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Michael Cowling
Stefano Meda
Roberta Pasquale

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RIESZ POTENTIALS AND AMALGAMS

by M. COWLING, S. MEDA & R. PASQUALE

0. Introduction.

Let M be Euclidean space $\mathbb{R}^n$ and let $-\mathcal{L}$ be the usual Laplace operator (so that $\mathcal{L}$ is positive). The Hardy–Littlewood–Sobolev regularity theorem states that, if $1 < p < q < \infty$ and $1/p - 1/q = 2/n$, then there is a constant C such that

$$\|u\|_q \leq C \, \|\mathcal{L}u\|_p \qquad \forall u \in C_c^2(M).$$

This may be expressed by stating that $\mathcal{L}^{-1}$ maps $L^p(M)$ into $L^q(M)$. More generally, one may consider the *Riesz potential operators* $\mathcal{L}^{-\alpha/2}$ when $0 < \alpha < n$; these are given by convolution with the tempered distributions whose Fourier transforms are $|\cdot|^{-\alpha}$. Then $\mathcal{L}^{-\alpha/2}$ maps $L^p(M)$ to $L^q(M)$ when $1 < p < q < \infty$ and $1/p - 1/q = \alpha/n$.

Now take M to be the infinite cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$, equipped with the restriction to M of the Euclidean metric, and let $\mathcal{L}$ be the Laplace–Beltrami operator of M. A natural question is whether the Riesz potential operators $\mathcal{L}^{-\alpha/2}$ map $L^p(M)$ to $L^q(M)$ for some positive real α. As is well known, and as we shall see shortly, the answer is no, essentially because M is two-dimensional but "seems" one-dimensional globally. Nevertheless, the Riesz potential operators do have some smoothing properties which may be formulated in terms of amalgams. We will now study this example in some detail, as it will be a paradigm

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for the rest of this paper. We parametrise the point $(\cos\theta, \sin\theta, z)$ of M by (θ, z), where $-\pi < \theta \le \pi$ and $-\infty < z < \infty$.

The heat equation on M has a fundamental solution H_t, given by the formula

$$H_t(\theta, z) = \frac{1}{4\pi t} \sum_{k\in\mathbb{Z}} \exp\Big(-\frac{(\theta+k)^2+z^2}{4t}\Big).$$

Then if f is a reasonably regular function on M, the solution of the heat equation in $M \times \mathbb{R}^+$ with initial data f is the convolution $H_t * f$, where

$$H_t * f(\theta, z) = \int_0^{2\pi} \int_{-\infty}^{\infty} H_t(\theta-\phi, z-w)\, f(\phi, w)\, \mathrm{d}\phi\, \mathrm{d}w.$$

Fix α in $(0, 1/2)$. The Riesz potential operator $\mathcal{L}^{-\alpha/2}$ is given by convolution with the kernel R_α, where

$$R_\alpha = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1}\, H_t\, \mathrm{d}t.$$

It is easy to see that, for all θ in $(-\pi, \pi]$ and z in $\mathbb{R}$,

$$H_t(\theta, z) \sim \begin{cases} t^{-1}\exp(-(\theta^2+z^2)/4t) & \text{when } 0 < t \le 1 \\ t^{-1/2}\exp(-z^2/4t) & \text{when } 1 \le t < \infty. \end{cases} \tag{0.1}$$

This means, for instance, that there are positive constants C and C' such that

$$C \le \frac{tH_t(\theta, z)}{\exp(-(\theta^2+z^2)/4t)} \le C',$$

for all θ in $(-\pi, \pi]$, z in $\mathbb{R}$, and t in $(0, 1]$. It follows immediately that

$$R_\alpha(\theta, z) \sim \begin{cases} (\theta^2+z^2)^{(\alpha-2)/2} & \text{when } \theta^2+z^2 \le 1 \\ (1+z^2)^{(\alpha-1)/2} & \text{when } \theta^2+z^2 \ge 1. \end{cases}$$

Let $B(o, \rho)$ be the ball in M centred at $(1, 0, 0)$ of radius ρ and $\chi_B(o, \rho)$ be its characteristic function. Then if $\rho \le 1$,

$$R_\alpha * \chi_{B(o,\rho)}(\theta, z) \sim \begin{cases} \rho^2(\rho^2+\theta^2+z^2)^{(\alpha-2)/2} & \text{when } \theta^2+z^2 \le 1 \\ \rho^2(1+z^2)^{(\alpha-1)/2} & \text{when } \theta^2+z^2 \ge 1, \end{cases}$$

while if $\rho \ge 1$,

$$R_\alpha * \chi_{B(o,\rho)}(\theta, z) \sim \rho(\rho^2+z^2)^{(\alpha-1)/2}.$$

If $\mathcal{L}^{-\alpha/2}$ were bounded from $L^p(M)$ to $L^q(M)$, then it would follow that $q(\alpha-1) < -1$ since $R_\alpha * \chi_{B(o,\rho)} \in L^q(M)$, and furthermore the ratio $\|R_\alpha * \chi_{B(o,\rho)}\|_q / \|\chi_{B(o,\rho)}\|_p$ would be uniformly bounded in ρ. Now

$$\frac{\|R_\alpha * \chi_{B(o,\rho)}\|_q}{\|\chi_{B(o,\rho)}\|_p} \sim \begin{cases} \rho^{\alpha+2/q-2/p} & \text{when } \rho \le 1 \\ \rho^{\alpha+1/q-1/p} & \text{when } \rho \ge 1, \end{cases}$$

and this cannot be uniformly bounded in ρ. Thus the Riesz potential operator $\mathcal{L}^{-\alpha/2}$ cannot be bounded from $L^p(M)$ to $L^q(M)$. A shorter way to prove this result is to appeal to [6], Theorem 1 once formula (0.1) is known. We have given the details here because this example motivates and informs the use of amalgams.

Nevertheless, the Riesz potential operator $\mathcal{L}^{-\alpha/2}$ is regularising when $\mathrm{Re}(\alpha) > 0$. We may express this using *amalgams*. For any integer j, define E_j to be

$$\{(x, y, z) \in M : |z - j| \leq 1/2\},$$

and χ_{E_j} to be its characteristic function. For p and q in $[1, \infty]$, let $A_p^q(M)$ be the Banach space of all measurable functions f on M such that $\|f\|_{A_p^q} < \infty$, where

$$\|f\|_{A_p^q} = \Big(\sum_{j\in\mathbb{Z}} \|f\chi_{E_j}\|_p^q\Big)^{1/q}$$

if $q < \infty$, and

$$\|f\|_{A_p^\infty} = \sup\big\{\|f\chi_{E_j}\|_p : j \in \mathbb{Z}\big\}.$$

THEOREM. — *Suppose that $0 < \alpha < 1$, $1 < p < r < \infty$, $1/p - 1/r = \alpha/2$, $1 < q < s < \infty$, and $1/q - 1/s = \alpha$. Then the Riesz potential operator $\mathcal{L}^{-\alpha/2}$ is bounded from $A_p^q(M)$ to $A_r^s(M)$.*

The proof of this theorem is omitted, as we shall prove a more general result later. The point is that the Riesz potential operators may not be L^p–L^q bounded, but that boundedness between amalgams may be a substitute. This theorem is formulated with the conditions $1/p - 1/r = \alpha/2$ and $1/q - 1/s = \alpha/1$, where the "local dimension" 2 is involved with the "local indices" p and r, while the "global dimension" 1 is involved with the "global indices" q and r. Thus the mapping properties between amalgams seem to encapsulate the local and the global nature of M quite well.

The aim of this paper is to prove a version of this theorem involving more general spaces M and operators $\mathcal{L}$. In Section 1, we discuss the definition of amalgams, and prove an $A_p^q - A_r^s$ mapping theorem for kernel operators. In Section 2, we consider "Gaussian" semigroups. In Sections 3 and 4, we consider examples involving Riemannian manifolds and Lie groups of polynomial growth respectively.

