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HYPERELLIPTIC ACTION INTEGRAL

by Bernhard ELSNER

Introduction.

Suppose a Riemann surface X with a holomorphic 1-form ω is given, and you are interested in the sheet structure of the primitive $s = \int \omega$. In fact there are two sheet structures to look at: the one above X and the one above the complex plane of values of s. The first is determined by the period-group of ω. The second is the subject of this article, but only in the special setting where X is a (hyper)elliptic curve given by

$$y^2 = x^m + a_{m-2}x^{m-2} + \cdots + a_0$$

and ω is the Liouville-form $y\,\mathrm{d}x$. The interest in this topic stems from exact WKB theory (see [1], [2], [8]).

The article is divided into two parts. In the first we give a method how to construct a certain type of Riemann surface F^∞ with a projection $\rho^\infty : F^\infty \to \mathbb{C}$. Since these constructions can be done (at least mentally) with paper, scissors and glue, they give a complete description of the sheet structure of (F^∞, ρ^∞) above $\mathbb{C}$. In the second part we prove that for any (F^∞, ρ^∞) constructed by this method there exists a (hyper)elliptic curve given by $y^2 = x^m + a_{m-2}x^{m-2} + \cdots + a_0$ such that F^∞ is the Riemann surface of the multivalued function $s = \int y\,\mathrm{d}x$ and such that the sheet structure of s above its value-plane is described by ρ^∞.

The method for constructing the "action-domain" (F^∞, ρ^∞) consists of several steps. First, in Section 1.3, we build an "action-element"

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(F, ρ), whose shape is inspired by what the complex plane is mapped to by $s = \int y\,\mathrm{d}x$, after cutting it along Stokes lines (Section 2.3). Then, in Section 1.4, we glue infinitely many copies of (F, ρ) together. This can be done in three natural ways, each one corresponding to one of the three groups introduced in 1.2; one of them yields (F^∞, ρ^∞). In 2.2 we shall finally see that there is a group of "translations" acting on F^∞ and leaving $\mathrm{d}\rho^\infty$ invariant, and that the quotient of F^∞ by this group is an (hyper)elliptic curve and the quotient of $\mathrm{d}\rho^\infty$ is the Liouville-form. At one point a technical problem arises: we have to show that the qotient of $\mathrm{d}\rho^\infty$ does not present an essential singularity at infinity. Section 1.5 is dedicated to this task and leads quite naturally to an elementary conjecture stated at the end of the article.

1. Constructions with paper, scissors and glue.

1.1. Integration.

For a Riemann surface X let $\mathcal{M}$ (resp. $\widetilde{\mathcal{M}}$) be the sheaves of meromorphic functions resp. meromorphic 1-forms of second kind on X. From $H^1(X, \mathcal{M}) = 0$ and the exact sequence

$$0 \to \mathbb{C} \longrightarrow \mathcal{M} \xrightarrow{\mathrm{d}} \widetilde{\mathcal{M}} \to 0,$$

it follows that the "meromorphic de Rham group"

$$\mathrm{Rh}^1_{\mathcal{M}} := \widetilde{\mathcal{M}}(X) / \,\mathrm{d}\mathcal{M}(X)$$

is naturally isomorphic to $H^1(X, \mathbb{C})$.

For a pair (X, ω), where $\omega \in \widetilde{\mathcal{M}}(X)$, denote by $[X, \omega]$ the isomorphism class of (X, ω), *i.e.*, the class of all (X', ω') such that there is an isomorphism $f : X \to X$ with $f^*\omega' = \omega$.

The *integration operator* I associates to each class $[X, \omega]$ a class $I[X, \omega]$ defined as follows. The total space $|\mathcal{M}|$ of the sheaf of meromorphic functions on X is an étale space over X: the étale map $\Lambda : |\mathcal{M}| \to X$ sends a germ on its center. Choose a connected component Y that contains a germ of primitive of ω. Let $\mathrm{V} : Y \to \mathbb{P}^1$ be the "evaluation" map sending a germ to the value on its center. Define

$$I[X, \omega] := [Y, \mathrm{dV}].$$

It is easy to check that this definition does not depend on the various choices made.

The connected component Y is the Riemann surface of the primitive $s = \int \omega$. The map $\Lambda_{|Y}$ gives the sheet structure of s above X, the space of the variable, while V gives the sheet structure above $\mathbb{P}^1$, the space of values. Construe the class $[Y, \mathrm{dV}]$ as a projected space (Y, V) given up to translation.

Clearly one has $\Lambda_{|Y}{}^*\omega = \mathrm{dV}$. This leads to an equivalent definition of the integration I: via the isomorphism $\mathrm{Rh}^1_{\mathcal{M}} \simeq H^1(X, \mathbb{C})$ the form ω corresponds to the group homomorphism

$$[\omega]: \pi_1(X) \longrightarrow \mathbb{C}, \quad \lambda \longmapsto \int_\lambda \omega.$$

Let $p: Y \to X$ be the Galois covering of X with characteristic subgroup $\ker[\omega]$. Then

$$I[X, \omega] = [Y, p^*\omega].$$

The *period-group* $[\omega](\pi_1(X)) \subset \mathbb{C}$ operates on $p: Y \to X$ as the group of deck-transformations.

If ω, $\omega' \in \widetilde{\mathcal{M}}(X)$ are cohomologous, and if (Y, σ) is an element of $I[X, \omega]$ and (Y', σ') an element of $I[X, \omega']$, then Y and Y' are isomorphic, but in general $I[X, \omega] \neq I[X, \omega']$. Also note that $I^2 = I$.

1.2. Three groups.

For an integer $m \geq 0$ let $\mathbf{F}_m$ be the free group with generators $f_1, \dots, f_m$. The group $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ operates on $\mathbf{F}_m$ by changing sign; more precisely there is a homomorphism

$$\varphi: \mathbb{Z}_2 \longrightarrow \mathbf{Aut}(\mathbf{F}_m), \quad k \longmapsto \varphi_k,$$

where φ_k is defined as follows on the level of generators:

$$\varphi_k(f_j) := f_j^{(-1)^k}, \quad k \in \mathbb{Z}_2,\ j = 1, \dots, m.$$

Every element $w \in \mathbf{F}_m$ can be written as a word

$$w = f_{j_1}^{n_1} \cdots f_{j_s}^{n_s}, \quad s \in \mathbb{N},\ n_1, \dots, n_s \in \mathbb{Z}^*,\ j_1, \dots, j_s \in \{1, \dots, m\}.$$

This word can be uttered uniquely without stuttering, *i.e.*, $j_k \neq j_{k+1}$; thus we can define its *length* as being the integer $|n_1| + \cdots + |n_s|$. Denote by $\mathbf{R}^1_m$ (resp. $\mathbf{R}_m$) the normal subgroup of $\mathbf{F}_m$ generated by the elements f_j^2, $j = 1, \dots, m$ (resp. by all the elements w^2, where $w \in \mathbf{F}_m$ has odd length).

Of course

$$\mathbf{R}^1_m \lhd \mathbf{R}_m.$$

Denote by g_j the coset $f_j\mathbf{R}^1_m$. Let us write $\mathbf{G}_m$ for the quotient group $\mathbf{F}_m/\mathbf{R}^1_m$; clearly it is isomorphic to the m-fold free product $\mathbb{Z}_2 * \cdots * \mathbb{Z}_2$.

PROPOSITION 1. — *The map*

$$\Theta : \mathbf{F}_{m-1} \rtimes_\varphi \mathbb{Z}_2 \longrightarrow \mathbf{G}_m,$$

$$(f_{j_1}^{n_1} \cdots f_{j_s}^{n_s}, k) \longmapsto (g_{j_1}g_m)^{n_1} \cdots (g_{j_s}g_m)^{n_s} g_m^k,$$

is an isomorphism of groups. It carries $\mathbf{K}(\mathbf{F}_{m-1}) \times 0$ *to* $\mathbf{R}_m/\mathbf{R}^1_m$. *(As usual* $\mathbf{K}$ *denotes the commutator subgroup.)*

Proof. — Clearly the map Θ is well-defined. We shall prove that it is a homomorphism, and we do it on the level of generators of $\mathbf{F}_{m-1} \rtimes_\varphi \mathbb{Z}_2$. First note that for all $k, n \in \mathbb{Z}$, $j, \ell = 1, \ldots, m$, one has

$$g_\ell^k (g_j g_\ell)^n = (g_j g_\ell)^{(-1)^k n} g_\ell^k. \tag{1}$$

Now let (f_j, k) and (f_ℓ, k') be two generators of $\mathbf{F}_{m-1} \rtimes_\varphi \mathbb{Z}_2$. Then using (1) we get

$$\begin{aligned}
\Theta\big((f_j,k)(f_\ell,k')\big) &= \Theta(f_j f_\ell^{(-1)^k}, k+k') = g_j g_m (g_\ell g_m)^{(-1)^k} g_m^{k+k'} \\
&= g_j g_m g_m^k (g_\ell g_m) g_m^{k'} \quad = (g_j g_m^{k+1})(g_\ell g_m^{k'+1}) \\
&= \Theta(g_j, k)\Theta(g_\ell, k'),
\end{aligned}$$

and

$$\begin{aligned}
\Theta\big((f_j,k)^{-1}\big) &= \Theta(f_j^{(-1)^{k+1}}, -k) \qquad = (g_j g_m)^{(-1)^{k+1}} g_m^{-k} \\
&= (g_j g_m)^{(-1)^{k+1}} g_m^{k+1} g_m = g_m^{k+1}(g_j g_m) g_m \\
&= g_m^{k+1} g_j = (g_j g_m^{k+1})^{-1} = \big(\Theta(f_j,k)\big)^{-1}.
\end{aligned}$$

Clearly Θ is a surjective map since every g_j, $j = 1, \ldots, m$, is attained. To see injectivity write $w = f_{j_1}^{n_1} \cdots f_{j_s}^{n_s} \in \mathbf{F}_{m-1}$ without stuttering; then $(w, k) \in \ker(\Theta)$ means that the word

$$w' = (f_{j_1} f_m)^{n_1} \cdots (f_{j_s} f_m)^{n_s} f_m^k$$

can be reduced to the empty word using only the relations $\mathbf{R}^1_m$. But the use of these relations does not change the parity of the number of letters in a word. This implies that the number of letters in w' is even; this number is $2(|n_1| + \cdots + |n_s|) + k$, so $k = 0$. Now if $s \geq 1$, then we can not reduce the word $(f_{j_1} f_m)^{n_1} \cdots (f_{j_s} f_m)^{n_s}$ anymore, because all neighbouring letters are distinct. Hence Θ is an isomorphism.

