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LUIS PARIS
DALE ROLFSEN

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GEOMETRIC SUBGROUPS OF SURFACE BRAID GROUPS

by L. PARIS and D. ROLFSEN

1. Introduction.

The classical braid groups B_m were introduced by Artin in 1926 (see [Ar1], [Ar2]) and have played a remarkable rôle in topology, algebra, analysis, and physics. A natural generalization to braids on surfaces was introduced by Fox and Neuwirth [FoN] in 1962. The surface braid groups, for closed surfaces, were calculated in terms of generators and relations during the ensuing decade (see [Bi1], [Sc], [Va], [FaV]). Since then, most progress in this subject has been in its relation with mapping class groups and the general theory of configuration spaces (see the surveys [Bi3], [Co]). However, recently there is renewed interest in these fascinating groups in their own right, in part because of the action of surface braid groups on certain topological quantum field theories.

The purpose of this paper is to continue the study of the structure of the surface braid groups, with emphasis on certain naturally-occuring subgroups. A subsurface of a surface gives rise to inclusion maps between their braid groups. We determine necessary and sufficient conditions that these inclusion-induced maps are injective in Section 2. The remainder of the paper is devoted to a detailed study of these "geometric" subgroups. In particular, we calculate their centralizers, normalizers and commensurators in the larger surface braid group. Commensurators, in infinite groups, are of importance in their (unitary) representation theory. It is our hope that these results will be useful in the further study of surface braid groups,

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their representations and applications. In the remainder of this introductory section we present definitions, basic properties of surface braid groups, and a brief review of the literature.

1.1. Surface braids and configuration spaces.

Let M be a topological manifold and choose distinct points $P_1, \ldots, P_m \in M$ (later we will specialize to $\dim(M) = 2$). A *braid* with m *strings* on M based at $(P_1, \ldots, P_m)$ is an m-tuple $b = (b_1, \ldots, b_m)$ of paths, $b_i\colon [0,1] \to M$, such that

1) $b_i(0) = P_i$ and $b_i(1) \in \{P_1, \ldots, P_m\}$ for all $i \in \{1, \ldots, m\}$,

2) $b_i(t) \neq b_j(t)$ for $i, j \in \{1, \ldots, m\}$, $i \neq j$, and for $t \in [0,1]$.

There is a natural notion of *homotopy* of braids. The *braid group* with m *strings* on M based at $(P_1, \ldots, P_m)$ is the group

$$B_mM = B_mM(P_1, \ldots, P_m)$$

of homotopy classes of braids based at $(P_1, \ldots, P_m)$. The group operation is concatenation of braids, generalizing the construction of the fundamental group. Indeed, for the case $m = 1$ we clearly have $B_1M(P_1) = \pi_1(M, P_1)$. For $m > 1$, it is useful to consider the class of *pure* braids, which have the property $b_i(1) = P_i$. These form a subgroup of B_m which we will denote by

$$PB_mM = PB_mM(P_1, \ldots, P_m).$$

Let Σ_m be the group of permutations of $\{P_1, \ldots, P_m\}$. There is a natural epimorphism $\sigma\colon B_mM \to \Sigma_m$; its kernel is the pure braid group, so we have an exact sequence

$$1 \to PB_mM \longrightarrow B_mM \xrightarrow{\sigma} \Sigma_m \to 1.$$

Note that, if M is a connected manifold of dimension at least two, then B_mM and PB_mM do not depend (up to isomorphism) on the choice of $P_1, \ldots, P_m$. An m-braid naturally gives rise to m different paths in M under the map $b \mapsto (b_1, \ldots, b_m)$. In the case of *pure* braids these are loops, so there is a natural homomorphism

$$PB_mM \longrightarrow \pi_1(M, P_1) \times \cdots \times \pi_1(M, P_m) \cong \pi_1(M^m),$$

where M^m denotes the m-fold cartesian power.

PROPOSITION 1.1 (see [Bi1]). — *If M is a connected manifold with* $\dim(M) > 2$, *the above map is an isomorphism. For* $\dim(M) = 2$ *it is surjective.*

The proof is straightforward when one views braids from the configuration space point of view (see [FoN], [FaN].) Let $F_m M$ denote the space of (ordered) distinct points of M, in other words $F_m M = (M^m)\backslash V$, where V is the big diagonal, consisting of m-tuples $x = (x_1, \ldots, x_m)$ for which $x_i = x_j$ for some $i \neq j$. Then we clearly have an isomorphism

$$PB_m M \cong \pi_1(F_m M).$$

Proof of Proposition 1.1. — The map in question is induced by the inclusion $F_m M = (M)^m \backslash V \to (M)^m$. Noting that $V = \bigcup_{1 \leq i < j \leq m} \{x_i = x_j\}$ is a union of submanifolds of codimension $\dim(M)$, the proposition follows from well-known general position arguments. □

Because of Proposition 1.1, braid theory (as formulated here) is of marginal interest for dimension ≥ 3 and we concentrate on dimension two, *i.e. surface* braid groups.

In the remainder of the paper, M will denote a connected surface, possibly with boundary and possibly nonorientable. To avoid pathology, we will assume M is either compact, or at least that it is a "punctured" compact manifold, *i.e.* M is homeomorphic to a compact 2-manifold, possibly with a finite set of points removed.

By permuting coordinates, there is a natural action of Σ_m upon $F_m M$ and we denote the orbit space, the space of *unordered* m-tuples, or configuration space, by

$$\widehat{F}_m M = F_m M / \Sigma_m.$$

We may view the full braid group as its fundamental group

$$B_m M \cong \pi_1(\widehat{F}_m M).$$

The inclusion $PB_m M \subseteq B_m M$ may thus be interpreted as the mapping induced by the covering space map $F_m M \longrightarrow \widehat{F}_m M$, which has fiber Σ_m. Fox and Neuwirth noted that $B_m(D)$, the braid groups of the disk D^2, coincide with the Artin braid groups.

One of the most useful tools in studying braid groups is the Fadell-Neuwirth fibration and its generalizations. As observed in [FaN], if M is a manifold and $1 \leq n < m$ the map $\rho: F_m M \to F_n M$ defined by

$$\rho(x_1, \dots, x_m) = (x_1, \dots, x_n)$$

is a (locally trivial) fibration which has the fiber $F_{m-n}(M \backslash \{P_1, \dots, P_n\})$. This gives rise to a long exact sequence of homotopy groups of these spaces. For example, in the case $n = m - 1$ we have the exact sequence

$$\cdots \longrightarrow \pi_2 F_m M \longrightarrow \pi_2 F_{m-1} M \longrightarrow \pi_1\big(M \backslash \{P_1, \dots, P_{m-1}\}\big) \longrightarrow PB_m M \to PB_{m-1} M \to 1.$$

The punctured surface $M \backslash \{P_1, \dots, P_{m-1}\}$ has the homotopy type of a one-dimensional complex, and we see immediately from the above long exact sequence that

$$\pi_k(F_m M) \cong \pi_k(F_{m-1} M) \cong \cdots \cong \pi_k(M), \quad k \geq 3$$

and

$$\pi_2(F_m M) \subset \pi_2(F_{m-1} M) \subset \cdots \subset \pi_2(M).$$

Because they are the only surfaces with nontrivial higher homotopy groups, the sphere S^2 and the projective plane P^2 are exceptional cases in the general theory.

Proposition 1.2. — *Suppose that M is a connected surface, $M \neq S^2$ or P^2, and $k \geq 2$. Then $\pi_k F_m M$ and $\pi_k \widehat{F}_m M$ are trivial groups.*

Proof. — Since $F_m M \to \widehat{F}_m M$ is a covering map, it suffices to prove the proposition for $F_m M$. But this follows from the observations made above, since $\pi_k(M) = 1$ for $k \geq 2$. □

Combining this with the Fadell-Neuwirth fibration:

Proposition 1.3. — *Suppose that M is a connected surface, $M \neq S^2$ or P^2, and $1 \leq n < m$. There is an exact sequence*

$$1 \to PB_{m-n} M \backslash \{P_1, \dots, P_n\} \longrightarrow PB_m M \xrightarrow{\rho} PB_n M \to 1. \qquad \square$$

Let Σ_n be the group of permutations of $\{P_1, \ldots, P_n\}$ and let Σ_{m-n} be the group of permutations of $\{P_{n+1}, \ldots, P_m\}$. The Fadell-Neuwirth map gives rise to a (locally trivial) fibration

$$\hat{\rho}: F_m M/(\Sigma_n \times \Sigma_{m-n}) \longrightarrow F_n M/\Sigma_n = \widehat{F}_n M$$

which has the fiber

$$\big(F_{m-n} M \backslash \{P_1, \ldots, P_n\}\big)/\Sigma_{m-n} = \widehat{F}_{m-n} M \backslash \{P_1, \ldots, P_n\}.$$

So:

PROPOSITION 1.4. — *Suppose that M is a connected surface, $M \neq S^2$ or P^2, and $1 \leq n < m$. There is an exact sequence*

$$1 \to B_{m-n} M \backslash \{P_1, \ldots, P_n\} \longrightarrow \sigma^{-1}(\Sigma_n \times \Sigma_{m-n}) \longrightarrow B_n M \to 1. \qquad \square$$

1.2. Torsion.

Except for $M = S^2, P^2$, the configuration space $\widehat{F}_m M$ is an Eilenberg-Maclane space, *i.e.* a classifying space for $B_m M$. As is well-known, a group which has elements of finite order must have an infinite-dimensional classifying space (see, *e.g.* [Br, Chap. VIII]). Since $\widehat{F}_m M$ has dimension $2m$, we can then conclude.

PROPOSITION 1.5. — *If M is a connected surface, $M \neq S^2$ or P^2, then its braid groups $B_m M$ have no elements of finite order.*

The braid groups of S^2 and P^2 do have torsion (with the exception the trivial group $B_1(S^2)$). We give a quick review of these, following [FaV] and [Va]. For S^2 take all the basepoints to lie in a disk $D^2 \subseteq S^2$ and let $\sigma_1, \ldots, \sigma_{m-1}$ be the standard braid generators of $B_m(D^2)$; σ_i exchanges P_i and P_{i+1}. They satisfy the famous braid relations

$$(*) \qquad \begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i, & |i-j| \geq 2; \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & 1 \leq i \leq m-2. \end{cases}$$

The same σ_i can also be taken to be generators of $B_m(S^2)$ where they still satisfy these relations. The word $\sigma_1 \sigma_2 \cdots \sigma_{m-1} \sigma_{m-1} \cdots \sigma_2 \sigma_1$ may be interpreted as the (pure) braid in which P_1 circles around $P_2, \ldots, P_m$, while those points stay fixed. This is clearly homotopic in S^2 to the identity braid, so we have the additional relation

$$\sigma_1\sigma_2\cdots\sigma_{m-1}\sigma_{m-1}\cdots\sigma_2\sigma_1 = 1.$$

It is shown in [FaV] that this, together with (*) are *defining* relations for $B_m(S^2)$. The element $\tau = \sigma_1\sigma_2\cdots\sigma_{m-1}$ has order $2m$ in $B_m(S^2)$; it can be pictured as a simple braid which permutes the basepoints cyclically.

For the projective plane, take σ_i as above corresponding to a disk engulfing the basepoints, and let ρ_j to be a braid in which the basepoint P_j travels along a nontrivial loop in P^2 while the other basepoints sit still. See [Va] for a more precise description and a proof that $B_n(P^2)$ is presented by the $2m-1$ generators $\sigma_1,\ldots,\sigma_{m-1},\rho_1,\ldots,\rho_m$ and relations (*) together with

$$\begin{cases} \sigma_i\rho_j = \rho_j\sigma_i, \quad j \neq i, i+1, \\ \rho_i = \sigma_i\rho_{i+1}\sigma_i, \\ \sigma_i^2 = \rho_{i+1}^{-1}\rho_i^{-1}\rho_{i+1}\rho_i, \\ \rho_1^2 = \sigma_1\sigma_2\cdots\sigma_{m-1}\sigma_{m-1}\cdots\sigma_2\sigma_1. \end{cases}$$

The element τ as defined above, but considered an element of $B_m(P^2)$, again has order $2m$. Thus we have the theorem of Van Buskirk, that for each $m \geq 2$, the surface braid group B_mM has elements of finite order if and only if $M = S^2$ or P^2.

Some of these braid groups are actually finite:

- $B_2(S^2) \cong \mathbb{Z}/2\mathbb{Z}, B_3(S^2)$ has order 12,
- $B_1(P^2) \cong \mathbb{Z}/2\mathbb{Z}$ and
- $B_2(P^2)$ is a group of order 16 whose subgroup $PB_2(P^2)$ is isomorphic with the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$;

But $B_3(P^2)$ is infinite, as are all the other higher braid groups of P^2 and S^2.

For the braid groups of higher genus closed surfaces, we refer the reader to [Sc], and only mention how to produce a generating set. After removing a disk from a surface M of genus g, the remainder can be modelled as a disk with g twisted bands attached, in the nonorientable case, or $2g$ bands if the surface is orientable. Then $B_m(M)$ is generated by $\sigma_1,\ldots,\sigma_{m-1}$ as above, plus ρ_{ij} which represents the basepoint P_i running once around the j_{th} band, while the others are fixed. A finite set of relations can be found in [Sc].

1.3. Centers and large surfaces.

The center $Z(G)$ of a group G is the subgroup of elements which commute with all elements of the group. Chow [Ch] proved that the groups $B_m = B_m(D^2)$ have infinite cyclic center, for $m \geq 2$. Some other surface braid groups also have nontrivial centers: those of S^2 [GV], P^2 [Va]. If τ is defined as in the preceding section, the element τ^m is central in $B_m S^2$. Birman stated in [Bi2] that the torus braid groups $B_m T^2$ have center which is free abelian with two generators, but did not include a complete proof. We will prove this, and also calculate the center of the braid groups of the annulus $S^1 \times I$ in Section 4. However, apart from these and a few other exceptions, most surface braid groups have no center. Our proof is the same as given in [Bi2].

DEFINITION. — A compact surface M will be called *large* if

$$M \neq S^2,\ P^2,\ D^2,\ S^1 \times I,\ T^2 = S^1 \times S^1,$$
$$\text{Möbius strip } S^1 \widehat{\times} I, \text{ or Klein bottle } S^1 \widehat{\times} S^1.$$

In other words, we call a surface large if its fundamental group has no finite index abelian subgroup.

PROPOSITION 1.6. — *Let M be a large compact surface. Then the center $Z(B_m(M))$ is a trivial group.*

Proof. — First, we prove by induction on m that $Z(PB_m M) = \{1\}$. The case $m = 1$ is well-known: the only surfaces whose fundamental groups have nontrivial centers are P^2, $S^1 \times I$, T^2, the Möbius strip, and Klein bottle.

Let $m > 1$ and M large. We consider the following exact sequence:

$$1 \to \pi_1(M \backslash \{P_1, \ldots, P_{m-1}\}) \longrightarrow PB_m M \xrightarrow{\rho} PB_{m-1} M \to 1.$$

Since ρ is surjective, it takes center into center, and by induction, $Z(PB_{m-1}M) = \{1\}$. So

$$Z(PB_m M) \subseteq \pi_1(M \backslash \{P_1, \ldots, P_{m-1}\}).$$

But this latter group has trivial center, so $Z(PB_m M) = \{1\}$. Now, let $g \in Z(B_m M)$. There exists an integer $k > 0$ such that $g^k \in PB_m M$. Then $g^k \in Z(PB_m M)$, thus $g^k = 1$. By Proposition 1.5, $g = 1$. $\square$

2. Subsurfaces.

A *subsurface* N of a surface M is the closure of an open subset of M. For simplicity we make the extra assumption that every boundary component of N either is a boundary component of M or lies in the interior of M.

Let $P_1 \in N$. The inclusion $N \subseteq M$ induces a morphism

$$\psi: \pi_1(N, P_1) \longrightarrow \pi_1(M, P_1).$$

The following proposition is well-known.

PROPOSITION 2.1. — *Let N be a connected subsurface of M such that $\pi_1(N, P_1) \neq \{1\}$. The morphism $\psi: \pi_1(N, P_1) \to \pi_1(M, P_1)$ is injective if and only if none of the connected components of the closure $\overline{M \setminus N}$ of $M \setminus N$ is a disk.* □

Let $P_1, \dots, P_n \in N$, and let $P_{n+1}, \dots, P_m \in M \setminus N$. The inclusion $N \subseteq M$ induces a morphism

$$\psi: B_n N \longrightarrow B_m M.$$

PROPOSITION 2.2. — *Let M be different from the sphere and from the projective plane, and let N be such that none of the connected components of $\overline{M \setminus N}$ is a disk. Then the morphism $\psi: B_n N \to B_m M$ is injective.*

Remark. — Proposition 2.2 is proved in [Go] in the particular case where N is a disk.

Proof. — Let $P\psi_n: PB_n N \to PB_n M$ be the morphism induced by the inclusion $N \subseteq M$. We prove that $P\psi_n$ is injective by induction on n. The case $n = 1$ is a consequence of Proposition 2.1.

Let $n > 1$. By Proposition 2.1, the inclusion

$$N \setminus \{P_1, \dots, P_{n-1}\} \subseteq M \setminus \{P_1, \dots, P_{n-1}\}$$

induces a monomorphism

$$\alpha: \pi_1\big(N \setminus \{P_1, \dots, P_{n-1}\}\big) \longrightarrow \pi_1\big(M \setminus \{P_1, \dots, P_{n-1}\}\big).$$

The following diagram commutes:

$$\begin{array}{ccccccccc}
1 & \to & \pi_1(N\backslash\{P_1,\dots,P_{n-1}\}) & \longrightarrow & PB_nN & \xrightarrow{\rho} & PB_{n-1}N & \to & 1 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle P\psi_n} & & \downarrow{\scriptstyle P\psi_{n-1}} & & \\
1 & \to & \pi_1(M\backslash\{P_1,\dots,P_{n-1}\}) & \longrightarrow & PB_nM & \xrightarrow{\rho} & PB_{n-1}M & \to & 1.
\end{array}$$

By induction, $P\psi_{n-1}$ is injective. By the five lemma, $P\psi_n$ is injective, too.

