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ALGEBRAIC EQUIVALENCE OF REAL ALGEBRAIC CYCLES

by M. ABÁNADES & W. KUCHARZ

1. Introduction.

Let X be a nonsingular, n-dimensional, quasiprojective variety over $\mathbb{R}$ (that is, an irreducible, n-dimensional, quasiprojective scheme over $\mathbb{R}$, smooth over $\mathbb{R}$). We endow the set $X(\mathbb{R})$ of $\mathbb{R}$-rational points of X with the topology induced by the usual metric topology on $\mathbb{R}$, and assume that $X(\mathbb{R})$ is nonempty and compact. Thus $X(\mathbb{R})$ is a C^∞, closed, n-dimensional manifold. Given a nonnegative integer k, we let $Z^k(X)$ denote the group of algebraic $(n-k)$-cycles on X (that is, the free Abelian group on the set of closed, $(n-k)$-dimensional subvarieties of X). There exists a unique group homomorphism

$$\mathrm{cl}_{\mathbb{R}} : Z^k(X) \to H^k(X(\mathbb{R}), \mathbb{Z}/2)$$

such that for every closed, $(n-k)$-dimensional subvariety V of X, the cohomology class $\mathrm{cl}_{\mathbb{R}}(V)$ is Poincaré dual to the homology class in $H_{n-k}(X(\mathbb{R}), \mathbb{Z}/2)$ determined by $V(\mathbb{R})$ (*cf.* [5] for the definition of this homology class). In the present paper we study the cohomology classes of the form $\mathrm{cl}_{\mathbb{R}}(z)$, where z is a cycle in $Z^k(X)$ algebraically equivalent to 0 (we refer to [7] for the theory of algebraic equivalence of cycles). Such cohomology classes need not be trivial, but as we shall see below they must satisfy quite restrictive conditions.

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The extreme cases, $k = 0$ and $k = n$, are easy to analyze. Obviously, a cycle z in $Z^0(X)$ is algebraically equivalent to 0 if and only if $z = 0$. On the other hand, every cycle in $Z^n(X)$ of the form $x_0 - x_1$, where x_0 and x_1 are points in $X(\mathbb{R})$, is algebraically equivalent to 0. We have $\text{cl}_{\mathbb{R}}(x_0 - x_1) \neq 0$ whenever x_0 and x_1 belong to distinct connected components of $X(\mathbb{R})$. It follows that a cohomology class u in $H^n(X(\mathbb{R}), \mathbb{Z}/2)$ can be written as $u = \text{cl}_{\mathbb{R}}(z)$ for some cycle z in $Z^n(X)$ algebraically equivalent to 0 if and only if the homology class in $H_0(X(\mathbb{R}), \mathbb{Z}/2)$ Poincaré dual to u is represented by an even number of points of $X(\mathbb{R})$. In view of these facts, we concentrate our attention on the intermediate cases, $1 \leq k \leq n-1$.

Given a continuous map $f : M \to N$ between topological spaces, we denote by $H^k(f) : H^k(N, \mathbb{Z}/2) \to H^k(M, \mathbb{Z}/2)$ the homomorphism induced by f. Recall that a cohomology class u in $H^k(M, \mathbb{Z}/2)$ with $k \geq 1$ is said to be *spherical* if $u = H^k(f)(c)$, where $f : M \to S^k$ is a continuous map into the unit k-sphere S^k, and c is the generator of $H^k(S^k, \mathbb{Z}/2) \cong \mathbb{Z}/2$. We denote, as usual, by $\cup$ and $< -, - >$ the cup product of cohomology classes and the Kronecker index (pairing) of cohomology and homology classes, *cf.* [11]. If M is a C^∞, closed manifold of dimension n, we denote by $w_k(M)$ the kth Stiefel-Whitney class of M and by μ_M the fundamental homology class of M in $H_n(M, \mathbb{Z}/2)$.