It remains to show the equality

$$\mathbf{I} = \mathbf{R}_m/\mathbf{R}_m^1, \quad \text{where} \quad \mathbf{I} := \Theta\big(\mathbf{K}(\mathbf{F}_{m-1}) \times 0\big).$$

We begin with "$\subset$". The commutator $\mathbf{K}(\mathbf{F}_{m-1})$ is the smallest normal subgroup of $\mathbf{F}_{m-1}$ containing the elements $f_j f_\ell f_j^{-1} f_\ell^{-1}$, $j, \ell = 1, \dots, m-1$. It follows that $\mathbf{K}(\mathbf{F}_{m-1}) \times 0$ is the smallest normal subgroup of $\mathbf{F}_{m-1} \rtimes_\varphi \mathbb{Z}_2$ containing $(f_j f_\ell f_j^{-1} f_\ell^{-1}, 0)$, $j, \ell = 1, \dots, m-1$,

$$\begin{aligned} (2) \qquad \Theta(f_j f_\ell f_j^{-1} f_\ell^{-1}, 0) &= g_j g_m g_\ell g_m (g_j g_m)^{-1} (g_\ell g_m)^{-1} \\ &= g_j g_m g_\ell g_m g_m g_j g_m g_\ell \\ &= (g_j g_m g_\ell)^2 \in \mathbf{R}_m/\mathbf{R}_m^1, \quad j, \ell = 1, \dots, m-1. \end{aligned}$$

Since $\mathbf{R}_m/\mathbf{R}_m^1 \lhd \mathbf{F}_m/\mathbf{R}_m^1$, the inclusion $\mathbf{I} \subset \mathbf{R}_m/\mathbf{R}_m^1$ is proven.

The group $\mathbf{I}$ is a normal subgroup of $\mathbf{R}_m/\mathbf{R}_m^1$. In order to show that $\mathbf{I} = \mathbf{R}_m/\mathbf{R}_m^1$ we look at the quotient of $\mathbf{R}_m/\mathbf{R}_m^1$ by the relations $\mathbf{I}$ and show that it is trivial. Proceeding in two steps we first show that

$$(3) \qquad (g_j g_k g_\ell)^2 = e \bmod \mathbf{I} \quad \text{for all } j, k, \ell \in \{1, \dots, m\}.$$

For $j = k$ or $k = \ell$ this is trivial, and for $k = m$ it was computed in (2). Therefore we show that $(g_j g_m g_\ell)^2 = e \bmod \mathbf{I}$ for all $j, \ell = 1, \dots, m$ implies (3); this is equivalent to showing that the relations

$$g_j g_m g_\ell = g_\ell g_m g_j, \quad j, \ell = 1, \dots, m,$$

induce the relations

$$(4) \qquad g_j g_k g_\ell = g_\ell g_k g_j, \quad j, k, \ell = 1, \dots, m.$$

This is seen as follows:

$$\begin{aligned} g_j g_k g_\ell = g_j g_m g_m g_k g_m g_m g_\ell &= g_j g_m g_\ell g_\ell g_m g_k g_m g_j g_j g_m g_\ell \\ = g_\ell g_m g_j g_k g_m g_\ell g_m g_j g_\ell g_m g_j &= g_\ell g_m g_j g_k g_m g_j g_m g_\ell g_\ell g_m g_j \\ = g_\ell g_m g_j g_k g_m g_j g_j &= g_\ell g_m g_j g_j g_m g_k g_j = g_\ell g_k g_j. \end{aligned}$$

The second step is to observe that $\mathbf{R}_m/\mathbf{R}_m^1$ is the smallest normal subgroup of $\mathbf{F}_m$ containing all the elements $(g_{j_1} \cdots g_{j_s})^2$, where s is odd and $j_1, \dots, j_s \in \{1, \dots, m\}$. This means that the triviality of $(\mathbf{R}_m/\mathbf{R}_m^1)/\mathbf{I}$ is equivalent to the fact that the relations (4) induce the relations

$$(5) \qquad g_{j_1} \cdots g_{j_s} = g_{j_s} \cdots g_{j_1}, \quad s \text{ odd}, j_1, \dots, j_s \in \{1, \dots, m\}.$$

Clearly from (4) it follows that $g_j g_k g_\ell g_n = g_\ell g_n g_j g_k$ and with the help of this it is an easy induction on s to show (4) $\Rightarrow$ (5). $\square$

Let us write $\mathbf{G}_m$ for the quotient group $\mathbf{F}_m/\mathbf{R}_m^1$, and $\mathbf{S}_m$ for the quotient group $\mathbf{F}_m/\mathbf{R}_m$. We denote by σ_j the coset $f_j\mathbf{R}_m$ and call $\mathbf{S}_m$ the *group of m free symmetries.* (For a reason of this name see Lemma 1 below.)

Let $\mathbf{T}_{m-1}$ be the free abelian group with generators $t_1, \dots, t_{m-1}$. As above in the non-abelian case there is a group homomorphism $\phi : \mathbb{Z}_2 \to \mathbf{Aut}(\mathbf{T}_{m-1})$, $k \mapsto \phi_k$, where ϕ_k is defined as follows on the level of generators:

$$\phi_k(t_j) := (-1)^k t_j, \quad k \in \mathbb{Z}_2,\ j = 1, \dots, m-1.$$

There are natural projections

$$\begin{aligned} \Psi &: \mathbf{F}_{m-1} \longrightarrow \mathbf{T}_{m-1}, \quad && f_j \longmapsto t_j, \\ \overline{\Psi} &: \mathbf{G}_m \longrightarrow \mathbf{S}_m, && g_j \longmapsto \sigma_j. \end{aligned}$$

The upshot of Proposition 1 is the following commuting diagram of parallel exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{R}_m/\mathbf{R}_m^1 & \longrightarrow & \mathbf{G}_m = \mathbf{F}_m/\mathbf{R}_m^1 & \xrightarrow{\overline{\Psi}} & \mathbf{S}_m = \mathbf{F}_m/\mathbf{R}_m & \longrightarrow & 0 \\ & & \wr\big\uparrow & & \wr\big\uparrow \Theta & & \wr\big\uparrow \Omega & & \\ 0 & \longrightarrow & \mathbf{K}(\mathbf{F}_{m-1}) \times 0 & \longrightarrow & \mathbf{F}_{m-1} \rtimes_{\varphi} \mathbb{Z}_2 & \xrightarrow{\Psi \times \mathrm{id}} & \mathbf{T}_{m-1} \rtimes_{\phi} \mathbb{Z}_2 & \longrightarrow & 0. \end{array}$$

The homomorphism Ω is given by

$$\Omega(n_1 t_1 + \cdots + n_{m-1} t_{m-1}, k) = (g_1 g_m)^{n_1} \cdots (g_{m-1} g_m)^{n_{m-1}} g_m^k.$$

We shall repeatedly make use of this diagram and abuse notation by identifying via Θ (resp. Ω) the sub-group $\mathbf{F}_{m-1} = \mathbf{F}_{m-1} \times 0$ (resp. $\mathbf{T}_{m-1} = \mathbf{T}_{m-1} \times 0$) with the sub-group of $\mathbf{G}_m$ (resp. $\mathbf{S}_m$) formed by the elements of even length.

For $s, v \in \mathbb{C}$ denote:

- $\sigma_s : \mathbb{C} \to \mathbb{C}$, $z \mapsto 2s - z$, the symmetry of the complex plane with center s, and by

- $\tau_v : \mathbb{C} \to \mathbb{C}$, $z \mapsto z + v$, the translation by vector v.

We consider symmetries and translations as elements of the group $\mathbf{Aut}(\mathbb{C})$. The composition of an odd number of symmetries is again a symmetry, while the composition of an even number of symmetries is a translation. More precisely, the following holds.

Remark 1. — For $s_1, \dots, s_n \in \mathbb{C}$ one has

$$\sigma_{s_1} \cdots \sigma_{s_n} = \begin{cases} \tau_{2(s_1-s_2)+\cdots+2(s_{n-1}-s_n)} & \text{if } n \text{ is even,} \\ \sigma_{s_1-s_2+s_3-\cdots+s_n} & \text{if } n \text{ is odd.} \end{cases}$$

For any $s \in \mathbb{C}$ the following are equivalent:

1) $\sigma_s \in \langle \sigma_{s_1}, \dots, \sigma_{s_n} \rangle$;

2) $s = \sum n_j s_j$, where $n_j \in \mathbb{Z}$, $\sum n_j = 1$.

In particular, if at least four elements among $s_1, \dots, s_n$ are affinely free over $\mathbb{Q}$, then the centers of the symmetries in $\langle \sigma_{s_1}, \dots, \sigma_{s_n} \rangle$ are dense in $\mathbb{C}$.

LEMMA 1. — *Let $s_1, \dots, s_m \in \mathbb{C}$. Then*

$$\psi : \mathbf{S}_m \longrightarrow \langle \sigma_{s_1}, \dots, \sigma_{s_m} \rangle, \quad \sigma_j \longmapsto \sigma_{s_j},$$

is an epimorphism which is an isomorphism exactly if $s_1, \dots, s_m$ are affinely free over $\mathbb{Q}$.

Proof. — Remark 1 shows that ψ is well defined, *i.e.*, the relations $\mathbf{R}_m$ hold for symmetries. Obviously ψ is surjective. The map $\psi \circ \Omega$ is given by

$$\mathbf{T}_{m-1} \rtimes_\phi \mathbb{Z}_2 \ni (n_1 t_1 + \cdots + n_{m-1} t_{m-1}, k) \longmapsto \tau_v (\sigma_{s_m})^k,$$

where $v = 2 \sum_{j=1}^{m-1} n_j (s_j - s_m)$. But one has equivalence between $v = 0$ and $n_1 = \cdots = n_{m-1} = 0$ if and only if the vectors $s_1 - s_m, \dots, s_{m-1} - s_m$ are free over $\mathbb{Q}$. □

The three groups $\mathbf{G}_m$, $\mathbf{S}_m$ and $\langle \sigma_{s_1}, \dots, \sigma_{s_m} \rangle$ will play an essential role in Section 1.4.

