S_{D_{\mathfrak{q}}}$ on the domain $Q(D_{\mathfrak{q}})$.

Now let $q_{D_{\mathfrak{q}}} : \Xi(D_{\mathfrak{q}}) \to Q(D_{\mathfrak{q}})$ denote the covering map discussed in Theorem 3.1 and note that it extends to a covering map $q_W : \Xi(W) \to Q(W)$. According to [KNÓ97], Prop. IV.4, the action of $\Gamma_G(\widehat{W})$ on $Q(W)$ lifts to an action on $\Xi(W)$. Now the first part of the proof asserts that the subsemigroup $\Gamma_G(\lim \widehat{D}_{\mathfrak{q}})$ preserves the domain $Q(D_{\mathfrak{q}})$, hence that the lifted action preserves its inverse image $\Xi(D_{\mathfrak{q}})$. □

If $(S, *)$ is an *involutive semigroup*, i.e., a semigroup S endowed with an involutive antiautomorphism $s \mapsto s^*$, then a function $\alpha : S \to \mathbb{R}^+$ is called an *absolute value* if

$$\alpha(s) = \alpha(s^*) \quad \text{and} \quad \alpha(st) \leq \alpha(s)\alpha(t)$$

holds for all $s, t \in S$.

LEMMA 5.5. — *Let $\phi : \Xi(D_{\mathfrak{q}}) \to \mathbb{R}$ be a G-invariant function for which the function $\psi : D_{\mathfrak{q}} \to \mathbb{R}, X \mapsto \phi([\mathbf{1}, X])$ is convex. Further let $\widehat{\psi} : i\widehat{D}_{\mathfrak{q}} \to \mathbb{R}$ be a $G \cup \{-\tau\}$-invariant extension, and put*

$$\alpha(g \operatorname{Exp} X) := e^{\sup \langle d\widehat{\psi}(i\widehat{D}_{\mathfrak{q}}), X\rangle}$$

for $g \operatorname{Exp} X \in S_{D_{\mathfrak{q}}}$. Then α is an absolute value on $S_{D_{\mathfrak{q}}}$, and

$$\phi(s.x) \leq \phi(x) + \log \alpha(s)$$

holds for $x \in Q(D_{\mathfrak{q}})$ and $s \in S_{D_{\mathfrak{q}}}$.

Proof. — Let $\widehat{\phi} : \Gamma_G(\widehat{D}_{\mathfrak{q}}) \to \mathbb{R}$ be the G-biinvariant extension of ϕ defined by $\widehat{\phi}(g \operatorname{Exp} X) := \widehat{\psi}(\frac{1}{2}X)$ for $g \in G$, $X \in i\widehat{D}_{\mathfrak{q}}$.

Then [Ne97], Cor. II.8, implies that α is an absolute value on the semigroup $S_{D_{\mathfrak{q}}}$ satisfying

$$\widehat{\phi}(s_1 x s_2) \leq \widehat{\phi}(x) + \frac{1}{2}\log \alpha(s_1) + \frac{1}{2}\log \alpha(s_2)$$

for $x \in \Gamma_G(\widehat{D}_{\mathfrak{q}})$ and $s_1, s_2 \in S_{D_{\mathfrak{q}}}$. For $x \in Q(D_{\mathfrak{q}})$ and $s \in S_{D_{\mathfrak{q}}}$ we therefore have

$$\phi(s.x) = \phi(sxs^\sharp) \leq \phi(x) + \frac{1}{2}\log \alpha(s) + \frac{1}{2}\log \alpha(s^\sharp) = \phi(x) + \log \alpha(s)$$

because the invariance of $\widehat{\psi}$ under $-\tau$ implies that $d\widehat{\psi}(X) \circ (-\tau) = d\widehat{\psi}(-\tau.X)$, hence that the set $d\widehat{\psi}(\widehat{D}_{\mathfrak{q}})$ is $-\tau$-invariant, and therefore that $\alpha(s^\sharp) = \alpha(s)$ holds for all $s \in S_{D_{\mathfrak{q}}}$. □

If we restrict the action of the semigroup $S_{D_{\mathfrak{q}}}$ to the subsemigroup $\Gamma_{G^\tau}(\lim D_{\mathfrak{q}}) = \{s \in S_{D_{\mathfrak{q}}} : s^\sharp = s\}$, then one would like to use Lemma 5.5 without referring to any extension $\widehat{\psi}$ of ψ, i.e., one would like to compute $\sup\langle d\widehat{\psi}(i\widehat{D}_{\mathfrak{q}}), X\rangle$ for $X \in \lim(D_{\mathfrak{q}})$ without having information on the function $\widehat{\psi}$. That this is possible follows from the following remarkable proposition on convex function.

Proposition 5.6. — *Let V be a real vector space, $\Omega \subseteq V$ a domain, and $\psi : \Omega \to \mathbb{R}$ a convex function. Then for $X \in \lim\Omega$ and $Y \in \Omega$ the following numbers in $\mathbb{R} \cup \{\infty\}$ coincide:*

(1) $\sup\langle d\psi(\Omega), X\rangle$.