THEOREM 1.1. — *Let X be a nonsingular, n-dimensional, quasiprojective variety over $\mathbb{R}$ with $X(\mathbb{R})$ nonempty and compact. Let z be a cycle in $Z^k(X)$ that is algebraically equivalent to 0. Then the cohomology class $\text{cl}_{\mathbb{R}}(z)$ in $H^k(X(\mathbb{R}), \mathbb{Z}/2)$ satisfies $\text{cl}_{\mathbb{R}}(z) \cup \text{cl}_{\mathbb{R}}(z) = 0$ in $H^{2k}(X(\mathbb{R}), \mathbb{Z}/2)$ and*

$$< \text{cl}_{\mathbb{R}}(z) \cup w_{i_1}(X(\mathbb{R})) \cup \ldots \cup w_{i_r}(X(\mathbb{R})),\ \mu_{X(\mathbb{R})} > = 0$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = n-k$. Furthermore, if $k = 1$ or if $k = n-1 \geq 1$ with $X(\mathbb{R})$ connected, then the cohomology class $\text{cl}_{\mathbb{R}}(z)$ is spherical.

Let us note that, in general, the cohomology class $\text{cl}_{\mathbb{R}}(z)$ of Theorem 1.1 need not be spherical. Indeed, suppose $X = X' \times X''$ (product over Spec$\mathbb{R}$), where X' and X'' are nonsingular, projective varieties over $\mathbb{R}$ such that $X'(\mathbb{R})$ is nonempty and $X''(\mathbb{R})$ is disconnected. Let z' be any algebraic cycle on X'. Choose two points p_0 and p_1 in $X''(\mathbb{R})$ that belong to distinct connected components. Since the 0-cycle $z'' = p_0 - p_1$ on X'' is algebraically equivalent to 0, the cycle $z' \times z''$ on X is algebraically equivalent to 0 as well. Furthermore, the cohomology class $\text{cl}_{\mathbb{R}}(z' \times z'') = \text{cl}_{\mathbb{R}}(z') \times \text{cl}_{\mathbb{R}}(z'')$ is spherical if and only if the cohomology class $\mathrm{cl}_{\mathbb{R}}(z')$ is spherical (for p_0 and p_1 belong to distinct connected components of $X''(\mathbb{R})$). Taking $X' = \mathbb{P}^m_{\mathbb{R}}$, we have $\mathrm{cl}_{\mathbb{R}}(Z^k(X')) = H^k(X'(\mathbb{R}), \mathbb{Z}/2)$, and the unique nontrivial cohomology class in $H^k(X'(\mathbb{R}), \mathbb{Z}/2) \cong \mathbb{Z}/2$ is not spherical, provided that $1 \leq k \leq m-1$ and m is even. In particular, "connected" cannot be omitted in the last part of Theorem 1.1.

If $\mathrm{cl}_{\mathbb{R}}(z)$ is spherical, then $\mathrm{cl}_{\mathbb{R}}(z) \cup \mathrm{cl}_{\mathbb{R}}(z) = 0$ is automatically satisfied, and Theorem 1.1 is in some sense the best possible result. More precisely, we have the following.

THEOREM 1.2. — *Let M be a C^∞, closed, n-dimensional manifold and let u be a spherical cohomology class in $H^k(M, \mathbb{Z}/2)$ with $1 \leq k \leq n-1$. Then the following conditions are equivalent :*

(a) *There exist a nonsingular, projective algebraic variety X over $\mathbb{R}$ and a C^∞ diffeomorphism $\varphi : X(\mathbb{R}) \to M$ such that $H^k(\varphi)(u) = \mathrm{cl}_{\mathbb{R}}(z)$ for some cycle z in $Z^k(X)$ algebraically equivalent to 0;*

(b) $< u \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M),\ \mu_M >= 0$ *for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = n - k$.*

Let us mention that Theorem 1.2 is an improvement upon inefficient [10], Theorem 2.4.