1.3. Action-elements.

An *action-element of order m* is a pair (F, ρ) of a Riemann surface with boundary F together with a projection $\rho : F \to \mathbb{C}$, that is constructed as follows by induction on m:

• *An action-element (F, ρ) of order* 1. — Choose a point s_1 on the complex plane $\mathbb{C}$ and cut the plane along the horizontal half-line $s_1 + x$, $x \leq 0$, from s_1 towards $-\infty$, thus creating an upper and a lower border along this cut. Then take a copy of the upper half plane $\overline{H} := \{s \in \mathbb{C} : \mathrm{Im}(s) \geq 0\}$ and glue it to the the cut plane by identifying each point $s_1 + x$, $x \leq 0$, of the lower border with the point x in $\overline{H}$. The surface F obtained this way comes with a natural projection $\rho : F \to \mathbb{C}$.

• *An action-element (F, ρ) of order $m \geq 2$.* — Take an action-element (F', ρ') of order $m-1$ and a point $s_m \in F'$, such that s_m is neither one of the points $s_1, \ldots, s_{m-1}$ of F' nor "sees" one of those points in horizontal direction. Now, as above, cut F' along the horizontal half-line from s_m towards $-\infty$. Then glue a copy of $\overline{H}$ to the cut surface F' by identifying the lower border of the cut in F' with $]-\infty, 0] \subset \overline{H}$ in the same fashion as above. The surface F thus obtained comes with a natural projection ρ (the one which continues ρ').

The boundary ∂F of an action-element (F, ρ) of order m is the subset where ρ is not a local homeomorphism. ∂F has m components, each one containing one of the points $s_1, \ldots, s_m$, at which ρ has a "ramification of angle 3π"; denote by s_j^- resp. s_j^+ the left resp. right border leaving s_j: the ray s_j^+ corresponds to the angle 0 around s_j, while the ray s_j^- corresponds to the angle 3π around s_j. With these notations

$$\partial F = \bigcup_{j=1}^{m} (s_j^- \cup s_j^+) \quad \text{and} \quad s_j^- \cap s_j^+ = \{s_j\}.$$

We shall always use the notation $s_1, \ldots, s_m$ for the "ramification-points" and s_j^-, s_j^+ for the corresponding borders; the numbering is of no importance, so we shall often use the one most convenient at the given moment. Note that for $j \neq k$ one may have $\mathrm{Im}(\rho(s_j)) = \mathrm{Im}(\rho(s_k))$ and even $\rho(s_j) = \rho(s_k)$, but not when s_j, s_k are on the same sheet ("see each other").

Let $\mathcal{F}_m$ be the set of all action-elements of order m. Note that $\mathcal{F}_m$ is in a natural way a complex manifold of dimension m; on each component of $\mathcal{F}_m$ the projection of the ramification-points $s_1, \ldots, s_m$ gives a chart. Instead of expliciting the dependence on F by writing s_j^F and ρ^F we shall simply write s_j and ρ.

Equivalence with a tree. — Let $(F, \rho) \in \mathcal{F}_m$. We shall associate to it a tree $T \subset F$. Precisely one of the $s_1, \ldots, s_m \in F$ has minimal $\mathrm{Im}(\rho(s_j))$; say it is s_1. Among the $s_2, \ldots, s_m$ there is at least one seen from s_1 along a ray of angle $\theta \in]0, \pi[\cup]2\pi, 3\pi[$; we say that it is *seen on the left* if $\theta \in]0, \pi[$, and *seen on the right* if $\theta \in]2\pi, 3\pi[$.

Among the $s_2, \dots, s_m$ seen on the left (resp. right) there is exactly one, say s_ℓ (resp. s_r), such that $\mathrm{Im}(\rho(s_\ell))$ (resp. $\mathrm{Im}(\rho(s_r))$) is minimal (as observed above, at least one of the two exists). Draw a *blue* (resp. *red*) edge oriented from s_1 to s_ℓ (resp. s_r). Now repeat this procedure by doing the same starting from s_ℓ and/or s_r instead of s_1, *etc.* Thus we get a coloured tree $T \subset F$ with m vertices $s_1, \dots, s_m$ and $m-1$ oriented edges (blue for seeing left, red for seeing right.) The tree T is a deformation retract of F.

On the other hand, the tree T contains all the information necessary to reconstruct F. More precisely, call a connected tree T with m vertices and $m-1$ blue or red oriented edges an *action-tree of order* m if:

- it comes equipped with a projection $T \to \mathbb{C}$, such that the oriented edges are mapped to segments on which the imaginary part strictly increases;
- from each vertex leaves at most one edge of each colour;
- there is exactly one vertex, the *root*, at which no edge arrives. On each other vertex arrives exactly one edge.

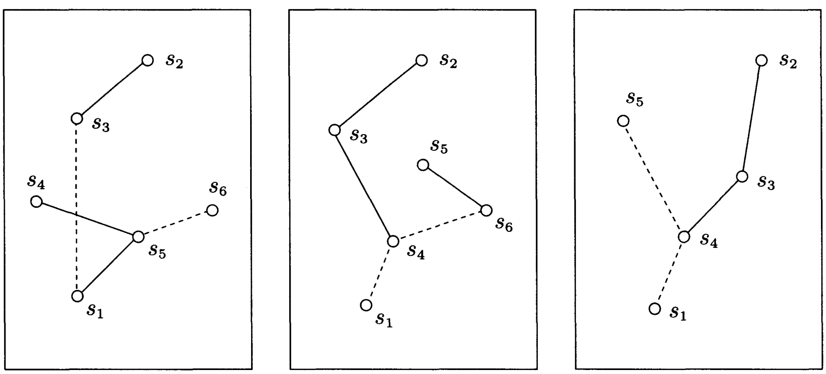

Figure 1: examples of action-trees. Edges are oriented upwards, blue = broken lines, red = solid lines. Numbering of vertices according to (6), *beginning with the root. The third action-tree corresponds to the type of Stokes-pattern on Fig.* 2.

There is a bijective correspondence between action-elements and action-trees of given order.

Note that any sub-tree of an action tree is again an action-tree. Moreover, an action-tree of order k can be glued together with one of order n to obtain an action-tree of order $k+n+1$: simply take any point below both roots as a new root and join it with different coulours to both roots. Clearly every action-tree of order m can be obtained by glueing together in this way an action-tree of order k with one of order $m-k-1$. With this in mind, one easily proves by induction on m the

Remark 2. — Each component of $\mathcal{F}_m$ is contractible. The number c_m of components of $\mathcal{F}_m$ satisfies

$$c_0 = 1 \quad \text{and} \quad c_{m+1} = \sum_{0\le k\le m} c_k c_{m-k}.$$

Moreover c_m coincides with the number of possible parenthizations of a product of $m+1$ factors. In combinatorics the c_m are called *Catalan numbers*.

Big arcs and the residue of an action-element. — The *residue* of $(F,\rho) \in \mathcal{F}_m$ will be a complex number defined up to sign. For $R \gg 0$ let

$$C^R := \rho^{-1}\big(\{|z| = R\}\big).$$

One sees by induction on m that:

• C^R has m connected components $C_1^R, \ldots, C_m^R$. One of them looks like an arc $R\,e^{i\vartheta}$, where ϑ runs through an interval tending to $[0, 3\pi]$ as $R \to +\infty$; the other $m-1$ look like arcs $R\,e^{i\vartheta}$, where ϑ runs through an interval tending to $[0,\pi]$ as $R \to +\infty$.

• Orientate each arc C_j^R in the direction of growing argument. We can (re)-number the C_j^R and the s_j such that (read indices modulo m)

$$\begin{cases} s_{j-1}^+ & \text{contains the initial-point of } C_j^R, \\ s_j^- & \text{contains the end-point of } C_j^R. \end{cases} \tag{6}$$

This numbering is unique up to circular permutations of the form $(1,\ldots,m) \mapsto (k,\ldots,m,1,\ldots,k-1)$.

Using the numbering (6), set

$$\operatorname{res}(F,\rho) := \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \pm 2\big((s_1 - s_2) + \cdots + (s_{m-1} - s_m)\big) & \text{if } m \text{ is even.} \end{cases}$$

DEFINITION 1. — *For an action-element $(F,\rho) \in \mathcal{F}_m$ denote by $\underline{F}$ the Riemann surface obtained when gluing each s_j^- to s_j^+, $j = 1,\dots,m$, by identifying points opposite with respect to s_j.*

The image of C^R in the quotient $\underline{F}$ is not necessarily a closed loop, because the end-point of C_j^R (which is on s_j^-) is not in general glued to the initial point of C_{j+1}^R (which is on s_j^+). To realize the deficiency is to remedy it: on each s_j^- there is a "linking" segment L_j^R such that the image in $\underline{F}$ of C^R together with $L_1^R,\dots,L_m^R$ is a closed loop λ^R in $\underline{F}$.