(2) $\lim_{t\to\infty} d\psi(Y + tX)(X)$.

(3) $\sup_{t>0} \psi\big(Y + (t+1)X\big) - \psi(Y + tX)$.

(4) $\sup_{Z\in\Omega} \big(\psi(Z + X) - \psi(Z)\big)$.

Proof. — Let a_1, a_2, a_3, a_4 denote the numbers defined in (1)-(4).

First we note that for each convex function $g :]0, \infty[\to \mathbb{R}^+$ the functions $g(t + h) - g(t)$ are increasing which follows from the inequalities

$$\frac{g(t+h) - g(t)}{h} \leq \frac{g(s+h) - g(t)}{s - t + h} \leq \frac{g(s+h) - g(s)}{h}$$

which in turn are direct consequences of the convexity of g. We conclude that the function

$$f :]0, \infty[\to \mathbb{R}, \quad t \mapsto d\psi(Y + tX)(X)$$

is increasing because $f(t) = \lim_{h\to 0+} \frac{1}{h} f_h(t)$ holds with $f_h(t) = \psi(Y + tX + hX) - \psi(Y + tX)$. This shows that a_2 exists.

It is clear that $a_1 \geq a_2$, and that $a_4 \geq a_3$. Further [Ne97], Lemma II.6, implies that $a_1 = a_4$ and, applying this argument to the function $\widetilde{\psi}(t) := \psi(Y + tX)$, we also obtain $a_2 = a_3$. So it remains to show that $a_4 \leq a_3$.

To this end we may w.l.o.g. assume that $a_3 < \infty$, otherwise there is nothing to show. Inductively we obtain

$$\psi(Y + nx) \leq \psi(Y) + na_3$$

for all $n \in \mathbb{N}$, i.e.,

$$\big(Y + nX, \psi(Y) + na_3\big) \in \operatorname{epi}(\psi),$$

so that $\big(Y + tX, \psi(Y) + ta_3\big) \in \operatorname{epi}(\psi)$ for all $t \in \mathbb{R}^+$ implies that $(X, a_3) \in \lim \overline{\operatorname{epi}(\psi)} = \lim \operatorname{epi}(\overline{\psi})$, where the closure $\overline{\psi}$ is the uniquely determined function whose epigraph coincides with the closure of the epigraph of ψ (*cf.* [Ne99], Prop. V.3.7). Hence $\overline{\psi}(Z + X) \le \overline{\psi}(Z) + a_3$ holds for each $Z \in \Omega$, so that $\overline{\psi}|_\Omega = \psi$ entails that $a_4 \le a_3$. □

For $X \in i\lim(D_\mathfrak{q}) \subseteq i\overline{W} \subseteq i\mathfrak{q}$ the preceding lemma implies in particular for $Y \in D_\mathfrak{q}$ that

$$\sup\langle d\widehat{\psi}(i\widehat{D}_\mathfrak{q}), X\rangle = \lim_{t\to\infty} d\widehat{\psi}(Y + tX)(X) = \lim_{t\to\infty} d\psi(Y + tX)(X)$$
$$= \sup\langle d\psi(D_\mathfrak{q}), X\rangle.$$

Hence

$$\log \alpha(h \operatorname{Exp} X) = \sup\langle d\psi(D_\mathfrak{q}), X\rangle$$

holds for each element $h \operatorname{Exp} X \in \Gamma_{G^\tau}(\lim D_\mathfrak{q}) \subseteq S_{D_\mathfrak{q}}$ and hence does not depend on the extension $\widehat{\psi}$.

As in Section 1, let $\widehat{\mathfrak{a}} \subseteq i\mathfrak{g}$ be an abelian maximal hyperbolic subspace and $\widehat{\Delta}^+$ a $\mathfrak{p}$-adapted positive system with $\widehat{W} \cap \widehat{\mathfrak{a}} = \widehat{C}^0_{\max}$. We write $p_\mathfrak{a} : i\mathfrak{g} \to \widehat{\mathfrak{a}}$ for the projection onto $\widehat{\mathfrak{a}}$ with kernel $[\widehat{\mathfrak{a}}, \mathfrak{g}]$. Let

$$\widehat{W}_{\min} := \{X \in i\mathfrak{g} : p_\mathfrak{a}(G.X) \subseteq \widehat{C}_{\min}\}.$$

Then $\widehat{W}_{\min} = \bigcap_{g\in G} \operatorname{Ad}(g).p_\mathfrak{a}^{-1}(C_{\min})$ shows that $\widehat{W}_{\min}$ is an invariant closed convex cone in $i\mathfrak{g}$, and $\widehat{W}_{\min} \subseteq \lim C$ holds for any open invariant convex subset $C \subseteq \widehat{W}$ (*cf.* [Ne97], Prop. I.10).