It should be pointed out that the novelty of this paper lies in the marriage between the idea of an amalgam and the study of heat-type semigroups. These latter have already been intensively studied, and closely related results have been proved, in particular by T. Coulhon, L. Saloff-Coste and N.Th. Varopoulos (see, e.g., [6], [7], [8], [9], [22], [23], [24]).

We will need a bit of notation. Given a function f on a measure space M and a subset E of M, we shall write χ_E for the characteristic function of E, and f_E for the pointwise product $f\chi_E$. We use the letters A, B, C, etc., to denote positive constants. These may vary from one occurrence to another, but do not vary within the enunciation and proof of a result.

1. Amalgams.

Throughout this section, we assume that (M, d, μ) is a *measured metric space*, by which we mean a metric space (M, d) equipped with a Borel measure μ. We further assume that (M, d, μ) satisfies the uniform ball size conditions

$$\begin{aligned} \inf\{\mu(B(x,\rho)) : x \in M\} &= v(\rho) > 0 \\ \sup\{\mu(B(x,\rho)) : x \in M\} &= V(\rho) < \infty, \end{aligned} \tag{1.1}$$

where $B(x, \rho)$ denotes the open ball in M with centre x and radius ρ.

For ε in $\mathbb{R}^+$, a discrete subset $\mathcal{M}$ of M is called an *ε-discretisation* of M if

$$d(x, x') \geq \varepsilon \qquad \forall x \in \mathcal{M} \quad \forall x' \in \mathcal{M} \setminus \{x\} \tag{1.2}$$

$$d(x, \mathcal{M}) < \varepsilon \qquad \forall x \in M. \tag{1.3}$$

It is easy to see that ε-discretisations always exist. Indeed, the class of all the discrete subsets of M which possess property (1.2) is clearly partially ordered by inclusion and nonvoid. Every totally ordered subset of this class admits a maximal element by the Hausdorff maximality theorem. Now any such maximal element $\mathcal{M}$ satisfies condition (1.3): for otherwise, there would be a point x in M such that $d(x, \mathcal{M}) \geq \varepsilon$, and then $\mathcal{M} \cup \{x\}$ would contain $\mathcal{M}$ and still satisfy condition (1.2), contradicting the maximality of $\mathcal{M}$.

The existence of ε-discretisations may also be proved more constructively if desired.

We denote by $\mathcal{B}(x, \rho)$ the set of all y in $\mathcal{M}$ such that $d(x, y) < \rho$.

LEMMA 1.1. — *Suppose that $\mathcal{M}$ is an ε-discretisation of M. Then if $\rho \in \mathbb{R}^+$,*

$$\#\,\mathcal{B}(x, \rho) \leq \frac{V(\rho + \varepsilon/2)}{v(\varepsilon/2)} \qquad \forall x \in M,$$

and if also $\rho > \varepsilon$, then

$$\#\mathcal{B}(x,\rho) \geq \frac{v(\rho-\varepsilon)}{V(\varepsilon)} \qquad \forall x \in M.$$

Proof. — Fix $x \in M$ and ρ in $\mathbb{R}^+$. Then the balls $B(y,\varepsilon/2)$ for y in $\mathcal{B}(x,\rho)$ are all disjoint, and all are contained in $B(x,\rho+\varepsilon/2)$. Consequently,

$$\#\mathcal{B}(x,\rho)\, v(\varepsilon/2) \leq \sum_{y\in\mathcal{B}(x,\rho)} \mu\big(B(y,\varepsilon/2)\big) \leq \mu\big(B(x,\rho+\varepsilon/2)\big) \leq V(\rho+\varepsilon/2),$$

proving the first inequality. Similarly, the balls $B(y,\varepsilon)$ for y in $\mathcal{B}(x,\rho)$ cover $B(x,\rho-\varepsilon)$, and

$$\#\mathcal{B}(x,\rho)\, V(\varepsilon) \geq \sum_{y\in\mathcal{B}(x,\rho)} \mu\big(B(y,\varepsilon)\big) \geq \mu\big(B(x,\rho-\varepsilon)\big) \geq v(\rho-\varepsilon),$$

proving the second inequality.

Fix an ε-discretisation $\mathcal{M}$ of M. For p and q in $[1,\infty]$, let $A^q_q(M,\mathcal{M})$ be the set of all measurable functions f on M such that $\|f\|_{A^q_p} < \infty$, where

$$\|f\|_{A^q_p} = \Big(\sum_{x\in\mathcal{M}} \|f_{B(x,\varepsilon)}\|_p^q\Big)^{1/q}$$

if $q<\infty$, and

$$\|f\|_{A^\infty_p} = \sup\big\{\|f_{B(x,\varepsilon)}\|_p : x\in\mathcal{M}\big\}.$$

Then $A^q_p(M,\mathcal{M})$ is a Banach space (when functions which coincide almost everywhere are identified) called an *amalgam.* Amalgams have been studied in various degrees of generality by various authors, including J.-P. Bertrandias, C. Datry and C. Dupuis, J.J.F. Fournier and J. Stewart, and F. Holland [1], [15], [16].

We now clarify the sense in which $A^q_p(M,\mathcal{M})$ looks locally like $L^p(M)$ and globally like $L^q(M)$.

PROPOSITION 1.2. — *Suppose that $\mathcal{M}$ is an ε-discretisation of M, that $1\leq p,q\leq\infty$, and that $\delta\in\mathbb{R}^+$. Define a to be $\max\{1/p-1/q,0\}$, b to be $\max\{1/q-1/p,0\}$, $\mathcal{V}(\varepsilon)$ to be $\sup\{\#\mathcal{B}(y,\varepsilon) : y\in M\}$, and N_0 to be $\#\mathcal{B}(x_0,\varepsilon+\delta)$.*

(i) *If $f : M\to\mathbb{C}$ is measurable and supported in $B(x_0,\delta)$, then $f\in L^p(M)$ if and only if $f\in A^q_p(M,\mathcal{M})$, and*

$$\|f\|_p \leq N_0^a\|f\|_{A^q_p} \quad\text{and}\quad \|f\|_{A^q_p} \leq N_0^b\,\mathcal{V}(\varepsilon)^{1/p}\|f\|_p.$$

(ii) *If* $f = \chi_{B(x_0,\delta)}$ *and* $\delta > 2\varepsilon$, *then*

$$\frac{v(\varepsilon)^{1/p}\, v(\delta-2\varepsilon)^{1/q}}{V(\varepsilon)^{1/q}} \le \|f\|_{A^q_p} \le \frac{V(\varepsilon)^{1/p}\, V(\delta+3\varepsilon/2)^{1/q}}{v(\varepsilon/2)^{1/q}}.$$

(iii) *If* $f : M \to \mathbb{C}$ *is measurable and there exists a constant* C *such that*

$$\sup\{|f(y)| : y \in B(x,\varepsilon)\} \le C \inf\{|f(y)| : y \in B(x,\varepsilon)\} \qquad \forall x \in M, \tag{1.4}$$

then $f \in L^q(M)$ *if and only if* $f \in A^q_p(M,\mathcal{M})$, *and if* $p \le q$, *then*

$$\|f\|_q \le C^{1/p-1/q}\, v(\varepsilon)^{1/q-1/p}\, \|f\|_{A^q_p} \text{ and } \|f\|_{A^q_p} \le \mathcal{V}(\varepsilon)^{1/q}\, V(\varepsilon)^{1/p-1/q}\, \|f\|_q$$

while if $p \ge q$, *then*

$$\|f\|_q \le V(\varepsilon)^{1/q-1/p}\, \|f\|_{A^q_p} \text{ and } \|f\|_{A^q_p} \le C^{1/q-1/p}\, \mathcal{V}(\varepsilon)^{1/q}\, v(\varepsilon)^{1/p-1/q}\, \|f\|_q.$$

(iv) *If* $f : M \to \mathbb{C}$ *is measurable, then* $f \in L^p(M)$ *if and only if* $f \in A^q_p(M,\mathcal{M})$, *and*

$$\|f\|_p \le \|f\|_{A^p_p} \le \mathcal{V}(\varepsilon)^{1/p}\, \|f\|_p.$$

(v) *If* $1 \le r \le p \le \infty$ *and* $1 \le q \le s \le \infty$, *then* $A^q_p(M,\mathcal{M}) \subseteq A^s_r(M,\mathcal{M})$, *and*

$$\|f\|_{A^s_r} \le V(\varepsilon)^{1/r-1/p}\, \|f\|_{A^q_p} \qquad \forall f \in A^q_p(M,\mathcal{M}).$$

Proof. — In this proof, we write η for $\varepsilon + \delta$.