PROPOSITION 2. — *The Riemann surface $\underline{F}$ is biholomorphic to $\mathbb{C}$.*

Proof. — The idea of the proof is due to M. Zaidenberg. Since $\underline{F}$ is a simply-connected Riemann surface, it is isomorphic to either $\mathbb{P}^1$, $\mathbb{C}$ or the unit-disk $\mathbb{P}^1$. Obviously $\underline{F}$ is non-compact and thus can not be isomorphic to $\mathbb{P}^1$; to finish the proof we will show that $\underline{F}$ is not isomorphic to D. The difference between $\mathbb{C}$ and D lies in their corresponding Poincaré-pseudo-metrics (*cf.* [4]). On $\mathbb{C}$ it is zero, whereas on D it is the metric given by

$$\mathrm{d}s_D^2 = \frac{\mathrm{d}z\,\mathrm{d}\overline{z}}{(1-|z|^2)^2}\cdot$$

Therefore to show $\underline{F} \not\simeq D$ it is sufficient to show that there exists a sequence (λ_n) of closed loops in $\underline{F}$ tending to infinity, such that the sequence of the length of λ_n does not tend to infinity. (A sequence of closed loops $\lambda_n \subset \underline{F}$ is "tending to infinity" if there is an exhausting sequence of compacts $K_n \subset \underline{F}$ such that λ_n is in $\underline{F} - K_n$ and its homotopy class generates $\pi_1(\underline{F} - K_n)$.) Using the λ^R constructed in the lines following definition 1, clearly

$$\lambda_n := \lambda^n \quad \text{for} \quad n \gg 0$$

is a sequence of closed loops in $\underline{F}$ tending to infinity. The lengths don't tend to infinity; this is seen as follows: on the upper half-plane

$$H := \{z = x + iy \in \mathbb{C} : y > 0\}$$

the Poincaré-metric is

$$\mathrm{d}s_H^2 = \frac{\mathrm{d}z\,\mathrm{d}\overline{z}}{4y^2}\cdot \tag{7}$$

By the construction of $\underline{F}$ one can exhibit a finite number of half-planes $H_1,\dots,H_p$ in $\underline{F}$ such that their union contains λ_n for all $n \gg 0$. Moreover, letting the half-planes overlap, we can cut λ_n in p pieces $\lambda_n^1, \ldots, \lambda_n^p$, such that λ_n^k is contained in H_k. Using the explicit expression (7) for the Poincaré-metric in a half-plane and the fact that each λ_n^k is an arc (plus eventually a linking segment), one easily shows that the length in H_k of λ_n^k is bounded as $n \to +\infty$. Now with help of the general fact, that every holomorphic map between Riemann surfaces is distance-decreasing with respect to the corresponding Poincaré-pseudo-metrics, one concludes that the length of λ_n^k in $\underline{F}$ is bounded as $n \to +\infty$; so the same holds for the length of λ_n. $\square$

1.4. Action-domains.

Fix $(F, \rho) \in \mathcal{F}_m$ and denote, as before, its branch-points by $s_1, \ldots, s_m$. For each of the three groups $\mathbf{G}_m$, $\mathbf{S}_m$ and $\langle \sigma_{\rho(s_1)}, \ldots, \sigma_{\rho(s_m)} \rangle$, we shall construct an infinitely-sheeted ramified covering over $\mathbb{C}$, such that outside the branch-points the group acts freely and such that F is "nearly" a fundamental domain under this action.

We begin with the group $\mathbf{G}_m$. The idea is this: take a copy of F and turn it by angle π around one of the points s_j. Then glue the border s_j^- (resp. s_j^+) of the copy to the border s_j^+ (resp. s_j^-) of the original. Taking new copies do the same procedure at each of the other points s_k, $k \neq j$. Repeat this *ad infinitum*.

To formalize this idea consider the disjoint union

$$\dot{\bigcup_{g \in \mathbf{G}_m}} gF.$$

Introduce a glueing relation $\sim$ on this sum: every two summands of the form gF and gg_jF, $g \in \mathbf{G}_m$, $j = 1, \ldots, m$, are glued together by identifying borders

$$gs_j^- \equiv gg_js_j^+ \quad \text{and} \quad gs_j^+ \equiv gg_js_j^- .$$

Now define the quotient surface

$$F_e := \dot{\bigcup_{g \in \mathbf{G}_m}} gF / \sim .$$

There is exactly one natural projection $\rho_e : F_e \to \mathbb{C}$, which extends ρ. To define it we introduce the following homomorphism:

$$\psi_F : \mathbf{S}_m \longrightarrow \mathbf{Aut}(\mathbb{C}), \quad \sigma_j \longmapsto \sigma_{\rho(s_j)}.$$

Then define

$$\rho_e(gx) := \big(\psi_F\overline{\Psi}(g)\big)\big(\rho(x)\big), \quad g \in \mathbf{G}_m,\ x \in F.$$

To check that ρ_e is well-defined, we have to show that

$$\rho_e(gx) = \rho_e(gg_jy)$$

whenever $j = 1, \dots, m$ and $gx \sim gg_jy$; but $gx \sim gg_jy$ implies that $x, y \in F$ are "opposite points with respect to s_j", *i.e.*, $\sigma_{\rho(s_j)}(\rho(y)) = \rho(x)$. Therefore

$$\begin{aligned}\rho_e(gg_jy) &= \big(\psi_F\overline{\Psi}(gg_j)\big)\big(\rho(y)\big)\\ &= \big(\psi_F\overline{\Psi}(g)\big)\big(\psi_F\overline{\Psi}(g_j)\big)\big(\rho(y)\big)\\ &= \big(\psi_F\overline{\Psi}(g)\big)\sigma_{\rho(s_j)}\big(\rho(y)\big)\\ &= \big(\psi_F\overline{\Psi}(g)\big)\rho(x) = \rho_e(gx).\end{aligned}$$

The next assertions are immediate consequences of the constructions carried out.

1) F_e is a Riemann surface, and $\rho_e : F_e \to \mathbb{C}$ is an infinitely-sheeted branched covering.

2) There is a natural group-action

$$\mathbf{G}_m \times F_e \longrightarrow F_e, \quad (g, g'x) \longmapsto gg'x. \tag{8}$$

A fundamental domain under this action is $(F - \partial F) \cup s_1^{\epsilon_1} \cup \cdots \cup s_m^{\epsilon_m}$, $\epsilon_j = \pm$.

3) The branch-points of ρ_e are precisely the points, where the action is not properly discontinous, namely the points $gs_j = gg_js_j$, $j = 1, \dots, m$, $g \in \mathbf{G}_m$. These are the m orbits $\mathbf{G}_m s_j$ under the action (8).

4) All the branch-points of ρ_e have order 3. A point $s \in \mathbb{C}$ is the image of a branch-point precisely if $s = \sum n_j\rho(s_j)$, where $n_j \in \mathbb{Z}$, $\sum n_j = 1$ (*cf.* Remark 1). In particular, when $m \geq 4$ then, in general, the projection of the branch-points is dense in $\mathbb{C}$.

5) The action (8) and the isomorphism Θ induce an action

$$(\mathbf{F}_{m-1} \rtimes_\varphi \mathbb{Z}_2) \times F_e \longrightarrow F_e. \tag{9}$$

The biggest subgroup of $\mathbf{F}_{m-1} \rtimes_\varphi \mathbb{Z}_2$ acting properly discontinuously on F_e is $\mathbf{F}_{m-1} = \mathbf{F}_{m-1} \times 0$. Via Θ this group is the group of words of even length in $\mathbf{G}_m$. They leave the 1-form $\mathrm{d}\rho_e$ invariant, while all the other elements change $\mathrm{d}\rho_e$ to $-\mathrm{d}\rho_e$.

Note that the construction only depends *a priori* on the numbering of $s_1, \ldots, s_m$, since any permutation of the generators $g_1, \ldots, g_m$ of $\mathbf{G}_m$ induces an automorphism of $\mathbf{G}_m$.

Consider the action-tree $T \subset F$ and its image $T_e \subset F_e$ under the action (8); thus T_e is the connected graph which arises when identifying in the formal sum $\dot{\bigcup\limits_{g \in \mathbf{G}_m}} gT$ the points gs_j and gg_js_j. Every $g \in \mathbf{G}_m$ can be uniquely written without stuttering: $g = g_{j_1} \cdots g_{j_s}$, $j_k \neq j_{k+1}$. From this it follows that T_e is a tree. Since T is a deformation retract of F and since the retraction can be chosen with identical "speed" on s_j^- and s_j^+, one can construct a deformation retract from T_e onto F_e, whence

Remark 3. — The Riemann surface F_e is simply connected.

(There is a relation with the Cayley graph of $\mathbf{G}_m$, *cf.* [5]. Let Γ_m be the Cayley graph of $\mathbf{G}_m$. All edges come in pairs. If we identify each pair, then we get a tree Γ'. It is dual to the tree T_e in the following sense: fix a point x in T which is not a vertex. There is precisely one path γ_j from x to each vertex s_j. Now consider the tree T_e' that as a set is the same as T_e, but the vertices of T_e' are the points gx, $g \in \mathbf{G}_m$, and the edges are the paths $g\gamma_j \cup gg_j\gamma_j$. Then T_e' and Γ' are equivalent trees.)

It follows from the definition of the projection ρ_e that it is invariant under the action of $\ker(\psi_F\overline{\Psi}) \subset \mathbf{G}_m$. Clearly $\ker(\psi_F\overline{\Psi})$ is contained in the subgroup of words of even length. Therefore, by point 5) above, every subgroup $\mathbf{N} \subset \ker(\psi_F\overline{\Psi})$ acts properly discontinously on F_e leaving ρ_e invariant. Setting

$$(F_{\mathbf{N}}, \rho_{\mathbf{N}}) := (F_e, \rho_e)/\mathbf{N} := (F_e/\mathbf{N}, \rho_e/\mathbf{N}),$$

the quotient map

$$\begin{array}{c}(F_e, \rho_e)\\ \downarrow\\ (F_{\mathbf{N}}, \rho_{\mathbf{N}})\end{array}$$

is a covering. In view of Remark 3 the fundamental group of $F_{\mathbf{N}}$ is $\mathbf{N}$. For the special case $\mathbf{N} = \ker(\psi_F\overline{\Psi})$ (resp. $\mathbf{R}_m/\mathbf{R}_m^1$) we write superscript ∞ (resp. $\mathbf{s}$) instead of subscript $\mathbf{N}$ and we call (F^∞, ρ^∞) an *action-domain.*

Let us look back at the glueing idea at the begin of this section: while we obtain (F_e, ρ_e) by glueing copies of (F, ρ) without identifying any two of them, we do the same glueing procedure for getting (F^∞, ρ^∞), but we identify any two copies that happen to lie exactly one above the other. $(F^{\mathbf{s}}, \rho^{\mathbf{s}})$ is constructed the same way, but in general there are less copies identified than in (F^∞, ρ^∞). More precisely, one has

$$(F^\infty, \rho^\infty) = (F^{\mathbf{s}}, \rho^{\mathbf{s}})$$

exactly if $\rho(s_1), \dots, \rho(s_m)$ are affinely free over $\mathbb{Q}$, see Lemma 1.

The action (8) of $\mathbf{G}_m$ on F_e quotients to an action of $\mathbf{G}_m/\mathbf{N}$ on $F_{\mathbf{N}}$. In particular $\langle \sigma_{\rho(s_1)}, \dots, \sigma_{\rho(s_m)} \rangle$ (resp. $\mathbf{S}_m$) acts on F^∞ (resp. $F^{\mathbf{s}}$). The analogues of the five assertions above hold.