THEOREM 5.7. — *If $\mathcal{H}_K \subseteq \operatorname{Hol}\big(\Xi(D_\mathfrak{q})\big)$ is an invariant Hilbert space and $\widehat{D}_\mathfrak{q}$ an extension of $D_\mathfrak{q}$, then $S_{\min} := \Gamma_G(\widehat{W}_{\min})$ acts on $\mathcal{H}$ by contractions via $(s.f)(z) = f(s^\sharp.z)$.*

Proof. — According to Lemma 5.1, we may w.l.o.g. assume that $D_\mathfrak{q}$ is convex. If $\widetilde{G}$ denotes the universal covering of G, and $q : \widetilde{\Xi}(D_\mathfrak{q}) := \widetilde{G} \times_{\widetilde{G}^\tau} D_\mathfrak{q} \to \Xi(D_\mathfrak{q})$ the corresponding holomorphic covering, then we may pull back the invariant Hilbert space $\mathcal{H}_K$ to a $\widetilde{G}$-invariant Hilbert space on $\widetilde{\Xi}(D_\mathfrak{q})$. Since q is equivariant with respect to the action of $\Gamma_{\widetilde{G}}(\lim D_\mathfrak{q})$, we may therefore assume that G is simply connected and therefore that $H = G^\tau$.

First we observe that Lemma 5.2 shows that $\phi(z) := \log K(z, z)$ is a G-invariant smooth plurisubharmonic function on $\Xi(D_\mathfrak{q})$. Hence the function $\psi : D_\mathfrak{q} \to \mathbb{R}, X \mapsto \phi([\mathbf{1}, X])$ is convex (Theorem 4.3), and [Ne97], Prop. I.10(ii), further shows that $d\widehat{\psi}(D_{\mathfrak{g}}) \subseteq -\widehat{W}_{\min}^{\star}$. Now Lemma 5.5 yields $\phi(s.z) \leq \phi(z)$ for $z \in \Xi(D_{\mathfrak{q}})$ and $s \in S_{\min}$.

On the other hand the invariance of the kernel K under G and the uniqueness of holomorphic extension (cf. [Ne99], Lemma XI.2.2) show that the kernel K on $D \times D$ is also $S_{\min}$-invariant in the sense that

$$K(s.z, w) = K(z, s^*.w)$$

for all $z, w \in \Xi(D_{\mathfrak{q}})$ and $s \in S_{\min}$.

Now [Ne99], Prop. II.4.3, Th. II.4.4, implies that $S_{\min}$ acts on the dense subspace $\mathcal{H}_K^0 := \operatorname{span}\{K_z : z \in \Xi(D_{\mathfrak{q}})\}$ in such a way that

$$\|\pi_K(s)\|^2 = \sup_{z \in \Xi(D_{\mathfrak{q}})} \frac{\exp\phi(s.z)}{\exp\phi(z)} \leq 1.$$

We conclude that $S_{\min}$ leaves $\mathcal{H}_K$ invariant and that it acts by contractions on this space. □

DEFINITION 5.8. — *Let $W \subseteq i\mathfrak{g}$ be an invariant convex cone. A unitary representation $(\pi, \mathcal{H})$ of G is called W-dissipative if its convex moment set I_π is contained in $W^\star$, i.e., if all the operators $d\pi(X)$, $X \in W$, are negative selfadjoint operators on $\mathcal{H}$ (cf. [Ne99], Prop. X.1.5). It is called absolutely $\widehat{W}_{\min}$-dissipative if the corresponding representation of the quotient algebra $\mathfrak{g}_1 := \mathfrak{g}/\ker d\pi$ is $\widehat{W}_{\min,1}$-dissipative with respect to a compatible $\mathfrak{p}$-adapted positive system Δ_1^+ (cf. [Ne97], Lemma I.6).* □

DEFINITION 5.9. — *Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan subalgebra, $\Delta \subseteq i\mathfrak{t}^*$ the corresponding system of roots, and $\Delta^+ \subseteq \Delta$ a positive system.*

(a) *A non-zero element v of a $\mathfrak{g}_{\mathbb{C}}$-module V is called primitive (with respect to Δ^+) if $\mathfrak{b}.v \subseteq \mathbb{C}.v$ holds for $\mathfrak{b} := \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\mathbb{C}}^{\alpha}$.*

(b) *For a $\mathfrak{g}_{\mathbb{C}}$-module V and $\mu \in \mathfrak{t}_{\mathbb{C}}^*$ we write $V^\mu := \{v \in V : (\forall X \in \mathfrak{t}_{\mathbb{C}}) X.v = \mu(X)v\}$ for the weight space of weight μ and call μ a weight of V if $V^\mu \neq \{0\}$.*

(c) *A $\mathfrak{g}_{\mathbb{C}}$-module V is called a highest weight module with highest weight λ (with respect to Δ^+) if it is generated by a primitive element of weight λ.*

(d) *Let $T = \exp_G i\widehat{\mathfrak{a}} \subseteq G$. A unitary highest weight representation $(\pi, \mathcal{H})$ of G is a unitary representation for which the $\mathfrak{g}_{\mathbb{C}}$-module $\mathcal{H}^T$ of smooth T-finite vectors in $\mathcal{H}$ is a highest weight module.* □

The following theorem is one of the main results of [Ne97] (*cf.* [Ne97], Th. III.14), where it has been used to study biinvariant Hilbert spaces on biinvariant domains in $\Gamma_G(\widehat{W})$. Here we will use it for invariant Hilbert spaces on domains of the type $\Xi(D_{\mathfrak{q}})$.