Suppose first that f is supported in $B(x_0,\delta)$. Then on the one hand,

$$\begin{aligned}
\|f\|_p &= \left(\int_M |f(y)|^p \,\mathrm{d}\mu(y) \right)^{1/p} \\
&\le \left(\int_M \sum_{x \in \mathcal{B}(x_0,\eta)} |f_{B(x,\varepsilon)}(y)^p \,\mathrm{d}\mu(y) \right)^{1/p} \\
&= \left(\sum_{x \in \mathcal{B}(x_0,\eta)} \|f_{B(x,\varepsilon)}\|_p^p \right)^{1/p} \\
&\le N_0^a \left(\sum_{x \in \mathcal{B}(x_0,\eta)} \|f_{B(x,\varepsilon)}\|_p^q \right)^{1/q} \\
&= N_0^a\, \|f\|_{A^q_p}.
\end{aligned} \tag{1.5}$$

On the other hand,

$$
\begin{aligned}
\|f\|_{A_p^q} &= \Big(\sum_{x\in\mathcal{B}(x_0,\eta)} \|f_{B(x,\varepsilon)}\|_p^q\Big)^{1/q} \\
&\le N_0^b \Big(\sum_{x\in\mathcal{B}(x_0,\eta)} \|f_{B(x,\varepsilon)}\|_p^p\Big)^{1/p} \qquad (1.6)\\
&= N_0^b \Big(\int_M \sum_{x\in\mathcal{B}(x_0,\eta)} |f_{B(x,\varepsilon)}(y)|^p \,\mathrm{d}\mu(y)\Big)^{1/p} \\
&\le N_0^b\, \mathcal{V}(\varepsilon)^{1/p}\, |f|_p,
\end{aligned}
$$

proving (i).

If f satisfies the hypotheses of (iii) and $1 \le r, s \le \infty$, then

$$
\|f_{B(x,\varepsilon)}\|_r \le \begin{cases} \mu\big(B(x,\varepsilon)\big)^{1/r-1/s}\, \|f_{B(x,\varepsilon)}\|_s & \text{when } r \le s \\ C^{1/s-1/r}\, \mu\big(B(x,\varepsilon)\big)^{1/r-1/s}\, \|f_{B(x,\varepsilon)}\|_s & \text{when } r \ge s. \end{cases}
$$

Therefore, on the one hand,

$$
\begin{aligned}
\|f\|_q &\le \Big(\int_M \sum_{x\in\mathcal{M}} |f_{B(x,\varepsilon)}(y)|^q \,\mathrm{d}\mu(y)\Big)^{1/q} \\
&= \Big(\sum_{x\in\mathcal{M}} \|f_{B(x,\varepsilon)}\|_q^q\Big)^{1/q} \\
&\le C^a \sup\big\{\mu\big(B(x,\varepsilon)\big)^{1/q-1/p} : x\in\mathcal{M}\big\} \Big(\sum_{x\in\mathcal{M}} \|f_{B(x,\varepsilon)}\|_p^q\Big)^{1/q} \\
&= C^a \sup\big\{\mu\big(B(x,\varepsilon)\big)^{1/q-1/p} : x\in\mathcal{M}\big\}\, \|f\|_{A_p^q}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\|f\|_{A_p^q} &= \Big(\sum_{x\in\mathcal{M}} \|f_{B(x,\varepsilon)}\|_p^q\Big)^{1/q} \\
&\le C^b \sup\big\{\mu\big(B(x,\varepsilon)\big)^{1/p-1/q} : x\in\mathcal{M}\big\} \Big(\sum_{x\in\mathcal{M}} \|f_{B(x,\varepsilon)}\|_q^q\Big)^{1/q} \\
&\le C^b\, \mathcal{V}(\varepsilon)^{1/q} \sup\big\{\mu\big(B(x,\varepsilon)\big)^{1/p-1/q} : x\in\mathcal{M}\big\}\, \|f\|_q,
\end{aligned}
$$

proving (iii).

The inequalities (1.5) and (1.6), with $\mathcal{B}(x_0,\eta)$ replaced by $\mathcal{M}$ and q replaced by p, show that $A_p^p(M,\mathcal{M})$ and $L^p(M)$ are isomorphic Banach spaces, and prove (iv).

Finally, (v) is a simple consequence of Hölder's inequality. $\square$

Remark 1.3. — Note that the constants in Proposition 1.2 (i) remain bounded when δ is bounded. Thus any compactly supported function is in $A_p^q(M,\mathcal{M})$ if and only if it is in $L^p(M)$. Similarly, when the absolute value of a function varies slowly, (1.4) holds, and (iii) applies.

A natural question is how $A_p^q(M,\mathcal{M})$ depends on the discretisation $\mathcal{M}$ used. This is answered in the next proposition.

PROPOSITION 1.4. — *Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are ε- and ε'-discretisations of M, where ε and ε' are positive real numbers. Then, for every p and q in $[1,\infty]$, the Banach spaces $A_p^q(M,\mathcal{M})$ and $A_p^q(M,\mathcal{M}')$ are isomorphic.*

Proof. — First, recall that $\mathcal{B}(y,\rho)$ is defined to be $\{x \in \mathcal{M} : d(x,y) < \rho\}$; similarly we define $\mathcal{B}'(y,\rho)$ to be $\{x' \in \mathcal{M}' : d(x',y) < \rho\}$. Write ε'' for $\varepsilon + \varepsilon'$. From Lemma 1.1, there exists N in $\mathbb{N}$ such that

$$\sup\{\# \mathcal{B}(y,\varepsilon'') : y \in M\} \leq N \quad \text{and} \quad \sup\{\# \mathcal{B}'(y,\varepsilon'') : y \in M\} \leq N.$$

Suppose that f is in $A_p^q(M,\mathcal{M}')$. Since $\chi_{B(x,\varepsilon)} \leq \sum_{x' \in \mathcal{B}'(x,\varepsilon'')} \chi_{B(x',\varepsilon')}$, the triangle inequality, Hölder's inequality, and Fubini's theorem imply that

$$\begin{aligned}
\Big(\sum_{x\in\mathcal{M}} \|f_{B(x,\varepsilon)}\|_p^q\Big)^{1/q} &\leq \Big(\sum_{x\in\mathcal{M}} \Big\| \sum_{x'\in\mathcal{B}'(x,\varepsilon'')} f_{B(x',\varepsilon')}\Big\|_p^q\Big)^{1/q} \\
&\leq \Big(\sum_{x\in\mathcal{M}} \Big(\sum_{x'\in\mathcal{B}'(x,\varepsilon'')} \|f_{B(x',\varepsilon')}\|_p\Big)^q\Big)^{1/q} \\
&\leq N^{1/q'} \Big(\sum_{x\in\mathcal{M}} \sum_{x'\in\mathcal{B}'(x,\varepsilon'')} \|f_{B(x',\varepsilon')}\|_p^q\Big)^{1/q} \\
&\leq N \Big(\sum_{x'\in\mathcal{M}'} \|f_{B(x',\varepsilon')}\|_p^q\Big)^{1/q},
\end{aligned}$$

proving that $A_p^q(M,\mathcal{M}')$ is continuously embedded in $A_p^q(M,\mathcal{M})$. The reverse inclusion is proved by exchanging the roles of $\mathcal{M}$ and $\mathcal{M}'$ in the above argument. $\square$

Since the amalgam does not depend substantially on the choice of ε-discretisation, we may ignore this choice in what follows; we use the notation $A^q_p(M)$ to denote the linear space with any one of the possible choices of discretisations and corresponding norms.