1.5. Spirals and points at infinity.

The Riemann surface F_e is equipped with the projection ρ_e on the plane $\mathbb{C}$. We shall investigate what happens around infinity and show that the quotient of $(F_e, \mathrm{d}\rho_e)$ by translations does not have an essential singularity at infinity. The same will hold for (F^∞, ρ^∞) and $(F^{\mathbf{s}}, \rho^{\mathbf{s}})$.

For $v \in \mathbb{C}$ set

$$\Sigma_v := \{(\xi, \eta) \in \mathbb{R}^2 : \xi > \ln |v|\}$$

(read $\Sigma_0 = \mathbb{R}^2$) and

$$\rho_v : \Sigma_v \longrightarrow \mathbb{C}, \quad (\xi, \eta) \longmapsto \mathrm{e}^{\xi + i\eta} + iv\eta.$$

LEMMA 2. — *(Σ_v, ρ_v) is an étale surface. In particular ρ_v induces a complex structure on Σ_v.*

Proof. — Writing $v = a + ib$ one has

$$\rho_v(\xi, \eta) = \mathrm{e}^\xi \cos\eta - b\eta + i(\mathrm{e}^\xi \sin\eta + a\eta).$$

From this one computes the (real) jacobian determinant of ρ_v as

$$\mathrm{e}^\xi(\mathrm{e}^\xi + a\cos\eta + b\sin\eta).$$

Since $|a\cos\eta + b\sin\eta| \le \sqrt{a^2 + b^2} = |v| < \mathrm{e}^\xi$, the jacobian determinant does not vanish. $\square$

We shall always equip Σ_v with the complex structure induced by the étale projection ρ_v. The vertical lines $\xi = \mathrm{C}^{\mathrm{te}}$ in Σ_v are projected on concentric spirals of radius e^{ξ} whose common center moves with velocity v. Therefore imagine (Σ_v, ρ_v) as the surface spread outside the spiral of radius $|v|$ whose center moves with speed v. In particular (Σ_0, ρ_0) is the universal covering of $\mathbb{C}^*$. For any $r > \ln|v|$ (read $\ln 0 = -\infty$) we write (Σ_v^r, ρ_v) for the subspace $\xi > r$.

Note that

$$[\Sigma_v^r, \mathrm{d}\rho_v] = [\Sigma_{-v}^r, \mathrm{d}\rho_{-v}]$$

(an isomorphism is given by $(\xi, \eta) \mapsto (\xi, \eta + \pi)$), therefore it is sufficient to give the speed v up to sign.

Definition 2. — *Let (X, ω) be a Riemann surface with meromorphic 1-form. We say that $U \subset X$ is a spiral neighborhood at infinity of radius $R > 0$ and speed $v \in \mathbb{C}$ if $[U, \omega_{|U}] = [\Sigma_v^r, \mathrm{d}\rho_v]$, where $\mathrm{e}^r = R$.*

The spirals (Σ_v^r, ρ_v) are easy to imagine, but they turn out to be unpractical for computations, because the complex structure on Σ_v^r does not coincide with $x + iy$ unless $v = 0$. Consider the group action of $\mathbb{Z}$ on Σ_v^r given by translation: $n \cdot (\xi, \eta) := (\xi, \eta + 2n\pi)$. Then

$$\rho_v\big(n \cdot (\xi, \eta)\big) = \rho_v(\xi, \eta) + 2ni\pi v,$$

so the form $\mathrm{d}\rho_v$ and the complex structure on Σ_v^r are invariant under this action. In order to say something about the quotient $(\Sigma_v^r, \mathrm{d}\rho_v)/n\mathbb{Z}$ we must give a more practical version of the spirals.

For $r \in \mathbb{R}$ let

$$H^r := \big\{z = x + iy \in \mathbb{C} :\ x > r\big\}$$

and for $v \in \mathbb{C}$ with $r > \ln|v|$

$$p_v : H^r \longrightarrow \mathbb{C}, \quad z \longmapsto \mathrm{e}^z + vz.$$

(H^r, p_v) also is an étale space, and again $\mathbb{Z}$ operates (H^r, p_v) by translation:

$$n \cdot z = z + 2ni\pi \quad \text{and} \quad p_v(n \cdot z) = p_v(z) + 2ni\pi v,$$

leaving the complex structure and $\mathrm{d}p_v$ invariant.

PROPOSITION 3. — *For $v \in \mathbb{C}$ and $r > \ln|v|$ there is a diffeomorphism $h : H^r \longrightarrow \Sigma_v^r$ such that the following two squares commute for any $n \in \mathbb{Z}$:*

$$\begin{array}{ccc} H^r & \xrightarrow{\ h\ } & \Sigma_v^r \\ {\scriptstyle n\cdot}\uparrow & & \uparrow{\scriptstyle n\cdot} \\ H^r & \xrightarrow{\ h\ } & \Sigma_v^r \\ {\scriptstyle p_v}\downarrow & & \downarrow{\scriptstyle \rho_v} \\ \mathbb{C} & \xrightarrow{\ \tau_{-rv}\ } & \mathbb{C}, \end{array}$$

where τ_{-rv} is the translation $z \mapsto z - rv$. In particular h is a biholomorphic map and

$$[H^r, \mathrm{d}p_v] = [\Sigma_v^r, \mathrm{d}\rho_v].$$

Proof. — Observe that a vertical line $x = c$ in H^r projects to the spiral $R\,\mathrm{e}^{iy} + v(c+iy)$, $R = \mathrm{e}^c$, $y \in \mathbb{R}$, and that a vertical line $\xi = c$ in Σ_v^r projects to the same spiral shifted by $-cv$. Therefore, talking "handwaving", both étale spaces look the same up to translation by $-rv$.

We shall formalize this idea and pull back the shift from the value-plane to the space H^r. Denote

$$\jmath : H^r \longrightarrow \Sigma_v^r$$

the map $z = x + iy \mapsto (x, y)$. For $v = 0$ one takes $h := \jmath$. Now let $v \neq 0$. Consider the family of maps for $t \in \mathbb{R}$:

$$F_t : H^r \longrightarrow \mathbb{C}, \quad z = x + iy \longmapsto \mathrm{e}^z + v\big(iy + (1-t)(x-r)\big).$$

One has $F_0 = \tau_{-rv} p_v$ and $F_1 = \rho_v \jmath$. Write the differential equation $\mathrm{d}F_t(z(t))/\,\mathrm{d}t = 0$:

$$(10) \qquad \dot{z}\,\mathrm{e}^z + v\big(\dot{z} - t\dot{x} - (x-r)\big) = 0.$$

If $\gamma(t)$ is an integral curve of this time-dependent differential equation then

$$\tau_{-rv} p_v\big(\gamma(0)\big) = F_0\big(\gamma(0)\big) = F_1\big(\gamma(1)\big) = \rho_v \jmath\big(\gamma(1)\big).$$

Hence, noting $\phi(t, z)$ the flow of (10), we take $h(z) := \phi(1, z)$ to make the lower square commute; since (10) is invariant under the translation $z \mapsto z + 2i\pi$, the upper square commutes as well.

We show now that the flow of (10) is defined for any time $t \in \mathbb{R}$. Writing $v = \mathrm{e}^{a+ib}$ and making the variable change $x+iy \mapsto (x-r)+i(y-b)$ brings (10) into the form

$$\dot{z} = \frac{t\dot{x}+x}{c\,\mathrm{e}^{z}+1}, \quad c := \mathrm{e}^{r-a} > 1. \tag{11}$$

This is a non-singular time-dependent differential equation on H_0; since it is invariant under the translation $z \mapsto z + 2i\pi$ we can consider (11) on the cylinder $H_0/2i\pi\mathbb{Z}$. Let

$$\gamma\colon [0,T[\longrightarrow H_0/2i\pi\mathbb{Z}, \quad T \in]0,+\infty],$$

be a maximal solution for positive time (the following argument also holds for negative time). Then either $T = +\infty$ or γ leaves any compact of $H_0/2i\pi\mathbb{Z}$ in a finite time.

Suppose now $T < +\infty$. We consider the two directions for leaving compacts: $x \longrightarrow 0$ and $x \longrightarrow +\infty$. We can get rid of the first by compactification of $H_0/2i\pi\mathbb{Z}$ "on the left": in fact (11) extends smoothly to the border $x = 0$ because $c > 1$ and any point on this border is a stationary solution. If γ leaves any compact subset as $x \longrightarrow +\infty$, then

$$\sup_{0\le t<T} \big|\dot{\gamma}(t)\big| = +\infty.$$

This is in contradiction with

$$\exists\, x_0 > 0,\ \forall\, z = x+iy \in \mathbb{C},\ x > x_0,\ \forall\, t \in [0,T[\ :\ \big|V(z,t)\big| < 1, \tag{12}$$

where V is the time-dependent vector field associated to (11). To prove assertion (12) let

$$w(z) = u(z) + iv(z) := (c\,\mathrm{e}^{z}+1)^{-1}.$$

Then (11) is equivalent to

$$\dot{x} = u(z)(t\dot{x}+x), \quad \dot{y} = v(z)(t\dot{x}+x),$$

from which follows

$$\big(1-tu(z)\big)\dot{x} = xu(z), \quad \big(1-tu(z)\big)\dot{y} = xv(z).$$

Hence

$$\big|1-tu(z)\big| \cdot |\dot{z}| = x \cdot \big|w(z)\big|. \tag{13}$$

Since $x(c\,\mathrm{e}^z+1)^{-1}\to 0$ as $x\to+\infty$, we can choose $x_0\gg 0$ such that for all $z=x+iy\in\mathbb{C}, x>x_0$ one has $x\cdot|w(z)|<\frac{1}{2}$ and $T|u(z)|<\frac{1}{2}$. The last estimation implies $|1-tu(z)|>\frac{1}{2}$ for any $z=x+iy\in\mathbb{C}$, $x>x_0$, $t\in[0,T[$. Thus we get from (13) that $|\dot{z}|<1$ whenever $z=x+iy\in\mathbb{C}$, $x>x_0$, $t\in[0,T[$. □

In view of Proposition 3, we shall rather use the spirals (H^r,p_v) instead of (Σ_v^r,ρ_v).

