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HODGE NUMBERS ATTACHED TO A POLYNOMIAL MAP

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1. Introduction.

(1.1) In the last years the behaviour at infinity of polynomial maps has been extensively studied, basically by means of topological invariants: Eisenbud-Neumann diagrams, monodromy at infinity, etc. In order to carry further this analysis of the asymptotic behaviour of polynomial maps we consider in this paper the following approach: One can attach to any polynomial map a geometrical variation of mixed Hodge structures (from now on, MHS; see (1.2) below for details). Using either the geometric methods of [24], [9] or Saito's theory of mixed Hodge modules ([17]), one gets that this variation gives rise to a functorial limit MHS. The equivariant Hodge numbers of this limit MHS are analytical invariants of the polynomial map under consideration and in this paper we compute them for a class of polynomials which was extensively studied in [5], [6] from a topological point of view. More precisely, we determine the equivariant Hodge numbers of this limit MHS in terms of equivariant Hodge numbers attached to some isolated hypersurface singularities and Hodge numbers of cyclic coverings of projective space branched along a hypersurface (see Theorem (3.1) for the precise statement). Both these local and global Hodge numbers can be explicitly computed in a number of cases, as shown in the examples in Section 7.

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(1.2) We give the precise description of the invariants we are going to study. Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial map. It is known that there is a finite set Γ_f such that f defines a locally trivial $\mathcal{C}^\infty$-fibration over $\mathbb{C} - \Gamma_f$ (cf. e.g. [15, Appendix]). Given a positive number $r \in \mathbb{R}$, denote by D_r the disk in the complex plane of center 0 and radius r, and set $D_r^* = D_r - \{0\}, Z_r = \mathbb{C}^{n+1} - f^{-1}(D_r)$. If $r > \max\{\, |x| \ : \ x \in \Gamma_f \,\}$ then the map

$$1/f : Z_r \longrightarrow D^*_{1/r}$$

is a locally trivial $\mathcal{C}^\infty$-fibration. Actually we have a projective system of equivalent fibrations indexed by r, thus we can assume r so big as necessary.

Now we compactify the map $1/f$ and we add a fiber over $0 \in \mathbb{C}$. That is, we take an analytical manifold X where $Z = Z_r$ is embedded as an open dense subset, $(X - Z)_{\text{red}}$ is a divisor with normal crossings and smooth irreducible components and the map $1/f$ extends to a flat projective map $p : X \longrightarrow D_{1/r}$, smooth over $D^*_{1/r}$. Set $X - Z = Y \cup \Delta$, where $Y = p^{-1}(0)$ and Δ is the union of the irreducible components of $X - Z$ not in Y. Increasing r if necessary, we can assume that the restriction of p to any intersection of components of Δ_{red} is smooth, so that Δ becomes a relative divisor with normal crossings. Let $\mathbb{H}$ be the universal covering space of $D^*_{1/r}$, set $\tilde{Z} = Z \times_{D_{1/r}} \mathbb{H}$. Notice that $\tilde{Z}$ has the homotopy type of a generic fiber of f. Then the cohomology groups $H^i(\tilde{Z}, \mathbb{Q})$ carry a limit MHS. This follows for example from [24, §5] or from [9, Théorème 5.13]. All MHS we will work with will be regarded as $\mathbb{Q}$-MHS.

One can prove using standard arguments that this MHS does not depend on the chosen compactification and is functorial on f (and therefore it is for example an invariant of polynomials with respect to algebraic changes of variables in $\mathbb{C}^{n+1}$).

(1.3) Definition. — *Given a polynomial map $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$, the limit MHS on $H^n(\tilde{Z}, \mathbb{Q})$ obtained in the way described above will be called the MHS at infinity of f.*

(1.4) The MHS at infinity should be viewed as a global analogue of the limit MHS associated to a hypersurface singularity in [22] and [9]. If f is a cohomologically tame polynomial (see [16]) there is another definition of a MHS at infinity due to C. Sabbah. For these polynomials, Sabbah has proved that the equivariant Hodge numbers of the two MHS's are the same. Unless otherwise stated, all (co)homology groups appearing in this paper will be assumed to be reduced and to have complex coefficients. We choose once and for all a root i of -1 in $\mathbb{C}$ (equivalently, an orientation on the complex plane) and from now on we will use the notation $e(r) := e^{2\pi i r}$, $r \in \mathbb{R}$.

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2. The main construction and exact sequences.

(2.1) Although the MHS at infinity is defined for any polynomial map, if one does not place some kind of restriction on f even the topological behaviour at infinity can be very complicated.

We will consider the class of polynomials $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ which satisfy the following condition:

$$(*) \begin{cases} \text{If } t \in \mathbb{C} \text{ is not a critical value of } f, \\ \text{then the closure in } \mathbb{P}^{n+1} \text{ of the affine} \\ \text{hypersurface } \{f = t\} \text{ is non-singular.} \end{cases}$$

Polynomials satisfying this condition will be called $(*)$-polynomials. In this section we recall some consequences of the $(*)$ condition and some exact sequences attached to a $(*)$-polynomial. For details, see [5], [6]. Let $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be a $(*)$-polynomial. Then

(a) Any fiber of f has at most isolated singularities and has the homotopy type of a bouquet of n-dimensional spheres. Consider the fibration at infinity of f:

$$f : f^{-1}(\partial \bar{D}_r) \cap B_R \longrightarrow \partial \bar{D}_r,$$

where D_r (resp. B_R) is a disc in $\mathbb{C}$ (resp. a ball in $\mathbb{C}^{n+1}$) of radius r (resp. R) and $0 << r << R$. This fibration is equivalent to the fibration $f : f^{-1}(\partial \bar{D}_r) \longrightarrow \partial \bar{D}_r$. For $t \in \partial \bar{D}_r$ set $X_t^0 = f^{-1}(t) \subseteq \mathbb{C}^{n+1}$. The geometric monodromy of this fibration (where $\partial \bar{D}_r$ is positively oriented) is denoted by T_{geom}. In [5], [6] we proved some basic properties of the *classical* algebraic monodromy $T_f^\infty : H^n(X_t^0) \longrightarrow H^n(X_t^0)$ which is defined by $T_f^\infty[\omega] = [T_{\text{geom}}^*(\omega)]$.

Let X_t denote the projective closure of the affine hypersurface X_t^0. The *classical* algebraic monodromy over $\partial \bar{D}_r$ on the n-th cohomology groups of the projective closures will be denoted by $T : H^n(X_t) \longrightarrow H^n(X_t)$ (and is defined also as a pull–back).

(b) For $R >> 0$, the Milnor fibration at infinity $\varphi = \dfrac{f}{\|f\|} : \partial \bar{B}_R - f^{-1}(0) \longrightarrow S^1$ exists and it is equivalent to the fibration at infinity of f. In particular, there is a well defined variation map $\mathrm{Var} : H^n(X_t^0) \longrightarrow H_c^n(X_t^0)$ ($\mathrm{Var}([\omega]) := [T^*_{\mathrm{geom}}(\omega) - \omega]$), which is an isomorphism and has the usual compatibility properties with the monodromy T_f^∞.

(c) The projective hypersurface $X^\infty := \{f_d = 0\} \subseteq \mathbb{P}^n$ has only isolated singularities. The semisimple part of T_f^∞ depends only on the local topological type of these singular points, but the nilpotent part of T_f^∞ (and also the intersection form on $H_n(X_t^0)$) depends also on the position of the singularities of X^∞.

In general, if one wants to relate the algebraic monodromy with Hodge theory, it is more convenient to use the algebraic monodromy induced on the (horizontal sections of the) cohomological Gauss–Manin connection (see e.g. [1] §12). This is the inverse dual of the classical homological monodromy [loc. cit 13.1.A]. In this paper they will be denoted by $M_f^\infty = (T_f^\infty)^{-1}$ acting on $H^n(X_t^0)$, $M = T^{-1}$ acting on $H^n(X_t)$; and we will call them simply (algebraic) monodromies. Obviously, $W([\omega]) := [(T^*_{\mathrm{geom}})^{-1}(\omega) - \omega]$ has the well–known compatibility with M_f^∞ (see 2.3). We will use the corresponding similar notations in all local situations which will appear in the body of the paper.

(2.2) In order to study the MHS at infinity of a $(*)$-polynomial we will consider a concrete compactification of the map $1/f$ (see [5], [6]).

Let $f = f_d + f_{d-1} + \ldots$ denote the decomposition of f into homogeneous components, let D denote a disc in the complex plane with center at the origin and sufficiently small radius. Set

$$\mathcal{X} = \{([z], t) \in \mathbb{P}^{n+1} \times D \mid t(f_d + z_0 f_{d-1} + \ldots + z_0^d f_0) = z_0^d\}$$

where $[z_0 : z_1 : \ldots : z_{n+1}]$ are homogeneous coordinates in $\mathbb{P}^{n+1}$.

The map $\pi : \mathcal{X} \longrightarrow D$ given by $\pi([z], t) = t$ induces a locally trivial $\mathcal{C}^\infty$ filtration over $D - \{0\}$ and the fibers of π are exactly the projective closures of the fibers of f. The monodromy of π over ∂D (with positive orientation) is M.

Then by [5] we have a commutative diagram

$$
\text{(E.0)} \qquad \begin{array}{ccccccccc}
0 \to & P^n_{X^\infty}(X_t) & \to & P^n(X_t) & \to & H^n(X_t^0) & \to & P^{n+1}_{X^\infty}(X_t) & \to 0 \\
 & \downarrow \mathrm{id} & & \downarrow M & & \downarrow M_f^\infty & & \downarrow \mathrm{id} & \\
0 \to & P^n_{X^\infty}(X_t) & \to & P^n(X_t) & \to & H^n(X_t^0) & \to & P^{n+1}_{X^\infty}(X_t) & \to 0
\end{array}
$$

where P denotes primitive cohomology (i.e. for $X \subset \mathbb{P}^N$ one has $P^k(X) = \mathrm{Coker}(H^k(\mathbb{P}^N) \to H^k(X))$). The above spaces carry natural MHS's: $P^n(X_t)$ carries Schmid's MHS, $H^n(X_t^0)$ the MHS introduced in Section 1 and all arrows are morphisms of MHS's.

The compactification provided by π is not such a good one since its center fiber $\pi^{-1}(0)$ is not reduced (it has multiplicity d) and $\mathcal{X}$ is very singular. Therefore, we will consider the normalization of the d-fold covering of π. More precisely, let D' be again a small disk, $\delta : D' \longrightarrow D$ the map given by $\delta(t) = t^d$. Define $\mathcal{X}'$ as the normalization of $\mathcal{X} \times_\delta D'$ and let $\pi' : \mathcal{X}' \longrightarrow D'$ be the natural projection. The map π' gives a fibration over $D' - \{0\}$ with the same fiber as π and classical monodromy T^{-d}. Both $\mathcal{X}'$ and the central fiber $X_0' = (\pi')^{-1}(0)$ have only isolated singularities. Set $\mathrm{Sing}\, X^\infty = \{p_1, \dots, p_k\}$ and let F_j, M_j be the local Milnor fiber and the algebraic monodromy $M_j : H^{n-1}(F_j) \longrightarrow H^{n-1}(F_j)$ of the hypersurface singularity $(X^\infty, p_j) \subset (\mathbb{P}^n, p_j)$ for $1 \leqslant j \leqslant k$ (Here $\mathbb{P}^n$ is the hyperplane at infinity).

Then $\mathrm{Sing}\,\mathcal{X}' = \mathrm{Sing}\, X^\infty \times \{0\}$ and the central fiber X_0' is the d-fold cyclic covering of $\mathbb{P}^n$ branched along X^∞. In particular, if we set $\mathrm{Sing}\, X_0' = \{p_1', \dots, p_k'\}$ then the isolated singularities (X_0', p_j') are the d-th suspensions of the singularities (X^∞, p_j).