We now consider operators on amalgams.

THEOREM 1.5. — *Suppose that $K : M \times M \longrightarrow \mathbb{C}$ is a measurable function (defined almost everywhere on $M \times M$), and that*

$$|K(x,y)| \leq \begin{cases} D\, d(x,y)^{-a} & \text{when } d(x,y) \leq 1 \\ D\, d(x,y)^{-b} & \text{when } d(x,y) \geq 1, \end{cases}$$

where $a, b > 0$. Suppose also that $m, n > 0$ and

$$\mu\big(B(x,\rho)\big) \leq \begin{cases} C_1\, \rho^m & \text{when } \rho \leq 1 \\ C_1\, \rho^n & \text{when } \rho \geq 1. \end{cases}$$

Then the kernel operator $\mathcal{K}$ with kernel K is bounded from $A^q_p(M)$ to $A^s_r(M)$, provided that $1 < p < r < \infty$, $1/p - 1/r \leq 1 - m/a$, $1 < q < s < \infty$ and $1/q - 1/s \geq 1 - n/b$.

Proof. — The key to the proof of this theorem is the following well known generalisation of Young's convolution inequality (see, e.g., [14] for an example of its use). Suppose that N is a measure space with measure ν, and let $G : N \times N \longrightarrow \mathbb{C}$ be a function such that

$$\nu(\{y \in N : |G(x,y)| + |G(y,x)| > \lambda\}) \leq C\, \lambda^{-t} \qquad \forall \lambda \in \mathbb{R}^+ \quad \forall x \in N.$$

Then the operator $\mathcal{G}$, defined by the rule

$$\mathcal{G}f(x) = \int_N G(x,y)\, f(y)\, \mathrm{d}\nu(y) \qquad \forall x \in N,$$

for all integrable simple functions f, extends to a bounded operator from $L^p(N)$ to $L^q(N)$ provided that $1 < p < q < \infty$ and $1/p - 1/q = 1 - 1/t$. The norm of this operator depends on p, q and C.

Now let $\mathcal{M}$ be a 1-discretisation of M. Let Q be the quasi-diagonal subset of $M \times M$

$$\{(x,y) \in M \times M : d(x,y) < 1\},$$

and let d_Q be the function $d\chi_Q$. Since

$$|K(x,y)| \leq D\,(d_Q(x,y))^{-a} + 2^b D\,(1 + d(x,y))^{-b},$$

it suffices to show that the operators $\mathcal{S}$ and $\mathcal{T}$ with kernels d_Q^{-a} and $(1+d)^{-b}$ map $A^q_p(M)$ into $A^s_p(M)$. We treat these two operators separately.

First, for any x in M,

$$\begin{aligned} \mu\big(\{y \in M : |d_Q(x,y)^{-a}| > \lambda\}\big) &= \mu\big(B(x,1) \cap B(x, \lambda^{-1/a})\big) \\ &\leq C_1 \min\{1, \lambda^{-1/a}\}^m \\ &\leq C_1\, \lambda^{-m/a} \qquad \forall \lambda \in \mathbb{R}^+, \end{aligned}$$

and similarly for any y in M,

$$\mu\big(\{x \in M : |d_Q(x,y)^{-a}| > \lambda\}\big) \le C_1\,\lambda^{-m/a} \qquad \forall \lambda \in \mathbb{R}^+.$$

From the generalisation of Young's theorem, the kernel operator $\mathcal{S}$ is bounded from $L^p(M)$ to $L^r(M)$. Further, the kernel of $\mathcal{S}$ vanishes off the quasi-diagonal Q, so that

$$\begin{aligned}&\big(\mathcal{S}(f_{B(y_0,1)})\big)_{B(x_0,1)}(x)\\ &\qquad = \int_M \chi_{B(x_0,1)}(x)\, d(x,y)^{-a}\chi_Q(x,y)\,\chi_{B(y_0,1)}(y)\, f(y)\,\mathrm{d}y = 0\end{aligned}$$

unless $d(x_0,y_0) \le 3$. Thus

$$\begin{aligned}\Big(\sum_{x\in\mathcal{M}} \|(\mathcal{S}f)_{B(x,1)}\|_r^s\Big)^{1/s} &= \Big(\sum_{x\in\mathcal{M}} \Big\|\Big(\mathcal{S}\Big(\sum_{y\in\mathcal{M}} f_{B(y,1)}\Big)\Big)_{B(x,1)}\Big\|_r^s\Big)^{1/s}\\
&\le \Big(\sum_{x\in\mathcal{M}} \Big\|\sum_{y\in\mathcal{B}(x,3)} \mathcal{S}(f_{B(y,1)})\Big\|_r^s\Big)^{1/s}\\
&\le \Big(\sum_{x\in\mathcal{M}} \Big(\sum_{y\in\mathcal{B}(x,3)} \|\mathcal{S}(f_{B(y,1)})\|_r\Big)^s\Big)^{1/s}\\
&\le E\Big(\sum_{x\in\mathcal{M}} \Big(\sum_{y\in\mathcal{B}(x,3)} \|f_{B(y,1)}\|_p\Big)^s\Big)^{1/s}\\
&\le E\Big(\sum_{x\in\mathcal{M}} \sum_{y\in\mathcal{B}(x,3)} \|f_{B(y,1)}\|_p^s(\#\,\mathcal{B}(x,3))^{s/s'}\Big)^{1/s}\\
&\le E\Big(\sum_{x\in\mathcal{M}} \|f_{B(x,1)}\|_p^s\,(\#\,\mathcal{B}(x,3))^{1+s/s'}\Big)^{1/s}\\
&\le E\,\frac{V(7/2)}{v(1/2)}\Big(\sum_{x\in\mathcal{M}} \|f_{B(x,1)}\|_p^s\Big)^{1/s}\\
&\le E\,\frac{V(7/2)}{v(1/2)}\|f\|_{A_p^q}.\end{aligned}$$

To treat the operator $\mathcal{T}$, we argue similarly. If $x \in B(x_0,1)$ and $y \in B(y_0,1)$, then

$$1 + d(x_0,y_0) \le 3 + d(x,y) \le 3(1 + d(x,y)).$$

Hence, for x_0 and y_0 in $\mathcal{M}$ and x in M,

$$\begin{aligned}
&|(\mathcal{T}(f_{B(y_0,1)}))_{B(x_0,1)}(x)| \\
&= \left| \int_M (1+d(x,y))^{-b}\, \chi_{B(x_0,1)}(x)\, \chi_{B(y_0,1)}(y)\, f(y)\, \mathrm{d}y \right| \\
&\le 3^b \int_M (1+d(x_0,y_0))^{-b}\, \chi_{B(x_0,1)}(x)\, \chi_{B(y_0,1)}(y)\, |f(y)| \mathrm{d}y \\
&= 3^b\, (1+d(x_0,y_0))^{-b}\, \chi_{B(x_0,1)}(x)\, \|f_{B(y_0,1)}\|_1 \\
&\le 3^b\, V(1)^{1/r'}\, (1+d(x_0,y_0))^{-b}\, \chi_{B(x_0,1)}(x)\, \|f_{B(y_0,1)}\|_r.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\mathcal{T}f\|_{A_r^s} &= \Big(\sum_{x\in\mathcal{M}} \Big\| \Big(\mathcal{T}\Big(\sum_{y\in\mathcal{M}} f_{B(y,1)}\Big)\Big)_{B(x,1)} \Big\|_r^s \Big)^{1/s} \\
&\le \Big(\sum_{x\in\mathcal{M}} \Big(\sum_{y\in\mathcal{M}} \|(\mathcal{T}(f_{B(y,1)}))_{B(x,1)}\|_r \Big)^s \Big)^{1/s} \\
&\le 3^b\, V(1)^{1/r'} \Big(\sum_{x\in\mathcal{M}} \Big(\sum_{y\in\mathcal{M}} (1+d(x,y))^{-b} \|f_{B(y,1)}\|_1 \Big)^s \Big)^{1/s}.
\end{aligned}$$