2. Proofs.

Let X be a nonsingular, n-dimensional, quasiprojective algebraic variety over $\mathbb{R}$ with $X(\mathbb{R})$ nonempty and compact. Recall that if an algebraic cycle z in $Z^k(X)$ is rationally equivalent to 0, then $\mathrm{cl}_{\mathbb{R}}(z) = 0$ (*cf.* [5], 5.13) and hence $\mathrm{cl}_{\mathbb{R}}$ induces a homomorphism, also denoted by $\mathrm{cl}_{\mathbb{R}}$, from the Chow group $A^k(X)$ of X into $H^k(X(\mathbb{R}), \mathbb{Z}/2)$. It is known that $\mathrm{cl}_{\mathbb{R}} : A^*(X) \to H^*(X(\mathbb{R}), \mathbb{Z}/2)$ is a homomorphism of graded rings [5], p. 495. Thus

$$H^*_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = \mathrm{cl}_{\mathbb{R}}(Z^*(X)) = \mathrm{cl}_{\mathbb{R}}(A^*(X))$$

is a graded subring of $H^*(X(\mathbb{R}), \mathbb{Z}/2)$. We shall need the following result [10], Theorem 2.1 :

$$(1) \qquad < \mathrm{cl}_{\mathbb{R}}(z) \cup v,\ \mu_{X(\mathbb{R})} >= 0$$

for all cycles z in $Z^k(X)$ algebraically equivalent to 0 and all v in $H^{n-k}_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$.

Assume now that X is projective. Then the set $X(\mathbb{C})$ of $\mathbb{C}$-rational points of X is a compact complex manifold of complex dimension n. There exists a unique group homomorphism

$$\mathrm{cl}_{\mathbb{C}} : Z^k(X) \to H^{2k}(X(\mathbb{C}), \mathbb{Z})$$

such that for every closed, $(n - k)$-dimensional subvariety V of X, the cohomology class $\mathrm{cl}_{\mathbb{C}}(V)$ is Poincaré dual to the homology class in $H_{2n-2k}(X(\mathbb{C}), \mathbb{Z})$ determined by $V(\mathbb{C})$ (*cf.* [5] for the definition of this homology class). In other words, if $\pi : X_{\mathbb{C}} = X \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C} \to X$ is the canonical projection, then $\mathrm{cl}_{\mathbb{C}}(z)$ is the cohomology class corresponding to the pullback algebraic cycle $\pi^*(z)$ on $X_{\mathbb{C}}$, *cf.* [5], 4.2 or [7], Chapter 19. In particular,

$$\mathrm{cl}_{\mathbb{C}}(z) = 0 \tag{2}$$

for all cycles z in $Z^k(X)$ algebraically equivalent to 0, *cf.* [5], 4.14 or [7], Proposition 19.1.1. Furthermore, it follows from the proof of [2], Theorem A that

$$\mathrm{cl}_{\mathbb{R}}(z) \cup \mathrm{cl}_{\mathbb{R}}(z) = \text{ the reduction modulo 2 of } r(\mathrm{cl}_{\mathbb{C}}(z)) \tag{3}$$

for all z in $Z^k(X)$, where $r : H^{2k}(X(\mathbb{C}), \mathbb{Z}) \to H^{2k}(X(\mathbb{R}), \mathbb{Z})$ is the homomorphism induced by the inclusion map $X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$.

Proof of Theorem 1.1. — By Hironaka's resolution of singularities theorem [8], 3, we may assume that X is projective.

We obtain $\mathrm{cl}_{\mathbb{R}}(z) \cup \mathrm{cl}_{\mathbb{R}}(z) = 0$ directly from (2) and (3).