Let F_j' (resp. M_j') be the Milnor fiber (resp. the monodromy) of the local smoothing $\pi' : (\mathcal{X}', p_j') \longrightarrow (D', 0)$. We remark that this smoothing is not the (natural) smoothing of (X_0', p_j') with total space smooth. The following commutative diagram is the exact sequence of vanishing cycles corresponding to π' together with the monodromy action. It is a diagram of MHS's (cf. [5, (10)] or [6, (2.2)]).

$$
\text{(E.1)} \qquad \begin{array}{ccccccccc}
0 \to & P^n(X_0') & \to & P^n(X_t) & \to & \bigoplus_{j=1}^k H^n(F_j') & \to & P^{n+1}(X_0') & \to 0 \\
 & \downarrow \mathrm{id} & & \downarrow M^d & & \downarrow \oplus (M_j')^{-1} & & \downarrow \mathrm{id} & \\
0 \to & P^n(X_0') & \to & P^n(X_t) & \to & \bigoplus_{j=1}^k H^n(F_j') & \to & P^{n+1}(X_0') & \to 0.
\end{array}
$$

Notice that $X'_0 \subset \mathbb{P}^{n+1}$ is given by the equation $z_0^d = f_d(z_1, \ldots, z_{n+1})$. There is a Galois action of $\mathbb{Z}/d\mathbb{Z}$ on X'_0 generated by the automorphism $[z_0 : \ldots : z_{n+1}] \longmapsto [e(1/d)\, z_0 : \ldots : z_{n+1}]$ which induces at the cohomology level an automorphism which will be denoted $G^q : P^q(X'_0) \longrightarrow P^q(X'_0)$.

The disadvantage of (E.1) is that the monodromy action at the level of $P^n(X_t)$ is M^d instead of M. In [6] we constructed an automorphism of $\mathcal{X}'$ over D' which is a "lifting" of the geometric monodromy of π. In [loc. cit.] we identified $(H^n(F'_j), M'_j)$ with

$$\left(H^{n-1}(F_j)^{\oplus (d-1)},\ c_{d-1}(M_j)^d\right)$$

where for an operator $\varphi : V \longrightarrow V$ and an integer $\ell \geqslant 1$ wé denote $c_\ell(\varphi) : V^{\oplus \ell} \longrightarrow V^{\oplus \ell}$ the operator defined by $c_\ell(\varphi)\ (x_1, \ldots, x_\ell) = (\varphi(x_\ell), x_1, \ldots, x_{\ell-1})$. Now the "lifted monodromy action" induces the following commutative diagram (of vector spaces):
(E.1′)

$$\begin{array}{ccccccccc}
0 \to & P^n(X'_0) & \to & P^n(X_t) & \to & \bigoplus_{j=1}^{k} H^{n-1}(F_j)^{\oplus (d-1)} & \to & P^{n+1}(X'_0) & \to 0 \\
& \downarrow G^n & & \downarrow M & & \downarrow \oplus\, c_{d-1}(M_j^{-1}) & & \downarrow G^{n+1} & \\
0 \to & P^n(X'_0) & \to & P^n(X_t) & \to & \bigoplus_{j=1}^{k} H^{n-1}(F_j)^{\oplus(d-1)} & \to & P^{n+1}(X'_0) & \to 0.
\end{array}$$

(In [loc. cit.] it is not clarified the MHS-meaning of this diagram, this will be done in the next section. That discussion will show that (E.1′) is actually a diagram of MHS's. One should be aware of the fact that if we consider on $H^n(F'_j)$ and $H^{n-1}(F_j)$ their natural MHS, then $H^n(F'_j) \neq H^{n-1}(F_j)^{\oplus(d-1)}$ as MHS's.)

Theorem 2 in the Appendix of [5] together with the "lifted monodromy action" gives the following commutative diagram of MHS (cf. [6]):
(E.2)

$$\begin{array}{ccccccc}
0 \to & P^n(X'_0) & \to & \operatorname{Ker}(M^d - \mathrm{Id} \mid P^n(X_t)) & \to & \bigoplus_{j=1}^{k} H^{n+1}_{\{p'_j\}}(\mathcal{X}') & \to 0 \\
& \downarrow G^n & & \downarrow M & & \downarrow \mathrm{id} & \\
0 \to & P^n(X'_0) & \to & \operatorname{Ker}(M^d - \mathrm{id} \mid P^n(X_t)) & \to & \bigoplus_{j=1}^{k} H^{n+1}_{\{p'_j\}}(\mathcal{X}') & \to 0.
\end{array}$$

(2.3) There is a nice connection between the variation map and the diagram (E.0) (for details, see [5, pp. 217–218]). There is an orthogonal decomposition $P^n(X_t)_1 = P^n_{X^\infty}(X_t) \oplus P^n_{X^\infty}(X_t)^\perp$, $M_1 = \mathrm{Id} \oplus M'$ (where

$P^n(X_t)_1$ denotes the space of generalized eigenvectors of $P^n(X_t)$ of eigenvalue 1 with respect to M, we will use the same notation for the other groups as well).

We identify $P^n_{X^\infty}(X_t)^\perp$ with the image of $P^n\,(X_t) \longrightarrow H^n\,(X_t^0)$. Then the composed map

$$w : H^n\,(X_t^0)_1 \xrightarrow{\mathrm{W}_1} H^n_c\,(X_t^0)_1 \longrightarrow H^n\,(X_t^0)_1$$

(where W was defined in (2.1)) has range exactly $P^n_{X^\infty}(X_t)^\perp$ and makes the following diagram (of vector spaces) commutative:

$$\begin{tikzcd}
0 \arrow[r] & P^n_{X^\infty}(X_t)^\perp \arrow[r] \arrow[d, "M' - \mathrm{id}"] & H^n(X_t^0)_1 \arrow[r] \arrow[d, "(M_f^\infty)_1 - \mathrm{id}"] \arrow[dl, "w"'] & P^{n+1}_{X^\infty}(X_t) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & P^n_{X^\infty}(X_t)^\perp \arrow[r] & H^n(X_t^0)_1 \arrow[r] & P^{n+1}_{X^\infty}(X_t) \arrow[r] & 0.
\end{tikzcd}$$

3. The mixed Hodge structure at infinity in the $(*)$-case.

In this section we describe the equivariant Hodge numbers of the MHS at infinity of a $(*)$-polynomial.

Let f be a $(*)$-polynomial and keep the notations of section 2. Let $(M_f^\infty)_{ss}$ (resp. $(M_f^\infty)_u$) denote the semisimple (resp. the unipotent) part of the monodromy at infinity M_f^∞. We recall ([9, Théorème 15]) that $(M_f^\infty)_{ss}$ is a MHS-automorphism of $H^n(X_t^0)$ of type $(0,0)$ and $N^0 = \log\,(M_f^\infty)_u$ is a MHS-automorphism of type $(-1,-1)$. We will denote $H^n(X_t^0)_\xi$ the ξ-eigenspace of $H^n(X_t^0)$ with respect to M_f^∞. For ξ a d-th root of unity, we denote by $P^\ell(X_0')_\xi$ the ξ-eigenspace of $P^\ell(X_0')$ with respect to the Galois action G^ℓ. Obviously, this eigenspace decomposition is compatible with the MHS of $P^\ell(X_0')$.

We recall the definition of primitive spaces with respect to a nilpotent endomorphism. (cf. e.g. [10, (5.3)], [19, (6.4)]): Let H be a finitely dimensional $\mathbb{Q}$-vector space and $N : H \to H$ a nilpotent endomorphism of H. Let W denote the weight filtration on H corresponding to N and centered at m. Assume there is a fitration F on the complexification of H such that (H, W, F) is a MHS. For $r \geqslant 0$ one sets

$$P_r := \mathrm{Ker} N^{r+1} : \mathrm{Gr}^W_{m+r} H \to \mathrm{Gr}^W_{m-r-2} H.$$

Then P^r carries a MHS of weight $m+r$. For $p+q \geqslant m$ we will call $P^{p,q}_{p+q-m}$ the primitive space of type (p,q).

The local vanishing cohomology $H^{n-1}(F_j)$ of (X^∞, p_j) has a MHS (cf. [22]). For $\xi \neq 1$ (resp. for $\xi = 1$) the weight filtration of the generalized ξ-eigenspace is the monodromy weight filtration centered at $n-1$ (resp. at n). In this case, we will denote the dimension of the corresponding primitive spaces (with respect to the infinitesimal nilpotent monodromy) by $p^{pq}_\xi(X^\infty, p_j)$. The main theorem of this section is the following

(3.1) THEOREM. — a) *The weight filtration of $H^n(X^0_t)_1$ is the monodromy weight filtration given by N^0_1 with center $n+1$. The dimensions of the primitive spaces of type (p,q) $(p+q \geqslant n+1)$ are*

$$p^{p,q}_1(f) = h^{n-q,\, n-p}\,(P^{n-1}(X^\infty)).$$

b) *The weight filtration of $H^n(X^0_t)_{\neq 1}$ is the monodromy weight filtration given by $N^0_{\neq 1}$ with center n. The dimensions of the primitive spaces of type (p,q) are given as follows:*

i) *If $\xi^d = 1$ and $\xi \neq 1$, then*

$$p^{p,q}_\xi(f) = h^{n-q,\, n-p}\,(P^n(X'_0)_\xi).$$

ii) *If $\xi^d \neq 1$, then*

$$p^{p,q}_\xi(f) = \sum_{j=1}^{k} p^{p-[\beta+\gamma],\; q-\delta+[\beta+\gamma]}_{\xi^{1-d}}\;(X^\infty, p_j)$$

where $\xi = e^{-2\pi i\beta}(0 < \beta < 1)$, $\gamma = \{(d-1)\beta\}$, and $\delta = 0$ if $\gamma = 0$ and $\delta = 1$ if $\gamma \neq 0$.

(Here $[\;\;]$ denotes integer part and for $\alpha \in \mathbb{R}$, $\{\alpha\} = \alpha - [\alpha]$).

Notice that for $\xi^d \neq 1$, $p^{p,q}_\xi(f)$ depends only on the type of the local singularities $(X^\infty, p_j)^k_{j=1}$ (and not on their position).

(3.2) *Remarks.* — a) The equivariant Hodge numbers $h^{a,b}_\lambda$ of $H^n(X^0_t)$ can be computed from the formula

$$h^{a,b}_\lambda = \sum_{\ell \geqslant 0} p^{a+\ell,\; b+\ell}_\lambda\;(f)$$

where $a+b \geqslant$ center of the weight filtration $:= c$, and from $h^{a,b}_\lambda = h^{c-a,\; c-b}_{\bar\lambda} = h^{c-b,\; c-a}_\lambda$ for $a+b \leqslant c$ (cf. [10, (5.3)]).

b) The above theorem shows that the weight filtration of $H^n(X_t^0)$ has a behaviour similar to that of the weight filtration of the MHS on the vanishing cohomology corresponding to an isolated hypersurface singularity (namely, it is centered at n for $\lambda \neq 1$ and at $n+1$ for $\lambda = 1$). For $\lambda \neq 1$ this can be proved using Schmid's results [Sch] and for $\lambda = 1$ follows basically from the fact that the variation map is an isomorphism.

c) It follows also from the theorem that the maximal size of a Jordan block of $(M_f^\infty)_\xi$ is $n+1$ if $\xi^d = 1$ and $\xi \neq 1$, otherwise is n (cf. [6, p. 4]).

d) If f and f' are two $(*)$-polynomials with the same highest degree form, their equivariant Hodge numbers are the same.

Proof of Theorem (3.1). — We will say that two MHS's H_1 and H_2 are numerically equivalent (and we will denote this $H_1 \overset{n}{=} H_2$) if their Hodge numbers are the same. The primitive decomposition theorem gives the following fact, which will be used several times.

(3.3) LEMMA. — *Let H_i $(i = 1,2)$ be a MHS with a nilpotent $(-1,-1)$-morphism N_i such that the weight filtration of H_i is the monodromy weight filtration of N_i (with the same fixed center). If* $\mathrm{Ker} N_1 \overset{n}{=} \mathrm{Ker} N_2$, *then* $H_1 \overset{n}{=} H_2$.

We divide the proof of (3.1) in the following cases:

Case $\xi = 1$.

In order to prove this case we study first the MHS's of $P^*(X^\infty)$ and $P^*_{X^\infty}(X_t)$. (Some of the results of this subsection will be used in Section 4 too.)