Let $P\psi: PB_nN \to PB_mM$ be the morphism induced by the inclusion $N \subseteq M$. The following diagram commutes:

$$\begin{array}{ccc}
PB_nN & \xrightarrow{P\psi} & PB_mM \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \rho} \\
PB_nN & \xrightarrow{P\psi_n} & PB_nM.
\end{array}$$

The morphism $P\psi_n$ is injective, thus $P\psi$ is injective, too.

Let $\iota: \Sigma_n \to \Sigma_m$ be the inclusion. The following diagram commutes:

$$\begin{array}{ccccccccc}
1 & \to & PB_nN & \longrightarrow & B_nN & \xrightarrow{\sigma} & \Sigma_n & \to & 1 \\
 & & \downarrow{\scriptstyle P\psi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \iota} & & \\
1 & \to & PB_mM & \longrightarrow & B_mM & \xrightarrow{\sigma} & \Sigma_m & \to & 1.
\end{array}$$

Both $P\psi$ and ι are injective, so, by the five lemma, ψ is injective, too. □

Let $N_1, \dots, N_r$ be the connected components of $\overline{M\backslash N}$. For $i = 1, \dots, r$, we write

$$\mathcal{P}_i = \{P_{n+1}, \dots, P_m\} \cap N_i.$$

THEOREM 2.3. — *Let M be different from the sphere and from the projective plane. The morphism $\psi: B_nN \to B_mM$ is injective if and only if either N_i is not a disk or $\mathcal{P}_i \neq \emptyset$, for all $i = 1, \dots, r$.*

Proof. — We suppose that there exists $i \in \{1, \dots, r\}$ such that N_i is a disk and such that $\mathcal{P}_i = \emptyset$. We consider the following commutative diagram:

$$\begin{array}{ccc}
\pi_1\big(N\backslash\{P_2,\dots,P_n\}\big) & \longrightarrow & B_nN \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\
\pi_1\big(M\backslash\{P_2,\dots,P_m\}\big) & \longrightarrow & B_mM.
\end{array}$$

By [FaN], the morphism $\pi_1(N \backslash \{P_2, \dots, P_n\}) \to B_n N$ is injective. On the other hand, the morphism $\psi: \pi_1(N \backslash \{P_2, \dots, P_n\}) \to \pi_1(M \backslash \{P_2, \dots, P_m\})$ is clearly not injective. Thus $\psi: B_n N \to B_m M$ is not injective.

We suppose that either N_i is not a disk or $\mathcal{P}_i \neq \emptyset$, for all $i = 1, \dots, r$. We consider the following commutative diagram:

$$\begin{array}{ccc} B_n N & \xrightarrow{\psi} & B_n M \backslash \{P_{n+1}, \dots, P_m\} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ B_n N & \xrightarrow{\psi} & B_m M. \end{array}$$

By Proposition 2.2, the morphism $\psi: B_n N \to B_n M \backslash \{P_{n+1}, \dots, P_m\}$ is injective. By [FaN], the morphism $B_n M \backslash \{P_{n+1}, \dots, P_m\} \to B_m M$ is injective. Thus $\psi: B_n N \to B_m M$ is injective. □

3. Commensurator, normalizer, and centralizer of $\pi_1 N$ in $\pi_1 M$.

Let N be a subsurface of a connected surface M. We say that N is a *Möbius collar* in M if N is a cylinder $S^1 \times I$ and $\overline{M \backslash N}$ has two components N_1, N_2 with one of them, say N_1, a Möbius strip (see Figure 3.1). Then $M_0 = N \cup N_1$ will be called the *Möbius strip collared* by N in M.

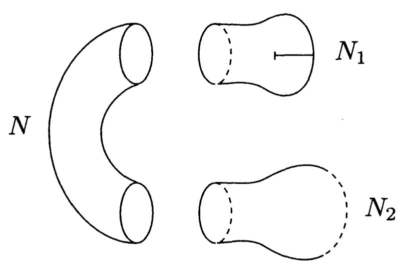

Figure 3.1

Let G be a group, and let H be a subgroup of G. We denote by

- $C_G(H)$ the commensurator of H in G, by
- $N_G(H)$ the normalizer of H in G, and by
- $Z_G(H)$ the centralizer of H in G.

That is,

$$Z_G(H) = \{g \in G\colon\ gh = hg \text{ for all } h \in H\},$$
$$N_G(H) = \{g \in G\colon\ gHg^{-1} = H\},$$
$$C_G(H) = \{g \in G\colon\ gHg^{-1} \cap H \text{ has finite index in } gHg^{-1} \text{ and } H\}.$$

The goal of this section is to prove the following theorem.

THEOREM 3.1. — *Let $P_0 \in N$. We write*

$$\pi_1 M = \pi_1(M,P_0) \quad \text{and} \quad \pi_1 N = \pi_1(N,P_0).$$

(i) *If M is not large or if $\pi_1 N = \{1\}$, then $C_{\pi_1 M}(\pi_1 N) = \pi_1 M$.*

(ii) *If M is large, if $\pi_1 N \neq \{1\}$, and if N is not a Möbius collar in M, then $C_{\pi_1 M}(\pi_1 N) = \pi_1 N$.*

(iii) *If M is large and if N is a Möbius collar in M, then $C_{\pi_1 M}(\pi_1 N) = \pi_1 M_0$, where M_0 is the Möbius strip collared by N in M.*

COROLLARY 3.2.

(i) *If M is either a cylinder, or a torus, or a Möbius strip, then*

$$C_{\pi_1 M}(\pi_1 N) = N_{\pi_1 M}(\pi_1 N) = Z_{\pi_1 M}(\pi_1 N) = \pi_1 M.$$

(ii) *If M is large, if N is not a Möbius collar in M, if $\pi_1 N \neq \{1\}$, and if N is not large, then*

$$C_{\pi_1 M}(\pi_1 N) = N_{\pi_1 M}(\pi_1 N) = Z_{\pi_1 M}(\pi_1 N) = Z(\pi_1 N) = \pi_1 N.$$

(iii) *If M and N are both large, then*

$$C_{\pi_1 M}(\pi_1 N) = N_{\pi_1 M}(\pi_1 N) = \pi_1 N,$$
$$Z_{\pi_1 M}(\pi_1 N) = Z(\pi_1 N) = \{1\}.$$

(iv) *If M is large and if N is a Möbius collar in M, then*

$$C_{\pi_1 M}(\pi_1 N) = N_{\pi_1 M}(\pi_1 N) = Z_{\pi_1 M}(\pi_1 N) = \pi_1 M_0,$$

where M_0 is the Möbius strip collared by N in M.

Before proving Theorem 3.1, we recall some well-known results on graphs of groups.

An (*oriented*) *graph* Γ is the following data:

1) A set $V(\Gamma)$ of *vertices*.

2) A set $A(\Gamma)$ of *arrows*.

3) A map $s: A(\Gamma) \to V(\Gamma)$ called *origin*, and a map $t: A(\Gamma) \to V(\Gamma)$ called *end*.

A *graph of groups* $G(\Gamma)$ on Γ is the following data.

1) A group G_v for all $v \in V(\Gamma)$.

2) A group G_a for all $a \in A(\Gamma)$.

3) Two monomorphisms $\phi_{a,s}: G_a \to G_{s(a)}$ and $\phi_{a,t}: G_a \to G_{t(a)}$ for all $a \in A(\Gamma)$.

We refer to [Se] for a general exposition on graphs of groups.

Let T be a maximal tree of Γ. The *fundamental group* $\pi_1(G(\Gamma), T)$ of $G(\Gamma)$ based at T is the (abstract) group given by the following presentation. The generating set of $\pi_1(G(\Gamma), T)$ is

$$\{e_a;\ a \in A(\Gamma)\} \cup \Big(\bigcup_{v \in V(\Gamma)} G_v \Big),$$

where $\{e_a; a \in A(\Gamma)\}$ is an abstract set in one-to-one correspondance with $A(\Gamma)$. The relations of $\pi_1(G(\Gamma), T)$ are

1) the relations of G_v for all $v \in V(\Gamma)$,

2) $e_a = 1$ for all $a \in A(T)$,

3) $e_a^{-1} \cdot \phi_{a,s}(g) \cdot e_a = \phi_{a,t}(g)$ for all $a \in A(\Gamma)$ and for all $g \in G_a$.

There is a morphism $\phi_v: G_v \to \pi_1(G(\Gamma), T)$ for all $v \in V(\Gamma)$. By [Se], this morphism is injective.

The fundamental group $\pi_1(\Gamma, T)$ of Γ based at T has the following presentation. The generating set of $\pi_1(\Gamma, T)$ is $\{e_a;\ a \in A(\Gamma)\}$. The set of relations of $\pi_1(\Gamma, T)$ is $\{e_a = 1;\ a \in A(T)\}$.

Let $p: \widetilde{\Gamma} \to \Gamma$ be the universal cover of Γ. Let $G(\widetilde{\Gamma})$ be the graph of groups on $\widetilde{\Gamma}$ defined as follows:

1) $G_{\tilde{v}} = G_{p(\tilde{v})}$ for all $\tilde{v} \in V(\widetilde{\Gamma})$.

2) $G_{\tilde{a}} = G_{p(\tilde{a})}$ for all $\tilde{a} \in A(\widetilde{\Gamma})$.

3) $\phi_{\tilde{a},s} = \phi_{p(\tilde{a}),s}$ and $\phi_{\tilde{a},t} = \phi_{p(\tilde{a}),t}$ for all $\tilde{a} \in A(\widetilde{\Gamma})$.

We fix a section $S: T \to \widetilde{\Gamma}$ of p over T. We extend S to a section $S: A(\Gamma) \to A(\widetilde{\Gamma})$ as follows. Let $a \in A(\Gamma)$. Then $S(a)$ is the unique lift of a such that $t(S(a)) = S(t(a))$.

We define an action of $\pi_1(\Gamma, T)$ on $\pi_1(G(\widetilde{\Gamma}), \widetilde{\Gamma})$ as follows. Let $\tilde{v} \in V(\widetilde{\Gamma})$, let $\tilde{g} \in G_{\tilde{v}}$, and let $u \in \pi_1(\Gamma, T)$. Then

$$u(\tilde{g}) = \tilde{g} \in G_{u(\tilde{v})}.$$

We consider the corresponding semidirect product $\pi_1(G(\widetilde{\Gamma}), \widetilde{\Gamma}) \rtimes \pi_1(\Gamma, T)$. By [Se], there is an isomorphism

$$\pi_1\big(G(\Gamma), T\big) \longrightarrow \pi_1\big(G(\widetilde{\Gamma}), \widetilde{\Gamma}\big) \rtimes \pi_1(\Gamma, T)$$

which sends G_v isomorphically on $G_{S(v)}$ for all $v \in V(\Gamma)$, and which sends e_a on e_a for all $a \in A(\Gamma)$. So, we can assume that

$$\pi_1\big(G(\Gamma), T\big) = \pi_1\big(G(\widetilde{\Gamma}), \widetilde{\Gamma}\big) \rtimes \pi_1(\Gamma, T),$$

that $G_v = G_{S(v)}$ for all $v \in V(\Gamma)$, and that $G_a = G_{S(a)}$ for all $a \in A(\Gamma)$.

Let $G = \pi_1(G(\Gamma), T)$, and let $\widetilde{G} = \pi_1(G(\widetilde{\Gamma}), \widetilde{\Gamma})$. The *universal cover* of $G(\Gamma)$ is the graph $\bar{\Gamma}$ defined as follows:

$$V(\bar{\Gamma}) = (V(\widetilde{\Gamma}) \times \widetilde{G}) / \sim,$$

where $\sim$ is the equivalence relation defined by

$$(\tilde{v}_1, \tilde{g}_1) \sim (\tilde{v}_2, \tilde{g}_2) \quad \text{if} \quad \tilde{v}_1 = \tilde{v}_2 = \tilde{v} \text{ and } \tilde{g}_2^{-1} \tilde{g}_1 \in G_{\tilde{v}}.$$

We denote by $[\tilde{v}, \tilde{g}]$ the equivalence class of $(\tilde{v}, \tilde{g})$.

$$A(\bar{\Gamma}) = (A(\widetilde{\Gamma}) \times \widetilde{G}) / \sim,$$

where $\sim$ is the equivalence relation defined by

$$(\tilde{a}_1, \tilde{g}_1) \sim (\tilde{a}_2, \tilde{g}_2) \quad \text{if} \quad \tilde{a}_1 = \tilde{a}_2 = \tilde{a} \text{ and } \tilde{g}_2^{-1} \tilde{g}_1 \in G_{\tilde{a}}.$$

We denote by $[\tilde{a}, \tilde{g}]$ the equivalence class of $(\tilde{a}, \tilde{g})$. The origin map $s: A(\bar{\Gamma}) \to V(\bar{\Gamma})$ is defined by

$$s\big([\tilde{a}, \tilde{g}]\big) = \big[s(\tilde{a}), \tilde{g}\big]$$

for $\tilde{a} \in A(\widetilde{\Gamma})$ and for $\tilde{g} \in \widetilde{G}$. The end map $t: A(\overline{\Gamma}) \to V(\overline{\Gamma})$ is defined by

$$t([\tilde{a}, \tilde{g}]) = [t(\tilde{a}), \tilde{g}]$$

for $\tilde{a} \in A(\widetilde{\Gamma})$ and for $\tilde{g} \in \widetilde{G}$. By [Se], $\overline{\Gamma}$ is a tree.

The group G acts on $\overline{\Gamma}$ as follows. Let $u \in \pi_1(\Gamma, T)$, let $\tilde{h}, \tilde{g} \in \widetilde{G}$, let $\tilde{v} \in V(\widetilde{\Gamma})$, and let $\tilde{a} \in A(\widetilde{\Gamma})$. Then

$$\tilde{h}([\tilde{v}, \tilde{g}]) = [\tilde{v}, \tilde{h}\tilde{g}], \quad \tilde{h}([\tilde{a}, \tilde{g}]) = [\tilde{a}, \tilde{h}\tilde{g}],$$
$$u([\tilde{v}, \tilde{g}]) = [u(\tilde{v}), u\tilde{g}u^{-1}], \quad u([\tilde{a}, \tilde{g}]) = [u(\tilde{a}), u\tilde{g}u^{-1}].$$

The *isotropy subgroup* of a vertex $\bar{v} \in V(\overline{\Gamma})$ is

$$\mathrm{Isot}(\bar{v}) = \{g \in G;\ g(\bar{v}) = \bar{v}\}.$$

The *isotropy subgroup* of an arrow $\bar{a} \in A(\overline{\Gamma})$ is

$$\mathrm{Isot}(\bar{a}) = \{g \in G;\ g(\bar{a}) = \bar{a}\}.$$

Let $v \in V(\Gamma)$ and let $a \in A(\Gamma)$. By [Se],

$$\mathrm{Isot}([S(v), 1]) = G_v, \quad \mathrm{Isot}([S(a), 1]) = G_a.$$

Now, we come back to our original assumptions. M is a surface (with boundary) different from the sphere and from the projective plane. N is a subsurface of M such that none of the connected components of $\overline{M \setminus N}$ is a disk. Without lost of generality, we can also assume that N is not a disk. Let $N_1, \dots, N_r$ be the connected components of $\overline{M \setminus N}$.

We define a graph Γ as follows. Let

$$V(\Gamma) = \{v_0, v_1, \dots, v_r\}.$$

For $i \in \{1, \dots, r\}$, we fix an abstract set $A_i(\Gamma)$ in one-to-one correspondance with the connected components of $N \cap N_i$. We set

$$A(\Gamma) = \bigcup_{i=1}^{r} A_i(\Gamma).$$

If $a \in A_i(\Gamma)$, then $s(a) = v_0$ and $t(a) = v_i$.

We define a graph of groups $G(\Gamma)$ on Γ as follows. Let $i \in \{1, \dots, r\}$. We fix a point $P_i \in N_i$ and we set

$$G_{v_i} = G_i = \pi_1(N_i, P_i).$$

We fix a point $P_0 \in N$ and we set

$$G_{v_0} = G_0 = \pi_1(N, P_0).$$

Let $a \in A_i(\Gamma)$. We denote by C_a the connected component of $N \cap N_i$ which corresponds to a. The set C_a is a boundary component of both N and N_i. We fix a point $P_a \in C_a$ and we set

$$G_a = \pi_1(C_a, P_a) \simeq \mathbb{Z}.$$

We fix a path $\gamma_{a,s}: [0,1] \to N$ from P_0 to P_a. This path induces a monomorphism $\phi_{a,s}: G_a \to G_0$. We fix a path $\gamma_{a,t}: [0,1] \to N_i$ from P_i to P_a. This path induces a monomorphism $\phi_{a,t}: G_a \to G_i$.

We fix an arrow $a_i \in A_i(\Gamma)$ for all $i \in \{1, \dots, r\}$. We consider the graph T defined as follows:

1) $V(T) = \{v_0, v_1, \dots, v_r\}$.

2) $A(T) = \{a_1, \dots, a_r\}$.

3) $s(a_i) = v_0$ and $t(a_i) = v_i$ for all $i \in \{1, \dots, r\}$.

The graph T is a maximal tree of Γ. We write

$$\gamma_a = \gamma_{a,s}\gamma_{a,t}^{-1}, \quad \beta_a = \gamma_a \gamma_{a_i}^{-1}$$

for all $a \in A_i(\Gamma)$. For $i = 1, \dots, r$, the path γ_{a_i} induces a morphism

$$\psi_i: G_i = \pi_1(N_i, P_i) \longrightarrow \pi_1(M, P_0).$$

We denote by

$$\psi_0: G_0 = \pi_1(N, P_0) \longrightarrow \pi_1(M, P_0)$$

the morphism induced by the inclusion $N \subseteq M$.