THEOREM 5.10. — *An irreducible absolutely $\widehat{W}_{\min}$-dissipative representation $(\pi, \mathcal{H})$ is a highest weight representation.* □

Our next step is to show that an irreducible representation of G in an invariant Hilbert subspace $\mathcal{H}_K \subseteq \mathrm{Hol}(D_{\mathfrak{q}})$ is absolutely $\widehat{W}_{\min}$-dissipative. In Theorem 5.7 we have already seen that it is $\widehat{W}_{\min}$-dissipative, so that it remains to check that we may reduce the problem to the situation where the representation has trivial kernel.

LEMMA 5.11. — *Let $D := \Xi(D_{\mathfrak{q}})$. If $(\pi_K, \mathcal{H}_K)$ is a representation of G in an invariant Hilbert subspace of* $\mathrm{Hol}(D)$, *then its kernel is τ-invariant.*

Proof. — Let K be the reproducing kernel of $\mathcal{H}_K$. Combining Lemma 5.2 and Lemma 5.3(ii), we see that $K(z,z) = K(z^*,z^*)$ holds for all $z \in D$. Hence the uniqueness of analytic continuation yields $K(z,w) = K(w^*,z^*)$ for all $z, w \in D$ (*cf.* [Ne96b], Example II.8(a)). We conclude that the involution on $\mathrm{Hol}(D)$ defined by $(\sigma.f)(z) := \overline{f(z^*)}$ preserves the Hilbert space $\mathcal{H}_K$ and induces an antilinear isometry of this space (*cf.* [Ne96b], Prop. I.6). For $g \in G$ we have

$$\begin{aligned}\big(\pi_K(g)\sigma.f\big)(z) = (\sigma.f)(g^\sharp.z) = (\sigma.f)(g^\sharp z g) = \overline{f(g^{-1}z^*g^{-\sharp})} = \overline{f(g^{-1}.z^*)}\\ = \sigma.\big(\pi_K(\tau.g).f\big)(z),\end{aligned}$$

i.e.,

$$\pi_K(g) \circ \sigma = \sigma \circ \pi_K(\tau.g).$$

From this relation it follows in particular that $\ker \pi_K$ is τ-invariant. □

Before we can prove Theorem 5.13 below, we need the following reduction lemma.

LEMMA 5.12. — *Let $\mathcal{H}_K \subseteq \mathrm{Hol}(D)$ be an invariant Hilbert space, $B \subseteq G$ a connected τ-invariant normal subgroup such that all functions in $\mathcal{H}_K$ are constant on the B-orbits in D, and $G_1 := G/B$. Let $\mathfrak{b}$ denote the Lie algebra of B, $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{b}$, and $p : \mathfrak{g}_{\mathbb{C}} \to (\mathfrak{g}_1)_{\mathbb{C}}$ the quotient map. Then the following assertions hold:*

(i) *There exists a unique involution τ_1 on $\mathfrak{g}_1$ turning p into a morphism of involutive Lie algebras, and $D_{\mathfrak{q},1} := p(D_{\mathfrak{q}}) \subseteq \mathfrak{q}_1$ is an open invariant domain.*

(ii) *If $H_1 := p(H)$ and $D_1 := \Xi(D_{\mathfrak{q},1}) = G_1 \times_{H_1} D_{\mathfrak{q},1}$, then p induces a holomorphic map*

$$p_D : D = \Xi(D_{\mathfrak{q}}) \to D_1 = \Xi(D_{\mathfrak{q},1}), \quad [g, X] \mapsto [p(g), p(X)].$$

(iii) *The holomorphic map $\Psi : D \to \mathcal{H}'_K$, $z \mapsto K_z$ factors over p_D, i.e., there exists a holomorphic map $\Psi_1 : D_1 \to \mathcal{H}'_K$ with $\Psi_1 \circ p_D = \Psi$.*

Proof. — (i) follows directly from the τ-invariance of B.

(ii) First we note that H_1 is an open subgroup of $G_1^{\tau_1}$ so that $\Xi(D_{\mathfrak{q},1})$ is well defined. Since $\Xi(D_{\mathfrak{q}})$ covers $Q(D_{\mathfrak{q}}) \subseteq S^\sharp$, $\Xi(D_{\mathfrak{q},1})$ covers $Q(D_{\mathfrak{q},1}) \subseteq S_1^\sharp$ (Theorem 3.1), and the corresponding map $S \to S_1$ is holomorphic, it follows that the mapping p_D is holomorphic.