(3.4) LEMMA. — *$P^{n+i}_{X^\infty}(X_t)$ and $P^{n-i}(X^\infty)$ are dual MHS's with respect to $\mathbb{Q}(-n)$. $P^n(X^\infty)$ is pure of weight n and $P^{n-1}(X^\infty) = W_{n-1}\, P^{n-1}(X^\infty)$.*

Proof of the lemma. — The duality statement follows quite directly from Saito's formalism: On the category of mixed Hodge modules one can define a duality functor which commutes with the nearby cycle functor up to a Tate twist (cf. [17], 2.6) and extends to a functor defined in the derived category, compatible with Verdier duality. In Saito's terminology and with the notations of (1.2), if $i : \Delta \hookrightarrow X$ is the inclusion map one has to apply this duality to the object $\mathbb{R}\Psi_p\, i_*\, i^!\, \mathbb{Q}_X[n+1]$ in the derived category of mixed Hodge modules. The purity of $P^n(X^\infty)$ is proved in [22, (4.7)].

(3.5) Consider now a 1-dimensional deformation of $X^\infty \subseteq \mathbb{P}^n$ by hypersurfaces, i.e. a family of hypersurfaces $X_s^\infty \subseteq \mathbb{P}^n$ of degree d such that $X_0^\infty \simeq X^\infty$, X_s^∞ is smooth for $s \neq 0$, and the total space of the deformation is smooth as well. Consider the exact sequence of vanishing cycles

$$(3.6)\quad 0 \to P^{n-1}(X^\infty) \to P^{n-1}(X_s^\infty)_1 \to \oplus_j H^{n-1}(F_j)_1 \to P^n(X^\infty) \to 0\,.$$

Here $P^*(X^\infty)$ carries Deligne's MHS, $P^{n-1}(X_s^\infty)_1$ Schmid's MHS and $H^{n-1}(F_j)_1$ Steenbrink's MHS. Moreover, the Hodge numbers of $H^{n-1}(F_j)_1$ do not depend on the choice of the smoothing (because in a μ–constant family of hypersurface singularities, the equivariant Hodge numbers are constant), in particular, they are equal to the Hodge numbers provided by the natural local smoothing. The sequence is compatible with the monodromy action, which is the identity on $P^*(X^\infty)$, M_j on $H^{n-1}(F_j)$ and it will be denoted $M_{g\ell}$ on $P^{n-1}(X_s^\infty)_1$.

By [19] (resp. [22]) the weight filtration of $P^{n-1}(X_s^\infty)_1$ (resp. $H^{n-1}(F_j)_1$) is the monodromy weight filtration with center $n-1$ (resp. n). By the invariant cycle theorem $P^{n-1}(X^\infty) = \operatorname{Ker}(M_{g\ell} - \mathrm{Id})$. From these facts it follows that we have, for $\ell \geqslant 1$

$$\begin{aligned} &\dim\, \mathrm{Gr}^W_{n-1-\ell}\, P^{n-1}(X^\infty) + \dim\, \mathrm{Gr}^W_{n-1-\ell} \oplus_j H^{n-1}(F_j)_1 \\ (3.7)\qquad &= \dim\, \mathrm{Gr}^W_{n-1+\ell}\, \oplus_j H^{n-1}(F_j)_1 - \dim\, \mathrm{Gr}^W_{n-1+\ell}\, P^n(X^\infty). \end{aligned}$$

Now we compare $\oplus_j H^{n-1}(F_j)_1$ and $P^n(X_t)_1$. The 1-eigenspaces in (E.2) provide the following MHS-identification (notice that G^n has no 1–eigenvector on $P^n(X'_0)$):

$$(3.8)\qquad \operatorname{Ker}\,(M - \mathrm{Id} \mid P^n(X_t)_1) = \bigoplus_{j=1}^{k} H^{n+1}_{\{p'_j\}}(\mathcal{X}').$$

Let g_i be a local equation defining the singularity germ (X^∞, p_j). The smoothings of the singularities $(\mathcal{X}', p'_j)$ are given locally by the equations

$$\hat{g}_j(x, x', x'') = g_j(x) + x'^{\,2} + x''^{\,2},\ (x, x', x'') \in (\mathbb{C}^n \times \mathbb{C} \times \mathbb{C},\ 0),$$

(cf. [5, p. 223], make the change of variables $2ix' = x_0 - t + x_0^{d-1}$, $2x'' = x_0 + t - x_0^{d-1}$). If $\hat{F}_j$ denotes the Milnor fiber of $\hat{g}_j$ then by Sebastiani-Thom (cf. [18]) one has $H^{n+1}(\hat{F}_j)_1 = H^{n-1}(F_j)_1(-1)$.

Let $V_1 : H^{n-1}(F_j)_1 \to H_c^{n-1}(F_j)_1(-1)$ be the infinitesimal variation map of type $(-1,\,-1)$ (cf. [23, (2.5)]), let $k : H_c^{n-1}(F_j)_1 \longrightarrow H^{n-1}(F_j)_1$ be the natural map, $M_{j,c} : H_c^{n-1}(F_j) \longrightarrow H_c^{n-1}(F_j)$ the monodromy at the level of cohomology with compact supports. Define $N_1 = \log(M_1)$, $N_{j,1} = \log(M_{j,1})$, and $N_{j,c,1} = \log(M_{j,c,1})$. Then from [23] one has

$$H^{n-1}_{\{p_j\}}(X^\infty) = \mathrm{Ker}\,(H_c^{n-1}(F_j)_1 \xrightarrow{k} H^{n-1}(F_j)_1)$$

and similarly for $\mathcal{X}'$, so one has

$$H^{n+1}_{\{p'_j\}}(\mathcal{X}') = H^{n-1}_{\{p_j\}}(X^\infty)(-1). \tag{3.9}$$

Since $N_{j,c,1} = V_1 \circ k$ and V_1 is an isomorphism, one obtains that $H^{n-1}_{\{p_j\}}(X^\infty)(-1) = \mathrm{Ker}\, k(-1) = \mathrm{Ker}\, N_{j,c,1}(-1) \simeq \mathrm{Ker}\, N_{j,1}$. The last isomorphism is given by V_1 and follows from the fact that $(M_{j,c})_1 \circ V_1 = V_1 \circ (M_j)_1$.

The above identities, together with (3.9) and (3.8) give

$$\mathrm{Ker}\,(N_1 \mid P^n(X_t)_1) \simeq \bigoplus_{j=1}^{k} \mathrm{Ker}\,(N_{j,1} \mid H^{n-1}(F_j)_1).$$

Now, (3.3) implies that

$$P^n(X_t)_1 \overset{n}{=} \bigoplus_{j=1}^{k} H^{n-1}(F_j)_1 \tag{3.10}$$

(i.e. they have the same Hodge numbers).

Now consider (E.0) and (2.3). The map $N_1^0 : H^n(X_t^0)_1 \longrightarrow H^n(X_t^0)_1$ is a morphism of type $(-1, -1)$ and by (2.3)

$$\mathrm{Im}\,(N_1^0) = \mathrm{Im}\,[P^n(X_t)_1 \to H^n(X_t^0)_1]. \tag{3.11}$$

Via (3.4) and (3.10), identity (3.7) reads

$$\begin{aligned} &\dim\, \mathrm{Gr}^W_{n+1+\ell}\, P^{n+1}_{X^\infty}(X_t) + \dim\, \mathrm{Gr}^W_{n+1+\ell}\, P^n(X_t)_1 \\ &= \dim\, \mathrm{Gr}^W_{n+1-\ell}\, P^n(X_t)_1 - \dim\, \mathrm{Gr}^W_{n+1-\ell}\, P^n_{X^\infty}(X_t). \end{aligned} \tag{3.12}$$

Using (E.0), this equality can be rewritten as

$$\dim\, \mathrm{Gr}^W_{n+1+\ell}\, H^n(X_t^0)_1 = \dim\, \mathrm{Gr}^W_{n+1-\ell}\, H^n(X_t^0)_1.$$

Together with (3.11), this implies that the weight filtration of $H^n(X_t^0)_1$ is the monodromy weight filtration centered at $n+1$ and its primitive spaces can be identified with $P^{n+1}_{X^\infty}(X_t)$, i.e. for $p+q \geqslant n+1$

$$p_1^{p,q}(f) = h^{p,q}(P^{n+1}_{X^\infty}(X_t)) = h^{n-p,n-q}(P^{n-1}(X^\infty)) = h^{n-q,n-p}(P^{n-1}(X^\infty)).$$

Case $\xi^d = 1,\ \xi \neq 1$.

From (E.0) $H^n(X_t^0)_\xi = P^n(X_t)_\xi$ and from (E.2)

$$\operatorname{Ker}\,(M - \xi\,\mathrm{Id} \mid P^n(X_t)\,) = P^n(X_0')_\xi.$$

Since the weight filtration of $P^n(X_t)_\xi$ is centered at n (cf. [19]),

$$p_\xi^{p,q}\,(f) = h^{n-q,\ n-p}\,(P^n(X_0')_\xi).$$

Case $\xi^d \neq 1$.

Given ξ with $\xi^d \neq 1$, from (E.0) we have $H^n(X_t^0)_\xi = P^n(X_t)_\xi$ as MHS's, and from (E.1') we have

$$P^n(X_t)_\xi = \Big(\bigoplus_j H^{n-1}(F_j)^{\oplus d-1},\ \oplus c_{d-1}(M_j)\Big)_{\xi^{-1}} \tag{3.13}$$

as vector spaces. Recall that, as vector spaces with automorphism we have

$$\big(H^n(F_j'), M_j'\big) \simeq \big(H^{n-1}(F_j)^{\oplus d-1},\ c_{d-1}(M_j)^d\big). \tag{3.14}$$

By (E.1), for $\eta \neq 1$ we have an isomorphism of MHS's

$$\bigoplus_{\xi^d=\eta} P^n(X_t)_\xi = \bigoplus_j \,(H^n(F_j'), M_j')_{\eta^{-1}} \tag{3.15}$$

where $(H^n(F_j'), M_j')$ is given by the local smoothing $\pi' : (\mathcal{X}', p_j') \longrightarrow (D', 0)$. In some local coordinates this can be identified with $\pi' : (\mathbb{Y}_j, 0) \longrightarrow (D', 0)$, where

$$\mathbb{Y}_j = \big\{(y, y_0, t) \mid g_j(y) + t\,y_0 = y_0^d\big\}$$

and $\pi'(y, y_0, t) = t$ (see [5]). We will compute the MHS of the vanishing cohomology of π' corresponding to eigenvalues $\neq 1$. This computation, and the above isomorphisms will provide the result.

The idea of the following construction comes from a recent paper of H. Hamm [8]. We thank J. Steenbrink for drawing our attention to that paper.

Fix an integer $m \geqslant 1$ such that for any eigenvalue λ of M_j one has $\lambda^m = 1$. Then, by (3.14), for any eigenvalue λ of M_j' one has $\lambda^{m(d-1)} = 1$.

Let $(\tilde{\mathbb{Y}}_j, 0)$ be the fibre product of $\pi' : (\mathbb{Y}_j, 0) \to (D', 0)$ and $(D'', 0) \to (D', 0)$ where the last arrow is $s \mapsto s^{m(d-1)} = t$ and D'' is again a small disk. So, $(\tilde{\mathbb{Y}}_j, 0)$ is locally given by the equation $g_j(y) + s^{m(d-1)} y_0 = y_0^d$.

Let $\pi'' : (\tilde{\mathbb{Y}}_j, 0) \to (D'', 0)$ be the projection $(y_0, y, s) \mapsto s$. Then the fiber of π'' is the same as the fiber of π', its monodromy is the $m(d-1)$-th power of the monodromy of π', in particular it is unipotent. Let $\tilde{K}$ be the real link of $(\tilde{\mathbb{Y}}_j, 0)$. On $\tilde{K}$ we define the natural Galois action induced by $(y_0, y, s) \mapsto \left(y_0, y, s\, e\Big(\dfrac{1}{m(d-1)}\Big)\right)$.

We denote by $V_j' : H^n(F_j') \longrightarrow H_c^n(F_j')(-1)$ the infinitesimal variation map ([23, (2.5)]). By [loc. cit., 2.6.b] one has the following exact sequence of MHS's:

$$0 \to H^n(\tilde{K}) \to H^n(F_j') \xrightarrow{V_j'} H_c^n(F_j')(-1) \to H^{n+1}(\tilde{K}) \to 0.$$

(3.16) LEMMA. — *This exact sequence is equivariant with respect to an action which at the level of $H^n(F_j')$ is the monodromy action M_j' of π' and on $H^n(\tilde{K})$ is the Galois action.*

Proof of the lemma. — If $T'_{j,\mathrm{geom}}$ denotes the geometric monodromy of π', then its lifting composed with the Galois action is isotopic to the identity (by a similar argument as in the proof of E.1' and E.2 in [6]). Therefore, the inverse of $(T'_{j,\mathrm{geom}})^*$ corresponds to the Galois action.