The following theorem is a well-known version of Van Kampen's theorem.

THEOREM 3.3. — *The map*

$$\begin{array}{ccc} \{e_a;\ a \in A(\Gamma)\} & \longrightarrow & \pi_1(M,P_0) \\ e_a & \longmapsto & \beta_a \end{array}$$

and the morphisms $\psi_i : G_i \to \pi_1(M,P_0)$ $(i = 0,1,\dots,r)$ *induce an isomorphism*

$$\psi : \pi_1\big(G(\Gamma),T\big) \longrightarrow \pi_1(M,P_0).$$

Let $\overline{\Gamma}$ be the universal cover of $G(\Gamma)$. Let $q : \overline{\Gamma} \to \Gamma$ be the map defined as follows. Let $\tilde{v} \in V(\widetilde{\Gamma})$, let $\tilde{a} \in A(\widetilde{\Gamma})$, and let $\tilde{g} \in \widetilde{G}$. Then

$$q\big([\tilde{v},\tilde{g}]\big) = p(\tilde{v}), \quad q\big([\tilde{a},\tilde{g}]\big) = p(\tilde{a}).$$

The following lemma is a preliminary result to the proof of Theorem 3.1.

LEMMA 3.4. — *Let* $i \in \{1,\dots,r\}$. *Let* $\bar{v} \in V(\overline{\Gamma})$ *be such that* $q(\bar{v}) = v_i$. *Let* $\bar{a},\bar{b} \in A(\overline{\Gamma})$ *be such that* $t(\bar{a}) = t(\bar{b}) = \bar{v}$ *(see Figure* 3.2*):*

(i) *If* $q(\bar{a}) = q(\bar{b})$ *and* $\mathrm{Isot}(\bar{a}) \cap \mathrm{Isot}(\bar{b}) \neq \{1\}$, *then* N_i *is a Möbius strip.*

(ii) *If* $q(\bar{a}) \neq q(\bar{b})$ *and* $\mathrm{Isot}(\bar{a}) \cap \mathrm{Isot}(\bar{b}) \neq \{1\}$, *then* N_i *is a cylinder and both boundary components of* N_i *are included in* $N \cap N_i$.

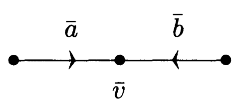

Figure 3.2

Proof. — (i) We suppose that $i = 1$ and that $\bar{a} = [S(a_1),1]$. Then

$$\bar{v} = t(\bar{a}) = \big[t(S(a_1)),1\big] = \big[S(v_1),1\big].$$

Let $\bar{b} = [\,\tilde{b},\tilde{g}]$. Then

$$t(\bar{b}) = \big[t(\tilde{b}),\tilde{g}\big] = \big[S(v_1),1\big],$$

thus $\tilde{b} = S(a_1)$ (since $q(\bar{a}) = q(\bar{b}) = a_1$), and $\tilde{g} \in G_{v_1} = G_1$. Note that $\tilde{g} \notin G_{a_1}$, otherwise

$$\bar{b} = \big[S(a_1),\tilde{g}\big] = \big[S(a_1),1\big] = \bar{a}.$$

So,

$$\mathrm{Isot}(\bar{a}) = G_{a_1} \quad \text{and} \quad \mathrm{Isot}(\bar{b}) = \tilde{g} G_{a_1} \tilde{g}^{-1}.$$

Let h_1 be a generator of G_{a_1}. There exist $k_1, k_2 \in \mathbb{Z} \backslash \{0\}$ such that

$$h_1^{k_1} = \tilde{g}\, h_1^{k_2} \tilde{g}^{-1}.$$

We suppose that N_1 is not a Möbius strip. Let F be the subgroup of G_1 generated by h_1 and $\tilde{g}$. The subsurface N_1 has non-empty boundary, thus G_1 is a free group, therefore F is a free group of rank either 1 or 2. Since F is a hopfian group (see [LS, Prop. 3.4]) and since $h_1^{k_1} = \tilde{g}\, h_1^{k_2} \tilde{g}^{-1}$, the group F has rank 1. By [Ep, Thm. 4.2], h_1 generates F. In particular, there exists $\ell \in \mathbb{Z}$ such that

$$\tilde{g} = h_1^{\ell} \in G_{a_1}.$$

This is a contradiction. So, N_1 is a Möbius strip.

(ii) We suppose that $i = 1$ and that $\bar{a} = [S(a_1), 1]$. Then

$$\bar{v} = t(\bar{a}) = \big[t(S(a_1)), 1\big] = \big[S(v_1), 1\big].$$

Let $\bar{b} = [\tilde{b}, \tilde{g}]$ and let $b = q(\bar{b}) \neq a_1$. Then

$$t(\bar{b}) = \big[t(\tilde{b}), \tilde{g}\big] = \big[S(v_1), 1\big],$$

thus $\tilde{b} = S(b)$ and $\tilde{g} \in G_{v_1} = G_1$. So,

$$\mathrm{Isot}(\bar{a}) = G_{a_1} \quad \text{and} \quad \mathrm{Isot}(\bar{b}) = \tilde{g}\, G_b \tilde{g}^{-1}.$$

Let h_1 be a generator of G_{a_1}, and let h be a generator of G_b. There exist $k_1, k_2 \in \mathbb{Z} \backslash \{0\}$ such that

$$h_1^{k_1} = \tilde{g}\, h^{k_2} \tilde{g}^{-1}.$$

Let F be the subgroup of G_1 generated by h_1 and $\tilde{g} h \tilde{g}^{-1}$. Since G_1 is a free group, F is a free group of rank either 1 or 2. Since F is a hopfian group and since $h_1^{k_1} = (\tilde{g} h \tilde{g}^{-1})^{k_2}$, the group F has rank 1. The subsurface N_1 has at least two boundary components, C_{a_1} and C_b, thus N_1 is not a Möbius strip. By [Ep, Thm. 4.2], both h_1 and $\tilde{g} h \tilde{g}^{-1}$ generate F. So, we can assume that

$$h_1 = \tilde{g} h \tilde{g}^{-1}.$$

By [Ep, Lemma 2.4], it follows that N_1 is a cylinder and that C_{a_1} and C_b are the boundary components of N_1. □

Proof of Theorem 3.1. — (i) It is obvious, as all the non-large surfaces have abelian fundamental groups, except the Klein bottle, which has an abelian subgroup of index 2.

(ii) We suppose that there exists $g \in C_{\pi_1 M}(\pi_1 N)$ such that $g \notin \pi_1 N$, and we prove that either M is not large, or N is a Möbius collar in M.

Let $\bar{v}_0 = [S(v_0), 1] \in V(\bar{\Gamma})$. We have $g(\bar{v}_0) \neq \bar{v}_0$ since $g \notin \pi_1 N = \mathrm{Isot}(\bar{v}_0)$. Let

$$\bar{a}_1^{\varepsilon_1} \bar{a}_2^{\varepsilon_2} \cdots \bar{a}_\ell^{\varepsilon_\ell} \quad \big(a_i \in A(\bar{\Gamma}) \text{ and } \varepsilon_i \in \{\pm 1\}\big)$$

be the (unique) reduced path of $\bar{\Gamma}$ from $\bar{v}_0$ to $g(\bar{v}_0)$ (see Figure 3.3). For $j = 1, \dots, \ell$ we denote by $\bar{v}_j$ the end of the path $\bar{a}_1^{\varepsilon_1} \cdots \bar{a}_j^{\varepsilon_j}$. Note that $\ell \geq 2$ since $q(g(\bar{v}_0)) = q(\bar{v}_0) = v_0$. If $h \in G_0 \cap gG_0g^{-1}$, then $h \in \mathrm{Isot}(\bar{v}_0)$ and $h \in \mathrm{Isot}(g(\bar{v}_0))$, thus $h \in \mathrm{Isot}(\bar{v}_j)$ and $h \in \mathrm{Isot}(\bar{a}_j)$ for all $j \in \{1, \dots, l\}$. We suppose that $q(\bar{v}_1) = v_1$.

$$\{1\} \neq G_0 \cap gG_0g^{-1} \subseteq \mathrm{Isot}(\bar{a}_1) \cap \mathrm{Isot}(\bar{a}_2),$$

thus, by Lemma 3.4, either N_1 is a Möbius strip, or N_1 is a cylinder and both boundary components of N_1 are included in $N \cap N_1$.

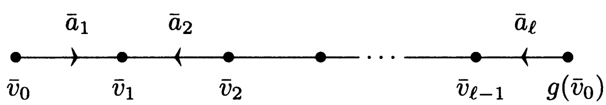

Figure 3.3

The group $G_0 \cap gG_0g^{-1}$ has finite index in $G_0 = \pi_1 N$, it is included in $\mathrm{Isot}(\bar{a}_1)$, and $\mathrm{Isot}(\bar{a}_1)$ is an infinite cyclic group. So, $\pi_1 N$ has an infinite cyclic subgroup of finite index, thus either N is a cylinder, or N is a Möbius strip.

If N is a Möbius strip, then N_1 is also a Möbius strip and $M = N \cup N_1$ is a Klein bottle (see Figure 3.4).

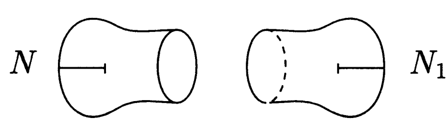

Figure 3.4

If N and N_1 are both cylinders, then $M = N \cup N_1$ is a torus (see Figure 3.5).

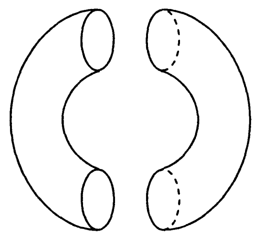

Figure 3.5

If N is a cylinder and if N_1 is a Möbius strip, then N is a Möbius collar in M (see Figure 3.6).

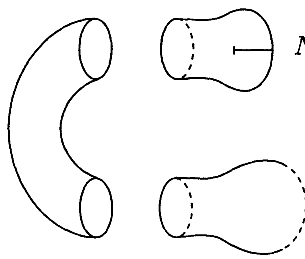

Figure 3.6

(iii) We suppose that N is a cylinder, that N_1 is a Möbius strip, and that M is large (see Figure 3.6). Let $M_0 = N \cup N_1$ be the Möbius strip collared by N in M. The subsurface M_0 is not a Möbius collar in M, thus, by (ii),

$$C_{\pi_1 M}(\pi_1 M_0) = \pi_1 M_0.$$

The group $\pi_1 N$ has finite index in $\pi_1 M_0$, thus

$$C_{\pi_1 M}(\pi_1 N) = C_{\pi_1 M}(\pi_1 M_0) = \pi_1 M_0. \qquad \square$$

4. Centers.

The goal of this section is to describe the center of $B_m M$, where M is either a cylinder or a torus.

Let C be a cylinder. We assume that

$$C = \{z \in \mathbb{C};\ 1 \leq |z| \leq 2\},$$

and that

$$P_i = 1 + \frac{i}{m+1} \quad \text{for } i = 1, \dots, m.$$

Let $d_i\colon [0,1] \to C$ be the path defined by

$$d_i(t) = \Big(1 + \frac{i}{m+1}\Big)\, \mathrm{e}^{2i\pi t} \quad \text{for } t \in [0,1].$$

Let α be the element of PB_mC represented by $d = (d_1, \dots, d_m)$ (see Figure 4.1).

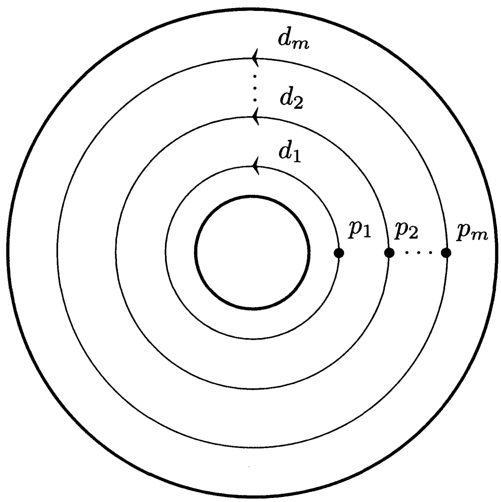

Figure 4.1

PROPOSITION 4.1. — *With the above assumptions, the center of B_mC is the infinite cyclic subgroup generated by α.*

Proof. — Let

$$D = \{z \in \mathbb{C};\ |z| \leq 2\}.$$

Let $P_0 = 0$. The inclusion $C \subseteq D \backslash \{P_0\}$ induces an isomorphism $B_mC \to B_m(D\backslash\{P_0\})$. Let Σ_{m+1} be the group of permutations of

$\{P_0, P_1, \dots, P_m\}$, and let Σ_m be the group of permutations of $\{P_1, \dots, P_m\}$. We consider the morphism $\sigma: B_{m+1}D \to \Sigma_{m+1}$. By Proposition 1.4, we have the following exact sequence:

$$1 \to B_m(D\backslash\{P_0\}) \longrightarrow \sigma^{-1}(\Sigma_m) \to \pi_1(D, P_0) \to 1.$$

Moreover, $\pi_1(D, P_0) = \{1\}$. Thus the inclusion $D\backslash\{P_0\} \subseteq D$ induces an isomorphism $B_m(D\backslash\{P_0\}) \to \sigma^{-1}(\Sigma_m)$. The image of α by this isomorphism is the element of $B_{m+1}D$, denoted by $\widetilde{\alpha}$, represented by the braid $\tilde{b} = (P_0, d_1, \dots, d_m)$. By [Ch], we have $Z(B_{m+1}D) = Z(PB_{m+1}D)$, and this group is the infinite cyclic subgroup generated by $\widetilde{\alpha}$. From the inclusions

$$PB_{m+1}D \subseteq \sigma^{-1}(\Sigma_m) \subseteq B_{m+1}D,$$

it follows that the center of $\sigma^{-1}(\Sigma_m)$ is equal to the center of $B_{m+1}D$ which is the cyclic subgroup generated by $\widetilde{\alpha}$. Thus, by the preceding isomorphism, the center of $B_mC = B_m(D\backslash\{P_0\})$ is the infinite cyclic subgroup generated by α. □

Now, we describe the center of B_mT, where T is a torus. We assume that

$$T = \mathbb{R}^2/\mathbb{Z}^2.$$

We denote by $\overline{(x, y)}$ the equivalence class of (x, y). We assume that

$$P_i = \overline{\left(\frac{i+1}{m+3}, \frac{i+1}{m+3}\right)} \quad \text{for } i = 1, \dots, m.$$

Let $a_i: [0, 1] \to T$ be the path defined by

$$a_i(t) = \overline{\left(\frac{i+1}{m+3} - t, \frac{i+1}{m+3}\right)} \quad \text{for } t \in [0, 1],$$

and let $b_i: [0, 1] \to T$ be the path defined by

$$b_i(t) = \overline{\left(\frac{i+1}{m+3}, \frac{i+1}{m+3} - t\right)} \quad \text{for } t \in [0, 1].$$

Let α be the element of PB_mT represented by $a = (a_1, \dots, a_m)$ (see Figure 4.2), and let β be the element of of PB_mT represented by $b = (b_1, \dots, b_m)$.

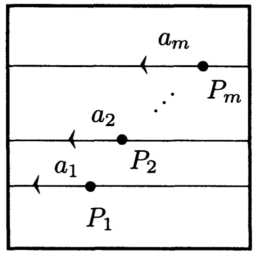

Figure 4.2

PROPOSITION 4.2. — *With the above assumptions, the center of B_mT is the subgroup generated by α and β. It is a free abelian group of rank 2.*

Proof. — The proof of Proposition 4.2 is divided into 4 steps. Let Z_m denote the subgroup of PB_mT generated by α and β.

Step 1. — Z_m is a free abelian group of rank 2.

By [Bil, Thm. 5], α and β commute, thus Z_m is an abelian group. We consider the following exact sequence:

$$1 \to PB_{m-1}T \backslash \{P_1\} \longrightarrow PB_mT \xrightarrow{\rho} \pi_1(T, P_1) \to 1.$$

The group $\pi_1(T, P_1)$ is a free abelian group of rank 2 and $\{\rho(\alpha), \rho(\beta)\}$ is a basis of $\pi_1(T, P_1)$, thus Z_m is also a free abelian group of rank 2.

Step 2. — $Z_m \subseteq Z(B_mT)$.

Let

$$D = \left[\frac{1}{m+3}, \frac{m+2}{m+3}\right] \times \left[\frac{1}{m+3}, \frac{m+2}{m+3}\right] \subseteq T.$$

By Proposition 2.2, the inclusion $D \subseteq T$ induces a monomorphism $B_mD \to B_mT$. The following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & PB_mD \arrow[r] \arrow[d] & B_mD \arrow[r, "\sigma"] \arrow[d] & \Sigma_m \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & PB_mT \arrow[r] & B_mT \arrow[r, "\sigma"] & \Sigma_m \arrow[r] & 1.
\end{tikzcd}$$

Thus B_mT is generated by $PB_mT \cup B_mD$.

By [Bil, Thm. 5], α commutes with all the elements of PB_mT.

Let

$$C = \left(\mathbb{R} \times \left[\frac{1}{m+3}, \frac{m+2}{m+3}\right]\right) / \mathbb{Z} \subseteq T.$$

By Proposition 2.2, the inclusion $C \subseteq T$ induces a monomorphism $B_m C \to B_m T$. Moreover, $\alpha \in B_m C$ and $B_m D \subseteq B_m C$. By Proposition 4.1, $Z(B_m C)$ is the infinite cyclic subgroup generated by α. So, α commutes with all the elements of $B_m D$.