(iii) We first consider the fibers of the map p_D. From $p_D([g, X]) = [p(g), p(X)]$ we conclude that $p_D([g, X]) = [g_1, X_1]$ if and only if there exists $h \in H$ with $q(g.h) = g_1$ and $X_1 = p(\mathrm{Ad}(h)^{-1}.X)$. Let us fix $g_0 \in G$ with $p_G(g_0) = g_1$ and $X_0 \in D_{\mathfrak{q}}$ with $p(X_0) = X_1$. Then $p_D([g, X]) = [g_1, X_1]$ holds if and only if $g \in g_0 B = B g_0$ and $p(X) = X_1$, i.e., $X \in (X_0 + \mathfrak{b} \cap \mathfrak{q}) \cap D_{\mathfrak{q}}$. This means that

$$p_D^{-1}([g_1, X_1]) = B.[g_0, (X_0 + \mathfrak{b} \cap \mathfrak{q}) \cap D_{\mathfrak{q}}].$$

Since by assumption Ψ is constant on the B-orbits in D, it remains to show that $\Psi([g_0, X_0]) = \Psi([g_0, X_0 + Y])$ holds for $Y \in \mathfrak{b} \cap \mathfrak{q}$ and $X_0 + Y \in D_{\mathfrak{q}}$. In view of the G-equivariance of Ψ, we may w.l.o.g. assume that $g_0 = \mathbf{1}$.

Let $X_t := X + tY$ for $t \in [0, 1]$ and put $\gamma(t) := [\mathbf{1}, X_t]$. We claim that $\gamma'(t) \in T_{\gamma(t)}(B.\gamma(t))$. To do this, we may w.l.o.g. pass to the domain $Q(D_{\mathfrak{q}})$ and consider the curve $\delta(t) := \mathrm{Exp}\, 2X_t$. Then

$$\delta'(t) = 2d\mathrm{Exp}(2X_t).Y \in d\mathrm{Exp}(2X_t).(\mathfrak{b} \cap \mathfrak{q}).$$

For $g = \mathrm{Exp}\, Z$ and $Y \in \mathfrak{b} \cap \mathfrak{q}$ we use [DN93], Lemma 2.6, to see that

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \exp(tY).g &= \frac{d}{dt}\Big|_{t=0} \exp(tY) g \exp tY^\sharp \\ &= \frac{d}{dt}\Big|_{t=0} \exp(tY)\, \mathrm{Exp}\, Z \exp tY^\sharp = d\mathrm{Exp}(Z)\tilde{g}(\mathrm{ad}\, Z).Y, \end{aligned}$$

where

$$\tilde{g}(z) = \frac{z \cosh \frac{z}{2}}{\sinh \frac{z}{2}}.$$

For $Z \in D_{\mathfrak{q}}$ and $Y \in \mathfrak{b} \cap \mathfrak{q}$ the fact that $\tilde{g}$ is an even function implies that $\tilde{g}(\operatorname{ad} Z).Y \in \mathfrak{b} \cap \mathfrak{q}$, and since $\tilde{g}(\operatorname{ad} Z)$ is invertible because $\operatorname{Spec} \operatorname{ad} Z \subseteq \mathbb{R}$, we see that $\tilde{g}(\operatorname{ad} Z).(\mathfrak{b} \cap \mathfrak{q}) = \mathfrak{b} \cap \mathfrak{q}$, and therefore that

$$T_g(B.g) \supseteq d\operatorname{Exp}(Z).(\mathfrak{b} \cap \mathfrak{q}).$$

This proves that $\gamma'(t) \in T_{\gamma(t)}\big(B.\gamma(t)\big) \subseteq \ker d\Psi\big(\gamma(t)\big)$ for all $t \in [0,1]$, and therefore that $\Psi \circ \gamma$ is constant, i.e., $\Psi([\mathbf{1}, X]) = \Psi([\mathbf{1}, X + Y])$. Thus Ψ is constant on the fibers of p_D, and the fact that $p_D : D \to D_1$ is a submersion implies that Ψ factors to a holomorphic map $\Psi_1 : D_1 \to \mathcal{H}_K'$ satisfying $\Psi_1 \circ p_D|_D = \Psi$. □

We note that Lemma 5.12 applies in particular to the *ideal of degeneracy* $\mathfrak{b} = H(\widehat{W}_{\min})$. In this case $\operatorname{Exp} \mathfrak{b}_{\mathbb{C}} = B_{\mathbb{C}} = H(S_{\min}) \subseteq \overline{S}$ is a complex subgroup acting on D. Since Lemma 5.12(iii) shows that the function Ψ factors over the domain $D_1 \cong D/B_{\mathbb{C}}$, we see that for the sake of studying G-invariant Hilbert spaces, we may w.l.o.g. assume that the cone $\widehat{W}_{\min}$ is pointed. We recall from [Ne97], Lemma I.4(v), that this implies that $\mathfrak{g}$ is an admissible Lie algebra.