Now for any $x \in \mathcal{M}$ and λ in $\mathbb{R}^+$,

$$\begin{aligned}
&\#\{y \in \mathcal{M} : (1+d(x,y))^{-b} > \lambda\} \\
&\quad = \#\mathcal{B}(x, \lambda^{-1/b} - 1) \\
&\quad \le \begin{cases} 0 & \text{when } \lambda \ge 1 \\ V(\lambda^{-1/b} - 1/2)/v(1/2) & \text{when } \lambda < 1 \end{cases}
\end{aligned}$$

by Lemma 1.1. Thus the assumption on the size of large balls implies that

$$\#\{y \in \mathcal{M} : (1+d(x,y))^{-b} > \lambda\} \le \frac{C_1}{v(1/2)}\, \lambda^{-n/b},$$

whence, by the generalization of Young's inequality,

$$\begin{aligned}
\Big(\sum_{x\in\mathcal{M}} \Big(\sum_{y\in\mathcal{M}} (1+d(x,y))^{-b}\, \|f_{B(y,1)}\|_1 \Big)^s \Big)^{1/s} &\le E \Big(\sum_{x\in\mathcal{M}} \|f_{B(y,1)}\|_1 \Big)^{1/q} \\
&\le E \Big(\sum_{x\in\mathcal{M}} \|f_{B(y,1)}\|_p^q \Big)^{1/q},
\end{aligned}$$

as required. □

2. Symmetric Gaussian semigroups.

In this section, we suppose that (M, d, μ) is a measured metric space. We shall further suppose that (M, d, μ) is *polynomial of order* $(m_0, m_1; n_0, n_1)$, by which we mean that

$$
\begin{aligned}
C_0\, \rho^{m_0} &\le v(\rho) \le V(\rho) \le C_1\, \rho^{m_1} \qquad \text{when } \rho \le 1 \\
C_0\, \rho^{n_0} &\le v(\rho) \le V(\rho) \le C_1\, \rho^{n_1} \qquad \text{when } \rho \ge 1.
\end{aligned} \tag{2.1}
$$

These conditions imply immediately that M satisfies the uniform ball size conditions (1.1). Consequently, the results of the previous section about amalgams on M hold.

Let $\mathcal{L}$ be a possibly unbounded positive self-adjoint operator on $L^2(M)$. Let $\{\mathcal{P}_\lambda\}_{\lambda>0}$ be the spectral resolution of the identity for which

$$\mathcal{L} = \int_0^\infty \lambda \, \mathrm{d}\mathcal{P}_\lambda.$$

For positive t, we define the *heat operator* $\mathcal{H}_t$ by the rule

$$\mathcal{H}_t f = \int_0^\infty e^{-t\lambda} \, \mathrm{d}\mathcal{P}_\lambda f \qquad \forall f \in L^2(M).$$

Clearly $\{\mathcal{H}_t\}_{t>0}$ is a contraction semigroup on $L^2(M)$. For such a semigroup, we define, for any complex number α, the *Riesz potential operator* $\mathcal{R}_\alpha$ of order α by the rule

$$\mathcal{R}_\alpha f = \int_0^\infty \lambda^{-\alpha/2} \, \mathrm{d}\mathcal{P}_\lambda f$$

for every f in the natural domain $\mathrm{Dom}(\mathcal{R}_\alpha)$, which is equal to

$$\Big\{ f : \int_0^\infty \lambda^{-\operatorname{Re}\alpha} \, \mathrm{d}(f, \mathcal{P}_\lambda f) < \infty \Big\}.$$

It is easy to check by spectral theory that

$$\mathcal{R}_\alpha f = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} \, \mathcal{H}_t f \, \mathrm{d}t \qquad \forall f \in \mathrm{Dom}(\mathcal{R}_\alpha).$$

This formula allows us to deduce information about $\mathcal{R}_\alpha$ from information about $\{\mathcal{H}_t\}_{t>0}$. (We only require that the generator be self-adjoint in order to avoid difficulties with the definitions of $\mathcal{H}_t$ and $\mathcal{R}_\alpha$. However, it would be enough to assume the existence of a reasonable functional calculus for $\mathcal{L}$.)

We say that a semigroup $\{\mathcal{H}_t\}_{t>0}$ on M is *sub-Gaussian* if there exist constants A_1 and B_1 such that its kernel H_t satisfies the inequality

$$|H_t(x,y)| \le \frac{A_1}{v(\sqrt{t})} \, e^{-B_1 d(x,y)^2/t} \qquad \forall x, y \in M \quad \forall t \in \mathbb{R}^+. \tag{2.2}$$

Similarly, we say that $\{\mathcal{H}_t\}_{t>0}$ on M is *super-Gaussian* if there exist constants A_0 and B_0 such that its kernel H_t is nonnegative and satisfies the inequality

$$H_t(x,y) \geq \frac{A_0}{V(\sqrt{t})}\, e^{-B_0\, d(x,y)^2/t} \qquad \forall x,y \in M \quad \forall t \in \mathbb{R}^+. \tag{2.3}$$

THEOREM 2.1. — *Suppose that (M,d,μ) is polynomial of order $(m_0,m_1;n_0,n_1)$ as in (2.1), and that the semigroup $\{\mathcal{H}_t\}_{t>0}$ on M is sub-Gaussian as in (2.2). Suppose further that $0 < \alpha < \min(m_0,n_0)$, $1 < p \leq r < \infty$, $1 < q \leq s < \infty$,*

$$\frac{1}{p} - \frac{1}{r} \leq \frac{\alpha + m_1 - m_0}{m_1}, \qquad \text{and} \qquad \frac{1}{q} - \frac{1}{s} \geq \frac{\alpha + n_1 - n_0}{n_1}.$$

Then the operator $\mathcal{R}_\alpha$ is bounded from $A^q_p(M)$ to $A^s_r(M)$.

Proof. — The main step of our proof is to show that there exists a constant D such that the kernel R_α of the Riesz potential $\mathcal{R}_\alpha$ satisfies the following inequalities:

$$|R_\alpha(x,y)| \leq D\, d(x,y)^{\alpha-m_0} \qquad \text{when } d(x,y) \leq 1 \tag{2.4}$$

$$|R_\alpha(x,y)| \leq D\, d(x,y)^{\alpha-n_0} \qquad \text{when } d(x,y) \geq 1. \tag{2.5}$$

The theorem then follows from Theorem 1.5.

Recall that R_α may be expressed in terms of the heat kernel by the formula

$$R_\alpha = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1}\, H_t \,\mathrm{d}t.$$

By the upper bound for the heat kernel (2.2), for all x and y in M, we have

$$
\begin{aligned}
|R_\alpha(x,y)| &\le \frac{A_1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1}\, v(\sqrt{t})^{-1}\, e^{-B_1 d(x,y)^2/t}\,\mathrm{d}t \\
&\le \frac{A_1}{C_0\,\Gamma(\alpha/2)} \int_0^1 t^{(\alpha-m_0)/2-1}\, e^{-B_1 d(x,y)^2/t}\,\mathrm{d}t \\
&\quad + \frac{A_1}{C_0\,\Gamma(\alpha/2)} \int_1^\infty t^{(\alpha-n_0)/2-1}\, e^{-B_1 d(x,y)^2/t}\,\mathrm{d}t \\
&= \frac{A_1}{C_0\,\Gamma(\alpha/2)} (B_1 d(x,y)^2)^{(\alpha-m_0)/2} \int_{B_1 d(x,y)^2}^\infty t^{(m_0-\alpha)/2-1}\, e^{-t}\,\mathrm{d}t \\
&\quad + \frac{A_1}{C_0\,\Gamma(\alpha/2)} (B_1 d(x,y)^2)^{(\alpha-n_0)/2} \int_0^{B_1 d(x,y)^2} t^{(n_0-\alpha)/2-1} e^{-t}\mathrm{d}t \\
&= \frac{A_1\, B_1^{(\alpha-m_0)/2}}{C_0\,\Gamma(\alpha/2)} d(x,y)^{\alpha-m_0} \int_{B_1 d(x,y)^2}^\infty t^{(m_0-\alpha)/2-1}\, e^{-t}\,\mathrm{d}t \\
&\quad + \frac{A_1\, B_1^{(\alpha-n_0)/2}}{C_0\,\Gamma(\alpha/2)} d(x,y)^{\alpha-n_0} \int_0^{B_1 d(x,y)^2} t^{(n_0-\alpha)/2-1}\, e^{-t}\,\mathrm{d}t \\
&= \frac{A_1\, B_1^{(\alpha-m_0)/2}}{C_0\,\Gamma(\alpha/2)}\, d(x,y)^{\alpha-m_0} I_0(d(x,y)) \\
&\quad + \frac{A_1\, B_1^{(\alpha-n_0)/2}}{C_0\,\Gamma(\alpha/2)}\, d(x,y)^{\alpha-n_0} I_\infty(d(x,y)),
\end{aligned}
$$