This shows that $\alpha \in Z(B_m T)$. Similarly, $\beta \in Z(B_m T)$.

Step 3. — $Z(PB_m T) \subseteq Z_m$.

We prove Step 3 by induction on m. Let $m = 1$. Then $PB_1 T = \pi_1(T, P_1) = Z_1$, thus $Z(PB_1 T) = Z_1$.

Let $m > 1$. Let $g \in Z(PB_m T)$. We consider the following exact sequence:

$$1 \to \pi_1\big(T \backslash \{P_1, \dots, P_{m-1}\}\big) \longrightarrow PB_m T \xrightarrow{\rho} PB_{m-1} T \to 1.$$

We have $\rho(g) \in Z(PB_{m-1} T)$. By induction, $Z(PB_{m-1} T) \subseteq Z_{m-1}$. Moreover, $\rho(Z_m) = Z_{m-1}$. Thus we can choose $h \in Z_m$ such that $\rho(h) = \rho(g)$. We write $g' = gh^{-1}$. Then $g' \in Z(PB_m T)$ and $g' \in \pi_1(T \backslash \{P_1, \dots, P_{m-1}\})$ (since $\rho(g') = 1$), thus $g' \in Z(\pi_1(T \backslash \{P_1, \dots, P_{m-1}\})) = \{1\}$, thus $g' = gh^{-1} = 1$, therefore $g = h \in Z_m$.

Step 4. — $Z(B_m T) \subseteq PB_m T$.

Let $g \in B_m T$. We suppose that there exist $i, j \in \{1, \dots, m\}$, $i \neq j$ such that $\sigma(g)(P_i) = P_j$, and we prove that $g \notin Z(B_m T)$.

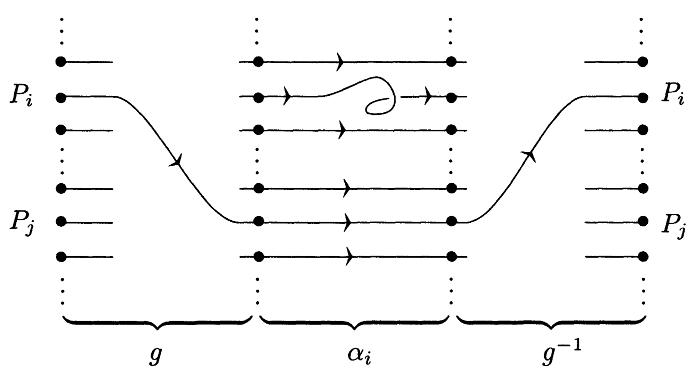

Figure 4.3

Let $\alpha_i \in PB_mT$ represented by $(P_1, \dots, P_{i-1}, a_i, P_{i+1}, \dots, P_m)$, where P_k denotes the constant path on P_k for $k = 1, \dots, m$ and a_i is as above. We consider the following exact sequence:

$$1 \to PB_{m-1}T \setminus \{P_i\} \longrightarrow PB_mT \xrightarrow{\rho_i} \pi_1(T, P_i) \to 1$$

Then $\rho_i(\alpha_i) \neq 1$ and $\rho_i(g\alpha_i g^{-1}) = 1$ (see Figure 4.3), thus $g\alpha_i g^{-1} \neq \alpha_i$, therefore $g \notin Z(B_mT)$. □

5. Commensurator, normalizer, and centralizer of B_nD in B_mM.

Let M be an oriented surface different from the sphere, and let $D \subseteq M$ be a disk embedded in M. Let $n \geq 2$, let $P_1, \dots, P_n \in D$, and let $P_{n+1}, \dots, P_m \in M \setminus D$. The goal of this section is to describe the commensurator, the normalizer, and the centralizer of B_nD in B_mM. Note that, if $n = 1$, then $B_1D = \{1\}$, thus

$$C_{B_mM}(B_1D) = N_{B_mM}(B_1D) = Z_{B_mM}(B_1D) = B_mM.$$

This section is divided into two subsections. We state our results in Subsection 5.1, and we prove them in Subsection 5.2.

5.1. Statements.

A *tunnel* on M based at $(D; P_{n+1}, \dots, P_m)$ is a map

$$H: D \cup \{P_{n+1}, \dots, P_m\} \times [0,1] \longrightarrow M$$

such that

1) $H(x, 0) = H(x, 1) = x$ for all $x \in D$,

2) $H(P_i, 0) = P_i$ and $H(P_i, 1) \in \{P_{n+1}, \dots, P_m\}$ for all $P_i \in \{P_{n+1}, \dots, P_m\}$,

3) $H(x, t) \neq H(y, t)$ for $x, y \in D \cup \{P_{n+1}, \dots, P_m\}$, $x \neq y$, and for $t \in [0, 1]$.

There is a natural notion of *homotopy of tunnels*. The *tunnel group* on M based at $(D; P_{n+1}, \dots, P_m)$ is the group

$$T_{m-n}M = T_{m-n}M(D; P_{n+1}, \dots, P_m)$$

of homotopy classes of tunnels on M based at $(D;P_{n+1},\dots,P_m)$. Multiplication is concatenation, as with braids.

We define a morphism

$$\tau\colon T_{m-n}M \times B_nD \longrightarrow B_mM$$

as follows. Let $h \in T_{m-n}M$ and let $f \in B_nD$. Let H be a tunnel on M based at $(D;P_{n+1},\dots,P_m)$ which represents h, and let $b=(b_1,\dots,b_n)$ be a braid on D based at $(P_1,\dots,P_n)$ which represents f. Let $\tilde{b}=(\tilde{b}_1,\dots,\tilde{b}_n,\tilde{b}_{n+1},\dots,\tilde{b}_m)$ be the braid on M defined by

- $\tilde{b}_i(t)=H(b_i(t),t)$ for $i\in\{1,\dots,n\}$ and for $t\in[0,1]$,
- $\tilde{b}_i(t)=H(P_i,t)$ for $i\in\{n+1,\dots,m\}$ and for $t\in[0,1]$.

Then $\tau(h,f)$ is the element of B_mM represented by $\tilde{b}$.

Remark. — This is related to the tensor product operation for the classical braid groups (see [Co]).

We denote by $C_{n,m}M$ the image of τ. Let $h\in T_{m-n}M$ and let $f,f'\in B_nD$. Then

$$\tau(h,f)\cdot f'\cdot\tau(h,f)^{-1}=\tau(1,ff'f^{-1})=ff'f^{-1}.$$

In particular,

$$C_{n,m}M\subseteq N_{B_mM}(B_nD).$$

THEOREM 5.1. — *Let $n\geq 2$, and M be an orientable surface, $M\neq S^2$. Then*

$$C_{B_mM}(B_nD)=C_{n,m}M.$$

Let $Z_{n,m}M$ denote the image by τ of $T_{m-n}M\times Z(B_nD)$.

COROLLARY 5.2. — *Let $n\geq 2$. Then*

$$C_{B_mM}(B_nD)=N_{B_mM}(B_nD)=C_{n,m}M,$$
$$Z_{B_mM}(B_nD)=Z_{n,m}M.$$

Remarks.

(i) We do not know whether a similar result holds for non-orientable surfaces.

(ii) Corollary 5.2 generalizes [FRZ, Thm. 4.2].

Let

$$B^1_{m-n+1}M = B^1_{m-n+1}M(P_1;P_{n+1},\dots,P_m)$$

denote the subgroup of $B_{m-n+1}M = B_{m-n+1}M(P_1,P_{n+1},\dots,P_m)$ consisting of $g \in B_{m-n+1}M$ such that $\sigma(g)(P_1) = P_1$. We define a morphism

$$\kappa: T_{m-n}M \longrightarrow B^1_{m-n+1}M$$

as follows. Let $h \in T_{m-n}M$. Let H be a tunnel on M based at $(D;P_{n+1},\dots,P_m)$ which represents h. Let $b = (b_1,b_{n+1},\dots,b_m)$ be the braid defined by

$$b_i(t) = H(P_i,t) \text{ for } i \in \{1,n+1,\dots,m\} \text{ and for } t \in [0,1].$$

Then $\kappa(h)$ is the element of $B^1_{m-n+1}M$ represented by b.

THEOREM 5.3. — *Let $n \geq 2$. There exists a morphism $\delta: C_{n,m}M \to B^1_{m-n+1}M$ such that*

$$\delta\big(\tau(h,f)\big) = \kappa(h)$$

for all $h \in T_{m-n}M$ and for all $f \in B_nD$. Moreover, we have the following exact sequences:

$$1 \to B_nD \longrightarrow C_{n,m}M \xrightarrow{\delta} B^1_{m-n+1}M \to 1,$$

$$1 \to Z(B_nD) \longrightarrow Z_{n,m}M \xrightarrow{\delta} B^1_{m-n+1}M \to 1.$$

THEOREM 5.4. — *Let $n \geq 2$. Let M be either with non-empty boundary or a torus. There exists a morphism $\iota: B^1_{m-n+1}M \to Z_{n,m}M$ such that $\delta \circ \iota = \mathrm{id}$. In particular,*

$$C_{n,m}M \simeq B^1_{m-n+1}M \times B_nD,$$

$$Z_{n,m}M \simeq B^1_{m-n+1}M \times Z(B_nD).$$

Remark. — Theorem 5.4 generalizes [FRZ, Thm. 4.3] and [Ro, Thm. 3].

5.2. Proofs.

LEMMA 5.5. — *We consider an exact sequence*

$$1 \to G_1 \longrightarrow G_2 \xrightarrow{\phi} G_3 \to 1.$$

Let $H_2 \subseteq G_2$ be a subgroup, let $H_3 = \phi(H_2)$, and let $H_1 = H_2 \cap G_1$. Then

$$\phi(C_{G_2}(H_2)) \subseteq C_{G_3}(H_3),$$
$$C_{G_2}(H_2) \cap G_1 \subseteq C_{G_1}(H_1).$$

Proof. — Let $g \in C_{G_2}(H_2)$. We write

$$F_2 = H_2 \cap gH_2g^{-1}.$$

Let $h_1, \dots, h_k \in H_2$ be such that

$$H_2 = F_2 \cup h_1F_2 \cup \dots \cup h_kF_2.$$

Then

$$\phi(H_2) = H_3 = \phi(F_2) \cup \phi(h_1)\phi(F_2) \cup \dots \cup \phi(h_k)\phi(F_2).$$

So, $\phi(F_2)$ has finite index in H_3. Moreover,

$$\phi(F_2) = \phi(H_2 \cap gH_2g^{-1}) \subseteq \phi(H_2) \cap \phi(gH_2g^{-1}) = H_3 \cap \phi(g)H_3\phi(g)^{-1},$$

thus $H_3 \cap \phi(g)H_3\phi(g)^{-1}$ has finite index in H_3. Similarly, $H_3 \cap \phi(g)H_3\phi(g)^{-1}$ has finite index in $\phi(g)H_3\phi(g)^{-1}$. So, $\phi(g) \in C_{G_3}(H_3)$.

Let $g \in C_{G_2}(H_2) \cap G_1$. We write

$$F_2 = H_2 \cap gH_2g^{-1}.$$

Let $h_1, \dots, h_k \in H_2$ be such that

$$H_2 = F_2 \cup h_1F_2 \cup \dots \cup h_kF_2.$$

We assume that

$$h_iF_2 \cap H_1 \neq \emptyset \qquad \text{for } i = 1, \dots, \ell,$$
$$h_iF_2 \cap H_1 = \emptyset \qquad \text{for } i = \ell+1, \dots, k.$$

We can also assume that $h_i \in H_1$ for $i = 1, \dots, \ell$. Then

$$H_1 = (F_2 \cap H_1) \cup h_1(F_2 \cap H_1) \cup \dots \cup h_\ell(F_2 \cap H_1).$$

Moreover,

$$F_2 \cap H_1 = H_2 \cap gH_2g^{-1} \cap H_1 = H_1 \cap gH_1g^{-1}.$$

Thus $H_1 \cap gH_1g^{-1}$ has finite index in H_1. Similarly, $H_1 \cap gH_1g^{-1}$ has finite index in gH_1g^{-1}. So, $g \in C_{G_1}(H_1)$. □

LEMMA 5.6. — *Let M be either with non-empty boundary or a torus. There exists a morphism $\iota_0: B^1_{m-n+1}M \to T_{m-n}M$ such that $\kappa \circ \iota_0 = \mathrm{id}$.*

Proof. — Let TM be the tangent space of M. It is known that $TM = \mathbb{R}^2 \times M$. We provide M with the flat Riemannian metric. Namely, for all $x \in M$, the metric $\langle\ ,\ \rangle_x$ on x is the standard scalar product on $\mathbb{R}^2$ (which does not depend on x). Furthermore, we set the following assumptions:

1) There is no closed geodesic of length ≤ 4.

2) D is the disk of radius 1 centred at P_1.

3) $d(P_1, P_i) \geq 2$ for all $i \in \{n+1, \dots, m\}$.

4) Let $C_1, \dots, C_q$ be the boundary components of M. Then $d(P_1, C_j) \geq 2$ for all $j \in \{1, \dots, q\}$.

Now, let $f \in B^1_{m-n+1}M$. Let $b = (b_1, b_{n+1}, \dots, b_m)$ be a braid based at $(P_1, P_{n+1}, \dots, P_m)$ which represents f. For $t \in [0,1]$, we write

$$r(t) = \inf\Big\{\frac{1}{2}d\big(b_1(t), b_{n+1}(t)\big), \dots, \frac{1}{2}d\big(b_1(t), b_m(t)\big),$$
$$\frac{1}{2}d\big(b_1(t), C_1\big), \dots, \frac{1}{2}d\big(b_1(t), C_q\big), 1\Big\}.$$

Then $r: [0,1] \to \mathbb{R}$ is a continuous map and $r(t) > 0$ for all $t \in [0,1]$. Let

$$D_0 = \big\{(X_1, X_2) \in \mathbb{R}^2;\ X_1^2 + X_2^2 \leq 1\big\}.$$

Let $H_0: D_0 \times [0,1] \to M$ be the map defined by

$$H_0(X,t) = \exp_{b_1(t)}\big(r(t)X\big) \quad \text{for } X \in D_0 \text{ and for } t \in [0,1].$$

Let $F: D_0 \to D$ be the diffeomorphism defined by

$$F(X) = \exp_{P_1} X \quad \text{for } X \in D_0.$$

Let

$$H: D \cup \{P_{n+1}, \dots, P_m\} \times [0,1] \longrightarrow M$$

be the map defined by

$$H(x,t) = H_0(F^{-1}(x), t) \quad \text{for } x \in D \text{ and for } t \in [0,1],$$

$$H(P_i, t) = b_i(t) \quad \text{for } P_i \in \{P_{n+1}, \dots, P_m\} \text{ and for } t \in [0,1].$$

The map H is a tunnel on M based at $(D; P_{n+1}, \dots, P_m)$. We define $\iota_0(f)$ to be the element of $T_{m-n}M$ represented by H.

One can easily verify that ι_0 is well-defined, that ι_0 is a morphism, and that $\kappa \circ \iota_0 = \mathrm{id}$. $\square$

LEMMA 5.7. — *The morphism $\kappa: T_{m-n}M \to B^1_{m-n+1}M$ is surjective.*

Remark. — We do not know whether a similar result holds for non-orientable surfaces.

Proof. — We choose an open disk K_0 embedded in $M\backslash D$ and which does not contain any P_i for $i = n+1, \dots, m$. The inclusion $M\backslash K_0 \subseteq M$ induces an epimorphism $\phi: B^1_{m-n+1}M\backslash K_0 \to B^1_{m-n+1}M$. The following diagram commutes:

$$\begin{array}{ccc} T_{m-n}M\backslash K_0 & \xrightarrow{\kappa} & B^1_{m-n+1}M\backslash K_0 \\ \downarrow & & \downarrow{\scriptstyle\phi} \\ T_{m-n}M & \xrightarrow{\kappa} & B^1_{m-n+1}M. \end{array}$$

By Lemma 5.6, $\kappa: T_{m-n}M\backslash K_0 \to B^1_{m-n+1}M\backslash K_0$ is surjective. It follows that $\kappa: T_{m-n}M \to B^1_{m-n+1}M$ is surjective, too. $\square$

From now on, we fix a (set) section $\iota_0: B^1_{m-n+1}M \to T_{m-n}M$ of κ. Moreover, we assume that ι_0 is a morphism if M is either with non-empty boundary or a torus, and that $\iota_0(1) = 1$.

Theorem 5.1 is a direct consequence of the following lemma.

LEMMA 5.8. — *Let $n \geq 2$. Let $g \in C_{B_mM}(B_nD)$. There exist $u \in B^1_{m-n+1}M$ and $f \in B_nD$ such that*

$$g = \tau\big(\iota_0(u), f\big).$$

The following lemmas 5.9 and 5.10 are preliminary results to the proof of Lemma 5.8.

Recall that Σ_m denotes the group of permutations of $\{P_1, \dots, P_m\}$, that Σ_n denotes the group of permutations of $\{P_1, \dots, P_n\}$, and that Σ_{m-n} denotes the group of permutations of $\{P_{n+1}, \dots, P_m\}$. We write

$$B^n_mM = \sigma^{-1}(\Sigma_{m-n}).$$

LEMMA 5.9. — *Let $n \geq 1$. Let $g \in C_{B^n_mM}(PB_nD)$. There exist $u \in B^1_{m-n+1}M$ and $f \in PB_nD$ such that*

$$g = \tau\big(\iota_0(u), f\big).$$

Proof. — We prove Lemma 5.9 by induction on n. Let $n = 1$. Then $PB_1D = \{1\}$, thus

$$C_{B_m^1 M}(PB_1D) = B_m^1 M.$$

On the other hand, if $u \in B_m^1 M$, then

$$u = \tau\big(\iota_0(u), P_1\big),$$

where P_1 denotes the constant path on P_1.