THEOREM 5.13. — *Let $D = \Xi(D_{\mathfrak{q}})$ be an invariant domain. Then an irreducible representation $(\pi_K, \mathcal{H}_K)$ of G on an invariant Hilbert subspace $\mathcal{H}_K \subseteq \operatorname{Hol}(D)$ given by $\big(\pi_K(g).f\big)(z) = f(g^\sharp.z)$ is a unitary highest weight representation.*

Proof. — Let $\mathfrak{b} := \ker d\pi$ and $B := (\ker \pi)_0$. According to Lemma 5.11, the group B is τ-invariant, so that the quotient Lie algebra $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{b}$ inherits the structure of a symmetric Lie algebra $(\mathfrak{g}_1, \tau_1)$. Let $p_1 : G \to G_1 := G/B$ denote the canonical quotient map and consider the invariant cone $W_1 := dp_1(0)(W) \subseteq \mathfrak{g}_1$ which is an open $-\tau$-invariant convex cone.

Then Lemma 5.12(iii) shows that Ψ factors to a holomorphic map $\Psi_1 : D_1 \to \mathcal{H}'$ satisfying $\Psi_1 \circ p_D|_D = \Psi$. Hence the mapping

$$\mathcal{H} \to \operatorname{Hol}(D_1), \quad f \mapsto \big(z \mapsto \langle f, \Psi_1(z)\rangle\big)$$

yields a G-invariant realization of $\mathcal{H}$ as an invariant Hilbert space of holomorphic functions on D_1.

Now Theorem 5.7 applied to the action of G, resp. G_1, on $\mathcal{H} \subseteq \operatorname{Hol}(D_1)$ shows that the representation of G_1 on this space is $\widehat{W}_{\min,1}$-dissipative because the one-parameter semigroups given by $\big(\operatorname{Exp}(tX).f\big)(z) = f(\operatorname{Exp} tX^\sharp z \operatorname{Exp} tX)$, $X \in i\widehat{W}_{\min,1}$, $t \in \mathbb{R}^+$, are contractive. Hence $(\pi, \mathcal{H})$

is absolutely $\widehat{W}_{\min}$-dissipative and therefore a highest weight representation (Theorem 5.10). $\square$

So far we have seen that each irreducible representation of G in an invariant Hilbert space is a highest weight representation. The next step is to restrict the class of those highest weight representations which may occur.

To get more information on these highest weight representations, we ask the more general question which highest weight representations of G can be found in the space $\operatorname{Hol}\big(Q(D_{\mathfrak{q}})\big) \cong \operatorname{Hol}\big(\Gamma_G(\widehat{D}_{\mathfrak{q}})\big)^H$ (cf. Theorem 3.3). Since $D = \Xi(D_{\mathfrak{q}})$ is covered by the domain $\widetilde{G} \times_{\widetilde{H}} D_{\mathfrak{q}}$, we may w.l.o.g. assume that G is simply connected and that $H = G^\tau$.

Let $G_{\mathbb{C}}$ be a simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and $\eta : G \to G_{\mathbb{C}}$ the natural map induced by the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$, and $\eta_K : K_{\mathbb{C}} \to G_{\mathbb{C}}$ the map obtained from the universality of $K_{\mathbb{C}}$ and the restriction $\eta|_K$, and likewise $\eta_H : H_{\mathbb{C}} \to G_{\mathbb{C}}$. We also put $G_1 := \eta(G)$, $K_{\mathbb{C},1} := \eta_K(K_{\mathbb{C}})$, and $H_1 := \eta(H)$, as well as $H_{\mathbb{C},1} := \eta_H(H_{\mathbb{C}})$. As we have seen in [KNÓ97], Prop. II.6, $\Omega := P^- K_{\mathbb{C},1} H_{\mathbb{C},1} \subseteq G_{\mathbb{C}}$ is an open domain which can be identified with

$$\big((P^- \rtimes K_{\mathbb{C}}) \times H_{\mathbb{C}}\big)/K^{\mathbf{1}},$$

where $K^{\mathbf{1}} = \{(k,h) \in K_{\mathbb{C}} \times H_{\mathbb{C}} : \eta_H(h) = \eta_K(k)\}$. Moreover $\Gamma_{G_1}(\overline{\widehat{W}}) \subseteq \Omega$ follows from $(P^+)^\sharp = P^-$ (cf. [KNÓ97], Lemma III.7). We conclude that the universal covering space $\widetilde{\Omega}$ of Ω can be identified with

$$\widetilde{\Omega} := \big((P^- \rtimes K_{\mathbb{C}}) \times \widetilde{H}_{\mathbb{C}}\big)/K_{\mathbb{C}}^\tau,$$

and that the covering map $\widetilde{\Omega} \to \Omega$ is given by the action of the direct product group $(P^- \rtimes K_{\mathbb{C}}) \times \widetilde{H}_{\mathbb{C}}$ by $(p,k,h).g := p\eta_K(k)g\eta_H(h)^{-1}$ as the orbit map of the identity element $\mathbf{1} \in G_{\mathbb{C}}$.