It follows from [5], p. 498 that $w_i(X(\mathbb{R}))$ is in $H^i_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$, and hence if $i_1, \ldots, i_r$ are nonnegative integers with $i_1 + \cdots + i_r = n - k$, then the cohomology class

$$v = w_{i_1}(X(\mathbb{R})) \cup \ldots \cup w_{i_r}(X(\mathbb{R}))$$

belongs to $H^{n-k}_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$. In view of (1), we have $< \mathrm{cl}_{\mathbb{R}}(z) \cup v, \mu_{X(\mathbb{R})} >= 0$, which completes the proof of the first part of the theorem.

Given an invertible sheaf $\mathcal{L}$ on X, we denote by $\mathcal{L}_{\mathbb{R}}$ (resp. $\mathcal{L}_{\mathbb{C}}$) the topological real (resp. complex) line bundle on $X(\mathbb{R})$ (resp. $X(\mathbb{C})$) determined by $\mathcal{L}$ in the usual way. If $\mathcal{L}$ corresponds to a Weil divisor D on X, then

$$w_1(\mathcal{L}_{\mathbb{R}}) = \mathrm{cl}_{\mathbb{R}}(D) \quad \text{and} \quad c_1(\mathcal{L}_{\mathbb{C}}) = \mathrm{cl}_{\mathbb{C}}(D),$$

where $w_1(-)$ and $c_1(-)$ stand for the first Stiefel-Whitney class and the first Chern class, respectively, *cf.* [5], p. 498, p. 489. Note that the restriction $\mathcal{L}_{\mathbb{C}}|X(\mathbb{R})$ of $\mathcal{L}_{\mathbb{C}}$ to $X(\mathbb{R})$ is the complexification of $\mathcal{L}_{\mathbb{R}}$, and hence

$$c_1(\mathcal{L}_{\mathbb{C}}|X(\mathbb{R})) = \beta(w_1(\mathcal{L}_{\mathbb{R}})),$$

where $\beta : H^1(X(\mathbb{R}), \mathbb{Z}/2) \to H^2(X(\mathbb{R}), \mathbb{Z})$ is the Bockstein homomorphism that appears in the long exact sequence

$$\ldots \to H^1(X(\mathbb{R}), \mathbb{Z}) \xrightarrow{2} H^1(X(\mathbb{R}), \mathbb{Z}) \to H^1(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\beta} H^2(X(\mathbb{R}), \mathbb{Z}) \to \ldots,$$

cf. [11], Problems 15-C and D. The last equality can be written in an equivalent form

$$r(\mathrm{cl}_{\mathbb{C}}(D)) = \beta(\mathrm{cl}_{\mathbb{R}}(D)), \tag{4}$$

where $r : H^2(X(\mathbb{C}), \mathbb{Z}) \to H^2(X(\mathbb{R}), \mathbb{Z})$ is the homomorphism induced by the inclusion map $X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$.

Suppose now that $k = 1$, that is, z is a Weil divisor on X. By (2) and (4), we have $\beta(\mathrm{cl}_{\mathbb{R}}(z)) = 0$, which means that $\mathrm{cl}_{\mathbb{R}}(z)$ is the reduction modulo 2 of a cohomology class in $H^1(X(\mathbb{R}), \mathbb{Z})$. This last fact implies that the cohomology class $\mathrm{cl}_{\mathbb{R}}(z)$ is spherical, *cf.* [9], p. 49.

Let us now assume that $k = n - 1 \geq 1$ and $X(\mathbb{R})$ is connected. We already know that $< \mathrm{cl}_{\mathbb{R}}(z) \cup w_1(X(\mathbb{R})), \ \mu_{X(\mathbb{R})} >= 0$, which in view of the connectedness of $X(\mathbb{R})$ is equivalent to $\mathrm{cl}_{\mathbb{R}}(z) \cup w_1(X(\mathbb{R})) = 0$. The last condition implies that the homology class in $H_1(X(\mathbb{R}), \mathbb{Z}/2)$ Poincaré dual to $\mathrm{cl}_{\mathbb{R}}(z)$ can be represented by a C^∞, closed curve in $X(\mathbb{R})$, with trivial normal vector bundle, *cf.* for example [4], p. 599. This in turn implies that $\mathrm{cl}_{\mathbb{R}}(z)$ is spherical, *cf.* [12], Théorème II.1. Thus the proof is complete. □

Proof of Theorem 1.2. — By Theorem 1.1, (a) implies (b), and we show below that (b) implies (a).