say. It is easy to check that $I_0(d)$ converges to $\Gamma((m_0-\alpha)/2)$ as d tends to $0+$ and decays exponentially as d tends to ∞. Similarly, $I_\infty(d)$ converges to $\Gamma((n_0-\alpha)/2)$ as d tends to ∞ and decays $d^{n_0-\alpha}$ as d tends to $0+$. This proves the estimate (2.4), and hence the theorem. □

Remark 2.2. — *Mutatis mutandis*, the same argument also shows that R_α is bounded from $A^q_p(M)$ to $A^s_r(M)$ for complex α. More precisely, the argument used to show (2.4) also shows that

$$
\begin{aligned}
|R_\alpha(x,y)| &\le D\, d(x,y)^{\mathrm{Re}\,\alpha-m_0} \qquad \text{when } d(x,y)\le 1 \\
|R_\alpha(x,y)| &\le D\, d(x,y)^{\mathrm{Re}\,\alpha-n_0} \qquad \text{when } d(x,y)\ge 1,
\end{aligned}
$$

and the boundedness from $A^q_p(M)$ to $A^s_r(M)$ follows again by Theorem 1.5.

Remark 2.3. — Similarly, if the semigroup satisfies the super-Gaussian bound (2.3), then, for real α, the kernel of the Riesz potential has a lower bound of the form

$$
\begin{aligned}
R_\alpha(x,y) &\ge D'\, d(x,y)^{\alpha-m_1} \qquad \text{when } d(x,y)\le 1 \\
R_\alpha(x,y) &\ge D'\, d(x,y)^{\alpha-n_1} \qquad \text{when } d(x,y)\ge 1.
\end{aligned}
\tag{2.6}
$$

Suppose that $\mathcal{R}_\alpha$ is bounded from $A^q_p(M)$ to $A^s_r(M)$ and that formula (2.6) holds. Take a 1-discretisation $\mathcal{M}$ of M, x_0 in $\mathcal{M}$, and ρ_0 in $(0,1)$. Then

$$\begin{aligned}\mathcal{R}_\alpha\chi_{B(x_0,\rho_0)}(x) &= \int_{B(x_0,\rho_0)} R_\alpha(x,y)\,\mathrm{d}\mu(y)\\ &\geq D''\mu\big(B(x_0,\rho_0)\big)\,\big(d(x,x_0)+\rho_0\big)^{\alpha-m_1}\end{aligned}$$

if $d(x,x_0)\leq 1$. It follows from Proposition 1.2 and formula (2.6) that

$$\begin{aligned}&\|\mathcal{R}_\alpha\chi_{B(x_0,\rho_0)}\|_{A^s_r}\\ &\geq C\,\big\|\big(\mathcal{R}_\alpha\chi_{B(x_0,\rho_0)}\big)_{B(x_0,1)}\big\|_r\\ &\geq D''\mu\big(B(x_0,\rho_0)\big)\left(\int_{B(x_0,1)}\big(d(x,x_0)+\rho_0\big)^{r(\alpha-m_1)}\,\mathrm{d}\mu(x)\right)^{1/r}\\ &= D''\mu\big(B(x_0,\rho_0)\big)\Big(r(\alpha-m_1)\int_0^1(\rho+\rho_0)^{r(\alpha-m_1)-1}\,\mu(B(x_0,\rho))\,\mathrm{d}\rho\\ &\qquad\qquad + (1+\rho_0)^{r(\alpha-m_1)}\mu(B(x_0,1))\Big)^{1/r}\\ &\geq D''\mu\big(B(x_0,\rho_0)\big)\Big(r(\alpha-m_1)\int_0^1(\rho+\rho_0)^{r(\alpha-m_1)-1}\,C_0\,\rho^{m_0}\,\mathrm{d}\rho\Big)^{1/r}\\ &= D''\mu\big(B(x_0,\rho_0)\big)\Big(r(\alpha-m_1)\,C_0\,\rho_0^{m_0+r(\alpha-m_1)}\\ &\qquad\qquad \int_0^{1/\rho_0}(1+t)^{r(\alpha-m_1)-1}\,t^{m_0}\,\mathrm{d}t\Big)^{1/r}.\end{aligned}$$

As ρ_0 tends to $0+$, the last integral tends to $B(r(\alpha-m_1)-m_0, m_0+1)$ (the Euler Beta function), provided that $r(\alpha-m_1)>m_0$, whence

$$\|\mathcal{R}_\alpha\chi_{B(x_0,\rho_0)}\|_{A^s_r}\geq D'''\mu\big(B(x_0,\rho_0)\big)\,\rho_0^{m_0/r+\alpha-m_1}.$$

Hence, by using Proposition 1.2 again, we find that

$$\begin{aligned}\limsup_{\rho_0\to 0}\frac{\|\mathcal{R}_\alpha\chi_{B(x_0,\rho_0)}\|_{A^s_r}}{\|\chi_{B(x_0,\rho_0)}\|_{A^q_p}} &\geq \limsup_{\rho_0\to 0}\frac{D'''\mu(B(x_0,\rho_0))\,\rho_0^{m_0/r+\alpha-m_1}}{C'\mu(B(x_0,\rho_0))^{1/p}}\\ &\geq \limsup_{\rho_0\to 0} D'''C_0^{1-1/p}\,C'^{-1}\,\rho_0^{m_0(1-1/p+1/r)+\alpha-m_1}.\end{aligned}$$

Since $\mathcal{R}_\alpha$ is bounded from $A^q_p(M)$ to $A^s_r(M)$, $m_0(1-1/p+1/r)+\alpha-m_1\geq 0$, i.e.,

$$\frac{1}{p}-\frac{1}{r}\leq\frac{\alpha+m_0-m_1}{m_0}.$$

In the same way, by considering $\mathcal{R}_\alpha\chi_{B(x_0,\rho_0)}$ as ρ tends to infinity and using Proposition 1.2 to estimate the amalgam norms in terms of Lebesgue norms, we may also show that

$$\frac{1}{q}-\frac{1}{s}\geq\frac{\alpha+n_0-n_1}{n_0}.$$

Corollary 2.4. — *Suppose that M and $\{\mathcal{H}_t\}_{t>0}$ satisfy the hypotheses of Theorem 2.1. Suppose also that there exist positive m and n such that*

$$\mu(B(x,\rho)) \sim \rho^m \quad \forall \rho \in (0,1] \qquad \text{and} \qquad \mu(B(x,\rho)) \sim \rho^n \quad \forall \rho \in [1,\infty).$$

If $0 < \alpha < \min(m,n)$, $1 < p \leq r < \infty$, $1 < q \leq s < \infty$,

$$\frac{1}{p} - \frac{1}{r} \leq \frac{\alpha}{m} \qquad \text{and} \qquad \frac{1}{q} - \frac{1}{s} \geq \frac{\alpha}{n},$$

then the operator $\mathcal{R}_\alpha$ is bounded from $A_p^q(M)$ to $A_r^s(M)$.

