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ON THE HILBERT SCHEME OF POINTS OF AN ALMOST COMPLEX FOURFOLD

by Claire VOISIN

1. Introduction.

If X is an algebraic variety, we can define for each k an algebraic variety $\mathrm{Hilb}^k(X)$. It is defined as the set of length k coherent quotient sheaves of $\mathcal{O}_X$. Such quotient is the structural sheaf $\mathcal{O}_Z$ of a 0-dimensional subscheme Z of X. $\mathrm{Hilb}^k(X)$ contains naturally the open subset $X_0^{(k)}$ of the symmetric product $X^{(k)}$ of X parametrizing unordered sets of k distinct points.

If $\dim X = 1$ and X is smooth, one has $\mathrm{Hilb}^k(X) = X^{(k)}$ for any k. In general it is easy to see that $X_0^{(k)}$ is open in $\mathrm{Hilb}^k(X)$ so that its closure is a schematic component of $\mathrm{Hilb}^k(X)$, but the following becomes false in dimension ≥ 3

THEOREM 1 (Fogarty), [5]. — *If X is a smooth surface,* $\mathrm{Hilb}^k(X)$ *is smooth and irreducible. Furthermore the Hilbert-Chow map*

$$c : \mathrm{Hilb}^k(X) \to X^{(k)},$$

which to a length k subscheme $Z \subset X$ associates its cycle $c(Z) = \sum_{x\in X} \ell_x(Z)x$ is a birational morphism and an isomorphism over $X_0^{(k)}$.

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These Hilbert schemes have recently attracted a lot of attention. Many beautiful works ([3], [7], [8], [1]) have been done towards the understanding of their cohomology and Hodge structures-it appears to be built by universal constructions on several copies of the tensor product of $H^*(X)$ with shifts of degree- and a very interesting structure in the infinite dimensional space $\oplus_k H^*(\mathrm{Hilb}^k(X))$, endowed with the actions of $H^*(X)$ on it induced by the natural incidence correspondences $Z_{k,n} \subset \mathrm{Hilb}^k(X) \times \mathrm{Hilb}^{k+n}(X) \times X$, has been found in [12].

Our main motivation however comes from the following theorem, due to Ellingsrud, Göttsche and Lehn [2]

THEOREM 2. — *The Chern numbers of* $\mathrm{Hilb}^k(X)$, *i.e. the integrals*

$$\int_{\mathrm{Hilb}^k(X)} P(c_1, \ldots, c_{2k}),$$

where P is a weighted homogeneous polynomial of degree $2k$, and the c_i's are the Chern classes of $\mathrm{Hilb}^k(X)$, *depend only on the Chern numbers of X, i.e.*

$$\int_{\mathrm{Hilb}^k(X)} P(c_1, \ldots, c_{2k}) = \Phi_P(c_1(X)^2, c_2(X))$$

for a function Φ_P which depends only on P.

If we work over $\mathbb{C}$, this theorem says that the complex cobordism class of $\mathrm{Hilb}^k(X)$ depends only on that of X. Now the complex cobordism class is determined by a much weaker structure than the complex structure, namely the almost complex structure (or more precisely the stable almost complex structure, that is a complex structure on $T_X \oplus T$, where T is a trivial vector bundle). Hence this result suggests that one should try to construct the Hilbert scheme, as a manifold or better as an almost complex manifold, for any almost complex structure on the underlying fourfold X, that is without using the integrability condition for the complex structure. The main result of this paper gives an answer to this problem

THEOREM 3. — *Let X be a $\mathcal{C}^\infty$ almost complex fourfold. Then there exists for each k a manifold* $\mathrm{Hilb}^k(X)$ *of real dimension $4k$ endowed with a stable almost complex structure, and a continuous map*

$$c : \mathrm{Hilb}^k(X) \to X^{(k)},$$

which is a diffeomorphism over $X_0^{(k)}$ and whose fibers over $z \in X^{(k)}$ are naturally homeomorphic to the fibers of the Hilbert-Chow morphism c over

z for any almost complex structure on X integrable in a neighbourhood of $\operatorname{Sup} z$.