Choose a nonsingular, irreducible algebraic subset W of $\mathbb{R}^{k+1}$, which has precisely two connected components W_0 and W_1, each diffeomorphic to the unit k-sphere S^k (for example, $W = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1^4 - 4x_1^2 + 1 + x_2^2 + \cdots + x_{k+1}^2 = 0\}$). Let c be the unique generator of the group $H^k(W_0, \mathbb{Z}/2) \cong \mathbb{Z}/2$, viewed as a subgroup of $H^k(W, \mathbb{Z}/2)$. Since the cohomology class u is spherical, there exists a C^∞ map $h : M \to W$ such that $h(M) \subseteq W_0$ and $u = H^k(h)(c)$. Choose a regular value y_0 of h in W_0. Then u is Poincaré dual to the homology class in $H_{n-k}(M, \mathbb{Z}/2)$ represented by the C^∞ submanifold $h^{-1}(y_0)$ of M, *cf.* [5], 2.15. Clearly, there exists a unique C^∞ map $f : M \to W$ such that for every connected component S of M and every point x in S, we have $f(x) = h(x)$ if $S \cap h^{-1}(y_0) \neq \emptyset$ and $f(x) = y_0$ if $S \cap h^{-1}(y_0) = \emptyset$. The map f satisfies

$$f(M) \subseteq W_0 \ \text{ and } \ u = H^k(f)(c). \tag{5}$$

Furthermore, each connected component of $f^{-1}(y_0)$ is a C^∞ submanifold of M of dimension either $n-k$ or n. Also, each connected component of M contains a connected component of $f^{-1}(y_0)$. Since $n-k \geq 1$, we can find a C^∞ closed curve C in M such that

$$f(C) = \{y_0\}, \tag{6}$$

the normal vector bundle of C in M is trivial, and each connected component of M contains a connected component of C. Choose an integer d with $2n+1 \leq d$ and let D be a compact, nonsingular, irreducible, 1-dimensional algebraic subset of $\mathbb{R}^d$ that has the same number of connected components as C. Replacing M by its image under a suitable C^∞ embedding into $\mathbb{R}^d$, we may assume that

$$D = C \subseteq M \subseteq \mathbb{R}^d. \tag{7}$$

By Tognoli's theorem [13] or [1], Corollary 2.8.6, there exists a nonsingular real algebraic subset A of $\mathbb{R}^p$, for some p, diffeomorphic to M. Consider the disjoint union $N = M \amalg A$ and the C^∞ map $F : N \to W$ defined by $F(x) = f(x)$ for x in M and $F(x) = y_0$ for x in A. We assert that if w is a cohomology class in $H^\ell(W, \mathbb{Z}/2)$ and if $j_1, \dots, j_s$ are nonnegative integers with $j_1 + \cdots + j_s = n - \ell$, then