Remark 2.5. — If the semigroup also satisfies the super-Gaussian bound (2.3), then from Remark 2.3, we see that $\mathcal{R}_\alpha$ is bounded from $A_p^q(M)$ to $A_r^s(M)$ if and only if the conditions

$$\frac{1}{p} - \frac{1}{r} \leq \frac{\alpha}{m} \qquad \text{and} \qquad \frac{1}{q} - \frac{1}{s} \geq \frac{\alpha}{n}$$

are satisfied.

In particular, since $A_p^p(M)$ is isomorphic to $L^p(M)$, Corollary 2.4 says that $\mathcal{R}_\alpha$ is bounded from $L^p(M)$ to $A_r^s(M)$ when

$$\frac{1}{p} - \frac{1}{r} = \frac{\alpha}{m} \qquad \text{and} \qquad \frac{1}{p} - \frac{1}{s} = \frac{\alpha}{n}.$$

Semigroup techniques [9] show that, if $m < n$, then $\mathcal{R}_\alpha$ is bounded from $L^p(M)$ to $L^r(M) \cap L^s(M)$. As the amalgam $A_r^s(M)$ is continuously (strictly) included in $L^r(M) \cap L^s(M)$, the use of amalgams in Corollary 2.4 enables a stronger and more succinct result to be formulated. On the other hand, if $m > n$, then $\mathcal{R}_\alpha$ is bounded from $L^p(M)$ to $L^r(M) + L^s(M)$, and this is a sharper result than is obtained using amalgams.

Remark 2.6. — A specialisation of the result of Corollary 2.4 states that if $\min(m,n) > 1$ and

$$\frac{1}{r} = \frac{1}{2} - \frac{1}{m} \qquad \text{and} \qquad \frac{1}{s} = \frac{1}{2} - \frac{1}{n},$$

then the operator $\mathcal{R}_\alpha$ is bounded from $L^2(M)$ to $A_r^s(M)$. This may be interpreted as a Sobolev embedding result: functions whose (distributional) Laplacian lies in $L^2(M)$ must lie in $A_r^s(M)$.

Remark 2.7. — Finally, it is tempting to speculate that this theorem must have a converse, at least for certain semigroups. If one could prove that boundedness between amalgams implies a good diagonal estimate for the heat kernel, i.e., an estimate of the form

$$|H_t(x,x)| \leq \frac{A_1}{v(\sqrt{t})} \qquad \forall x \in M \quad \forall t \in \mathbb{R}^+,$$

then now-standard techniques would show that the semigroup is sub-Gaussian.

3. Riemannian manifolds.

Throughout this section, unless stated otherwise, we suppose that M is an m-dimensional complete connected noncompact Riemannian manifold, with Ricci curvature Ric_M bounded below by $(m-1)H$, where $H \in \mathbb{R}$, and positive injectivity radius $i(M)$. Equip M with the Riemannian distance d and the Riemannian volume μ. The Laplace–Beltrami operator of M is denoted by $-\mathcal{L}_0$; the positive operator $\mathcal{L}_0$ has a self-adjoint extension on $L^2(M)$, denoted by $\mathcal{L}$.

The heat semigroup on a Riemannian manifold has been the object of investigation of many mathematicians in the last three decades, and we now know much about $\{\mathcal{H}_t\}_{t>0}$, some of which is summarised in the book of E.B. Davies [11].

In the case where $H = 0$, i.e., where $\mathrm{Ric}_M \geq 0$, P. Li and S.T. Yau [19] proved that there exist positive constants A_0, B_0, A_1 and B_1 such that

$$\frac{A_0}{V(\sqrt{t})}\, e^{-B_0\, d(x,y)^2/t} \leq H_t(x,y) \leq \frac{A_1}{v(\sqrt{t})}\, e^{-B_1 d(x,y)^2/t} \tag{3.1}$$

for all x, y in M and for all t in $\mathbb{R}^+$. The upper bound has been proved in greater generality. In particular, Davies [12] showed that, when the Ricci curvature Ric_M of the manifold M is bounded below, then for any ε in $\mathbb{R}^+$,

$$H_t(x,y) \leq \frac{A_\varepsilon}{v(\sqrt{t})}\, e^{\varepsilon t}\, e^{-B_1 d(x,y)^2/t}. \tag{3.2}$$

We will use the estimates on the size of balls to give information about the heat kernel.

Standard comparison theorems ([2], pp. 253–257) imply that

$$\mu(B(x,\rho)) \leq V_H(\rho) \qquad \forall \rho \in \mathbb{R}^+ \quad \forall x \in M, \tag{3.3}$$

where $V_H(\rho)$ denotes the volume of the ball of radius ρ in the simply connected complete Riemannian m-manifold with constant curvature H.

Furthermore, C.B. Croke ([10], Prop. 14) showed that there exists a constant C such that

$$\mu(B(x,\rho)) \geq C\,\rho^m \qquad \forall \rho \in (0, i(M)/2] \quad \forall x \in M.$$

These two estimates imply that

$$\mu(B(x,\rho)) \sim \rho^m \qquad \forall \rho \in (0,1]. \tag{3.4}$$

Finally, it follows from a result of Yau [25] (see also [4], Prop. 4.1 (iv)), that, if $\mathrm{Ric}_M \geq 0$, then

$$\mu(B(x,\rho)) \geq C\,\rho \qquad \forall \rho \in [i(M)/2, \infty) \quad \forall x \in M. \tag{3.5}$$

Clearly, the uniform ball size conditions (1.1) hold in this case.

The applications of amalgams we have in mind require further assumptions on M. We say that a Riemannian manifold is polynomial of order (n_0, n_1) at infinity if there exist constants C_0 and C_1 such that

$$C_0\,\rho^{n_0} \leq \mu(B(x,\rho)) \leq C_1\,\rho^{n_1} \qquad \forall \rho \in [1, \infty) \quad \forall x \in M. \tag{3.6}$$

In particular, if $\mathrm{Ric}_M \geq 0$ (and the other assumptions of this section hold), then formulae (3.3) and (3.5) imply that M is polynomial of order $(1, m)$ at infinity.

COROLLARY 3.1. — *Suppose that M is a complete, connected, noncompact manifold of dimension m and with nonnegative Ricci curvature, and that M is polynomial of order (n_0, n_1) at infinity as in formula (3.6). Suppose also that $0 < \alpha < \min(m, n_0)$, $1 < p \leq r < \infty$, $1 < q \leq s < \infty$,*

$$\frac{1}{p} - \frac{1}{r} \leq \frac{\alpha}{m} \qquad \text{and} \qquad \frac{1}{q} - \frac{1}{s} \leq \frac{\alpha + n_1 - n_0}{n_1}.$$

Then the operator $\mathcal{R}_\alpha$ is bounded from $A_p^q(M)$ to $A_r^s(M)$.

Proof. — The hypotheses ensure that the Li–Yau bounds (3.1) hold; Theorem 2.1 then implies the result.

We now extend this result to a more general setting. More precisely, we want to weaken the assumption that the manifold M has nonnegative Ricci curvature. We shall make use of the notion of quasi-isometry, discussed by M. Kanai (who uses the terminology "rough isometry") in [18].

Suppose that (M, d) and (M', d') are metric spaces. A (not necessarily continuous) map $\varphi : M \to M'$ is said to be a *quasi-isometry* if

$$\sup\left\{d'(\varphi(M), x') : x' \in M'\right\} < \infty, \tag{3.7}$$

and there exist constants A and B such that

$$A^{-1}\, d(x,y) - B \le d'(\varphi(x),\varphi(y)) \le A\, d(x,y) + B \qquad \forall x,y \in M. \tag{3.8}$$

Two metric spaces (M,d) and (M',d') are said to be *quasi-isometric* if there is a quasi-isometry between them. We refer the reader to [18] for comments on this definition as well as for bibliography.