If $H^n(\tilde{K})_\eta$ denotes the η-eigenspace ($\eta \neq 1$) with respect to this Galois action, then by (3.16) $H^n(\tilde{K})_\eta = \mathrm{Ker}\,(V_j')_\eta$. Recall that $k \circ V_j' = N_j'$ where $N_j' = \dfrac{1}{m(d-1)} \log\,(M_j')^{m(d-1)}$ and k is the natural map $H_c^n(F_j') \longrightarrow H^n(F_j')$. Since k_η is an isomorphism for $\eta \neq 1$, $\mathrm{Ker}\,(N_j')_\eta = \mathrm{Ker}\,(V_j')_\eta$. Therefore, for $\eta \neq 1$,

$$H^n(\tilde{K})_\eta = \mathrm{Ker}\,\big((N_j')_\eta \mid H^n(F_j')_\eta\big). \tag{3.17}$$

The advantage of this identification is the following: The map π' is defined on a hypersurface singularity germ, and there are very few methods to compute the MHS of its vanishing cohomology. On the other hand, $\tilde{K}$ can be represented as the link of a hypersurface singularity, and there are good methods to compute its MHS. Indeed, let $u : (\mathbb{C}^{n+2}, 0) \longrightarrow (\mathbb{C}, 0)$ be defined by $u\,(y, y_0, s) = g_j(y) + s^{m(d-1)}\, y_0 - y_0^d$. Let U be its Milnor fiber. Then by [23] we have the exact sequence of MHS's

$$0 \to H^n(\tilde{K}) \to H_c^{n+1}(U)_1 \xrightarrow{k_1} H^{n+1}(U)_1 \to H^{n+1}(\tilde{K}) \to 0. \tag{3.18}$$

Besides the monodromy action, on this exact sequence there is a Galois action induced by the transformation $s \mapsto s \cdot e\Big(\dfrac{1}{m(d-1)}\Big)$, which commutes with the monodromy action.

Let $H_c^{n+1}(U)_{1,\eta}$ be the generalized eigenspace corresponding to eigenvalue 1 with respect to the monodromy action and eigenvalue η with respect to the Galois action. Then (3.18) gives

$$H^n(\tilde{K})_\eta = \operatorname{Ker}(k_{1,\eta} \mid H_c^{n+1}(U)_{1,\eta}).$$

If $V_1 : H^{n+1}(U)_1 \longrightarrow H_c^{n+1}(U)_1(-1)$ is the infinitesimal variation map of u, then $V_1 \circ k = N_{c,1}$ (N_c being the logarithm of monodromy acting on cohomology with compact supports). Since V_1 is an isomorphism and $N_{c,1} \circ V_1 = V_1 \circ N_1$, we get

$$\begin{aligned}\operatorname{Ker} k_{1,\eta} &= \operatorname{Ker}(N_{c,1,\eta} \mid H_c^{n+1}(U)_{1,\eta}) \\ &= \operatorname{Ker}(N_{1,\eta} \mid H^{n+1}(U)_{1,\eta})(+1).\end{aligned}$$

Therefore, for $\eta \neq 1$:

$$H^n(\tilde{K})_\eta = \operatorname{Ker}\Big(N_{1,\eta} \mid H^{n+1}(U)_{1,\eta}\Big)(+1). \tag{3.19}$$

Now, (3.17), (3.19) and (3.3) give the *numerical* equivalence of $H^n(F_j')_\eta$ and $H^{n+1}(U)_{1,\eta}(+1)$, i.e.,

$$H^{pq}(F_j')_\eta \overset{n}{=} H^{p+1,q+1}\ (U)_{1,\eta}. \tag{3.20}$$

Let $w : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be given by $w(y_0, s) = s^{m(d-1)}y_0 - y_0^d$. Since w defines an isolated quasihomogeneous singularity, its equivariant Hodge numbers can be computed using [21] (see also [1]). Then, the Sebastiani-Thom result of [18] allows to express the equivariant Hodge numbers of $u = g_j + w$ in terms of those of g_j.

(3.21) PROPOSITION. — (a) *For each 1-eigenvector of $(H^{n-1}(F_j), M_j)$ of type (p,q) there are $d-2$ eigenvectors of $(H^n(F_j'), M_j')_{\neq 1}$ of type (p,q) and of eigenvalues $e\Big(\dfrac{k}{d-1}\Big)$ with $1 \leqslant k \leqslant d-2$.*

(b) *For each $e(i/d)$-eigenvector of type (p,q) of $(H^{n-1}(F_j), M_j)$ with $i \in \mathbb{Z}$, $0 < i < d$, there are $d-2$ eigenvectors $\{v_k\}_k$ of $(H^n(F_j'), M_j')_{\neq 1}$ where $k = 0, \ldots, \widehat{d-1-i}, \ldots d-2$, v_k has eigenvalue $e\Big(\dfrac{k+i}{d-1}\Big)$ and type*

$$\Big(p + \Big[\frac{k+i}{d-1}\Big],\ q + 1 - \Big[\frac{k+i}{d-1}\Big]\Big).$$

(c) *For each $e(\gamma)$-eigenvector of $(H^{n-1}(F_j), M_j)$ with $d\gamma \notin \mathbb{Z}$ and $\gamma \in (0,1)$, and type (p,q), there are $d-1$ eigenvectors $\{w_k\}_k$ of $(H^n(F_j'), M_j')_{\neq 1}$*

where $k = 0, \ldots, d-2$, w_k *has eigenvalue* $e\Big(\gamma + \dfrac{k+\gamma}{d-1}\Big)$ *and type*

$$\Big(p + \Big[\gamma + \frac{k+\gamma}{d-1}\Big],\ q+1-\Big[\gamma+\frac{k+\gamma}{d-1}\Big]\Big).$$

(d) *All eigenvectors of* $(H^{n-1}(F_j'), M_j)$ *corresponding to eigenvalues* $\neq 1$ *are obtained from a), b) or c) above.*

Proof. — In order to have a simpler form of Sebastiani-Thom, we will work with spectral pairs. A λ-eigenvector v of type (p,q) of $(H^{n-1}(F_j), M_j)$ gives a pair $S_v = (\alpha, p+q-s)$ where $\lambda = e(-\alpha)$, $n-1+[-\alpha] = p$ and $s = 0$ if $\alpha \notin \mathbb{Z}$ and $s = 1$ if $\alpha \in \mathbb{Z}$ (cf. §6).

One has $\mathrm{Spp}(w) = \sum\limits_{(i,j)} \Big(\dfrac{i}{d} + \dfrac{j}{md} - 1, 1\Big)$, where the residue classes of the monomials $\{y_0^{i-1}s^{j-1}\}$ form a base in the Milnor algebra $\mathbb{C}[y_0, s]/(\partial w)$. A possible choice of the monomials is the following: (i,j) either satisfies $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m(d-1)-1$, or $(i,j) = (1, m(d-1))$. Then v contributes to $\mathrm{Spp}(u)$ with pairs

$$\sum_{(i,j)} \Big(\alpha + \frac{i}{d} + \frac{j}{md},\ p+q-s+2\Big).$$

This gives eigenvectors of $H^{n+1}(U)_{1,\neq 1}$ if and only if

$$\alpha + \frac{i}{d} + \frac{j}{md} \in \mathbb{Z} \quad \text{and} \quad \frac{j}{m(d-1)} \notin \mathbb{Z}. \tag{3.22}$$

One has to search for solutions of (3.22) with the restrictions

$$1 \leqslant i \leqslant d\,, \quad 1 \leqslant j \leqslant m(d-1)-1. \tag{3.23}$$

In case (a) (resp. (b), resp. (c)) one has $S_v = (n-p-1,\ p+q-1)$ (resp. $S_v = (-i/d - p + n - 1,\ p+q)$, resp. $S_v = (-\gamma - p + n - 1,\ p+q)$). One gets (a), (b), (c) by solving (3.22), (3.23) in each case. The details of the computation are left to the reader.

Now we can finish the proof of theorem 3.1 (case $\xi^d \neq 1$). Recall the identities (3.13-14-15).

Subcase 1: Assume $\xi^{d-1} = 1$, i.e. $\xi = e\Big(-\dfrac{\ell}{d-1}\Big)$ with $1 \leqslant \ell \leqslant d-2$. Since ξ^{-1} is an eigenvalue of $c_{d-1}(M_j)$ (cf. 3.13), $\xi^{1-d} = 1$ is an eigenvalue of M_j. By (3.21.a), each 1-eigenvector of M_j of type (p,q) gives $d-2$ eigenvectors of M_j' of type (p,q), with eigenvalues $\eta^{-1} = e\Big(\dfrac{k}{d-1}\Big)$ $(1 \leqslant k \leqslant d-2)$. But (in (3.15)) only one corresponds to ξ (i.e. satisfies $e\Big(-\dfrac{k}{d-1}\Big) = \xi^d$), namely that with $k = \ell$. So, each 1-eigenvector of M_j of type (p,q) gives one eigenvector of $(H^n(X_t^0),\ M_f^\infty)_\xi$ of type (p,q). Therefore,

$$p_\xi^{p,q}(f) = \sum_j p_1^{p,q}(X^\infty, p_j).$$

Subcase 2: Assume $\xi^{(d-1)d} = 1$ but $\xi^{d-1} \neq 1$. Then $\xi = e(-\beta)$, where $\beta = \dfrac{\ell}{d-1} + \dfrac{i}{d(d-1)}$, with $0 < i < d$, $0 \leqslant \ell \leqslant d-2$. Then $\alpha = \xi^{1-d} = e\Big(\dfrac{i}{d}\Big)$ is an eigenvalue of M_j with $\alpha \neq 1$, $\alpha^d = 1$. Using (3.21.b) one gets eigenvalues $\eta^{-1} = e\Big(\dfrac{k+i}{d-1}\Big)$ for M_j', but only one corresponds to ξ (because $e\Big(-\dfrac{k+i}{d-1}\Big) = \xi^d$ implies $k = \ell$). Thus, from (3.21.b), (3.13) and (3.15) one has

$$p_\xi^{p,q}(f) = \sum_j p_{\xi^{1-d}}^{p-[\frac{\ell+i}{d-1}],\ q-1+[\frac{\ell+i}{d-1}]}(X^\infty, p_j).$$

Now, notice that $\gamma := \{(d-1)\beta\} = \dfrac{i}{d}$, hence $[\beta+\gamma] = \Big[\dfrac{\ell+i}{d-1}\Big]$.

Subcase 3: $\xi^{d(d-1)} \neq 1$.

Follows from (3.21.c) as in the subcases above (now $k = [(d-1)\beta]$).

(3.24) *Remark.* — The proof of Theorem (3.1) in case $\xi = 1$ suggests the existence of a natural perfect pairing

$$D : P^{n-1}(X_s^\infty)_1 \otimes H^n(X_t^0)_1 \longrightarrow \mathbb{Q}(-n).$$

Our proof gives this duality at the level of Hodge numbers and we expect that there is a geometrical construction of D. We expect that $H^n(X_t^0)_1$ is not pure if Var is not an isomorphism (cf. (3.2.b)). This suggests that D should be constructed from the Milnor fibration at infinity (Notice that the MHS on $P^{n-1}(X_s^\infty)_1$ depends on the choice of the smoothing $\{X_s\}_s$ of X^∞; also the MHS of $H^n(X_t^0)_1$ depends on the lower order terms $f_{d-1}+f_{d-2}+\cdots$ of f. But maybe a good pairing of these parameter spaces gives rise to a duality D).

4. The Hodge numbers of X^∞ and X_0'.

Theorem (3.1) gives the MHS of $\underset{\xi^d=1}{\oplus} H^n(X_t^0)_\xi$ in terms of the MHS of X^∞ and X_0'. In the sequel we would like to separate the local and the global information. We say that an invariant is local if it depends only on the type of the hypersurface singularities $\{(X^\infty, p_j)\}_{j=1}^k$ and some universal constant depending on n and d. If an invariant depends on the positions of the singularities of X^∞, then we say that it is global. For example, dim $P^n(X^\infty)$ and dim $P^{n-1}(X^\infty)$ are global, but their difference is local.