COROLLARY 1. — *Let $v\in\mathbb{C}$, $n\geq 1$, and $r>0$ with $r>\ln|v|$. Let $\overline{D}_R$ be the closed disk $|z|\leq R$, $R=\mathrm{e}^r$. Then*

1) $I[\mathbb{C}-\overline{D}_R, n(z^{n-1}+v/z)\,\mathrm{d}z]=[H^r,\,\mathrm{d}p_v]=[\Sigma_v^r,\mathrm{d}\rho_v]$, *if* $v\neq 0$;

2) $[\mathbb{C}-\overline{D}_R, n(z^{n-1}+v/z)\,\mathrm{d}z]=[(H^r,\,\mathrm{d}p_v)/n\mathbb{Z}]=[(\Sigma_v^r,\mathrm{d}\rho_v)/n\mathbb{Z}]$.

Proof. — The second equalities in both assertions follow from the proposition.

Let $v\neq 0$. Then $n(z^{n-1}+v/z)\,\mathrm{d}z$ does not have a single-valued primitive on $\mathbb{C}-\overline{D}_R$; the primitive lives on the universal covering. The map

$$p\colon H^{r/n}\longrightarrow\mathbb{C}-\overline{D}_R,\quad \zeta\longmapsto\mathrm{e}^{n\zeta}$$

is a universal covering. A primitive s of $p^*(n(z^{n-1}+v/z)\,\mathrm{d}z)$ is given by $s(\zeta)=\mathrm{e}^{n\zeta}+nv\zeta$. Therefore

$$I\big[\mathbb{C}-\overline{D}_R,n(z^{n-1}+v/z)\,\mathrm{d}z\big]=[H^{r/n},\mathrm{d}s].$$

On the other hand

$$[H^{r/n},\mathrm{d}s]=[H^r,\mathrm{d}p_v]$$

because of the isomorphism $H^{r/n}\to H^r$, $\zeta\mapsto n\zeta$, which transports s to ρ_v. This proves the first equality of assertion 1). The first equality of assertion 2) now follows because

$$\big[(H^r,\,\mathrm{d}p_v)/n\mathbb{Z}\big]=\big[(H^{r/n},\,\mathrm{d}s)/\mathbb{Z}\big]=\big[\mathbb{C}-\overline{D}_R,n(z^{n-1}+v/z)\mathrm{d}z\big].\quad\square$$

Let $(F,\rho)\in\mathcal{F}_m$. Recall the arcs $C_1^R,\dots,C_m^R$ and the segments $L_1^R,\dots,L_m^R$ defined in 1.3 for $R\gg 0$; we numerate them as in (6). Let $\mathbf{V}$ (resp. $\mathbf{W}$) be the infinite cyclic subgroup of $\mathbf{G}_m$ generated by $g_1\cdots g_m$ (resp. $(g_1\cdots g_m)^2$). Let $x\in F$ be the initial-point of C_1^R and $y\in F$ the end-point of L_m^R. Then there is a path C_e^R in F_e from x to $g_1\cdots g_{m-1}y=g_1\cdots g_m x$ given by

$$C_e^R:=\bigcup_{j=0,\dots,m-1} g_1\cdots g_j(C_{j+1}^R\cup L_{j+1}^R).$$

Define an infinite path in F_e

$$S_e^R := \bigcup_{n\in\mathbb{Z}} (g_1\cdots g_m)^n C_e^R = \bigcup_{v\in\mathbf{V}} vC_e^R.$$

The fact that the infinite path S_e^R exists for all $R \gg 0$ can be translated as follows: "when running through S_e^R in the positive sense, one never sees a singularity on the right-hand side", or, equivalently, $(F_e, \mathrm{d}\rho_e)$ contains a spiral neighborhood $(U, \mathrm{d}\rho_{e|U})$ at infinity. To determine its speed v we have to distinguish the cases m even or odd.

Let m be odd. $\mathbf{V}$ acts on U but only the elements in $\mathbf{W} \subset \mathbf{V}$ leave $\mathrm{d}\rho_{e|U}$ invariant (the others change sign). The path $C_e^R \cup (g_1\cdots g_m)C_e^R$ in F_e goes from x to $(g_1\cdots g_m)^2x$; the projections via ρ_e of these two points coincide, since $\overline{\Psi}(g_1\cdots g_m)$ is of order 2 in $\mathbf{S}_m$. Therefore the speed of $(U, \mathrm{d}\rho_{e|U})$ is $v = 0$. The sum of the angles of the arcs $C_1^R, \ldots, C_m^R$ tends to $(m+2)\pi$ when $R \to +\infty$; thus when going from a point $x \in U$ to $(g_1\cdots g_m)^2x \in U$, one describes $(m+2)$-times a big circle.

Let m be even. $\mathbf{V}$ acts on U, leaving $\mathrm{d}\rho_{e|U}$ invariant. The difference between the projections of a point $x \in U$ and $g_1\cdots g_m x$ is precisely $\pm\mathrm{res}(F,\rho)$. When going from x to $g_1\cdots g_m x$ one describes $\frac{1}{2}(m+2)$-times a big circle. Therefore the speed is $v = \pm 2\,\mathrm{res}(F,\rho)/(m+2)$.

Remark 4. — $(F_e,\ \mathrm{d}\rho_e)$ contains a spiral neighborhood $(U,\ \mathrm{d}\rho_{e|U})$ isomorphic to $(H^r,\ \mathrm{d}p_v)$, where $r \gg 0$ and $v = \pm 2\,\mathrm{res}(F,\rho)/(m+2)$. Moreover all spiral neigborhoods of $(F_e,\ \mathrm{d}\rho_e)$ are of the form gU with $g \in \mathbf{G}_m$.

THEOREM 1. — *For $(F,\rho) \in \mathcal{F}_m$ the quotient of $(F_e,\ \mathrm{d}\rho_e)/\mathbf{F}_{m-1}$ (cf. (9)) has one (resp. two) point(s) at infinity when m is odd (resp. even). If m is odd, $\mathrm{d}\rho_e/\mathbf{F}_{m-1}$ has a pole at infinity of order $m+3$ with residue 0; if m is even, $\mathrm{d}\rho_e/\mathbf{F}_{m-1}$ has poles of order $\frac{1}{2}m+2$ at infinity with residues $\pm\mathrm{res}(F,\rho)$.*

Proof. — We have to investigate the behavior of all spiral neighborhoods under the action of $\mathbf{F}_{m-1}$. As usual we identify $\mathbf{F}_{m-1} = \mathbf{F}_{m-1} \times 0$ via Θ with the subgroup of $\mathbf{G}_m$ formed by the elements of even length.

If U is a spiral neighborhood of $(F_e, \mathrm{d}\rho_e)$, then $[U, \mathrm{d}\rho_{e|U}] = [H^r, \mathrm{d}p_v]$, where $r \gg 0$ and $v = \pm 2\,\mathrm{res}(F,\rho)/(m+2)$. The collection of all spiral neighborhoods in $(F_e,\ \mathrm{d}\rho_e)$ is $\bigcup_{g\in\mathbf{G}_m} gU$.

Consider the case m even. Then $\mathbf{V} \subset \mathbf{F}_{m-1}$ and

$$\Big[\Big(\bigcup_{g\in\mathbf{G}_m} gU, \mathrm{d}\rho_e\Big)/\mathbf{F}_{m-1}\Big] = \big[(U \cup g_1U, \mathrm{d}\rho_e)/\mathbf{V}\big].$$

The action of $\mathbf{V}$ on $(U, \mathrm{d}\rho_{e|U})$ corresponds to the action of $\frac{1}{2}(m+2)\mathbb{Z}$ on $(H^r, \mathrm{d}p_v)$. Hence by Corollary 1

$$\Big[\Big(U, \mathrm{d}\rho_{e|U}\Big)/\mathbf{V}\Big] = \big[\mathbb{C} - \bar{D}_R, \tfrac{1}{2}(m+2)(z^{\frac{m+2}{2}-1} + v/z)\,\mathrm{d}z\big].$$

The form

$$\frac{m+2}{2}\Big(z^{\frac{m+2}{2}-1} + \frac{v}{z}\Big)\,\mathrm{d}z = \Big(\frac{m+2}{2}z^{m/2} \pm \frac{\mathrm{res}(F,\rho)}{z}\Big)\,\mathrm{d}z$$

has a pole of order $\frac{1}{2}m + 2$ at infinity with the desired residue. The same holds for the other component $(g_1U, \mathrm{d}\rho_{e|g_1U})/\mathbf{V}$ with opposite sign of v.

Let us now consider the case when m is odd. Here $\mathrm{res}(F,\rho) = 0$, so the spiral does not "move":

$$\begin{aligned}\Big[\Big(\bigcup_{g\in\mathbf{G}_m} gU, \mathrm{d}\rho_e\Big)/\mathbf{F}_{m-1}\Big] &= [(U, \mathrm{d}\rho_e)/\mathbf{W}]\\ &= [(H^r, \mathrm{d}p_0)/(m+2)\mathbb{Z}]\\ &= \big[\mathbb{C} - \bar{D}_R, (m+2)z^{m+1}\,\mathrm{d}z\big].\end{aligned}$$

The form $z^{m+1}\,\mathrm{d}z$ has a pole of order $m+3$ at infinity without residue. □

2. Integrating $(\mathcal{L}_a, y\mathrm{d}x)$.

2.1. Hyperelliptic curves with Liouville-form.

Let $\Delta \subset \mathbb{C}^{m-1}$ be the discriminant-subset formed by the points $a = (a_0, \ldots, a_{m-2})$ such that the polynomial

$$P_a(x) = x^m + a_{m-2}x^{m-2} + \cdots + a_0$$

has multiple roots. Let

$$\mathbb{C}^{m-1}_\Delta := \mathbb{C}^{m-1} - \Delta \quad \text{and} \quad \mathcal{L} := \{(x,y,a) \in \mathbb{C}^2 \times \mathbb{C}^{m-1}_\Delta : y^2 = P_a(x)\}.$$

Then the natural projection

$$\begin{array}{c}\mathcal{L}\\ \downarrow\\ \mathbb{C}^{m-1}_\Delta\end{array}$$

is a locally trivial fibration. Denote the fibers by $\mathcal{L}_a$, $a \in \mathbb{C}^{m-1}_\Delta$. Each projection $\pi_a : \mathcal{L}_a \to \mathbb{C}$, $(x,y) \mapsto x$, is a two-sheeted ramified covering with branch-points precisely above the roots of P_a. Therefore, if $m = 1$ (resp. 2), each fiber is $\mathbb{P}^1$ with one (resp. two) point(s) deleted. When $m \geq 3$ then each fiber is a (hyper)elliptic curve of genus $[\frac{1}{2}(m-1)]$ with one (resp. two) point(s) at infinity deleted if m is odd (resp. even).