COROLLARY 3.2. — *Suppose that M and M' are quasi-isometric, complete, connected, noncompact Riemannian manifolds of dimensions m and m', and with positive injectivity radius. Suppose also that M has nonnegative Ricci curvature and is polynomial of order (n_0,n_1) at infinity, and M' has Ricci curvature bounded below. If $0 < \alpha < \min(m',n_0)$, $1 < p \le r < \infty$, $1 < q \le s < \infty$,*

$$\frac{1}{p} - \frac{1}{r} \le \frac{\alpha}{m'} \qquad \text{and} \qquad \frac{1}{q} - \frac{1}{s} \le \frac{\alpha + n_1 - n_0}{n_1},$$

then the Riesz potential operator $\mathcal{R}'_\alpha$ associated to the Laplace–Beltrami operator on M' is bounded from $A^q_p(M')$ to $A^s_r(M')$.

Proof. — Since the manifolds M and M' are quasi-isometric, M' is polynomial of order (n_0,n_1) at infinity, by a result of Kanai ([18], Theorem 3.3). Hence the uniform ball size conditions (1.1) hold for M'.

The heat kernel H'_t on M' satisfies the upper bound

$$H'_t(x,y) \le \frac{A_1}{v(\sqrt{t})}\, e\, e^{-B_1 d'(x',y')^2/t} \qquad \forall x',y' \in M' \quad \forall t \in (0,1],$$

by Davies' result (3.2).

From the Li–Yau estimates (3.1), the heat kernel H_t associated to the Laplace–Beltrami operator on M satisfies the upper bound

$$\sup\big\{H_t(x,y) : x,y \in M\big\} \le C\, t^{-n_0/2} \qquad \forall t \in [1,\infty),$$

so the heat kernel H'_t associated to the Laplace–Beltrami operator on M' satisfies a similar upper bound

$$\sup\big\{H'_t(x',y') : x',y' \in M'\big\} \le C\, t^{-n_0/2} \qquad \forall t \in [1,\infty)$$

by results of Coulhon ([7], Proposition 3). From the work of Davies and M.M.H. Pang [13], we find the upper bound

$$H'_t(x',y') \le A_1\, t^{-n_0/2}\, e^{-B_1 d'(x',y')^2/t} \qquad \forall x',y' \in M' \quad \forall t \in [1,\infty).$$

The corollary now follows from Theorem 2.1. □

Such transfer results between quasi-isometric manifolds are not new. For instance, see [22] for a wide range of results about equivalent Riemannian metrics on a given manifold.

The infinite Loch Ness monster is quasi-isometric to the Euclidean plane, and the infinite jungle gym is quasi-isometric to the Euclidean three space (see [18], p. 393). Corollary 3.2 applies to these exotic examples.

4. Lie groups.

Suppose that G is a connected Lie group with Haar measure μ. We shall assume that there exist a nonvoid compact neighbourhood U of the origin and a positive real number n such that $\mu(U^j) \sim j^n$, for all positive integers j. It then follows that this holds for all neighbourhoods of the identity, and G is said to be *of polynomial growth of order n*. Such a group is unimodular.

Let $X_1, \ldots, X_N$ be left invariant vector fields on G, which, together with their commutators, generate the Lie algebra $\mathfrak{g}$ of all left invariant vector fields on G. By a celebrated theorem of L. Hörmander [17], the left invariant differential operator $\mathcal{L}_0$, which acts on smooth functions of compact support on G by the rule

$$\mathcal{L}_0 f(x) = -\sum_{j=1}^{N} X_j^2 f(x) \qquad \forall x \in G,$$

is hypoelliptic. There is a metric related to $\mathcal{L}_0$, commonly referred to as the *control distance* associated to $\mathcal{L}_0$, which we now briefly describe. For every pair of points x and y in G, let $P(x,y)$ be the set of all absolutely continuous paths $\gamma : [0,1] \to G$ such that $\gamma(0) = x$ and $\gamma(1) = y$, and

$$\gamma'(t) = \sum_{j=1}^{N} a_j(t)\, X_j(\gamma(t)) \qquad \text{for almost every } t \text{ in } [0,1].$$

Define the distance function d by the rule

$$d(x,y) = \inf\{|\gamma| : \gamma \in P(x,y)\},$$

where

$$|\gamma| = \int_0^1 \left(\sum_{j=1}^{N} |a_j(t)|^2 \right)^{1/2} dt.$$

This distance was first introduced by C. Carathéodory [3]. Subsequently, W.L. Chow [5] proved that $d(x,y)$ is finite for all x and y in G, and that the distance d induces the topology of G. A very readable account of the properties of the balls $B(x,\rho)$ associated to d is given by A. Nagel, E.M. Stein and S. Wainger [20], Chap. 1.

Clearly, d is a left invariant distance on G, so that the condition (1.1) is clearly satisfied—indeed, all balls of the same radius have the same measure. We shall denote $\mu(B(e,\rho))$ by $v(\rho)$. There exists a positive integer m such that

$$v(\rho) \sim \begin{cases} \rho^m & \text{when } \rho \in (0,1] \\ \rho^n & \text{when } \rho \in [1,\infty); \end{cases}$$

(see [20]). Let $\mathcal{L}'$ be another operator on G constructed from another set of left invariant vector fields satisfying Hörmander's condition, and denote by d' the associated control distance and by $B'(x,\rho)$ the associated ball. The distances d and d' need not be equivalent, as classical examples show. In particular, there is a positive number m', possibly different from m, such that

$$\mu(B'(e,\rho)) \sim \rho^{m'} \qquad \forall \rho \in (0,1].$$

However, d and d' are equivalent distances "at infinity". In particular

$$\mu(B'(e,\rho)) \sim \rho^{n} \qquad \forall \rho \in [1,\infty),$$

where n is the same as in (4.1).

The operator $\mathcal{L}_0$ is essentially self–adjoint; we denote by $\mathcal{L}$ its self–adjoint extension on $L^2(G)$. It generates a symmetric contraction semigroup $\mathcal{H}_t$, which has a smooth convolution kernel H_t, so that

$$\mathcal{H}_t f(x) = \int_G f(y)\, H_t(y^{-1}x)\, \mathrm{d}y \qquad \forall f \in L^p(G).$$

See, e.g., ([24], Theorem V.4.2) or ([21], Theorem IV.4.13). It is known (see [23], [21]) that, for suitable positive constants A_0, B_0, A_1 and B_1, H_t satisfies an estimate of the form

$$\frac{A_0}{v(\sqrt{t})}\, e^{-B_0\, d(x,y)^2/t} \le H_t(x,y) \le \frac{A_1}{v(\sqrt{t})}\, e^{-B_1 d(x,y)^2/t}$$

for all (x,t) in G and t in $\mathbb{R}^+$.

COROLLARY 4.1. — *Suppose that G is a connected Lie group of polynomial growth of order n. Let $\mathcal{L}$ be a self-adjoint invariant second order operator generated by a set of vector fields on G satisfying Hörmander's condition, and let m be as in (4.1). If $0 < \alpha < \min(m,n)$, $1 < p \le r < \infty$, $1 < q \le s < \infty$,*

$$\frac{1}{p} - \frac{1}{r} \le \frac{\alpha}{m} \qquad \text{and} \qquad \frac{1}{q} - \frac{1}{s} \ge \frac{\alpha}{n},$$

then the operator $\mathcal{R}_\alpha$ is bounded from $A^q_p(M)$ to $A^s_r(M)$.

Proof. — This follows from Corollary 2.4. □

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M. COWLING,
University of New South Wales
School of Mathematics
Sydney 2052 NSW (Australia).
m.cowling@unsw.edu.au

S. MEDA,
Università di Milano "Bicocca"
Dipartimento di Statistica
Viale Sarca 202
20100 Milano (Italia).
stemed@mate.polimi.it

R. PASQUALE,
Università di Milano
Dipartimento di Matematica
Via Saldini 50
20133 Milano (Italia).
pasquale@vmimat.mat.unimi.it