Let $n > 1$. Let $g \in C_{B_m^n M}(PB_nD)$. We write

$$M' = M \backslash \{P_1, \dots, P_{n-1}, P_{n+1}, \dots, P_m\},$$

and $D' = D\backslash\{P_1, \dots, P_{n-1}\}$. We consider the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & \pi_1 D' & \longrightarrow & PB_nD & \xrightarrow{\rho} & PB_{n-1}D & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_1 M' & \longrightarrow & B_m^n M & \xrightarrow{\rho} & B_{m-1}^{n-1} M & \to & 1. \end{array}$$

By Lemma 5.5, $\rho(g) \in C_{B_{m-1}^{n-1} M}(PB_{n-1}D)$. By induction, there exist $u \in B_{m-n+1}^1 M$ and $f_1 \in PB_{n-1}D$ such that

$$\rho(g) = \tau\big(\iota_0(u), f_1\big).$$

We choose $f_2 \in PB_nD$ such that $\rho(f_2) = f_1$ and we write

$$g' = g \cdot \tau\big(\iota_0(u), f_2\big)^{-1}.$$

We have $g' \in \pi_1 M'$ (since $\rho(g') = 1$) and $g' \in C_{B_m^n M}(PB_nD)$, thus, by Lemma 5.5,

$$g' \in C_{\pi_1 M'}(\pi_1 D').$$

If either $m \neq n$ or M is not a disk, then M' is large and D' is not a Möbius collar in M', thus, by Theorem 3.1,

$$C_{\pi_1 M'}(\pi_1 D') = \pi_1 D'.$$

If $m = n$ and M is a disk, then $\pi_1 M' = \pi_1 D'$, thus

$$C_{\pi_1 M'}(\pi_1 D') = \pi_1 D'.$$

If follows that

$$g' = f_3 \in \pi_1 D' \subseteq PB_nD.$$

So,

$$g = f_3 \cdot \tau\big(\iota_0(u), f_2\big) = \tau\big(\iota_0(u), f_3 f_2\big). \qquad \square$$

LEMMA 5.10. — *Let $n \geq 2$. Let $g \in C_{B_m M}(B_n D)$. Then $\sigma(g)$ is an element of $\Sigma_n \times \Sigma_{m-n}$.*

Proof. — Let $g \in C_{B_m M}(B_n D)$. We suppose that $\sigma(g)(P_{n+1}) = P_1$. Let $f \in \pi_1(D\backslash\{P_2, \dots, P_n\}, P_1)$, $f \neq 1$. The group PB_nD has finite index in B_nD, thus $C_{B_m M}(B_n D) = C_{B_m M}(PB_n D)$. Since $\pi_1(D\backslash\{P_2, \dots, P_n\}) \subseteq PB_nD$ and since $g \in C_{B_m M}(PB_n D)$, there exists an integer $k > 0$ such that

$$gf^k g^{-1} \in PB_n D.$$

We consider the following exact sequence:

$$1 \to \pi_1\big(M\backslash\{P_1, \dots, P_n, P_{n+2}, \dots, P_m\}\big) \longrightarrow PB_m M \xrightarrow{\rho} PB_{m-1} M \to 1.$$

The morphism ρ sends PB_nD isomorphically on PB_nD. On the other hand, $gf^kg^{-1} \neq 1$ (since $f \neq 1$ and B_mM is torsion free) and $\rho(gf^kg^{-1}) = 1$ (see Figure 5.1). This is a contradiction.

This proves that $\sigma(g) \in \Sigma_n \times \Sigma_{m-n}$. □

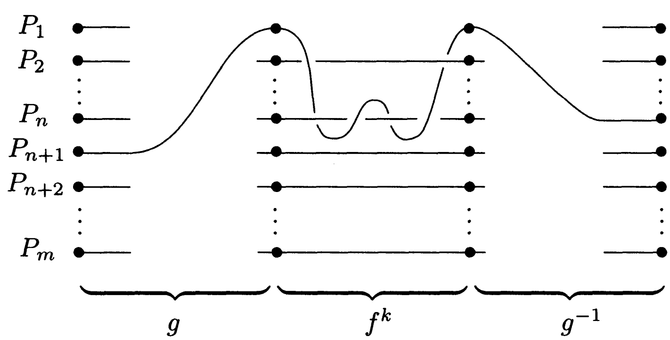

Figure 5.1

Proof of Lemma 5.8. — Let $g \in C_{B_m M}(B_n D)$. By Lemma 5.10, $\sigma(g) \in \Sigma_n \times \Sigma_{m-n}$. We choose $f_1 \in B_nD$ such that $\sigma(gf_1^{-1}) \in \Sigma_{m-n}$ and we write $g' = gf_1^{-1}$. Then $g' \in B_m^n M$, $g' \in C_{B_m M}(B_n D)$, and $C_{B_m M}(B_n D) = C_{B_m M}(PB_n D)$, thus

$$g' \in C_{B_m^n M}(PB_n D).$$

By Lemma 5.9, there exist $u \in B_{m-n+1}^1 M$ and $f_2 \in PB_nD$ such that

$$g' = \tau\big(\iota_0(u), f_2\big).$$

So,

$$g = \tau\big(\iota_0(u), f_2\big) \cdot f_1 = \tau\big(\iota_0(u), f_2 f_1\big). \qquad \square$$

Proof of Theorem 5.3. — The proof is divided into five steps.

Step 1. — Definition of δ.

We consider the natural morphism

$$\delta_0 : B_m^n M \longrightarrow B_{m-n+1}^1 M.$$

Let $g \in C_{n,m}M$. By Lemma 5.10, $\sigma(g) \in \Sigma_n \times \Sigma_{m-n}$. We choose $f \in B_n D$ such that $\sigma(gf^{-1}) \in \Sigma_{m-n}$ and we set

$$\delta(g) = \delta_0(gf^{-1}).$$

We prove that the definition of $\delta(g)$ does not depend on the choice of f. Let $f_1, f_2 \in B_n D$ be such that $\sigma(gf_1^{-1}) \in \Sigma_{m-n}$ and $\sigma(gf_2^{-1}) \in \Sigma_{m-n}$. Then

$$\delta_0(gf_2^{-1})^{-1}\delta_0(gf_1^{-1}) = \delta_0(f_2 g^{-1} g f_1^{-1}) = \delta_0(f_2 f_1^{-1}) = 1,$$

thus $\delta_0(gf_1^{-1}) = \delta_0(gf_2^{-1})$.

Step 2. — The map $\delta : C_{n,m}M \to B_{m-n+1}^1 M$ is a morphism.

Let $g_1, g_2 \in C_{n,m}M$. Let $f_1, f_2 \in B_n D$ be such that $\sigma(g_1 f_1^{-1}) \in \Sigma_{m-n}$ and $\sigma(g_2 f_2^{-1}) \in \Sigma_{m-n}$. By Corollary 5.2,

$$C_{n,m}M = N_{B_m M}(B_n D),$$

thus there exists $f_3 \in B_n D$ such that $g_2^{-1} f_1 g_2 = f_3$. Moreover,

$$\sigma((g_1 g_2)(f_2 f_3)^{-1}) = \sigma(g_1 f_1^{-1} g_2 f_2^{-1}) \in \Sigma_{m-n}.$$

So,

$$\begin{aligned} \delta(g_1)\delta(g_2) &= \delta_0(g_1 f_1^{-1})\delta_0(g_2 f_2^{-1}) = \delta_0(g_1 f_1^{-1} g_2 f_2^{-1}) \\ &= \delta_0((g_1 g_2)(f_2 f_3)^{-1}) \;\; = \delta(g_1 g_2). \end{aligned}$$

Step 3. — Let $h \in T_{m-n}M$ and let $f \in B_n D$. Then

$$\delta\big(\tau(h,f)\big) = \delta\big(\tau(h,1)\cdot\tau(1,f)\big) = \delta\big(\tau(h,1)\big)\cdot\delta(f) = \kappa(h).$$

Step 4. — We have the following exact sequence:

$$1 \to B_n D \longrightarrow C_{n,m}M \xrightarrow{\delta} B^1_{m-n+1}M \to 1.$$

Let $u \in B^1_{m-n+1}M$. Then

$$\delta\big(\tau(\iota_0(u),1)\big) = \kappa\big(\iota_0(u)\big) = u.$$

This shows that δ is surjective.

Let $g \in C_{n,m}M$. By Lemma 5.8, there exist $u \in B^1_{m-n+1}M$ and $f \in B_n D$ such that $g = \tau(\iota_0(u), f)$. If $g \in \ker\delta$, then

$$1 = \delta(g) = \kappa\big(\iota_0(u)\big) = u,$$

thus

$$g = \tau\big(\iota_0(u), f\big) = \tau(1,f) = f \in B_n D.$$

Step 5. — We have the following exact sequence:

$$1 \to Z(B_n D) \longrightarrow Z_{n,m}M \xrightarrow{\delta} B^1_{m-n+1}M \to 1.$$

By Step 4, it suffices to show that $\delta\colon Z_{n,m}M \to B^1_{m-n+1}M$ is surjective. Let $u \in B^1_{m-n+1}M$. Then $\tau(\iota_0(u),1) \in Z_{n,m}M$ and $\delta(\tau(\iota_0(u),1)) = u$. □

Proof of Theorem 5.4. — The morphism $\iota\colon B^1_{m-n+1}M \to Z_{n,m}M$ is defined by

$$\iota(u) = \tau\big(\iota_0(u),1\big) \quad \text{for } u \in B^1_{m-n+1}M.$$

Clearly, $\delta\circ\iota = \mathrm{id}$. □

6. Commensurator, normalizer, and centralizer of B_nN in B_mM.

Let M be a large surface, and let N be a subsurface of M such that N is neither a disk, nor a Möbius collar in M, and such that none of the connected components of $\overline{M\backslash N}$ is a disk. Let $N_1,\dots,N_r$ be the connected components of $\overline{M\backslash N}$. Let $P_1,\dots,P_n \in N$, and let $P_{n+1},\dots,P_m \in M\backslash N$. For $i=1,\dots,r$ we write

$$\mathcal{P}_i = \{P_{n+1},\dots,P_m\}\cap N_i, \quad B_{n_i}N_i = B_{n_i}N_i(\mathcal{P}_i),$$

where n_i denotes the cardinality of $\mathcal{P}_i$. If $n_i = 0$, we make the convention that $B_0N_i = \{1\}$.

The goal of this section is to prove the following theorem.

THEOREM 6.1. — *One has*

$$C_{B_mM}(B_nN) = B_nN \times B_{n_1}N_1 \times \cdots \times B_{n_r}N_r.$$

COROLLARY 6.2. — *One has*

$$C_{B_mM}(B_nN) = N_{B_mM}(B_nN) = B_nN \times B_{n_1}N_1 \times \cdots \times B_{n_r}N_r,$$
$$Z_{B_mM}(B_nN) = Z(B_nN) \times B_{n_1}N_1 \times \ldots \times B_{n_r}N_r.$$

The remains of this section are divided into two subsections. In Subsection 6.1 we study an action of $\pi_1 N$ on some groupoid $\Pi_1(M\backslash\{P_0\})$. In Subsection 6.2 we apply the results of Subsection 6.1 to prove Theorem 6.1.

6.1. Action of $\pi_1 N$ on $\Pi_1(M\backslash\{P_0\})$.

Throughout this subsection, we fix a point $P_0 \in N$ and a point $P_i \in N_i$ for all $i = 1, \ldots, r$. Moreover, we do not assume that none of the connected components of $\overline{M\backslash N}$ is a disk.

The *fundamental groupoid* of $M\backslash\{P_0\}$ based at $\{P_1, \ldots, P_r\}$ is the groupoid $\Pi_1(M\backslash\{P_0\})$ defined by the following data:

1) The set of objects of $\Pi_1(M\backslash\{P_0\})$ is $\{P_1, \ldots, P_r\}$.

2) Let $P_i, P_j \in \{P_1, \ldots, P_r\}$. The set of morphisms from P_i to P_j is the set $\Pi_1(M\backslash\{P_0\})[P_i, P_j]$ of homotopy classes of paths in $M\backslash\{P_0\}$ from P_i to P_j.

Let $P_i, P_j, P_k \in \{P_1, \ldots, P_r\}$. For convenience, we assume that the composition map goes from $\Pi_1(M\backslash\{P_0\})[P_i, P_j] \times \Pi_1(M\backslash\{P_0\})[P_j, P_k]$ to $\Pi_1(M\backslash\{P_0\})[P_i, P_k]$. Note that

$$\Pi_1(M\backslash\{P_0\})[P_i, P_i] = \pi_1(M\backslash\{P_0\}, P_i).$$

Moreover, if $x \in \Pi_1(M\backslash\{P_0\})[P_i, P_j]$, then the map

$$\begin{array}{rccc} \theta_x\colon & \pi_1(M\backslash\{P_0\}, P_i) & \longrightarrow & \Pi_1(M\backslash\{P_0\})[P_i, P_j] \\ & g & \longmapsto & gx \end{array}$$

is a bijection.

Let $P_i, P_j \in \{P_1, \ldots, P_r\}$. An *interbraid* on M based at $(P_0, [P_i, P_j])$ is a pair $b = (b_0, b_1)$ of paths, $b_k\colon [0,1] \to M$, such that

1) $b_0(0) = b_0(1) = P_0$, $b_1(0) = P_i$, and $b_1(1) = P_j$,

2) $b_0(t) \neq b_1(t)$ for $t \in [0,1]$.

There is a natural notion of *homotopy of interbraids*. The *interbraid groupoid* on M based at $(P_0, \{P_1, \dots, P_r\})$ is the groupoid

$$IB_2M = IB_2M\big(P_0, \{P_1, \dots, P_r\}\big)$$

defined by the following data:

1) The set of objects of IB_2M is $\{P_1, \dots, P_r\}$.

2) Let $P_i, P_j \in \{P_1, \dots, P_r\}$. The set of morphisms from P_i to P_j is the set $IB_2M[P_i, P_j]$ of homotopy classes of interbraids on M based at $(P_0, [P_i, P_j])$.

Let $P_i, P_j, P_k \in \{P_1, \dots, P_r\}$. For convenience, we assume that the composition map goes from $IB_2M[P_i, P_j] \times IB_2M[P_j, P_k]$ to $IB_2M[P_i, P_k]$. Note that

$$IB_2M[P_i, P_i] = PB_2M(P_0, P_i).$$

Moreover, if $X \in IB_2M[P_i, P_j]$, then the map

$$\begin{aligned} \Theta_X\colon PB_2M(P_0, P_i) &\longrightarrow IB_2M[P_i, P_j] \\ g &\longmapsto gX \end{aligned}$$

is a bijection.

Let $P_i, P_j \in \{P_1, \dots, P_r\}$. We consider the natural maps

$$\alpha\colon IB_2M[P_i, P_j] \longrightarrow \pi_1(M, P_0),$$

$$\beta\colon \Pi_1\big(M\backslash\{P_0\}\big)[P_i, P_j] \longrightarrow IB_2M[P_i, P_j].$$

Let $x \in \Pi_1(M\backslash\{P_0\})[P_i, P_j]$, and let $X = \beta(x)$. Then the following diagram commutes:

$$\begin{array}{ccccccc} 1 \to & \pi_1\big(M\backslash\{P_0\}, P_i\big) & \longrightarrow & PB_2M(P_0, P_i) & \xrightarrow{\rho} & \pi_1(M, P_0) & \to 1 \\ & \downarrow{\scriptstyle \theta_x} & & \downarrow{\scriptstyle \Theta_X} & & \downarrow{\scriptstyle \mathrm{id}} & \\ & \Pi_1\big(M\backslash\{P_0\}\big)[P_i, P_j] & \xrightarrow{\beta} & IB_2M[P_i, P_j] & \xrightarrow{\alpha} & \pi_1(M, P_0). & \end{array}$$

Thus, α is surjective, β is injective, and

$$\alpha^{-1}(1) = \beta\big(\Pi_1(M\backslash\{P_0\})[P_i, P_j]\big).$$

So, we can assume that

$$\Pi_1(M\backslash\{P_0\})[P_i,P_j] = \beta(\Pi_1(M\backslash\{P_0\})[P_i,P_j]) \subseteq IB_2M[P_i,P_j].$$

The inclusion $N \subseteq M$ induces a morphism

$$\psi_k : \pi_1(N,P_0) \longrightarrow PB_2M(P_0,P_k)$$

for all $k = 1,\ldots,r$. We define an action of $\pi_1(N,P_0)$ on $IB_2M[P_i,P_j]$ as follows. Let $u \in \pi_1(N,P_0)$ and let $X \in IB_2M[P_i,P_j]$. Then

$$u(X) = \psi_i(u) \cdot X \cdot \psi_j(u)^{-1}.$$

Let $x \in \Pi_1(M\backslash\{P_0\})[P_i,P_j]$ and let $u \in \pi_1(N,P_0)$. Then $\alpha(u(x)) = 1$, thus $u(x) \in \Pi_1(M\backslash\{P_0\})[P_i,P_j]$. So, the action of $\pi_1(N,P_0)$ on $IB_2M[P_i,P_j]$ induces an action of $\pi_1(N,P_0)$ on $\Pi_1(M\backslash\{P_0\})[P_i,P_j]$.

We denote by $S_N[P_i,P_j]$ the set of $x \in \Pi_1(M\backslash\{P_0\})[P_i,P_j]$ such that, for all $u \in \pi_1(N,P_0)$ there exists an integer $k > 0$ such that $u^k(x) = x$. The main result of Subsection 6.1 is the following proposition.

Proposition 6.3. — *Let $i,j \in \{1,\ldots,r\}$. Then*

$$S_N[P_i,P_j] = \begin{cases} \pi_1(N_i,P_i) = \pi_1(N_j,P_j) & \text{if } i = j, \\ \emptyset & \text{if } i \neq j. \end{cases}$$

Lemmas 6.4 to 6.7 are preliminary results to the proof of Proposition 6.3.