Conversely, each elliptic curve with one (resp. two) deleted points is isomorphic to a curve of the family $(\mathcal{L}_a)_{a\in\mathbb{C}^3_\Delta}$ (resp. $(\mathcal{L}_a)_{a\in\mathbb{C}^4_\Delta}$). Each hyperelliptic curve of genus g with one Weierstraß point (resp. two points in hyperelliptic involution) deleted is isomorphic to a curve of the family $(\mathcal{L}_a)_{a\in\mathbb{C}^{m-1}_\Delta}$, where $m = 2g+1$ (resp. $2g+2$).

LEMMA 3. — *Let $a \in \mathbb{C}^{m-1}_\Delta$. Then $\pi_1(\mathcal{L}_a) \simeq \mathbf{F}_{m-1}$. Choose any tree Γ in the x-plane with vertices the m roots of P_a and with $m-1$ edges d_j, $j = 1,\dots,m-1$. Let λ_j be the homotopy class of the closed loop $\pi_a^{-1}(d_j)$ (oriented arbitrarily). Then $\lambda_1,\dots,\lambda_{m-1}$ form a basis of $\pi_1(\mathcal{L}_a)$.*

Proof. — Any retraction by deformation from $\mathbb{C}$ onto Γ lifts to one from $\mathcal{L}_a$ onto $\pi_a^{-1}(\Gamma)$, and $\pi_a^{-1}(\Gamma)$ is a bouquet of the $m-1$ circles $\pi_a^{-1}(d_j)$. □

On each curve $\mathcal{L}_a$ consider the *Liouville-form* $y\,\mathrm{d}x$. Its divisor is

$$\operatorname{div}(y\,\mathrm{d}x) = \begin{cases} 2R_1 + \cdots + 2R_m - (m+3)R_\infty & \text{if } m \text{ is odd,} \\ 2R_1 + \cdots + 2R_m - \left(\frac{1}{2}m+2\right)(R_\infty + R'_\infty) & \\ & \text{if } m \text{ is even,} \end{cases}$$

where $R_1,\dots,R_m$ are the branch-points of π_a and R_∞, R'_∞ the points at infinity. The sum of the residues of a meromorphic 1-form on a compact surface being zero, $y\,\mathrm{d}x$ has no residue at R_∞ when m is odd, and residues of opposite sign at R_∞, R'_∞ when m is even. (In the case $m = 4$ on can show easily that the residues vanish precisely if the four roots of $P_a(x)$ form a parallelogram. For even $m \geq 4$ the subset of $\mathbb{C}^{m-1}_\Delta$ formed by the a such that $y\,\mathrm{d}x$ has no residue at the infinte pionts of $\mathcal{L}_a$ is an analytic hypersurface. In particular the development that M. Fedoryuk gives on page 85 of [3] nearly never exists.) Non-vanishing residues are certainly one obstruction for $y\,\mathrm{d}x$ to be exact; but even in the case of odd m, $y\,\mathrm{d}x$ is not exact:

LEMMA 4. — *For all $a \in \mathbb{C}^{m-1}_\Delta$ the Liouville-form $y\,\mathrm{d}x$ on $\mathcal{L}_a$ is not exact.*

Proof. — The cases $m = 1, 2$ are easy. Let us treat the cases $m > 2$. Suppose that there is $f : \mathcal{L}_a \to \mathbb{P}^1$ such that $\mathrm{d}f = y\,\mathrm{d}x$. Then one sees on the expression for $\operatorname{div}(y\,\mathrm{d}x)$ above that at infinity f has a pole of order $m+2$ when m is odd and two poles, each of order $\frac{1}{2}m+1$, when m is even. Therefore in any case the degree of f is $m+2$.

Now choose a tree in the x-plane such that each edge d_j, $j = 1, \ldots, m-1$, has $\pi_a(R_j)$ and $\pi_a(R_m)$ as vertices; then let $\lambda_1, \ldots, \lambda_{m-1}$ be the corresponding closed loops in $\mathcal{L}_a$ according to Lemma 3. Since $y\,\mathrm{d}x$ is exact one has $\int_{\lambda_j} y\,\mathrm{d}x = 0$. The form $y\,\mathrm{d}x$ changes sign under the involution $(x, y) \mapsto (x, -y)$ on $\mathcal{L}_a$. Therefore

$$\int_{\lambda_j} y\,\mathrm{d}x = \pm 2\big(f(R_m) - f(R_j)\big).$$

Hence f takes the value $f(R_m)$ at m different points with multiplicity 2, so the degree of f is at least $2m$; and $2m > m+2$ when $m > 2$. □

Using the quasi-homogeneous properties of $(\mathcal{L}_a, y\,\mathrm{d}x)_{a \in \mathbb{C}^{m-1}_\Delta}$ with respect to a it is not difficult to find the moduli-space of the family $[\mathcal{L}_a, y\,\mathrm{d}x]_{a \in \mathbb{C}^{m-1}_\Delta}$. In fact every $\lambda \in \mathbb{C}^*$ gives rise to an isomorphism

$$\hat{\lambda} : \mathcal{L}_a \longrightarrow \mathcal{L}_{\lambda \cdot a}, \quad (x, y) \longmapsto (\lambda x, \pm \lambda^{m/2} y), \tag{14}$$

where

$$\lambda \cdot (a_0, \ldots, a_{m_2}) := (\lambda^m a_0, \ldots, \lambda^2 a_{m-2}).$$

The pull-back via $\hat{\lambda}$ of the Liouville-form on $\mathcal{L}_{\lambda \cdot a}$ differs from the Liouville-form on $\mathcal{L}_a$ by a factor $\pm \lambda^{m/2+1}$; hence $\hat{\lambda}$ leaves the Liouville-form invariant precisely when λ is a $(m+2)$-th root of unity. Therefore the moduli-space of $[\mathcal{L}_a, y\,\mathrm{d}x]_{a \in \mathbb{C}^{m-1}_\Delta}$ is the quotient

$$\mathbb{C}^{m-1}_\Delta / \mathbb{Z}_{m+2} \tag{15}$$

under the group-action of $\mathbb{Z}_{m+2}$ on $\mathbb{C}^{m-1}_\Delta$ given by

$$k \cdot (a_0, \ldots, a_{m-2}) = (k \cdot a_0, \ldots, k \cdot a_{m-2}), \quad k \cdot a_j := \mathrm{e}^{\frac{m-j}{m+2} 2ki\pi} a_j.$$

(Closer investigation of this action shows without difficulty that the moduli-space (15) is singular whenever $m \geq 4$ is even; for odd $m > 1$ it is non-singular precisely when $m \equiv 1 \bmod 6$. In contrast to this, the moduli-space of hyperelliptic curves of genus g has singularities for any g; in fact, it is the quotient of an open subset of $\mathbb{C}^{2g-1}$ by a group-action, which has non-trivial stabilizors at certain points, *cf.* [6], p. 3.124. The reason for this difference is that our classification is "rigidized" by requiring the invariance of the form $y\,\mathrm{d}x$: we can't simply fix three of the Weierstraß-points at $0, 1, \infty$ as is done in [6].)

2.2. $[F^\infty, \mathrm{d}\rho^\infty]$ is the integration $I[\mathcal{L}_a, y\,\mathrm{d}x]$.

DEFINITION 3. — *For $(F, \rho) \in \mathcal{F}_m$ let $F^\div$ be the Riemann surface obtained by taking a copy F' of F and glueing the border s_j^+ (resp. s_j^-) of F' to the border s_j^- (resp. s_j^+) of F, $j = 1, \dots, m$. Let $\omega^\div$ be the (unique) holomorphic 1-form on $F^\div$ continuing $\mathrm{d}\rho_{|F-\partial F}$.*

THEOREM 2. — *Let $(F, \rho) \in \mathcal{F}_m$.*

1) $[F^\div, \omega^\div] = [\mathcal{L}_a, y\,\mathrm{d}x]$ *for some* $a \in \mathbb{C}_\Delta^{m-1}$.

2) *One has* $I[F^\div, \omega^\div] = [F^\infty, \mathrm{d}\rho^\infty]$.

Proof. — First note that $(F^\div, \omega^\div)$ is isomorphic to the quotient $(F_e, \mathrm{d}\rho_e)/\mathbf{F}_{m-1}$ of Theorem 1.

1) Associating to a point in the original $F \subset F^\div$ the corresponding point in the copy $F' \subset F^\div$ and vice versa, defines a group action of $\mathbb{Z}_2$ on $F^\div$. The quotient $F^\div/\mathbb{Z}_2$ is $\underline{F} \simeq \mathbb{C}$ (*cf.* Prop. 2), and the quotient map $F^\div \longrightarrow \underline{F}$ is a ramified two-sheeted covering with m branch-points $s_1, \dots, s_m$. Consider the polynomial whose roots are the images of $s_1, \dots, s_m$ under the isomorphism $\underline{F} \simeq \mathbb{C}$; we can suppose that the sum of the roots is zero, so the polynomial is in the family $(P_a)_{a \in \mathbb{C}_\Delta^{m-1}}$ giving rise to an isomorphism

$$h : \mathcal{L}_a \longrightarrow F^\div,$$

sending the branch-points of π_a to the branch-points $s_1, \dots, s_m$ of $F^\div \longrightarrow \underline{F}$. The form $\omega^\div$ vanishes precisely at the branch-points $s_1, \dots, s_m$ and does it with multiplicity 2. At infinity $\omega^\div$ does not have essential singularities: in Theorem 1 we proved that, in the case m odd, it has a pole of order $m+3$ at infinity, whereas in the case m even it has two poles of order $\frac{1}{2}m + 2$ at infinity. Comparing the divisors $\mathrm{div}(\omega^\div)$ and $\mathrm{div}(y\,\mathrm{d}x)$ and using the fact that two meromorphic 1-forms with the same divisor on a compact Riemann surface coincide up to a constant factor, one concludes that the pull-back $h^*(\omega^{\div})$ coincides up to a constant factor with the Liouville-form on $\mathcal{L}_a$. Now composing h with the isomorphism $\hat{\lambda}$ from (14) for a suitable $\lambda \in \mathbb{C}^*$ one gets the desired result.