(4.1) We start defining some universal constants: If $Q \in \mathbb{C}[z_1, \ldots, z_{n+1}]$ is an homogeneous polynomial of degree d which defines a smooth hypersurface $V = V(Q) \subseteq \mathbb{P}^n$, then the Hodge numbers of V can be computed as follows (cf. [7]): Let (∂Q) denote the ideal in $\mathbb{C}[z_1, \ldots, z_{n+1}]$ generated by the partial derivatives of Q and set $\mathbb{F}(Q) = \mathbb{C}[z_1, \ldots, z_{n+1}]/(\partial Q)$. Let $\mathbb{F}(Q)_\ell$ be the subspace generated by the monomials of degree ℓ, and set

$$\Omega_n = \sum_{i=1}^{n+1} (-1)^i \; z_i \; dz_1 \wedge \ldots \wedge \; \widehat{dz_i} \wedge \ldots \wedge \; dz_{n+1}.$$

If P is an homogeneous polynomial of degree $\deg P = (k+1)\,d - n - 1$ for some $k \geqslant 0$, then $P \cdot \Omega_n \,/\, Q^{k+1}$ defines a rational differential on $\mathbb{P}^n$ with poles along the hypersurface V. Its residue class $\mathrm{Res}(P\,\Omega_n \,/\, Q^{k+1})$ lives in the primitive cohomology $P^{n-1}(V)$ and one has an isomorphism (cf. [7, (4.6)])

$$\mathbb{F}(Q)_{(k+1)\,d-n-1} \xrightarrow{\sim} P^{n-1-k,\,k}\,(V) \tag{4.2}$$

given by $P \longmapsto \mathrm{Res}(P\,\Omega_n \,/\, Q^{k+1})$.

It is well known that the Hodge numbers of the cohomology of V (equivalently, dim $\mathbb{F}(Q)_\ell$) does not depend on the choice of Q as long as $V(Q) \subseteq \mathbb{P}^n$ is smooth. We denote

$$j_{k,s}^{n,d} = \dim\; \mathbb{F}(Q)_{(k+1)\,d-(n+1)-s} \tag{4.3}$$

where $0 \leqslant s < d$ and $k \geqslant 0$. The next theorem gives the Hodge numbers of $P^{n-1}(X^\infty)$ in terms of the Hodge numbers of $P^n(X^\infty)$ modulo local contributions. After this paper was written, we have got to know of a very recent paper of A. Dimca ([4]) where the same theorem is proved, under a slightly different form.

(4.4) THEOREM. — *Let X^∞, $\{p_1, \dots, p_k\}, F_1, \dots, F_k$ be as in the previous sections. Then*

(a) $h^{i,\, n-k-i}\,(P^{n-1}(X^\infty)) = \sum_j p_1^{k+i-1,\; n-1-i}\,(X^\infty, p_j)$ *if* $k \geqslant 3$.

(b) $h^{i,\, n-2-i}\,(P^{n-1}(X^\infty)) = \sum_j p_1^{i+1,\; n-1-i}\,(X^\infty, p_j) - h^{i+1,\; n-1-i}$ $(P^n(X^\infty))$.

(c) $h^{i,\, n-1-i}\,(P^{n-1}(X^\infty)) = j_{i,0}^{n,d} - \sum_j \dim\ \mathrm{Gr}_F^i\ H^{n-1}(X^\infty, p_j) - \sum_j \sum_{q>i} p_1^{q,\; n-1-i}\,(X^\infty, p_j) + h^{i,\, n-i}\,(P^n(X^\infty)) + h^{i+1,\; n-1-i}\,(P^n(X^\infty))$.

Proof. — (a) and (b) follow from (3.6) or (3.7). For (c), consider the following exact sequence (similarly as in (3.6), but now for all eigenvalues):

$$(4.5)\quad 0 \to P^{n-1}(X^\infty) \to P^{n-1}(X_s^\infty) \to \oplus_j\ H^{n-1}(F_j) \to P^n(X^\infty) \to 0.$$

Passing to the limit MHS does not change the dimension of the terms of the Hodge filtration, thus $\dim\ \mathrm{Gr}_F^i\ P^{n-1}(X_s^\infty) = j_{n-1-i,0}^{n,d} = j_{i,0}^{n,d}$ (cf. 4.2). Now take Gr_F^i on the sequence above and apply (a) and (b).

(4.6) COROLLARY. — *If $p + q \geqslant n + 3$, then $p_1^{p,q}(f)$ is local.*

Proof. — Use the theorem above and Theorem (3.1, a).

Let $X_0' = \{f_d(z_1, \dots, z_{n+1}) + z_{n+2}^d = 0\} \subseteq \mathbb{P}^{n+1}$ be the d-th fold cyclic covering of $\mathbb{P}^n$ branched along X^∞. The Galois action of $\mathbb{Z}/d\mathbb{Z}$ is given by $\xi * [z_1 : \dots : z_{n+2}] = [z_1 : \dots : \xi\ z_{n+2}]$ ($\xi^d = 1$). Recall that $P^*(X_0')_\xi$ denotes the ξ-eigenspace of $P^*(X_0')$ with respect to the Galois action.

The singularities Sing $X_0' = \{p_1', \dots, p_k'\}$ are the d-th suspensions of the singularities of X^∞. If (X^∞, p_j) is locally given in $\mathbb{C}^n$ by $g_j(x_2, \dots, x_{n+1}) = 0$, then (X_0', p_j') is given by $\tilde{g}_j := g_j(x_2, \dots, x_{n+1}) + x_{n+2}^d = 0$ in $\mathbb{C}^{n+1}$. Let $\tilde{F}_j$ be the Milnor fiber of $\tilde{g}_j$, $\tilde{M}_j$ its monodromy acting on $H^n(\tilde{F}_j)$. Then on $H^n(\tilde{F}_j)$ there is a Galois action (given by $\xi * (x_2, \dots, x_{n+2}) = (x_2, \dots, \xi\, x_{n+2})$ if $\xi^d = 1$) and this action commutes with $\tilde{M}_j$. $H^n(\tilde{F}_j)$ carries a MHS, we consider the space of primitive elements given by the nilpotent part of $\tilde{M}_j$. Let $P_1^{ab}(X_0', p_j')$ be the space of primitive vectors of type (a, b) corresponding to the eigenvalue 1 of $\tilde{M}_j$. Finally, let $p_1^{ab}(X_0', p_j')_\xi$ be the dimension of the ξ-eigenspace of the induced Galois action on $P_1^{a,b}(X_0', p_j')$. (In order to eliminate any confusion, we emphasize that below in (4.7.c) $H^n(\tilde{F}_j)_\xi$ is the ξ–eigenspace with respect to the Galois action.)

(4.7) THEOREM. — *Let $\xi = e(s/d)$, $0 < s < d$. Then with the above notations, one has*

(a) $h^{i,\,n+1-k-i}\,(P^n(X_0')_\xi) = \sum_j p_1^{k+i-1,\;n-i}\,(X_0', p_j')_\xi \qquad$ *if* $k \geqslant 3$

(b) $h^{i,\,n-1-i}\,(P^n(X_0')_\xi) = \sum_j p_1^{i+1,\;n-i}\,(X_0', p_j')_\xi - h^{i+1,\;n-i}\,(P^{n+1}(X_0')_\xi)$.

(c) $h^{i,\,n-i}\,(P^n(X_0')_\xi) = j^{n,d}_{n-i,s} - \sum_j \dim \mathrm{Gr}_F^i\; H^n(\tilde F_j)_\xi - \sum_j \sum_{q>i} p_1^{q,\;n-i}$ $(X_0', p_j)_\xi + h^{i,\;n+1-i}\,(P^{n+1}(X_0')_\xi) + h^{i+1,\;n-i}\,(P^{n+1}(X_0')_\xi)$.

Proof. — A deformation $(f_d)_s$ of f_d induces a deformation $(f_d)_s + z_{n+2}^d$ of $f_d + z_{n+2}^d$. Let $V\,(\,(f_d)_s + z_{n+2}^d) = X_s' \subseteq \mathbb{P}^{n+1}$. Then the exact sequence of vanishing cycles is

$$(4.8) \qquad 0 \to P^n(X_0') \to P^n(X_s') \to \bigoplus_j H^n(\tilde F_j) \to P^{n+1}(X_0') \to 0.$$

On this sequence there are a monodromy and a Galois action which commute. Now the proof of (4.7) is the same as the proof of (4.4) if we replace the exact sequence (4.5) by (4.8) (and we take the ξ-eigenspaces, with respect to the Galois action). The remaining part is the computation of $\dim \mathrm{Gr}_F^i\; P^n(X_s')_\xi$ (for $s \neq 0$ fixed). Take $i = n - k$. By the discussion in (4.1) and with the same notations, $H^{n-k,\;k}\,(P^n(\,V(Q + z_{n+2}^d)\,)_\xi)$ is given by the residues of rational differentials of type $\tilde P\,\Omega_{n+1}\,/\,(Q + z_{n+2}^d)^{k+1}$ where $\deg \tilde P + (n+2) = (k+1)\,d$. Since Ω_{n+1} is homogeneous of degree one in z_{n+2}, such a form lives in the ξ-eigenspace with respect to the Galois action if and only if $\tilde P = P\,(z_1, \ldots, z_{n+1}) z_{n+2}^{s-1}$, hence $P \in \mathbb{F}(Q)_{(k+1)d-(n+1)-s}$.

(4.9) COROLLARY. — *If $p + q \geqslant n + 2$, then $p_\xi^{p,q}\,(f)$ is local for any $\xi^d = 1$, $\xi \neq 1$.*

In the next section we will study the polarization properties of $H^n\,(X_t^0)$. This is not difficult for the generalized eigenspace $H^n\,(X_t^0)_{\neq 1}$, but for $H^n(X_t^0)_1$ we need a suplementary construction.

Notice first that if $f = f_d + \ldots$ is a $(*)$-polynomial, then $\tilde f = f\,(x_1, \ldots, x_{n+1}) + x_{n+2}^d : \mathbb{C}^{n+2} \to \mathbb{C}$ is again a $(*)$-polynomial. We will compare the equivariant Hodge numbers and the polarization properties of f for eigenvalue 1 and of $\tilde f$ corresponding to eigenvalues $\neq 1$. In order to do this, we need a Sebastiani-Thom-type property for the MHS at infinity of $\tilde f = f + x_{n+2}^d$ (at least for eigenvalues ξ with $\xi^d = 1$). This will be established now. Let $X_0'' = \{f_d\,(z_1, \ldots, z_{n+1}) + z_{n+2}^d + z_{n+3}^d = 0\} \subseteq \mathbb{P}^{n+2}$ be the d-fold cyclic covering of $\mathbb{P}^{n+1}$ branched along X_0'. The Galois action on X_0'' is given by multiplication by $e(1/d)$ on the last coordinate z_{n+3}. Then one has

(4.10) THEOREM. — *For $\xi = e(s/d)$, $0 < s < d$ and $k \in \{n, n-1\}$,*

$$h^{p+1,\, q+1}\,(P^{k+2}\,(X_0'')_\xi) = h^{p,q}\,(P^k\,(X^\infty)\,) + \sum_{\substack{0<t<d\\ t+s\neq d}} h^{p+[\frac{s+t}{d}],\, q+1-[\frac{s+t}{d}]}\,(P^{k+1}\,(X_0')_{e(\frac{s+t}{d})}).$$

Proof. — Similarly as in the proof of (4.7), consider a deformation $(f_d)_s$ of f_d, which gives deformations $(f_d)_s + z_{n+2}^d$, $(f_d)_s + z_{n+2}^d + z_{n+3}^d$. We consider the exact sequences (4.5), (4.8) and the analogous one for $(f_d)_s + z_{n+2}^d + z_{n+3}^d$ and we compare dim Gr_F^p of the corresponding terms.