THEOREM 5.14. — *The space* $\operatorname{Hol}\big(\Xi(D_{\mathfrak{q}})\big)$ *contains a* $\mathfrak{g}_{\mathbb{C}}$*-highest weight module* V_λ *with highest weight* λ *with respect to* Δ^+ *if and only if the* $\mathfrak{k}_{\mathbb{C}}$*-module* $F(\lambda)$ *of highest weight* λ *is spherical in the sense that it has a non-zero fixed vector for the subgroup* $K^\tau = K \cap H$. *In this case the embedding is unique up to a scalar factor.*

Proof. — First we show that a necessary condition for the existence of V_λ is that the space $F(\lambda) \cong V_\lambda^{\mathfrak{p}^+}$ which is a highest weight representation of K of highest weight λ is spherical. We recall from Theorem 3.3 that we may identify the spaces $\operatorname{Hol}\big(\Xi(D_{\mathfrak{q}})\big)$ and $\operatorname{Hol}\big(\Gamma_G(\widehat{D}_{\mathfrak{q}})\big)^H$.

Let $U \subseteq \widetilde{\Omega}$ be an open subset which can be written as $U = U_P U_K U_H$, where $U_P \subseteq P^-$ and $U_H \subseteq \widetilde{H}_{\mathbb{C}}$ are connected identity neighborhoods and $U_K \subseteq K_{\mathbb{C}}$ is a K-left- and K_H-right-invariant domain. Now the space of H-right-invariant holomorphic functions on U which are annihilated by the right-invariant vector fields corresponding to $\mathfrak{p}^-$ restricts to the space of right K_H-invariant holomorphic functions on the domain U_K. If $f \in \mathrm{Hol}(U_K)^{K_H}$ is contained in a finite dimensional $\mathfrak{k}$-invariant subspace, then the representation of $\mathfrak{k}$ on this space integrates to a holomorphic representation of $K_{\mathbb{C}}$, and this implies that f extends to a holomorphic function on $K_{\mathbb{C}}$. Since the representation of K on the K-finite functions in $\mathrm{Hol}(K_{\mathbb{C}})^{K_H}$ is multiplicity free, each irreducible subspace occurs with multiplicity at most one, and only the K_H-spherical irreducible representations of K occur in this space (*cf.* [Hel84]).

Now assume that we have a $\mathfrak{g}$-equivariant embedding

$$V_\lambda \hookrightarrow \mathrm{Hol}\left(\Gamma_G(\widehat{D}_{\mathfrak{q}})\right)^H.$$

Lifting the embedding $\Gamma_{G_1}(\widehat{W}) \hookrightarrow \Omega$ to a regular holomorphic map $\Gamma_G(\widehat{W}) \to \widetilde{\Omega}$, we also obtain a regular holomorphic map $\Gamma_G(\widehat{D}_{\mathfrak{q}}) \to \widetilde{\Omega}$. We claim that this map is injective. In fact, it suffices to show that the subgroup $K \subseteq G$ is mapped injectively into $\widetilde{\Omega}$ which follows from the fact that $K^{\mathbf{1}} \cap K_{\mathbb{C}} = \{\mathbf{1}\}$. Hence the preceding argument applies to an open domain U as above, which, in addition, is contained in $\Gamma_G(\widehat{D}_{\mathfrak{q}})$. This shows that the $\mathfrak{k}_{\mathbb{C}}$-module $F(\lambda)$ of highest weight λ has to be spherical.

Suppose, conversely, that $F(\lambda)$ is spherical. Then $F(\lambda)$ is K-isomorphic to a uniquely determined submodule of $\mathrm{Hol}(K_{\mathbb{C}})^{K_H}$ which in turn can be viewed as a space of left P^--invariant, right-H-invariant holomorphic functions on $\widetilde{\Omega}$. Using the lift $\Gamma_G(\widehat{W}) \to \widetilde{\Omega}$, we thus obtain a K-equivariant embedding

$$F(\lambda) \hookrightarrow \mathrm{Hol}\left(\Gamma_G(\widehat{W})\right)^{P^- \times H}.$$

The functions in the range of this mapping are annihilated by the right-invariant vector fields coming from $\mathfrak{p}^-$, hence corresponding to the operators coming from $\mathfrak{p}^+$ via

$$\left(d\pi(X).f\right) = \frac{d}{dt}\Big|_{t=0} f(\exp tX^\sharp.z).$$

Therefore the $\mathfrak{g}_{\mathbb{C}}$-submodule of $\mathrm{Hol}\left(\Gamma_G(W)\right)^H$ generated by $F(\lambda)$ is a highest weight module V_λ of highest weight λ. This shows that $\mathrm{Hol}\left(\Xi(D_{\mathfrak{q}})\right)$ contains a unique highest weight module V_λ of weight λ if and only if the $\mathfrak{k}$-module $F(\lambda)$ of highest weight λ is spherical. $\square$

Problem 5. — (a) Let $S = \Gamma_G(W)$ be an Ol'shanskiĭ semigroup and $D := \Gamma_G(D_{\mathfrak{g}}) \subseteq S$ a left-invariant domain. Suppose that K is a positive definite left-invariant kernel on $D \times \overline{D}$ which extends to a kernel $K^1 : D_1 \times \overline{D_1} \to \mathbb{C}$ on the bigger left-invariant domain $D_1 \subseteq S$. Is the kernel K^1 positive definite?