From now on and till the end of the proof of Lemma 6.7, we set the following assumptions (see Figure 6.1):

1) N is a sphere with $q+1$ holes ($q \geq 1$). We denote by $C_0, C_1, \ldots, C_q$ the boundary components of N.

2) $\overline{M\backslash N}$ has two connected components, N_1 and N_2.

3) $N \cap N_1 = C_1 \cup \ldots \cup C_q$, and $N \cap N_2 = C_0$.

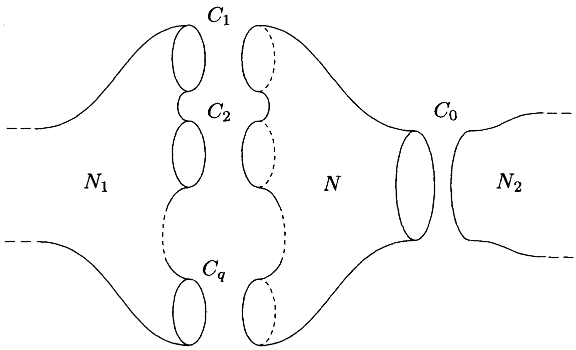

Figure 6.1

We choose a point $P_0' \in N$ different from P_0. We choose a point $Q_i \in C_i$ for all $i = 0, 1, \dots, q$. According to Figure 6.2,

1) we choose a path $\gamma_i^s\colon [0,1] \to N \backslash \{P_0\}$ from P_0' to Q_i for all $i = 0, 1, \dots, q$,

2) we choose a path $\gamma_i^t\colon [0,1] \to N_1$ from P_1 to Q_i for all $i = 1, \dots, q$,

3) we choose a path $\gamma_0^t\colon [0,1] \to N_2$ from P_2 to Q_0.

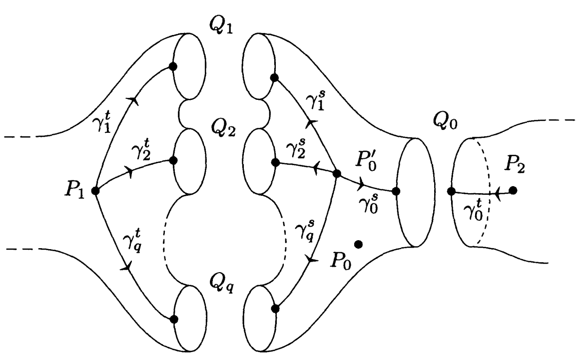

Figure 6.2

We write

$$
\begin{aligned}
\gamma_i &= \gamma_i^s (\gamma_i^t)^{-1} && \text{for } i = 0, 1, \dots, q, \\
\beta_i &= \gamma_1^{-1} \gamma_i \in \pi_1\big(M \backslash \{P_0\}, P_1\big) && \text{for } i = 1, \dots, q, \\
T &= \gamma_1^{-1} \gamma_0 \in \Pi_1\big(M \backslash \{P_0\}\big)[P_1, P_2]. &&
\end{aligned}
$$

Note that the path T induces a morphism

$$\begin{array}{ccc}\pi_1(N_2,P_2) & \longrightarrow & \pi_1(M\backslash\{P_0\},P_1),\\ g & \longmapsto & TgT^{-1}.\end{array}$$

The following lemma is a consequence of Van Kampen's theorem.

LEMMA 6.4. — *Let F be the subgroup of $\pi_1(M\backslash\{P_0\},P_1)$ generated by $\beta_2,\ldots,\beta_q$,*

$$\pi_1(M\backslash\{P_0\},P_1) = \pi_1(N_1,P_1) * (T\cdot\pi_1(N_2,P_2)\cdot T^{-1}) * F.$$

All these groups are free and $\{\beta_2,\ldots,\beta_q\}$ is a basis for F. □

According to Figure 6.3,

1) we choose a simple loop $\alpha_i\colon[0,1]\to C_i$ based at Q_i for all $i=0,1,\ldots,q$,

2) we choose a path $\delta_i\colon[0,1]\to N$ from P_0 to Q_i for all $i=0,1,\ldots,q$.

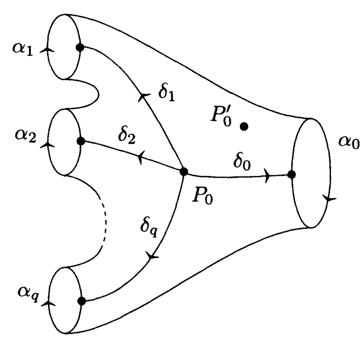

Figure 6.3

We write

$$\begin{aligned} h_i &= \gamma_i^t\alpha_i(\gamma_i^t)^{-1}\in\pi_1(N_1,P_1) &&\text{for } i=1,\ldots,q,\\ h_0 &= \gamma_0^t\alpha_0(\gamma_0^t)^{-1}\in\pi_1(N_2,P_2), &&\\ u_i &= \delta_i\alpha_i\delta_i^{-1}\in\pi_1(N,P_0) &&\text{for } i=0,1,\ldots,q.\end{aligned}$$

According to Figure 6.4, we choose a loop $\mu\colon[0,1]\to N\backslash\{P_0\}$ based at P_0' turning around P_0.

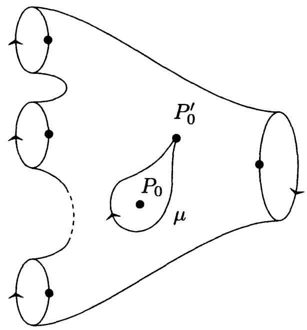

Figure 6.4

We write

$$h_c = \gamma_1^{-1}\mu\gamma_1 \in \pi_1\big(M\backslash\{P_0\}, P_1\big).$$

One can easily verify that

$$h_c = Th_0^{-1}T^{-1} \cdot h_1^{-1} \cdot \beta_2 h_2^{-1}\beta_2^{-1} \cdot \ldots \cdot \beta_q h_q^{-1}\beta_q^{-1}.$$

LEMMA 6.5. — *One has*

(i) $u_0(g) = g$ *for all* $g \in \pi_1(N_1,P_1)$,

(ii) $u_0(g) = g$ *for all* $g \in \pi_1(N_2,P_2)$,

(iii) $u_0(\beta_i) = \beta_i$ *for all* $\beta_i \in \{\beta_2,\ldots,\beta_q\}$,

(iv) $u_0(T) = h_c^{-1}T$.

Proof. — (i) We choose a loop $\zeta\colon[0,1] \to N_1$ based at P_1 which represents g. Then the image of ζ and the image of u_0 are disjoint (see Figure 6.5), thus $u_0(g) = g$.

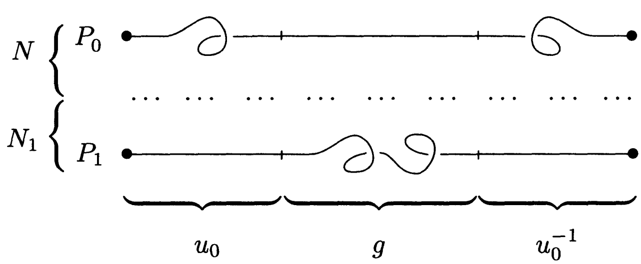

Figure 6.5

(ii) We choose a loop $\zeta\colon [0,1] \to N_2$ based at P_2 which represents g. The image of ζ and the image of u_0 are disjoint, thus $u_0(g) = g$.

(iii) The image of β_i and the image of u_0 are disjoint, thus $u_0(\beta_i) = \beta_i$.

(iv) In Figure 6.6, the interbraid drawn in (a) is homotopic to the interbraid drawn in (b), and the interbraid drawn in (b) is homotopic to the interbraid drawn in (c). The interbraid drawn in (a) represents $u_0(T)$, and the interbraid drawn in (c) represents

$$\gamma_1^{-1}\mu^{-1}\gamma_0.$$

It follows that

$$u_0(T) = \gamma_1^{-1}\mu^{-1}\gamma_0 = \gamma_1^{-1}\mu^{-1}\gamma_1 \cdot \gamma_1^{-1}\gamma_0 = h_c^{-1}T. \qquad \square$$

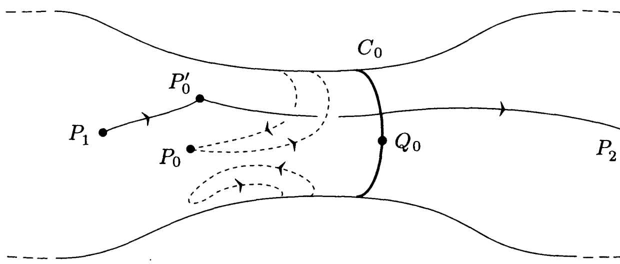

Figure 6.6.a

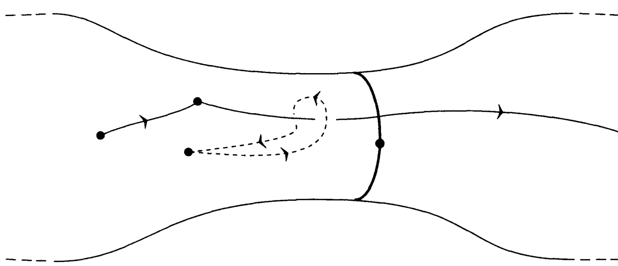

Figure 6.6.b

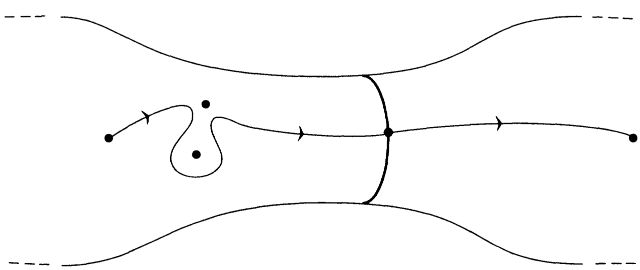

Figure 6.6.c

LEMMA 6.6. — *Let $k \in \{2, \dots, q\}$. Then*

(i) $u_k(g) = g$ *for all* $g \in \pi_1(N_1, P_1)$,

(ii) $u_k(g) = g$ *for all* $g \in \pi_1(N_2, P_2)$,

(iii) $u_k(T) = T$,

(iv) $u_k(\beta_i) = \beta_i$ *for all* $i \in \{2, \dots, k-1\}$,

(v) $u_k(\beta_k) = \beta_k h_k^{-1} \beta_k^{-1} h_c^{-1} \beta_k h_k$,

(vi) $u_k(h_c) = \beta_k h_k^{-1} \beta_k^{-1} h_c \beta_k h_k \beta_k^{-1}$.

Proof. — The statements (i) to (iv) can be proved with the same arguments as those given in the proofs of the statements (i) to (iii) of Lemma 6.5.

(v) In Figure 6.7, the braid drawn in (a) is homotopic to the braid drawn in (b), and the braid drawn in (b) is homotopic to the braid drawn in (c). The braid drawn in (a) represents $u_k(\beta_k)$, and the braid drawn in (c) represents

$$\gamma_1^{-1} \gamma_k^s \alpha_k^{-1} (\gamma_k^s)^{-1} \mu^{-1} \gamma_k^s \alpha_k (\gamma_k^t)^{-1}.$$

It follows that

$$\begin{aligned} u_k(\beta_k) &= \gamma_1^{-1} \gamma_k^s \alpha_k^{-1} (\gamma_k^s)^{-1} \mu^{-1} \gamma_k^s \alpha_k (\gamma_k^t)^{-1} \\ &= \gamma_1^{-1} \gamma_k^s (\gamma_k^t)^{-1} \cdot \gamma_k^t \alpha_k^{-1} (\gamma_k^t)^{-1} \cdot \gamma_k^t (\gamma_k^s)^{-1} \gamma_1 \cdot \gamma_1^{-1} \mu^{-1} \gamma_1 \\ &\qquad \cdot \gamma_1^{-1} \gamma_k^s (\gamma_k^t)^{-1} \cdot \gamma_k^t \alpha_k (\gamma_k^t)^{-1} \\ &= \beta_k h_k^{-1} \beta_k^{-1} h_c^{-1} \beta_k h_k. \end{aligned}$$

(vi) In Figure 6.8, the braid drawn in (a) is homotopic to the braid drawn in (b), the braid drawn in (b) is homotopic to the braid drawn in (c), and the braid drawn in (c) is homotopic to the braid drawn in (d). The braid drawn in (a) represents $u_k(h_c)$, and the braid drawn in (d) represents

$$\gamma_1^{-1} \gamma_k^s \alpha_k^{-1} (\gamma_k^s)^{-1} \mu \gamma_k^s \alpha_k (\gamma_k^s)^{-1} \gamma_1.$$

It follows that

$$\begin{aligned} u_k(h_c) &= \gamma_1^{-1} \gamma_k^s \alpha_k^{-1} (\gamma_k^s)^{-1} \mu \gamma_k^s \alpha_k (\gamma_k^s)^{-1} \gamma_1 \\ &= \gamma_1^{-1} \gamma_k^s (\gamma_k^t)^{-1} \cdot \gamma_k^t \alpha_k^{-1} (\gamma_k^t)^{-1} \cdot \gamma_k^t (\gamma_k^s)^{-1} \gamma_1 \cdot \gamma_1^{-1} \mu \gamma_1 \\ &\qquad \cdot \gamma_1^{-1} \gamma_k^s (\gamma_k^t)^{-1} \cdot \gamma_k^t \alpha_k (\gamma_k^t)^{-1} \cdot \gamma_k^t (\gamma_k^s)^{-1} \gamma_1 \\ &= \beta_k h_k^{-1} \beta_k^{-1} h_c \beta_k h_k \beta_k^{-1}. \end{aligned}$$

□

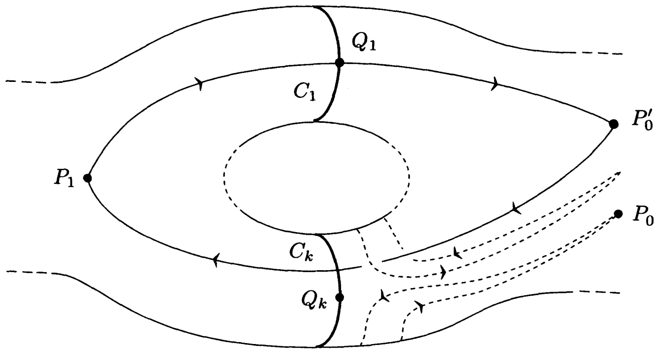

Figure 6.7.a

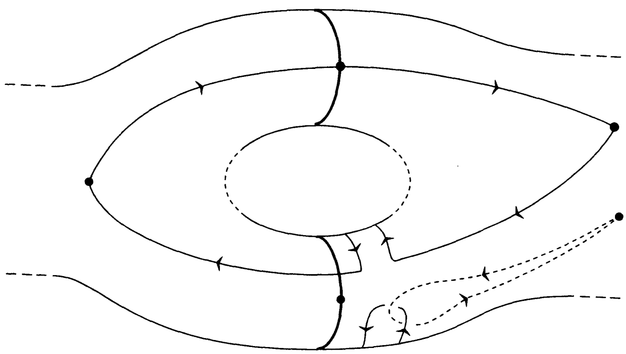

Figure 6.7.b

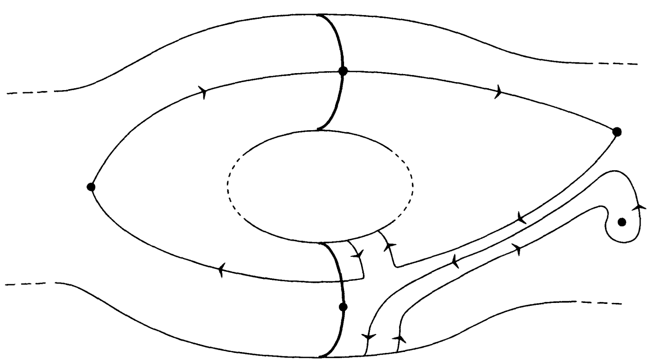

Figure 6.7.c

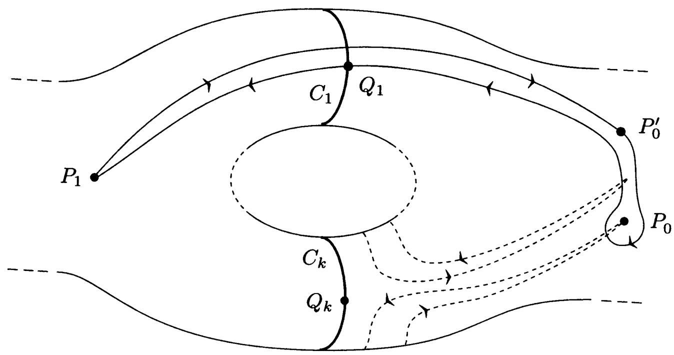

Figure 6.8.a

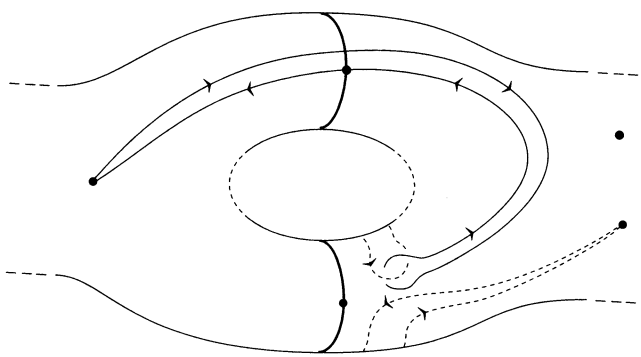

Figure 6.8.b

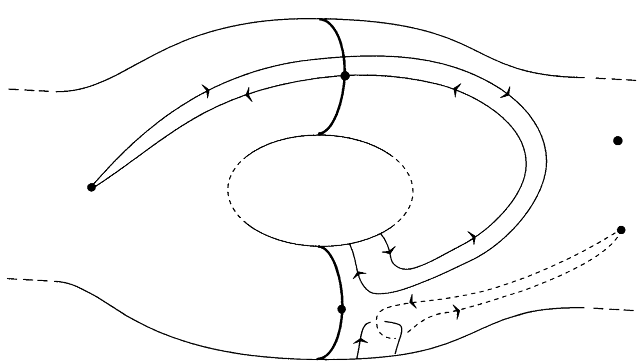

Figure 6.8.c

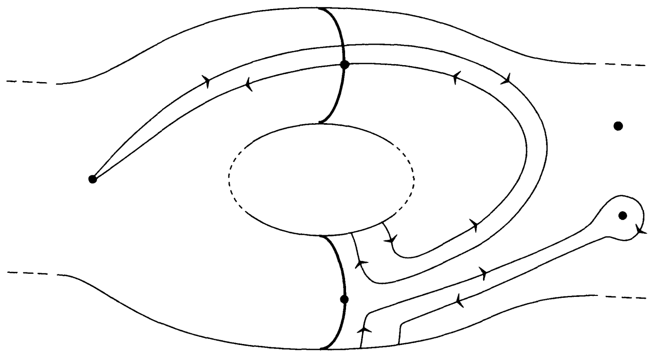

Figure 6.8.d

LEMMA 6.7. — *$S_N[P_1,P_1] = \pi_1(N_1,P_1)$ and $S_N[P_1,P_2] = \emptyset$.*

Proof. — The proof of Lemma 6.7 is divided into five steps.