$$< H^\ell(F)(w) \cup w_{j_1}(N) \cup \ldots \cup w_{j_s}(N),\ \mu_N >= 0.$$

Indeed, first note that $w = 0$, unless $\ell = 0$ or $\ell = k$. If $\ell = 0$, then either $H^0(F)(w) = 0$ or $H^0(F)(w) = 1$. In the latter case the assertion holds since M and A are diffeomorphic. If $\ell = k$, then either $H^k(F)(w) = 0$ or $H^k(F)(w) = u$ (we view $H^k(M, \mathbb{Z}/2)$ as a subgroup of $H^k(N, \mathbb{Z}/2)$). In the latter case the assertion follows from condition (b). Thus the assertion is proved. It implies that there exist a C^∞ compact manifold $\tilde{N}$ with boundary $\partial\tilde{N} = N$ and a C^∞ map $\tilde{F} : \tilde{N} \to W$ satisfying $\tilde{F}|N = F$, cf. [6], 17.3. In other words, the map $f : M \to W$ and the constant map $A \to W$, which sends A to y_0, represent the same class in the unoriented bordism group of W. By construction, the normal bundle of D in M is trivial, so the restriction $\nu|D$ to D of the normal bundle ν of M in $\mathbb{R}^d$ admits an algebraic structure. Therefore, by [1], Theorem 2.8.4 and in view of (6) and (7), one can find a nonnegative integer e, a nonsingular algebraic subset V of $\mathbb{R}^d \times \mathbb{R}^e$, a C^∞ diffeomorphism $\varphi : V \to M$, and a regular map $g : V \to W$ (the latter designates the restriction to V of a rational map from $\mathbb{R}^d \times \mathbb{R}^e$ into $\mathbb{R}^{k+1}$ which has no poles on V and maps V into W) such that g is homotopic to $f \circ \varphi$ and $D \times \{0\} \subseteq V$. Since D is irreducible and each connected component of V contains a connected component of

$D \times \{0\}$, it follows that V is irreducible as well (this is the only place where D is needed).

Irreducibility of V and W allows us to choose nonsingular, quasiprojective varieties T and Y over $\mathbb{R}$ with $T(\mathbb{R}) = V$ and $Y(\mathbb{R}) = W$. By Hironaka's resolution of singularities theorem [8], 3, we may assume that T and Y are projective (and still nonsingular). Let $\tilde{g} : U \to Y$ be an algebraic morphism over $\mathbb{R}$, defined on a Zariski open neighborhood of $T(\mathbb{R}) = V$ in T, such that $\tilde{g}|T(\mathbb{R}) = g$. By applying Hironaka's theorem on removing points of indeterminacy [8], 3, we can find a nonsingular, projective algebraic variety X over $\mathbb{R}$ and an algebraic morphism $G : X \to Y$ over $\mathbb{R}$ satisfying $X(\mathbb{R}) = T(\mathbb{R})$ and $G|X(\mathbb{R}) = g$.

Let y_1 be a point in W_1 and let β be the class in $A^k(Y)$ of the 0-cycle $y_0 - y_1$ on Y. By (5), $u = H^k(f)(\mathrm{cl}_{\mathbb{R}}(\beta))$ (although, of course, $\mathrm{cl}_{\mathbb{R}}(\beta) \neq c$). Since $G|X(\mathbb{R}) = g$ is homotopic to $f \circ \varphi$, we obtain

$$\begin{aligned} H^k(\varphi)(u) &= H^k(\varphi)(H^k(f)(\mathrm{cl}_{\mathbb{R}}(\beta))) = H^k(f \circ \varphi)(\mathrm{cl}_{\mathbb{R}}(\beta)) \\ &= H^k(G|X(\mathbb{R}))(\mathrm{cl}_{\mathbb{R}}(\beta)) = \mathrm{cl}_{\mathbb{R}}(G^*(\beta)), \end{aligned}$$

where the last equality is a consequence of the functorial property of $\mathrm{cl}_{\mathbb{R}} : A^* \to H^*$, *cf.* [5], 5.12. Let z be a cycle in $Z^k(X)$ that represents in the Chow group $A^k(X)$ the pullback class $G^*(\beta)$. Then $\mathrm{cl}_{\mathbb{R}}(z) = \mathrm{cl}_{\mathbb{R}}(G^*(\beta)) = H^k(\varphi)(u)$. The proof is now complete since the cycle $y_0 - y_1$ is algebraically equivalent to 0 on Y and hence the cycle z is algebraically equivalent to 0 on X. □

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