First we concentrate our attention on the second terms in these exact sequences: $P^{n-1}\,(X_s^\infty)$, $P^n\,(X_s')$, $P^{n+1}\,(X_s'')$. Since dim Gr_F^p remains constant when we pass to a limit MHS, in order to compute it we can replace this MHS by the pure Hodge structures given by a fixed s_0 ($s_0 \neq 0$). Set $Q = (f_d)_{s_0}$, $X = \{Q = 0\} \subseteq \mathbb{P}^n$, $X' = \{Q + z_{n+2}^d = 0\} \subseteq \mathbb{P}^{n+1}$, $X'' = \{Q + z_{n+2}^d + z_{n+3}^d\} \subseteq \mathbb{P}^{n+2}$. Notice that there is also an action of $\mathbb{Z}/d\mathbb{Z}$ on X' (resp. X'') given by multiplication of z_{n+2} (resp. z_{n+3}) by $e(1/d)$. Again by Griffiths' construction, $P^{n+1-(q+1),\, q+1}\,(X'')$ is freely generated by the classes $P\,\Omega_{n+2}\,(Q + z_{n+2}^d + z_{n+3}^d)^{-(q+2)}$ with $\deg P + n + 3 = d\,(q+2)$, $P \in \mathbb{F}\,(Q + z_{n+2}^d + z_{n+3}^d)$. Moreover, the ξ-eigenspace is given by the classes corresponding to polynomials of the form $P = z_{n+3}^{s-1}\,\tilde{P}\,(z_1, \ldots, z_{n+2})$. Write $\tilde{P} = \sum_{0<t<d} z_{n+2}^{t-1}\,P_{t-1}(z_1, \ldots, z_{n+1})$. Then $\deg\,P_{t-1} + t - 1 + s - 1 + n + 3 = d\,(q+2)$, and

$$(4.11) \qquad P^{n+1-(q+1),\, q+1}\,(X'')_\xi = \sum_{0<t<d} \mathbb{F}(Q)_{d\,(q+2)-n-1-t-s}.$$

If $t+s < d$ then it follows that $\deg\,P_{t-1} + t + s - 1 + n + 2 = d\,(q+2)$, hence these polynomials P_{t-1} provide rational differentials $z_{n+2}^{t+s-1}\,P_{t-1}\,\Omega_{n+1}\,(Q + z_{n+2}^d)^{-(q+2)}$ generating $P^{n-(q+1),\, q+1}\,(X')_{e(\frac{s+t}{d})}$. If $t + s > d$, then

$$\deg\,P_{t-1} + t + s - 1 - d + n + 2 = d\,(q+1)$$

hence these P_{t-1}'s provide forms $z_{n+2}^{t+s-1-d}\,P_{t-1}\,\Omega_{n+1}(Q + z_{n+2}^d)^{-(q+1)}$ generating $P^{n-q,\, q}\,(X')_{e(\frac{s+t}{d})}$. Finally, if $t + s = d$ then $\deg\,P_{t-1} + n + 1 = d\,(q+1)$ hence $P_{t-1}\,\Omega_n\,Q^{-(q+1)}$ are rational forms on $\mathbb{P}^n$ which generate

$P^{n-1-q,q}(X)$ via Griffith's isomorphism. Therefore

(4.12)
$$p^{p+1,\, q+1}(X'')_{e(s/d)} = p^{p,q}(X) + \sum_{\substack{0<t<d\\ t+s\neq d}} p^{p+[\frac{t+s}{d}],\, q+1-[\frac{t+s}{d}]}(X')_{e(\frac{t+s}{d})}.$$

Passing to the limit we obtain

$$\begin{aligned} \dim\ \mathrm{Gr}_F^{p+1}\ P^{n+1}(X_s'')_\xi &= \dim\ \mathrm{Gr}_F^p\ P^{n-1}(X_s^\infty) \\ &+ \sum_{\substack{0<t<d\\ t+s\neq d}} \dim\ \mathrm{Gr}_F^{p+[\frac{t+s}{d}]}\ P^n(X_s')_{e(\frac{t+s}{d})}. \end{aligned} \tag{4.13}$$

Now we consider the third (the local) terms in the sequences (4.5), (4.8) and the one for $(f_d)_s + z_{n+2}^d + z_{n+3}^d$. There is a local analogue of the above global argument where the rational differential forms are replaced by forms in the Brieskorn lattice.

(4.14) LEMMA. — *Consider an isolated singularity given by a map germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ (with local coordinates $(x_2, \dots, x_{n+1})$), set $g' = g + x_{n+2}^d$, $g'' = g + x_{n+2}^d + x_{n+3}^d$. The Galois group $\mathbb{Z}/d\mathbb{Z}$ acts on the vanishing cohomology corresponding to g' (resp. g''), at the level of coordinates this action is given by multiplication of x_{n+2} (resp. x_{n+3}) by $e(1/d)$. Let $H_\lambda^p(g')_\xi$ be the (λ, ξ)-generalized eigenspace of the p-th vanishing cohomology group of g' where ξ (resp. λ) is an eigenvalue of the Galois action (resp. of the monodromy), and similarly for g''. These actions are compatible with the* MHS *on the vanishing cohomology, let $p_\lambda^{pq}(g')_\xi$ be the dimension of the space of primitive elements of type (p,q). Then for $\xi = e(s/d)$ $(0 < s < d)$ and any λ one has*

$$p_\lambda^{p+1,\, q+1}(g'')_\xi = \sum_{\substack{0<t<d\\ t+s\neq d}} p_\lambda^{p+[\frac{t+s}{d}],\, q+1-[\frac{t+s}{d}]}(g')_{e(\frac{t+s}{d})} + p_\lambda^{pq}(g).$$

Proof. — Since the information provided by the equivariant primitive cohomology Hodge numbers is the same as the one in the set of spectral pairs, we will give the proof at the level of spectral pairs. If (H, T) is a MHS with a semisimple finite action, we refer to [22] (5.3) or [10] for the definition of the set of spectral pairs $\mathrm{Spp}\,(H, T)$ (or see Section 6).

By the Sebastiani-Thom formula proved in [18] (which obviously is compatible with the Galois action), we have

$$\mathrm{Spp}\, H_\lambda^{n+1}(g'')_\xi = \sum_{\beta+\gamma\equiv\alpha (\mathrm{mod}\ \mathbb{Z})} S_\beta(g) * \mathrm{Spp}\, H'_{e(-\gamma)}\,(x_{n+2}^d + x_{n+3}^d)_\xi \tag{4.15}$$

where $\lambda = e(-\alpha)$, $S_\beta(g) = \text{Spp } H^{n-1}_{e(-\beta)}(g)$ and $(\alpha,\omega) * (\alpha',\omega') = (\alpha+\alpha'+1,\ \omega+\omega'+1)$.

Since $h = x^d_{n+2} + x^d_{n+3}$ is homogeneous, its vanishing cohomology is generated (in the Brieskorn lattice) by the forms $\omega = x^{u-1}_{n+2}\ x^{v-1}_{n+3}$ $(h-1)^{[-\frac{u+v}{d}]}\ dx_{n+2} \wedge dx_{n+3}$, where $0 < u,v < d$. The Galois action on this form is $e(1/d) * \omega = e(v/d)\omega$ and the spectral pair provided by ω is $\left(\dfrac{u+v}{d} - 1,\ 1\right)$. Therefore

$$\text{Spp } H^1_{e(-\gamma)}\ (h)_\xi = \left\{\left(\frac{u+s}{d} - 1,\ 1\right)\right\}$$

where $\gamma = \dfrac{u+s}{d} - 1$. So (4.15) reads

$$\text{(4.16)} \qquad \text{Spp } H^{n+1}_\lambda\ (g'')_\xi = \sum S_\beta\,(g) * \left(\frac{t+s}{d} - 1,\ 1\right)$$

where the sum is over $0 < t < d$, $\beta + \dfrac{t+s}{d} - \alpha \in \mathbb{Z}$.

If we denote $T(a,b)$ the transformation on $\mathbb{Z}[\mathbb{Q} \times \mathbb{Z}]$ given by $T\,(a,b)\ (\alpha,\omega) = (\alpha+a,\ \omega+a+b)$, then (again by the Sebastiani–Thom theorem), we have

$$S_\beta(g)*\left(\frac{t+s}{d}-1,1\right) = \begin{cases} T([\frac{t+s}{d}],\, 1-[\frac{t+s}{d}])\,\text{Spp}\,H^n_\lambda(g')_{e(\frac{s+t}{d})} & \text{if } t+s \neq d \\ T\,(1,1)\,\text{Spp}\,H^{n-1}_\lambda(g) & \text{if } t+s = d. \end{cases}$$

This finishes the proof of (4.14).

We return now to the proof of (4.10). Consider the middle maps in the sequences (4.5), (4.8) and the one for $(f_d)_s + z^d_{n+2} + z^d_{n+3}$. Both the sources and the targets of these maps have decompositions compatible with them. Thus their kernels and cokernels also decompose, so for $k = n-1,\ n$ one has

$$\begin{aligned}\text{(4.17)}\ \dim\ Gr^{p+1}_F\ P^{k+2}(X''_0)_\xi = &\sum_{\substack{0<t<d\\ s+t\neq d}} \dim\ Gr^{p+[\frac{t+s}{d}]}_F\ P^{k+1}\,(X'_0)_{e(\frac{s+t}{d})}\\ &+\dim\ Gr^p_F\ P^k(X^\infty).\end{aligned}$$

If $k = n$, then (4.10) follows since the corresponding Hodge structures are pure. If $k = n-1$, then apply (4.7), (4.11), (4.14) and the case $k = n$.

5. Polarization. Connection with the real Seifert form at infinity.

(5.1) First we recall the classification of simple ε-hermitian variation structures. The basic reference is [10], see also [12].

Let U be the complexification of a finite dimensional real vector space, set $U^* = \mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C})$. We denote $\theta : U \longrightarrow U^{**}$ the natural isomorphism $(\theta(u)\ (\varphi) = \varphi(u))$. A bar over an element of U denotes complex conjugation.

If $\varphi \in \mathrm{Hom}_{\mathbb{C}}(U, U')$ then we denote $\bar{\varphi} \in \mathrm{Hom}_{\mathbb{C}}(U, U')$ the map $\bar{\varphi}(x) = \overline{\varphi(\bar{x})}$. The dual map $\varphi^* : U'^* \longrightarrow U^*$ of φ is defined by $\varphi^*(\psi) = \psi \circ \varphi$.

(5.2) DEFINITION ([10, (2.1)]). — *An ε-hermitian variation structure over $\mathbb{C}$ ($\varepsilon = \pm 1$), abbreviated in the sequel as ε-HVS, is a system (U, b, h, V) where U is the complexification of a finite dimensional real vector space and*

(a) $b : U \longrightarrow U^*$ *is a $\mathbb{C}$-linear morphism with* $\overline{b^* \circ \theta} = \varepsilon\, b$

(b) h *is a b-orthogonal automorphism of U, i.e.* $\overline{h^*} \circ b \circ h = b$

(c) $V : U^* \longrightarrow U$ *is a $\mathbb{C}$-linear morphism with* $\overline{\theta^{-1} \circ V^*} = -\varepsilon\, V$ *and* $V \circ b = h - \mathrm{Id}$.

If V is an isomorphism, then (U, b, h, V) is called simple and in this case $h = -\varepsilon\, V\, \overline{(\theta^{-1} \circ V^*)}^{-1}$ and $b = -V^{-1} - \varepsilon\, \overline{(\theta^{-1} \circ V^*)}^{-1}$.

(5.3) *Example.* — Let $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be a polynomial map which admits a Milnor fibration at infinity (e.g. f is a $(*)$-polynomial, cf. (2.1 b)). Let F be the fiber of the Milnor fibration, set $U_{\mathbb{R}} = H_n(F, \mathbb{R})$, $U_{\mathbb{R}}^* = H^n(F, \mathbb{R})$ (which is identified with $H_n(F, \partial F, \mathbb{R})$ via the perfect pairing $H_n(F, \partial F, \mathbb{R}) \otimes H^n(F, \mathbb{R}) \longrightarrow \mathbb{R}$). Let $h_{\mathbb{R}} : U_{\mathbb{R}} \longrightarrow U_{\mathbb{R}}$ be the real (classical) algebraic monodromy, $b_{\mathbb{R}} : U_{\mathbb{R}} \longrightarrow U_{\mathbb{R}}^*$ the intersection form and $\mathrm{Var} : U_{\mathbb{R}}^* \longrightarrow U_{\mathbb{R}}$ the variation map. Then Var is an isomorphism and $(U_{\mathbb{R}}, b_{\mathbb{R}}, h_{\mathbb{R}}, \mathrm{Var}) \otimes_{\mathbb{R}} \mathbb{C}$ is a $(-1)^n$-HVS which will be denoted $\mathcal{V}(f)$. Notice, that similarly as in the local case, Var is the inverse of the (real) Seifert form of the Milnor fibration at infinity.