(b) Let $\mathcal{H}_K \subseteq \mathrm{Hol}\big(\Xi(D_{\mathfrak{q}})\big)$ be an irreducible invariant subspace. Do all functions in $\mathcal{H}_K$ extend to the whole domain $\Xi(W)$? Note that the existence of the embedding $V_\lambda \hookrightarrow \mathrm{Hol}\big(\Xi(W)\big)$ implies that in this case $V_\lambda \cong L(\lambda)$, and that all functions in $L(\lambda)$ therefore extend to $\Xi(W)$. Since the representation of G on $\mathcal{H}_K$ is a unitary highest weight representation, it is at least clear that the semigroup $\Gamma_G(\widehat{W})$ acts holomorphically on $\mathcal{H}_K$.

(c) Let $\mathcal{H}_K \subseteq \Gamma_G(D_{\mathfrak{g}})$ be an irreducible left-invariant subspace. Do the functions in $\mathcal{H}_K$ extend to the whole semigroup $\Gamma_G(W)$?

(d) Suppose that $F(\lambda)$ is a spherical $\mathfrak{k}$-module. For which values of λ is the highest weight module $V_\lambda \subseteq \mathrm{Hol}\big(\Xi(D_{\mathfrak{q}})\big)$ irreducible? Is it always irreducible? Does the irreducibility imply that it is unitary? □

Additional remarks.

We have seen above that each invariant Hilbert subspace $\mathcal{H} \subseteq \mathrm{Hol}(D_{\mathfrak{q}})$ carries an isometric antilinear involution σ with $\sigma \circ \pi(g) = \pi(\tau.g) \circ \sigma$ for all $g \in G$. If the representation is irreducible, then it extends to a holomorphic representation of the semigroup $S = \Gamma(\widehat{W})$ which, in view of the uniqueness of analytic continuation, then satisfies

$$\sigma \circ \pi(s^*) = \pi(s^\sharp) \circ \sigma$$

for all $s \in S$ (cf. [Ne99], Ch. XI).

Since there exists an element $X \in D_{\mathfrak{q}} \cap \widehat{C}^0_{\max}$, and the operator $d\pi(X)$ commutes with σ, its eigenspace for the maximal eigenvalue which is isomorphic to some $F(\lambda)$ is invariant under σ. The existence of such an involution on $F(\lambda)$ is easily seen to be equivalent to

$$-\tau.\lambda \in \mathcal{W}_{\mathfrak{k}}.\lambda.$$

It is interesting that for the group case the preceding property is already sufficient to conclude that the unitary representation of G of highest weight λ is spherical, and this has been used in [Ne97] to describe all minimal biinvariant Hilbert subspace of domains of the type $\Gamma_G(\widehat{D}_{\mathfrak{q}})$.

In this case $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{h}$ and accordingly $\mathfrak{t}_{\mathfrak{g}} \cong \mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{t}_{\mathfrak{h}}$, where τ acts by the flip involution. Hence $-\tau.(\lambda, -\mu) = (\mu, -\lambda)$. We have $\mathcal{W}_{\mathfrak{k}} \cong \mathcal{W}_{\mathfrak{k}_{\mathfrak{h}}} \times \mathcal{W}_{\mathfrak{k}_{\mathfrak{h}}}$, and this group acts separately on each factor. If $(\lambda, -\mu)$ is a highest weight of a unitary representation with respect to a positive system Δ^+ with $-\tau.\Delta_p^+ = \Delta_p^+$, then λ and μ are dominant, and therefore the condition that $(\lambda, -\mu)$ is $\mathcal{W}_{\mathfrak{k}}$-conjugate to $(\mu, -\lambda)$ implies that μ is $\mathcal{W}_{\mathfrak{k}_{\mathfrak{h}}}$-conjugate to λ, hence that $\lambda = \mu$. Now $(\lambda, -\mu) = (\lambda, -\lambda)$, and this is the highest weight of the spherical representation of $G = H \times H$ on the space $B_2(\mathcal{H}_\lambda)$, where $(\pi_\lambda^H, \mathcal{H}_\lambda)$ is the unitary highest weight representation of H of highest weight λ and

$$\pi_{(\lambda,-\lambda)}(h_1, h_2).A = \pi_\lambda^H(h_1) A \pi_\lambda^H(h_2^{-1}).$$

The same argument applies in the general setup whenever $\mathfrak{a}$ contains regular elements of the Lie algebra $\mathfrak{k}_{\mathbb{C}}$.

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Karl-Hermann NEEB,
Technische Universität Darmstadt
Fachbereich Mathematik AG5
Schlossgartenstrasse 7
64289 Darmstadt (Germany).