Step 1. — $\pi_1(N_1,P_1) \subseteq S_N[P_1,P_1]$.

Let $g \in \pi_1(N_1,P_1)$ and let $u \in \pi_1(N,P_0)$. Let $\zeta\colon[0,1] \to N_1$ be a loop based at P_1 which represents g, and let $\xi\colon[0,1] \to N$ be a loop based at P_0 which represents u. The image of ζ and the image of ξ are disjoint, thus $u(g) = g$.

Step 2. — $S_N[P_1,P_1] \subseteq \pi_1(N_1,P_1) * F$.

Let

$$h'_c = \beta_q h_q \beta_q^{-1} \cdot \ldots \cdot \beta_2 h_2 \beta_2^{-1} \cdot h_1.$$

Then

$$h_c = T h_0^{-1} T^{-1} \cdot (h'_c)^{-1}, \quad h'_c \in \pi_1(N_1,P_1) * F.$$

Let $g \in \pi_1(M\backslash\{P_0\},P_1)$. By Lemma 6.4, g can be (uniquely) written

$$g = x_0 T y_1 T^{-1} x_1 \cdots T y_\ell T^{-1} x_\ell,$$

where

$$\begin{array}{ll} x_i \in \pi_1(N_1,P_1) * F & \text{for } i = 0,1,\ldots,\ell, \\ x_i \neq 1 & \text{for } i = 1,\ldots,\ell-1, \\ y_i \in \pi_1(N_2,P_2)\backslash\{1\} & \text{for } i = 1,\ldots,\ell. \end{array}$$

We suppose that $\ell \geq 1$. By Lemma 6.5,

$$\begin{aligned} u_0(g) &= x_0 h_c^{-1} T y_1 T^{-1} h_c x_1 \cdots h_c^{-1} T y_\ell T^{-1} h_c x_\ell \\ &= x_0 h_c' \cdot T \cdot h_0 y_1 h_0^{-1} \cdot T^{-1} \cdot (h_c')^{-1} x_1 h_c' \\ &\qquad \cdot \ldots \cdot T \cdot h_0 y_\ell h_0^{-1} \cdot T^{-1} \cdot (h_c')^{-1} x_\ell. \end{aligned}$$

It follows that, for an integer $k > 0$,

$$\begin{aligned} u_0^k(g) = x_0 (h_c')^k \cdot T \cdot h_0^k y_1 h_0^{-k} \cdot T^{-1} \cdot (h_c')^{-k} x_1 (h_c')^k \\ \cdot \ldots \cdot T \cdot h_0^k y_\ell h_0^{-k} \cdot T^{-1} \cdot (h_c')^{-k} x_\ell, \end{aligned}$$

thus $u_0^k(g) \neq g$.

So, if $g \in S_N[P_1, P_1]$, then there exists an integer $k > 0$ such that $u_0^k(g) = g$, thus $\ell = 0$, therefore $g \in \pi_1(N_1, P_1) * F$.

For $j = 2, \ldots, q$, we denote by $F(\beta_2, \ldots, \beta_j)$ the subgroup of F generated by $\{\beta_2, \ldots, \beta_j\}$.

Step 3. — $S_N[P_1, P_1] \subseteq \pi_1(N_1, P_1) * F(\beta_2, \ldots, \beta_{q-1})$.

Let

$$h' = \beta_{q-1} h_{q-1} \beta_{q-1}^{-1} \cdot \ldots \cdot \beta_2 h_2 \beta_2^{-1} \cdot h_1 \cdot T h_0 T^{-1}.$$

Then

$$\begin{gathered} h_c = (h')^{-1} \beta_q h_q^{-1} \beta_q^{-1}, \\ h' \in \pi_1(N_1, P_1) * \left(T \cdot \pi_1(N_2, P_2) \cdot T^{-1}\right) * F(\beta_2, \ldots, \beta_{q-1}). \end{gathered}$$

Let $g \in \pi_1(M \backslash \{P_0\}, P_1)$. By Lemma 6.4, g can be (uniquely) written

$$g = x_0 \beta_q^{\varepsilon_1} x_1 \cdots \beta_q^{\varepsilon_\ell} x_\ell,$$

where

$$\begin{aligned} & x_i \in \pi_1(N_1, P_1) * (T \cdot \pi_1(N_2, P_2) \cdot T^{-1}) * F(\beta_2, \ldots, \beta_{q-1}) \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{for } i = 0, 1, \ldots, \ell, \\ & \varepsilon_i \in \{\pm 1\} \quad \text{for } i = 1, \ldots, \ell, \\ & x_i \neq 1 \ \text{ if } \ \varepsilon_{i+1} = -\varepsilon_i \quad \text{for } i = 1, \ldots, \ell - 1. \end{aligned}$$

We call this expression a *relative reduced expression* of g with respect to β_q of *length* $\ell = \ell_q(g)$.

We suppose that $\ell \geq 1$. Let $k > 0$ be an integer. By Lemma 6.6,

$$u_q(\beta_q) = \beta_q h_q^{-1} \beta_q^{-1} h_c^{-1} \beta_q h_q = h' \beta_q h_q,$$
$$u_q(x_i) = x_i \quad \text{for } i = 0, 1, \ldots, \ell.$$

If $\varepsilon_i = \varepsilon_{i+1} = 1$, then

$$u_q^k(\beta_q x_i \beta_q) = (h')^k \cdot \beta_q \cdot h_q^k x_i (h')^k \cdot \beta_q \cdot h_q^k.$$

If $\varepsilon_i = 1$ and $\varepsilon_{i+1} = -1$, then

$$u_q^k(\beta_q x_i \beta_q^{-1}) = (h')^k \cdot \beta_q \cdot h_q^k x_i h_q^{-k} \cdot \beta_q^{-1} \cdot (h')^{-k},$$

and $h_q^k x_i h_q^{-k} \neq 1$ (since $x_i \neq 1$). If $\varepsilon_i = -1$ and $\varepsilon_{i+1} = 1$, then,

$$u_q^k(\beta_q^{-1} x_i \beta_q) = h_q^{-k} \cdot \beta_q^{-1} \cdot (h')^{-k} x_i (h')^k \cdot \beta_q \cdot h_q^k,$$

and $(h')^{-k} x_i (h')^k \neq 1$ (since $x_i \neq 1$). If $\varepsilon_i = \varepsilon_{i+1} = -1$, then

$$u_q^k(\beta_q^{-1} x_i \beta_q^{-1}) = h_q^{-k} \cdot \beta_q^{-1} \cdot (h')^{-k} x_i h_q^{-k} \cdot \beta_q^{-1} \cdot (h')^{-k}.$$

So, $u_q^k(g)$ has a relative reduced expression with respect to β_q of length ℓ, and this expression begins with either $x_0 (h')^k$ (if $\varepsilon_1 = 1$) or $x_0 h_q^{-k}$ (if $\varepsilon_1 = -1$). In particular, $u_q^k(g) \neq g$.

So, if $g \in S_N[P_1, P_1]$, then there exists an integer $k > 0$ such that $u_q^k(g) = g$, thus $\ell_q(g) = 0$, therefore

$$g \in \pi_1(N_1, P_1) * (T \cdot \pi_1(N_2, P_2) \cdot T^{-1}) * F(\beta_2, \ldots, \beta_{q-1}).$$

By Step 2, it follows that

$$g \in \pi_1(N_1, P_1) * F(\beta_2, \ldots, \beta_{q-1}).$$

Step 4. — $S_N[P_1, P_1] \subseteq \pi_1(N_1, P_1)$.

By Step 3,

$$S_N[P_1, P_1] \subseteq \pi_1(N_1, P_1) * F(\beta_2, \ldots, \beta_{q-1}).$$

Let $j \in \{2, \ldots, q-1\}$. We suppose that $S_N[P_1, P_1] \subseteq \pi_1(N_1, P_1) * F(\beta_2, \ldots, \beta_j)$ and we prove that $S_N[P_1, P_1] \subseteq \pi_1(N_1, P_1) * F(\beta_2, \ldots, \beta_{j-1})$.

Let R be the set of $g \in \pi_1(M \setminus \{P_0\}, P_1)$ which can be (uniquely) written

$$g = x_0 \beta_j^{\varepsilon_1} x_1 \cdots \beta_j^{\varepsilon_\ell} x_\ell,$$

where

$$\begin{aligned}
&\text{either } x_i \in \pi_1(N_1, P_1) * F(\beta_2, \ldots, \beta_{j-1}) \text{ or } x_i \in \{h_c, h_c^{-1}\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } i = 0, 1, \ldots, \ell, \\
&x_i \neq 1 \text{ if } \varepsilon_{i+1} = -\varepsilon_i \quad \text{for } i = 1, \ldots, \ell - 1, \\
&x_0, x_\ell \notin \{h_c, h_c^{-1}\}, \\
&\varepsilon_i = -1 \text{ and } \varepsilon_{i+1} = 1 \text{ if } x_i \in \{h_c, h_c^{-1}\} \quad \text{for } i = 1, \ldots, \ell - 1.
\end{aligned}$$

We write $\ell = \ell_R(g)$.

In order to be able to choose $j \in \{2, \ldots, q-1\}$, we first have to assume that $q \geq 3$. In particular, neither N_1, nor $N \cup N_2$ is a disk, thus $h_i \neq 1$ for all $i = 1, \ldots, q$. The uniqueness of the expression of g comes from the fact that h_c can be written

$$h_c = T h_0^{-1} T^{-1} \cdot h_1^{-1} \cdot \beta_2 h_2^{-1} \beta_2^{-1} \cdots \beta_j h_j^{-1} \beta_j^{-1} \cdot \beta_{j+1} h_{j+1}^{-1} \beta_{j+1}^{-1} \cdot \cdots \beta_q h_q^{-1} \beta_q^{-1}.$$

This kind of expression would not be necessarily unique if $j = q$.

We suppose that $\ell \geq 1$. If $\varepsilon_i = \varepsilon_{i+1} = 1$, then, by Lemma 6.6,

$$u_j(\beta_j x_i \beta_j) = \beta_j \cdot h_j^{-1} \cdot \beta_j^{-1} \cdot h_c^{-1} \cdot \beta_j \cdot h_j x_i \cdot \beta_j \cdot h_j^{-1} \cdot \beta_j^{-1} \cdot h_c^{-1} \cdot \beta_j \cdot h_j.$$

If $\varepsilon_i = 1$ and $\varepsilon_{i+1} = -1$, then, by Lemma 6.6,

$$u_j(\beta_j x_i \beta_j^{-1}) = \beta_j \cdot h_j^{-1} \cdot \beta_j^{-1} \cdot h_c^{-1} \cdot \beta_j \cdot h_j x_i h_j^{-1} \cdot \beta_j^{-1} \cdot h_c \cdot \beta_j \cdot h_j \cdot \beta_j^{-1},$$

and $h_j x_i h_j^{-1} \neq 1$ (since $x_i \neq 1$). If $\varepsilon_i = -1$, $\varepsilon_{i+1} = 1$, and $x_i \notin \{h_c, h_c^{-1}\}$, then, by Lemma 6.6,

$$u_j(\beta_j^{-1} x_i \beta_j) = h_j^{-1} \cdot \beta_j^{-1} \cdot h_c \cdot \beta_j \cdot h_j \cdot \beta_j^{-1} \cdot x_i \cdot \beta_j \cdot h_j^{-1} \cdot \beta_j^{-1} \cdot h_c^{-1} \cdot \beta_j \cdot h_j.$$

If $\varepsilon_i = \varepsilon_{i+1} = -1$, then, by Lemma 6.6,

$$u_j(\beta_j^{-1} x_i \beta_j^{-1}) = h_j^{-1} \cdot \beta_j^{-1} \cdot h_c \cdot \beta_j \cdot h_j \cdot \beta_j^{-1} \cdot x_i h_j^{-1} \cdot \beta_j^{-1} \cdot h_c \cdot \beta_j \cdot h_j \cdot \beta_j^{-1}.$$

If $\varepsilon_i = -1$, $\varepsilon_{i+1} = 1$ and $x_i = h_c^\varepsilon$ (where $\varepsilon \in \{\pm 1\}$), then, by Lemma 6.6,

$$\begin{aligned} u_j(\beta_j^{-1} h_c^\varepsilon \beta_j) &= h_j^{-1}\beta_j^{-1} h_c \beta_j h_j \beta_j^{-1} \cdot \beta_j h_j^{-1} \beta_j^{-1} h_c^\varepsilon \beta_j h_j \beta_j^{-1} \\ &\qquad \cdot \beta_j h_j^{-1} \beta_j^{-1} h_c^{-1} \beta_j h_j \\ &= h_j^{-1} \cdot \beta_j^{-1} \cdot h_c^\varepsilon \cdot \beta_j \cdot h_j. \end{aligned}$$

It follows that $u_j(g) \in R$, that $\ell_R(u_j(g)) \geq \ell$, and that $\ell_R(u_j(g)) > \ell$ if none of the x_i is included in $\{h_c, h_c^{-1}\}$ for $i = 1, \dots, \ell - 1$. This shows that $u_j^k(g) \neq g$ if $k > 0$ is an integer, if $g \in \pi_1(N_1, P_1) * F(\beta_2, \dots, \beta_j)$, and if $\ell_R(g) \geq 1$.

Let $g \in S_N[P_1, P_1]$. By hypothesis, $g \in \pi_1(N_1, P_1) * F(\beta_2, \dots, \beta_j)$. There exists an integer $k > 0$ such that $u_j^k(g) = g$, thus $l_R(g) = 0$, therefore $g \in \pi_1(N_1, P_1) * F(\beta_2, \dots, \beta_{j-1})$.

Step 5. — $S_N[P_1, P_2] = \emptyset$.

Let $g \in \Pi_1(M \backslash \{P_0\})[P_1, P_2]$. By Lemma 6.4, g can be (uniquely) written

$$g = x_0 T y_1 T^{-1} x_1 \cdots T y_\ell T^{-1} x_\ell T,$$

where

$$\begin{aligned} &x_i \in \pi_1(N_1, P_1) * F && \text{for } i = 0, 1, \dots, \ell, \\ &x_i \neq 1 && \text{for } i = 1, \dots, \ell - 1, \\ &y_i \in \pi_1(N_2, P_2) \backslash \{1\} && \text{for } i = 1, \dots, \ell. \end{aligned}$$

By Lemma 6.5,

$$\begin{aligned} u_0(g) &= x_0 h_c^{-1} T y_1 T^{-1} h_c x_1 \dots h_c^{-1} T y_\ell T^{-1} h_c x_\ell h_c^{-1} T \\ &= x_0 h_c' \cdot T \cdot h_0 y_1 h_0^{-1} \cdot T^{-1} \cdot (h_c')^{-1} x_1 h_c' \\ &\qquad \cdot \dots \cdot T \cdot h_0 y_\ell h_0^{-1} \cdot T^{-1} \cdot (h_c')^{-1} x_\ell h_c' \cdot T \cdot h_0. \end{aligned}$$

It follows that, for an integer $k > 0$,

$$\begin{aligned} u_0^k(g) =& x_0 (h_c')^k \cdot T \cdot h_0^k y_1 h_0^{-k} \cdot T^{-1} \cdot (h_c')^{-k} x_1 (h_c')^k \\ &\qquad \cdot \dots \cdot T \cdot h_0^k y_\ell h_0^{-k} \cdot T^{-1} \cdot (h_c')^{-k} x_\ell (h_c')^k \cdot T \cdot h_0^k, \end{aligned}$$

thus $u_0^k(g) \neq g$. □

Now, the special assumptions on N that we made just before Proposition 6.3 are dropped.

Proof of Proposition 6.3. — We prove that $S_N[P_1, P_1] = \pi_1(N_1, P_1)$ and that $S_N[P_1, P_2] = \emptyset$. The same argument works for any P_i and P_j.

Let $g \in \pi_1(N_1, P_1)$ and let $u \in \pi_1(N, P_0)$. Let $\zeta: [0,1] \to N_1$ be a loop based at P_1 which represents g, and let $\xi: [0,1] \to N$ be a loop based at P_0 which represents u. The image of ζ and the image of ξ are disjoint, thus $u(g) = g$. This shows that $\pi_1(N_1, P_1) \subseteq S_N[P_1, P_1]$.