2) For constructing a representative (Y, ω) of $I[F^{\div}, \omega^{\div}]$ we need the kernel of the period-homomorphism

$$\pi_1(F^{\div}) \longrightarrow \mathbb{C}, \quad \lambda \longmapsto \int_\lambda \omega^{\div}. \tag{16}$$

Then Y will be the covering of $F^{\div}$ with this kernel as characteristic sub-group and ω will be the pullback of $\omega^{\div}$. Let $\Gamma \subset F$ be a tree with vertices $s_1, \dots, s_m$ and edges $d_1, \dots, d_{m-1}$, such that the endpoints of the edge d_j are s_j, s_m. (In general Γ is not the action-tree T.) Under the isomorphism $\underline{F} \simeq \mathbb{C}$ the tree Γ is mapped to a tree on which we can apply Lemma 3 to get a basis $\lambda_1, \dots, \lambda_{m-1}$ of $\pi_1(F^{\div}) = \pi_1(\mathcal{L}_a)$. Choosing a convenient orientation of the λ_j the period-homomorphism (16) is the map

$$\lambda_j \longmapsto 2\big(\rho(s_j) - \rho(s_m)\big), \quad j = 1 \dots, m-1.$$

Therefore under the isomorphism $\mathbf{F}_{m-1} \to \pi_1(F^{\div})$, $\lambda_j \mapsto f_j$, the kernel of the period-homomorphism is formed by the elements $f \in \mathbf{F}_{m-1}$ such that $(f, 0) \in \ker(\psi_F \overline{\Psi} \Theta)$.

Now consider the action (9); the quotient $F_e/\mathbf{F}_{m-1}$ is $F^{\div}$. By Remark 3 the quotient map $F_e \to F^{\div}$ is the universal covering of $F^{\div}$. By definition F^∞ is the quotient $F_e/\ker(\psi_F \overline{\Psi} \Theta)$, *i.e.*, the covering $F^\infty \to F^{\div}$ has $\ker(\psi_F \overline{\Psi} \Theta)$ as characteristic sub-group. Clearly $\mathrm{d}\rho^\infty$ is the pullback of $\omega^{\div}$. □

Note that F^∞ also is the Riemann surface of $\int x \, \mathrm{d}y$, since $y \, \mathrm{d}x$ and $x \, \mathrm{d}y$ are cohomologous. But $[F^\infty, \mathrm{d}\rho^\infty] \neq I[\mathcal{L}_a, x \, \mathrm{d}y]$, see the end of the Section 1.1.

2.3. Stokes lines.

Consider the stationary Schrödinger equation

$$\Big(-\hbar^2 \Big(\frac{\mathrm{d}}{\mathrm{d}q}\Big)^2 + V(q)\Big)\psi(q) = E\psi(q) \tag{17}$$

in one dimension. The potential V determines the lagrangian curve

$$p^2 = E - V(q)$$

in (q, p)-phase-space, on which is defined the action-integral $\int p \, \mathrm{d}q$ as multivalued function.

Complexify the variables (q,p) to $(x,y) \in \mathbb{C}^2$, consider the case $V(x)$ a (complex) polynomial and write[(1)]

$$P_a(x) := V(x) - E.$$

Then (17) reads

$$\hbar^2 \Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^2 \psi(x) = P_a(x)\psi(x).$$

To the lagrangian curve above corresponds the (hyper)elliptic curve $\mathcal{L}_a$: $y^2 = P_a(x)$. (Note that with these conventions the multivalued action integral $S = \int y\,\mathrm{d}x$ on $\mathcal{L}_a$ differs with the "classical" action defined above by the factor i.)

The (real) curves on $\mathcal{L}_a$ along which S has constant imaginary part and which start at a branch-point of $\pi_a : \mathcal{L}_a \to \mathbb{C}$, $(x,y) \mapsto x$, are called *Stokes lines.* Since $y\,\mathrm{d}x$ simply changes sign under the involution $(x,y) \mapsto (x,-y)$, Stokes lines go to Stokes lines under this involution. Therefore they are completely determined by their projection via π_a on the x-plane, where they are also called Stokes lines. In [9] or [3] are many pictures of Stokes lines (the Stokes lines represented there are sometimes called *anti-Stokes lines* and they differ from our definition by rotation of 90°.) The importance of Stokes lines for asymptotic developments of the wave-function ψ is explained in [9], [3], and in terms of the "exact WKB method" it is exposed in [2], [8].

Consider the Stokes lines in $\mathbb{C}$ for P_a, $a \in \mathbb{C}^{m-1}_{\Delta}$. They have $m+2$ asymptotic directions $2n\pi/(m+2)$, $n = 0, \dots, m+1$, when tending to ∞ and they divide the plane into $m+2$ domains of half-plane type and k domains of strip type, where $0 \leq k \leq m-1$. Let $\Delta^+ \subset \mathbb{C}^{m-1}$ be the real hypersurface of parameters such that the number k of domains of strip type is maximal, *i.e.*, $k = m-1$. This means that there is no finite Stokes line. Here is an algorithm to make m cuts along Stokes lines such that on the cut x-plane the action-integral $S = \int y\,\mathrm{d}x$ is singlevalued and sends the cut plane biholomorphically to the interior $F - \partial F$ of an action-element $(F,\rho) \in \mathcal{F}_m$:

1) Cut along the Stokes line which is the first in counterclockwise sense asymptotic to the direction[(2)] $2\pi/(m+2)$. Call the corresponding

---

(1) It is possible to rescale V so that $a \in \mathbb{C}^{m-1}$, where $m = \deg(V)$. Moreover we suppose that $x \notin \Delta$.

(2) Asymptotic when going to ∞.

turning point x_1 and delete the two other Stokes lines that leave x_1. There remain $3(m-1)$ Stokes lines.

2) $n-1 \Rightarrow n$: go in counterclockwise sense to the next asymptotic direction containing one of the remaining $3(m-n+1)$ Stokes lines and cut along the first. Call the corresponding turning point x_n and delete the two other Stokes lines that leave x_n. There remain $3(m-n)$ Stokes lines.

The numbering $x_1, \ldots, x_m$ obtained this way corresponds precisely to (6) up to permutations of the form $(1, \ldots, m) \mapsto (k, \ldots, m, 1, \ldots, k-1)$. We can translate Remark 2 into the space of the parameter a as follows: $\mathbb{C}^{m-1} - \Delta^+$ has c_m components, they are in one-to-one correspondance with the topological types of Stokes-patterns without finite Stokes lines; each component is non-compact and contractible.

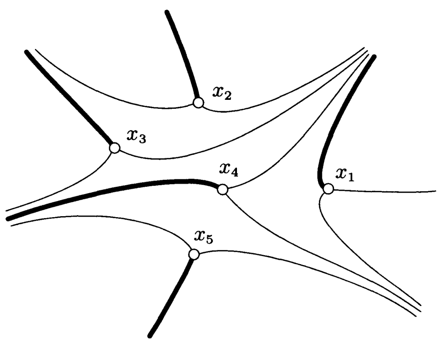

Figure 2: Stokes-pattern corresponding to the type of the third action-tree on Fig. 1. Heavy lines are cuts.

In all the preceding constructions the horizontal direction has been endowed with a privileged role: in the construction of action-elements in Section 1.3 we supposed that no point s_j "sees" another point s_k in the horizontal direction, or, equivalently, that action-trees are always growing strictly upwards. Of course, this choice is arbitrary, we can define a set $\mathcal{F}_m^\vartheta$ of action-elements in any direction $\vartheta \in \mathbb{R}/2\pi$ ($\mathcal{F}_m = \mathcal{F}_m^0$); as soon as we glue together to action domains F_e, F^{s} or F^∞ we can "move" the points s_j without hurting borders.

The converse to Theorem 2 holds as well: for each $a \in \mathbb{C}_\Delta^{m-1}$ there is a direction $\vartheta \in \mathbb{R}/2\pi$ and a $(F, \rho) \in \mathcal{F}_m^\vartheta$ such that $I[\mathcal{L}_a, y\,\mathrm{d}x] = [F^\infty, \mathrm{d}\rho^\infty]$. In fact, the period-group

$$\left\{ \int_\gamma y\,\mathrm{d}x : \gamma \in H^1(\mathcal{L}_a) \right\}$$

being a countable sub-set of $\mathbb{C}$, it is clear that there is a direction which no period takes.

The period-group never reduces to zero because of Lemma 4. For any non-trivial sub-group of $(\mathbb{C},+)$ generated by $m-1$ points $z_1,\dots,z_{m-1}$ there is $a\in\mathbb{C}^{m-1}_{\Delta}$ such that $(\mathcal{L}_a, y\,\mathrm{d}x)$ has this group as period-group: it suffices to take any action element $(F,\rho)\in\mathcal{F}^{\vartheta}_m$ such that $2(\rho(s_j)-\rho(s_m))=z_j$, $j=1,\dots,m-1$. This also follows from [7]. (A question is whether the parameters a for which the period-group of $(\mathcal{L}_a, y\,\mathrm{d}x)$ has a given rank $k\in\{1,\dots,m-1\}$ form an algebraic variety.)

Final remark. — The dicussion of spirals and points at infinity in Section 1.5 has mainly two aims: one is to show that a spiral quotiented by translations is biholomorphic to the punctured unit disk, the other is to prove that the quotient of the differential of the spiral's projection is meromorphic (*i.e.*, does not have an essential singularity at the center of the unit disk). The first can be proved like Proposition 2; the second results easily if the following beautiful conjecture is assumed to be true. (Then much of the cumbersome technics of Section 1.5 could be omitted.)

A conjecture. — Let $D^*=U_1\cup\dots\cup U_n$ be an open covering of the punctured unit-disk. Let $f_j:U_j\to\mathbb{C}, j=1,\dots,n$, be *schlicht* (*i.e.*, holomorphic and injective) such that $\mathrm{d}f_j=\mathrm{d}f_k$ on each intersection $U_j\cap U_k$. Then these differentials glue together to a meromorphic 1-form on the unit-disk D.

This conjecture is like a differential version of Picard's Theorem.

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