If f is a $(*)$-polynomial, then the Milnor fibration is equivalent to the fibration of f at infinity, hence F is diffeomorphic to the generic fiber of f and $(h_{\mathbb{R}}^*)^{-1}$ is exactly T_f^{∞}.

(5.4) In the sequel, if (U, b, h, V) is a ε-HVS we will assume that the eigenvalues λ of h satisfy $|\lambda| = 1$ (for the general case, see [10]). In the next examples, $\mathcal{J}_k$ denotes the $k \times k$ Jordan block with eigenvalue 1.

(a) If b is an isomorphism, then $V = (h - Id)\, b^{-1}$ is determined from h, b. Up to isomorphism, there are exactly two non-degenerate $k \times k$ matrices b such that $\bar{b}^* = \varepsilon\, b$ and $\mathcal{J}_k^*\ b\ \mathcal{J}_k = b$. They are determined by the sign of the entry $b_{1,k}$ and will be denoted $b_\pm^k$. If we set $\varepsilon = (-1)^n$, we can take $(b_\pm^k)_{1,k} = \pm i^{-n^2-k+1}$. In particular, for $\lambda \neq 1$ ($|\lambda| = 1$) there are exactly two simple $(-1)^n$ – HVS's:

$$W_\lambda^k(\pm 1) = (\mathbb{C}^k,\ b_\pm^k, \lambda\, \mathcal{J}_k,\ (\lambda\, \mathcal{J}_k - \mathrm{Id})\ (b_\pm^k)^{-1}).$$

(b) If $\lambda = 1$ and $k = 1$ there are two simple $(-1)^n$-HVS's

$$W_1^1\ (\pm 1) = (\mathbb{C}, 0, \mathrm{Id}_{\mathbb{C}}, \pm i^{n^2+1}).$$

(c) If $\lambda = 1$ and $k \geqslant 2$, then $h = \mathcal{J}_k$ and b is not an isomorphism.

Again, there are two simple $(-1)^n$-HVS's

$$W_1^k(\pm 1) = (\mathbb{C}^k, \tilde{b}_\pm^k, \mathcal{J}_k, \tilde{V}_\pm^k)$$

and they are determined by (the sign of) $(\tilde{b}_\pm^k)_{k,2} = \pm i^{-n^2-k+2}$. This structure can be recognized also from ($(\tilde{V}_\pm^k)^{-1})_{1,k} = \pm i^{-n^2-k}$.

(d) $-(U, b, h, V)$ denotes the structure $(U, -b, h, -V)$ (with the same ε).

The sign conventions are motivated by Hodge-theoretical reasons, they correspond to the sign conventions of the polarisations of Hodge structures. The structure theorem of ε-HVS is the following

(5.5) THEOREM ([10, (2.9)]). — *Any simple ε-HVS decomposes as a sum of indecomposable ones, the decomposition is unique up to isomorphism and order of summands. If the eigenvalues λ of h satisfy $|\lambda| = 1$, then the indecomposable structures are $W_\lambda^k(\pm 1)$ where $k \geqslant 1$.*

(5.6) *Remark.* — Two real variation structures are isomorphic over $\mathbb{C}$ if and only if they are isomorphic as real structures (cf. [10, (2.10)]. This implies that if f is a polynomial which admits a Milnor fibration at infinity the information contained in $\mathcal{V}(f)$ (see (5.3) above) is the same as the one contained in the real Seifert form of the Milnor fibration at infinity.

If f is a $(*)$-polynomial, the connection between $\mathcal{V}(f)$ and the Hodge theoretical invariants introduced in sections 1-4 is the following

(5.7) THEOREM *Let f be a $(*)$-polynomial. Then*

$$(-1)^n \cdot \mathcal{V}(f) = \bigoplus_{\lambda} \bigoplus_{2n \geqslant a+b \geqslant n+s} p_\lambda^{b,a}(f) \cdot \mathcal{W}_\lambda^{r+1}((-1)^b)$$

where $s = 0$ if $\lambda \neq 1$, $s = 1$ if $\lambda = 1$ and $r = a+b-n-s \geqslant 0$. In particular, the equivariant Hodge numbers of $H^n(X_t^0)$ determine completely the real Seifert form of the Milnor fibration at infinity.

Proof. — The proof follows closely that of Theorem (6.1) in [Ne1] (which gives the same relation in the case of hypersurface singularities. In [*loc.cit.*] there is a slightly different definition of the structures $\mathcal{W}_\lambda^k$, and in the present theorem we made a sign-correction.) Notice that $\mathcal{V}(f)$ has a direct sum decomposition $\mathcal{V}(f) = \oplus_\lambda \mathcal{V}_\lambda(f)$ given by the eigenvalues of the monodromy h. So, it will be enough to show that one has the identity

$$(-1)^n \cdot \mathcal{V}_\lambda(f) = \bigoplus_{a,b} p_\lambda^{b,a}(f)\ \mathcal{W}_\lambda^{r+1}((-1)^b). \tag{5.8}$$

In order to see this, first notice that one can define a cohomological variation structure $\mathcal{V}_{\mathrm{coh}}(f)$ of f by $U_{\mathbb{R}} = H_c^n(F, \mathbb{R})$, $U_{\mathbb{R}}^* = H^n(F, \mathbb{R})$ (via the perfect pairing $H_c^n(F, \mathbb{R}) \otimes H^n(F, \mathbb{R}) \longrightarrow \mathbb{R}$); $h_{\mathbb{R}} = T_{\mathrm{geom}}^*$ and $V([\omega]) = [Te_{\mathrm{geom}}^*(\omega) - \omega]$. It turns out that $(-1)^n \cdot \mathcal{V}(f) = \mathcal{V}_{\mathrm{coh}}(f)$.

Assume first that $\lambda \neq 1$. Then by (E.0) we have $H^n(X_t^0)_\lambda \simeq P^n(X_t)_\lambda$ as MHS, therefore $\mathcal{V}_{\mathrm{coh}}(f)_\lambda$ can be determined from the polarization properties of the limit MHS of Schmid. More precisely, if $N_\lambda^0(T_f^\infty)$ is the infinitesimal nilpotent monodromy operator associated with T_f^∞, then the polarization given by $N_\lambda^0(T_f^\infty)$ provides

$$\mathcal{V}_{\mathrm{coh}}(f)_\lambda = \bigoplus_{a,b} p_\lambda^{a,b}(T_f^\infty)\ \mathcal{W}_\lambda^{r+1}((-1)^b),$$

where $p_\lambda^{a,b}(T_f^\infty)$ are the equivariant primitive Hodge numbers determined by the classical monodromy T_f^∞, hence they are equal to $p_\lambda^{a,b}(f) = p_\lambda^{b,a}(f)$.

Suppose now that $\lambda = 1$. Consider the polynomial $f + x_{n+2}^d : \mathbb{C}^{n+2} \longrightarrow \mathbb{C}$. As remarked before, it is easy to see that $f + x_{n+2}^d$ is again a $(*)$-polynomial. We want to compare $\mathcal{V}(f + x_{n+2}^d)$ with $\mathcal{V}(f)$. The ε-HVS of the polynomial map $x_{n+2} \longmapsto x_{n+2}^d$ is $\bigoplus_{\substack{\eta^d=1 \\ \eta \neq 1}} \mathcal{W}_\eta^1(+1)$ (with $\varepsilon = (-1)^0 = +1$).

By [11] (which solves the Sebastiani–Thom problem for Seifert form at infinity associated with "good" polynomials) one has

$$\begin{aligned}\mathcal{V}_\xi(f+x_{n+2}^d) &= (-1)^{n+1}\Big[\mathcal{V}(f)\otimes\Big(\bigoplus_{\substack{\eta^d=1\\ \eta\neq 1}} W_\eta^1(+1)\Big)\Big]_\xi \\ &= (-1)^{n+1}\mathcal{V}_1(f)\otimes W_\xi^1(+1) + (-1)^{n+1}\bigoplus_{\substack{\eta\neq 1\\ \eta\neq\xi}} \mathcal{V}_{\eta^{-1}\xi}(f)\otimes W_\eta^1(+1).\end{aligned} \tag{5.9}$$

One can verify that if $\eta = e(\alpha)$ and $\xi = e(\beta)$ with $\alpha, \beta \in (0,1)$ and $\alpha+\beta\neq 1$, then

$$W_\eta^k(\pm 1)\otimes W_\xi^1(+1) = (-1)^n\cdot W_{\eta\xi}^k(\pm\kappa(\eta,\xi)),$$

where $\kappa(\eta,\xi) = +1$ if $\alpha+\beta<1$ and $\kappa(\eta,\xi)=-1$ if $\alpha+\beta>1$. One has as well

$$W_1^k(\pm 1)\otimes W_\xi^1(+1) = (-1)^n\cdot W_\xi^k(\pm 1).$$

From these formulas and (5.9), it follows that we can determine $\mathcal{V}_1(f)$ from $\mathcal{V}_{\lambda\neq 1}(f)$ and $\mathcal{V}_{\lambda\neq 1}(f+x_{n+2}^d)$. But, as showed above, these last ε-HVS can be computed in terms of equivariant Hodge numbers of f and $f+x_{n+2}^d$. These equivariant Hodge numbers are related to those of f for eigenvalue 1 (using (3.1) and (4.10), the latter applied to $k=n-1$), and then formula (5.8) for $\lambda=1$ follows, the details of the computation are left to the reader. This ends the proof.

(5.10) The formula given in this theorem is the exact analogue of the corresponding formula for isolated hypersurface singularities (cf. [10]). Thus, roughly speaking, any connection between Hodge numbers and topological invariants which holds for isolated hypersurface singularities holds also for $(*)$-polynomials if we replace the MHS of the local vanishing cohomology by the MHS at infinity, the Milnor fiber by the generic fiber and the local monodromy by the monodromy at infinity. We resume and illustrate this in the next theorem.

(5.11) THEOREM. — *Let $f:\mathbb{C}^{n+1}\longrightarrow\mathbb{C}$ be a $(*)$-polynomial.*

a) *$H^n(X_f^0)_{\neq 1}$ and $H_c^n(X_f^0)_{\neq 1}$ carry MHS's which are dual with respect to $\mathbb{Q}(-n)$, the weight filtration of $H^n(X_f^0)_{\neq 1}$ is the monodromy weight filtration of $N_{\neq 1}^0$ centered at n. This space has a natural $(-1)^n$-symetric polarization form (the Poincaré dual of the intersection form $\langle\,\cdot\,,\, j\,\cdot\,\rangle$) and it is polarized by the infinitesimal monodromy $-N_{\neq 1}^0$ (in the sense of Cattani-Kaplan).*

b) $H^n(X_f^0)_1$ *and* $H_c^n(X_f^0)_1$ *carry MHS's which are dual with respect to* $\mathbb{Q}(-n)$. *The weight filtration of* $H^n(X_f^0)_1$ *is the monodromy weight filtration of* N_1^0 *centered at* $n+1$. *The infinitesimal variation map ([23] (2.5)* $V_1 : H^n(X_f^0)_1 \longrightarrow H_c^n(X_f^0)_1(-1)$ *is an isomorphism of MHS's of type* $(-1,-1)$.$H^n(X_f^0)_1$ *has a natural* $(-1)^{n+1}$*-symmetric polarization form given by* $\langle \cdot, V_1 \cdot \rangle$ *and it is polarized by the monodromy* $-N_1^0$.

Proof. — Case a) follows from (E.0) and Schmid's results [19], which assures that the MHS is polarized by $\log(T_f^\infty)_u = -N^0$. For b), use the polarization properties of $W_1^{r+1}(\pm 1)$ which behaves as in the local case. For the local case see [23] and [21] or [13].

(5.12) Corollary. — *The equivariant signature* $\sigma_\lambda(f)$ *of the generic fiber of* f, *with respect to the monodromy at infinity, is given by*

$$\sigma_\lambda(f) = \sum_{2n \geqslant a+b \geqslant n+s} (-1)^b p_\lambda^{a,b}(f) \cdot \left(1 + (-1)^{a+b-n}\right)/2,$$

where $s = 0$ *if* $\lambda \neq 1$, *and* $s = 1$ *if* $\lambda = 1$.