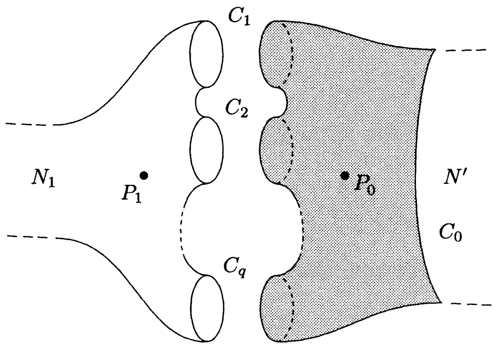

Figure 6.9

Now, let $C_1, \dots, C_q$ be the connected components of $N \cap N_1$. We choose a subsurface $N' \subseteq N$ (see Figure 6.9) such that

1) N' is a sphere with $q + 1$ holes,

2) $\overline{M \backslash N'}$ has two connected components, N_1 and

$$N_2' = \overline{N \backslash N'} \cup N_2 \cup \ldots \cup N_r,$$

3) $N' \cap N_1 = C_1 \cup \ldots \cup C_q$,

4) $N' \cap N_2'$ has a unique connected component that we denote by C_0,

5) $P_0 \in N'$.

Moreover, in the case where $r = 1$, we pick some point $P_2 \in N_2'$.

Let $S_{N'}[P_1, P_1]$ be the set of $g \in \pi_1(M \backslash \{P_0\}, P_1)$ such that, for all $u \in \pi_1(N', P_0)$, there exists an integer $k > 0$ such that $u^k(g) = g$. We have $S_N[P_1, P_1] \subseteq S_{N'}[P_1, P_1]$ (since $N \supseteq N'$), and, by Lemma 6.7, $S_{N'}[P_1, P_1] = \pi_1(N_1, P_1)$, thus $S_N[P_1, P_1] \subseteq \pi_1(N_1, P_1)$. It is clear by disjointness that $\pi_1(N_1, P_1) \subseteq S_N[P_1, P_1]$, so $\pi_1(N_1, P_1) = S_N[P_1, P_1]$.

Now, we assume that $r \geq 2$. Let $S_{N'}[P_1, P_2]$ be the set of $g \in \Pi_1(M \backslash \{P_0\})[P_1, P_2]$ such that, for all $u \in \pi_1(N', P_0)$, there exists an integer $k > 0$ such that $u^k(g) = g$. We have $S_N[P_1, P_2] \subseteq S_{N'}[P_1, P_2]$ (since $N \supseteq N'$), and, by Lemma 6.7, $S_{N'}[P_1, P_2] = \emptyset$, thus $S_N[P_1, P_2] = \emptyset$. □

6.2. Proof of Theorem 6.1.

Lemmas 6.8 to 6.12 are preliminary results to the proof of Theorem 6.1.

LEMMA 6.8. — *Let $m = 2$ and let $n = 1$. Let $i \in \{1, \dots, r\}$ be such that $P_2 \in N_i$. Then*

$$C_{PB_2M}(PB_1N) = C_{PB_2M}(\pi_1 N) = \pi_1(N, P_1) \times \pi_1(N_i, P_2).$$

Proof. — We assume that $i = 1$ (*i.e.* $P_2 \in N_1$). The inclusion

$$\pi_1(N, P_1) \times \pi_1(N_1, P_2) \subseteq C_{PB_2M}(\pi_1 N)$$

is obvious.

Let $g \in C_{PB_2M}(\pi_1 N)$. We consider the following exact sequence:

$$1 \to \pi_1(M \backslash \{P_1\}) \longrightarrow PB_2M \xrightarrow{\rho} \pi_1 M \to 1.$$

The morphism ρ sends $\pi_1(N, P_1)$ isomorphically on $\pi_1(N, P_1)$. By Lemma 5.5, $\rho(g) \in C_{\pi_1 M}(\pi_1 N)$. By Theorem 3.1, $\rho(g) = f \in \pi_1(N, P_1)$. We write $g' = f^{-1}g$. We have $g' \in \pi_1(M \backslash \{P_1\}, P_2)$ (since $\rho(g') = 1$) and $g' \in C_{PB_2M}(\pi_1 N)$.

Let $u \in \pi_1(N, P_1)$. Since $g' \in C_{PB_2M}(\pi_1 N)$, there exists an integer $k > 0$ such that

$$g' u^k (g')^{-1} \in \pi_1(N, P_1).$$

The morphism ρ sends $\pi_1(N, P_1)$ isomorphically on $\pi_1(N, P_1)$, thus

$$g' u^k (g')^{-1} = \rho\big(g' u^k (g')^{-1}\big) = \rho(g') \rho(u^k) \rho(g')^{-1} = u^k,$$

therefore

$$u^k g' u^{-k} = g'.$$

We write $Q_1 = P_2$ and we choose a point $Q_i \in N_i$ for all $i = 2, \dots, r$. Let $\Pi_1(M \backslash \{P_1\})$ be the fundamental groupoid on $M \backslash \{P_1\}$ based at $\{Q_1, Q_2, \dots, Q_r\}$. By the above considerations, $g' \in S_N[Q_1, Q_1]$, thus, by Proposition 6.3, $g' \in \pi_1(N_1, P_2)$. So,

$$g = f g' \in \pi_1(N, P_1) \times \pi_1(N_1, P_2).$$

□

LEMMA 6.9. — *One has*

$$C_{PB_mM}(PB_nN) = PB_nN \times PB_{n_1}N_1 \times \cdots \times PB_{n_r}N_r.$$

Proof. — The proof of Lemma 6.9 is divided into two steps.

Step 1. — Let $n = 1$. We prove by induction on m that

$$C_{PB_mM}(PB_1N) = C_{PB_mM}(\pi_1 N) = \pi_1 N \times PB_{n_1}N_1 \times \cdots \times PB_{n_r}N_r.$$

The case $m = 1$ is proved in Theorem 3.1, and the case $m = 2$ is proved in Lemma 6.8. Let $m \geq 3$. The inclusion

$$\pi_1 N \times PB_{n_1}N_1 \times \cdots \times PB_{n_r}N_r \subseteq C_{PB_mM}(\pi_1 N)$$

is obvious.

Let $g \in C_{PB_mM}(\pi_1 N)$. We consider the following exact sequence:

$$1 \to \pi_1\big(M\backslash\{P_1, P_2, \ldots, P_{m-1}\}\big) \longrightarrow PB_mM \xrightarrow{\rho} PB_{m-1}M \to 1.$$

The morphism ρ sends $\pi_1(N, P_1)$ isomorphically on $\pi_1(N, P_1)$. By Lemma 5.5, $\rho(g)$ is an element of $C_{PB_{m-1}M}(\pi_1 N)$. We assume that $P_m \in N_1$. By induction,

$$C_{PB_{m-1}M}(\pi_1 N) = \pi_1 N \times PB_{n_1-1}N_1 \times PB_{n_2}N_2 \times \cdots \times PB_{n_r}N_r.$$

Thus we can choose $f \in \pi_1(N, P_1)$, $h_1' \in PB_{n_1-1}N_1$, and $h_i \in PB_{n_i}N_i$ for all $i = 2, \ldots, r$ such that

$$\rho(g) = f h_1' h_2 \cdots h_r.$$

The morphism ρ sends $PB_{n_i}N_i$ isomorphically on $PB_{n_i}N_i$ for all $i = 2, \ldots, r$, and sends $PB_{n_1}N_1$ surjectively on $PB_{n_1-1}N_1$. We choose $h_1 \in PB_{n_1}N_1$ such that $\rho(h_1) = h_1'$ and we write

$$g' = g h_r^{-1} \cdots h_2^{-1} h_1^{-1} f^{-1}.$$

We have $g' \in \pi_1(M\backslash\{P_1, P_2, \ldots, P_{m-1}\}, P_m)$ (since $\rho(g') = 1$) and $g' \in C_{PB_mM}(\pi_1 N)$. We have the inclusions

$$\pi_1\big(M\backslash\{P_1, P_2, \ldots, P_{m-1}\}\big) \subseteq PB_2M\backslash\{P_2, \ldots, P_{m-1}\},$$
$$\pi_1 N \subseteq PB_2M\backslash\{P_2, \ldots, P_{m-1}\},$$

where $PB_2M\backslash\{P_2,\dots,P_{m-1}\}$ denotes the pure braid group on $M\backslash\{P_2,\dots,P_{m-1}\}$ based at (P_1,P_m). So,

$$g' \in C_{PB_2M\backslash\{P_2,\dots,P_{m-1}\}}(\pi_1 N),$$

thus, by Lemma 6.6,

$$g' \in \pi_1(N,P_1) \times \pi_1(N_1\backslash\mathcal{P}_1',P_m),$$

where $\mathcal{P}_1' = \mathcal{P}_1\backslash\{P_m\}$. Let $\bar{f} \in \pi_1(N,P_1)$ and let $\bar{h}_1 \in \pi_1(N_1\backslash\mathcal{P}_1',P_m)$ be such that $g' = \bar{f}\bar{h}_1$. Then

$$1 = \rho(g') = \rho(\bar{f})\rho(\bar{h}_1) = \bar{f},$$

thus $g' = \bar{h}_1 \in \pi_1(N_1\backslash\mathcal{P}_1',P_m)$. So,

$$\begin{aligned} g = g' \cdot fh_1h_2\cdots h_r &= f(g'h_1)h_2\dots h_r \\ &\in \pi_1N \times PB_{n_1}N_1 \times PB_{n_2}N_2 \times \cdots \times PB_{n_r}N_r. \end{aligned}$$

Step 2. — We prove by induction on n that

$$C_{PB_mM}(PB_nN) = PB_nN \times PB_{n_1}N_1 \times \cdots \times PB_{n_r}N_r.$$

The case $n=1$ is proved in Step 1. Let $n>1$. The inclusion

$$PB_nN \times PB_{n_1}N_1 \times \cdots \times PB_{n_r}N_r \subseteq C_{PB_mM}(PB_nN)$$

is obvious.

Let $g \in C_{PB_mM}(PB_nN)$. Let $N' = N\backslash\{P_1,\dots,P_{n-1}\}$ and let $M' = M\backslash\{P_1,\dots,P_{n-1},P_{n+1},\dots,P_m\}$. We consider the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & \pi_1N' & \longrightarrow & PB_nN & \xrightarrow{\rho} & PB_{n-1}N & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_1M' & \longrightarrow & PB_mM & \xrightarrow{\rho} & PB_{m-1}M & \to & 1. \end{array}$$

By Lemma 5.5, $\rho(g) \in C_{PB_{m-1}M}(PB_{n-1}N)$. By induction,

$$C_{PB_{m-1}M}(PB_{n-1}N) = PB_{n-1}N \times PB_{n_1}N_1 \times \cdots \times PB_{n_r}N_r.$$

Thus we can choose $f' \in PB_{n-1}N$ and $h_i \in PB_{n_i}N_i$ for all $i = 1, \dots, r$ such that

$$\rho(g) = f' h_1 \cdots h_r.$$

The morphism ρ sends $PB_{n_i}N_i$ isomorphically on $PB_{n_i}N_i$ for all $i = 1, \dots, r$, and sends PB_nN surjectively on $PB_{n-1}N$. We choose $f \in PB_nN$ such that $\rho(f) = f'$ and we write

$$g' = gh_r^{-1} \cdots h_1^{-1} f^{-1}.$$

We have $g' \in \pi_1(M', P_n)$ (since $\rho(g') = 1$) and $g' \in C_{PB_mM}(PB_nN)$, thus, by Lemma 5.5, $g' \in C_{\pi_1 M'}(\pi_1 N')$. By Theorem 3.1, g' is an element of $\pi_1(N', P_n) \subseteq PB_nN$. So,

$$g = (g'f)h_1 \cdots h_r \in PB_nN \times PB_{n_1}N_1 \times \cdots \times PB_{n_r}N_r. \qquad \square$$

LEMMA 6.10. — *Let $m = n$. Then*

$$C_{B_nM}(B_nN) = B_nN.$$

Proof. — The inclusion

$$B_nN \subseteq C_{B_nM}(B_nN)$$

is obvious.

Let $g \in C_{B_nM}(B_nN)$. We choose $f \in B_nN$ such that $\sigma(f) = \sigma(g)$ and we write $g' = gf^{-1}$. We have $g' \in PB_nM$ and $g' \in C_{B_nM}(B_nN) = C_{B_nM}(PB_nN)$, thus $g' \in C_{PB_nM}(PB_nN)$. By Lemma 6.9, $g' \in PB_nN$. So,

$$g = g'f \in B_nN. \qquad \square$$

Recall that Σ_m denotes the group of permutations of $\{P_1, \dots, P_m\}$, that Σ_n denotes the group of permutations of $\{P_1, \dots, P_n\}$, and that Σ_{m-n} denotes the group of permutations of $\{P_{n+1}, \dots, P_m\}$. The following lemma can be proved with the same arguments as those given in the proof of Lemma 5.10. Note that, since $\pi_1(N, P_1) \neq \{1\}$, we do not need to assume that $n \geq 2$ in Lemma 6.11.

LEMMA 6.11. — *Let $g \in C_{B_mM}(B_nN)$. Then $\sigma(g) \in \Sigma_n \times \Sigma_{m-n}$.* $\square$

Let Σ_{n_i} denote the group of permutations of $\mathcal{P}_i$ for $i = 1, \dots, r$.

LEMMA 6.12. — *Let $g \in C_{B_m M}(B_n N)$. Then*

$$\sigma(g) \in \Sigma_n \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_r}.$$

Proof. — Let $g \in C_{B_m M}(B_n N)$. By Lemma 6.11, $g \in \sigma^{-1}(\Sigma_n \times \Sigma_{m-n})$. We consider the following exact sequence:

$$1 \to B_{m-n}M \backslash \{P_1, \ldots, P_n\} \longrightarrow \sigma^{-1}(\Sigma_n \times \Sigma_{m-n}) \xrightarrow{\rho} B_n M \to 1.$$

The morphism ρ sends $B_n N$ isomorphically on $B_n N$. By Lemma 5.5, $\rho(g) \in C_{B_n M}(B_n N)$. By Lemma 6.10, $\rho(g) = f \in B_n N$. We write $g' = gf^{-1}$. We have $g' \in B_{m-n}M\backslash\{P_1, \ldots, P_n\}$ (since $\rho(g') = 1$) and $g' \in C_{B_m M}(B_n N)$.

Let $h \in B_n N$. Since $g' \in C_{B_m M}(B_n N)$, there exists an integer $k > 0$ such that

$$g' h^k (g')^{-1} \in B_n N.$$

Since ρ is an isomorphism on $B_n N$,

$$g' h^k (g')^{-1} = \rho(g' h^k (g')^{-1}) = \rho(g')\rho(h^k)\rho(g')^{-1} = h^k,$$

therefore

$$h^k g' h^{-k} = g'.$$

We suppose that $P_{n+1} \in N_1$, that $P_{n+2} \in N_2$, and that $\sigma(g)(P_{n+1}) = P_{n+2}$. We also have $\sigma(g')(P_{n+1}) = P_{n+2}$. We write $Q_1 = P_{n+1}$ and $Q_2 = P_{n+2}$. We choose a point $Q_i \in N_i$ for all $i = 3, \ldots, r$. Let $\Pi_1(M\backslash\{P_1\})$ be the fundamental groupoid on $M\backslash\{P_1\}$ based at $\{Q_1, \ldots, Q_r\}$. Let $b' = (b'_{n+1}, \ldots, b'_m)$ be a braid on $M\backslash\{P_1, \ldots, P_n\}$ based at $(P_{n+1}, \ldots, P_m)$ which represents g'. Let $x \in \Pi_1(M\backslash\{P_1\})[Q_1, Q_2]$ be represented by b'_{n+1}. By the above considerations, $x \in S_N[Q_1, Q_2]$. This contradicts Proposition 6.3.

So,

$$\sigma(g) \in \Sigma_n \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_r}. \qquad \square$$

Proof of Theorem 6.1. — The inclusion

$$B_n N \times B_{n_1} N_1 \times \cdots \times B_{n_r} N_r \subseteq C_{B_m M}(B_n N)$$

is obvious.

Let $g \in C_{B_m M}(B_n N)$. By Lemma 6.12,

$$\sigma(g) \in \Sigma_n \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_r}.$$

Thus we can choose $f \in B_n N$ and $h_i \in B_{n_i} N_i$ for all $i = 1, \dots, r$ such that

$$\sigma(g) = \sigma(f)\sigma(h_1) \cdots \sigma(h_r).$$

We write

$$g' = g h_r^{-1} \cdots h_1^{-1} f^{-1}.$$

We have $g' \in PB_m M$ and $g' \in C_{B_m M}(B_n N) = C_{B_m M}(PB_n N)$, thus $g' \in C_{PB_m M}(PB_n N)$. By Lemma 6.9, there exist $f' \in PB_n N$ and $h_i' \in PB_{n_i} N_i$ for all $i = 1, \dots, r$ such that

$$g' = f' h_1' \cdots h_r'.$$

So,

$$\begin{aligned} g = f' h_1' \cdots h_r' \cdot f h_1 \cdots h_r &= (f'f)(h_1' h_1) \cdots (h_r' h_r) \\ &\in B_n N \times B_{n_1} N_1 \times \cdots \times B_{n_r} N_r. \end{aligned}$$ □

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L. PARIS,
Université de Bourgogne
Laboratoire de Topologie
UMR 5584 du CNRS
BP 400
21011 Dijon Cedex (France).
lparis@satie.u–bourgogne.fr
&
D. ROLFSEN,
University of British Columbia
Department of Mathematics
Vancouver, B.C. V6T 1Z2 (Canada).
rolfsen@math.ubc.ca