The proof is analogous to that of [10, (6.6)], use the formula

$$\sigma(W_\lambda^k(\pm 1)) = \pm \frac{1 + (-1)^{k+1+s}}{2}.$$

6. The spectral pairs of a $(*)$-polynomial.

As in the local case (cf. [22], [18], see also [1, 13.3.A] and [10]) the equivariant Hodge numbers $h_\lambda^{p,q}$ of the MHS at infinity of a polynomial f can be codified in the set of spectral pairs $\mathrm{Spp}(f) \in \mathbb{Z}[\mathbb{Q} \times \mathbb{N}]$ defined by

$$\mathrm{Spp}(f) = \sum_{(\alpha,\omega)} h_{e(-\alpha)}^{n+[-\alpha],\, \omega + s_\alpha - n - [-\alpha]} (\alpha, \omega)$$

where $s_\alpha = 0$ if $\alpha \notin \mathbb{Z}$ and $s_\alpha = 1$ if $\alpha \in \mathbb{Z}$. The spectrum of f is defined by

$$\mathrm{Sp}(f) = \sum (\alpha) \in \mathbb{Z}[\mathbb{Q}]$$

where the sum is over the spectral pairs (α, ω).

In the local case the spectrum has three important properties: symmetry, Sebastiani–Thom property, and semicontinuity. In the sequel, we will discuss these properties briefly in our global situation (i.e. for (*)-polynomials).

The symmetry of the weight filtration gives $h_\lambda^{pq} = h_\lambda^{n+s-q,n+s-p}$, hence the invariance of Spp(f) with respect to the transformation $(\alpha, n+k) \leftrightarrow (\alpha+k,\ n-k)$. The complex conjugation (i.e. $h_\lambda^{pq} = h_{\bar\lambda}^{qp}$) composed with the first symmetry gives the second one: $(\alpha,\ n+k) \leftrightarrow (n-1-\alpha,\ n-k)$. This shows also that the spectrum is symmetric with respect to $(n-1)/2$ (for details in the local situation, see e.g. [1, 13.3.C]).

Moreover, we have also a (global) Sebastiani-Thom theorem: if (α,ω), $(\alpha',\omega') \in \mathbb{Q}\times\mathbb{N}$, define $(\alpha,\omega) * (\alpha',\omega') = (\alpha+\alpha'+1,\ \omega+\omega'+1)$, and extend $*$ to $\mathbb{Z}[\mathbb{Q}\times\mathbb{N}]$ by linearity. Set

$$S_d = \sum_{0<s<d} (-s/d,\ 0) \in \mathbb{Z}[\mathbb{Q}\times\mathbb{N}].$$

(6.1) THEOREM. — *If f is a (*)-polynomial of degree d, then*

$$\mathrm{Spp}(f+x^d) = \mathrm{Spp}(f) * S_d.$$

Proof. — For the spectral pairs of $f+x^d$ corresponding to eigenvalue 1 the result is easy, it uses (3.1.a) and the fact that

$$P^n(X_0') = \oplus_{\xi^d=1,\ \xi\neq 1}\ P^n(X_0')_\xi.$$

For eigenvalues $\xi \neq 1$ with $\xi^d = 1$ the result follows from (3.1.b) and (4.10) (cf. the proof of (5.7)). For $\xi^d \neq 1$ one uses (3.1.b) and the local Sebastiani-Thom theorem ([18]). In this later case the computation is rather long and tedious. The details are left to the reader. (For a different proof, in the context of "cohomologically tame" polynomials, see [14].)

The main application of (6.1) is a semicontinuity property of Spp(f). If f is a (*)-polynomial, we will consider deformations $f_\lambda : \mathbb{C}^{n+1} \to \mathbb{C}$ of f with $\lambda \in (\mathbb{C},0)$ and fixed degree d. Notice that since the (*) condition defines an open dense subset in the space of polynomials in $n+1$ variables of degree d, f_λ is also a (*)-polynomial (Actually, because of (3.2.d) it would be enough to consider deformations $f_\lambda = (f_d)_\lambda + f_{d-1} + \ldots$, where $(f_d)_\lambda$ is a deformation of f_d). Since in such a deformation one has $\mu^\infty(f_{\lambda\neq 0}) \geqslant \mu^\infty(f)$ (where $\mu^\infty(f)$ denotes the dimension of the middle cohomology group of the generic fibre of f), one cannot expect an upper semicontinuity for the spectrum as in the local case, but a lower semicontinuity.

(6.2) DEFINITION (cf. [25]). — *Given a deformation $(f)_\lambda$ of a $(*)$-polynomial f and a subset $A \subseteq \mathbb{R}$, we will say that A is a lower semicontinuity domain for $(f)_\lambda$ if the function which associates to $\lambda \in (\mathbb{C},0)$ the sum of the frequencies of spectral numbers of f_λ in A (denoted by $s_A(f_\lambda)$) is lower semicontinuous, i.e.,*

$$s_A(f_{\lambda\neq 0}) \geqslant s_A(f).$$

(6.3) THEOREM. — *Let f be a $(*)$-polynomial of degree d. Every half open interval $\left(\frac{k}{d}, \frac{k}{d}+1\right]$, $k \in \mathbb{Z}$, is a lower semicontinuity domain for deformations of f with fixed degree.*

Proof. — By (E.0) and (4.5), there is a $C \in \mathbb{N}$ depending only on n, d, p such that $\dim \mathrm{Gr}_F^p H^n(X_t^0) = C - \dim \mathrm{Gr}_F^p \oplus_j H^{n-1}(F_j)$. By the local upper-semicontinuity result ([25]) the theorem follows for the interval $(t, t+1]$, $t \in \mathbb{Z}$. This, together with (6.1), gives the result similarly as in the local case (see [loc.cit, (2.7)]).

(6.4) *Remarks.* — a) Actually, by a tedious combinatorial argument, using the corresponding local result and (3.1), one can show that any interval $(t, t+1]$, $t \in \mathbb{R}$ is a lower semicontinuity domain. We do not emphasize this proof because a more conceptual proof is in preparation, which works for any cohomologically tame polynomial ([14]).

b) The symmetry shows that $[t, t+1)$ is also a semicontinuity domain.

c) As in the local case, (5.12) has the following corollary: Let $\mathrm{Spp}\,(f) \in \mathbb{Z}[\mathbb{Q}\times\mathbb{N}]$ be the set of spectral pairs of a $(*)$-polynomial. Let $\mathrm{Spp}_{\mathrm{mod}-2}(f)$ be its projection in $\mathbb{Z}\,[\mathbb{Q}/2\,\mathbb{Z}\times\mathbb{N}]$. The knowledge of $\mathrm{Spp}_{\mathrm{mod}-2}(f)$ is equivalent to that of the real Seifert form of the Milnor fibration of f at infinity (cf. [10]).

7. Examples.

(7.1) *Case* $n = 1$.

Let $f \in \mathbb{C}[X,Y]$ be a $(*)$-polynomial with highest degree form f_d. Write

$$f_d = \prod_{j=1}^{m} l_j^{\alpha_j}$$

where the $l_j \in \mathbb{C}[X,Y]$ are distinct linear forms and $\sum \alpha_j = d$. Set $\alpha = \gcd(\alpha_1, \ldots, \alpha_m)$. For $x \in \mathbb{R}$, set $\delta(x) = 1$ if $x \in \mathbb{Z}$, $\delta(x) = 0$ if $x \notin \mathbb{Z}$. Denote by $\lceil x \rceil$ the smallest integer bigger or equal than x.

The equivariant Hodge numbers of $P^2(X'_0)$ can be computed directly, if $\xi = e(s/d)$ with $0 < s < d$ then $h^{1,1}(P^2(X'_0)_\xi) = \delta\Big(\dfrac{s\alpha}{d}\Big)$, the other numbers are zero. Then from theorem (4.7) (using also [1, p. 305, Théorème 6] or [21] for the local terms appearing in the formulas in (4.7)) one gets all equivariant Hodge numbers of $P^1(X'_0)$. Now one computes the equivariant Hodge numbers of the MHS at infinity using (3.1). One gets:

a) $p_1^{1,1}(f) = m - 1$.

b) If $\xi = e(-s/d)$, $0 < s < d$, then

$$p_\xi^{11}(f) = -\delta\Big(\frac{s\alpha}{d}\Big) + \sum_j \delta\Big(\frac{s\alpha_j}{d}\Big).$$

$$p_\xi^{0,1}(f) = -s - 1 + \delta\Big(\frac{\alpha s}{d}\Big) + \sum_j \Big\lceil \frac{s\alpha_j}{d} \Big\rceil.$$

$$p_\xi^{1,0}(f) = s - 1 + \delta\Big(\frac{\alpha s}{d}\Big) - \sum_j \Big[\frac{s\alpha_j}{d} \Big].$$

c) If $\xi^d \neq 1$, put $\xi = e(-\beta)$ with $0 < \beta < 1$ and set $\gamma = \{(d-1)\beta\}$. Notice that $\beta + \gamma \notin \mathbb{Z}$.

If $\xi^{d-1} = 1$, then $p_\xi^{*,*}(f) = 0$.

If $\xi^{d-1} \neq 1$ and $\beta + \gamma < 1$, then $p_\xi^{10}(f) = 0$ and $p_\xi^{0,1}(f) = \#\{j \mid \xi^{(d-1)\alpha_j} = 1\}$.

If $\xi^{d-1} \neq 1$ and $\beta + \gamma > 1$, then $p_\xi^{01}(f) = 0$ and $p_\xi^{1,0}(f) = \#\{j \mid \xi^{(d-1)\alpha_j} = 1\}$.

(All other primitive equivariant Hodge numbers vanish.)

Example.— If $f = x^2y^2 + (x+y)^3$, then $p_1^{11} = 1$, $p_{-1}^{11} = 1$ hence $h_{-1}^{00} = 1$ too, $p_{e(-1/6)}^{01} = p_{e(-5/6)}^{10} = 2$. In particular, the rank of the midle (co)homology of the generic fiber is 7, and the monodromy at infinity has a Jordan block of size two.

(7.2) *Zariski's examples.*

Let $f_6 \in \mathbb{C}[X,Y,Z]$ be a homogeneous polynomial of degree 6 which defines a sextic in $\mathbb{P}^2$ with six cusps and no other singularities. Let $f = f_6 + \ldots$ be any $(*)$- polynomial with highest degree form f_6.

The Hodge numbers of $X^\infty = \{f_6 = 0\} \subset \mathbb{P}^2$ are easy to compute. Those of X_0', the 6-fold cyclic covering of $\mathbb{P}^2$ branched along X^∞, depend on the position of the cusps. The equivariant Hodge numbers of $P^3(X_0')$ can be computed with the aid of [3, ch. 6, (4.9), see also (3.18.ii)] One has two cases:

Case i) If the six cusps are on a conic then

$$h^{2,1}_{e(1/6)}(P^3(X_0')) = 1 = h^{1,2}_{e(5/6)}(P^3(X_0')).$$

Case ii) If the six cusps are not on a conic then $P^3(X_0') = 0$. Now from Theorem (4.7) (and using again the results in [21] or [1, p. 305, Théorème 6] to compute the local terms appearing in (4.7)), one gets the equivariant Hodge numbers of $P^2(X_0')$. Finally, Theorem (3.1) gives the primitive equivariant Hodge numbers of the MHS at infinity of f. One has:

a) Both in cases (i) and (ii),

$$p_1^{1,2}(f) = p_1^{2,1}(f) = 4.$$

b) If $\xi = e(s/6)$ with $0 < s < 6$ the numbers $p^{p,q}(f)_\xi$ are:

(p,q)	$(1,1)$	$(2,0)$	$(0,2)$	$(1,2)$	$(2,1)$
$s=5$	4 (i) 3 (ii)	0	1 (i) 0 (ii)	5 (i) 6 (ii)	0
$s=4$	6	0	3	0	0
$s=3$	7	1	1	0	0
$s=2$	6	3	0	0	0
$s=1$	4 (i) 3 (ii)	1 (i) 0 (ii)	0	0	5 (i) 6 (ii)

c) Both in cases (i) and (ii), if $\xi = e(l/30)$ with $l \in \{1, 7, 11, 13, 17, 19, 23, 29\}$ then $p_\xi^{1,1}(f) = 6$.

All other primitive equivariant Hodge numbers vanish.

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