Our construction is not canonical, and provides in fact a family of such manifolds, parametrized by a contractible basis, so that the resulting manifold is well defined up to diffeomorphisms isotopic to identity.

This theorem is proved in Section 3. The construction, contrarily to the construction of the Hilbert scheme in the integrable case does not depend only on the almost complex structure, and involves the choice of supplementary parameters. It remains open whether a construction of a "pseudoholomorphic Hilbert scheme", depending canonically on the almost complex structure, is possible. In Section 2, we consider this problem and study another possible construction, which is canonical but unfortunately leads only to the construction of an open part of the Hilbert scheme; we study, when it is possible to define them, the "pseudoholomorphic finite subschemes" of X. It turns out that this allows to construct in the almost complex setting the curvilinear part of the Hilbert scheme, that we will denote by $\mathrm{Hilb}^k_{\mathrm{curv}}(X)$ and also the open set $\mathrm{Hilb}^k_{\mathrm{curv}}(X)'$ of $\mathrm{Hilb}^k(X)$ where at any of their points the schemes are curvilinear or with length ≤ 3. Since a scheme of length 3 is either curvilinear or the infinitesimal neighbourhood of a point $x \in X$, this last set is a partial compactification of $\mathrm{Hilb}^k_{\mathrm{curv}}(X)$ obtained by adding sets of the form

$$(X_0^{(l)} \times \mathrm{Hilb}^{k-3l}_{\mathrm{curv}}(X))_0,$$

where the first summand parametrizes infinitesimal neighbourhoods of points, and the two subscripts 0 again mean that we consider the open sets consisting of cycles with disjoint supports. The construction we give for $\mathrm{Hilb}^k_{\mathrm{curv}}(X)'$ can also be adapted to enlarge further our Hilbert scheme and construct the analogue of $\mathrm{Hilb}^k_{\mathrm{curv}}(X)''$, the open set of $\mathrm{Hilb}^k(X)$ where at any of their points the schemes are curvilinear or with length ≤ 4, but we do not include the proof here, since it is not very instructive. In any case it would be interesting to decide whether it is possible to define pseudoholomorphic finite subschemes in a more general situation than those we have been considering here.

Finally, this work provides a desingularization of the symmetric products of an almost complex fourfold, with fibers as in the integrable case. I have no idea whether a similar desingularization (a "generalized Hilbert scheme") exists for any fourfold.

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2. Pseudoholomorphic finite subschemes.

2.1. Pseudoholomorphic curvilinear subschemes.

Let X be a complex variety; we shall denote by $\mathrm{Hilb}^k_{\mathrm{curv}}(X)$ the (open) subset of $\mathrm{Hilb}^k(X)$ parametrizing curvilinear subschemes of X of length k. Our goal in this section is to prove the following

THEOREM 4. — *If (X, J) is an almost complex fourfold, the set $\mathrm{Hilb}^k_{\mathrm{curv}}(X)$ of pseudoholomorphic curvilinear finite subschemes of length k has a natural structure of (non compact) manifold. Furthermore the natural map*

$$c : \mathrm{Hilb}^k_{\mathrm{curv}}(X) \to X^{(k)}$$

has only smooth fibers and they are naturally diffeomorphic to the fibers of the corresponding Hilbert-Chow map for an integrable complex structure.

We first begin defining the pseudoholomorphic curvilinear finite subschemes of length k by noting that in the integrable case, $z \in \mathrm{Hilb}^k_{\mathrm{curv}}(X)$ if and only if z is locally (near each point of its support) contained in a smooth complex curve. So $z = \sqcup_i z_i$ and z_i is identified with the $n_i - 1$-th order infinitesimal neighbourhood of x_i in some complex curve $C_i \subset X$. The cycle $c(z)$ is then equal to $\sum_i n_i x_i$.

Choosing a uniformizing coordinate t on C_i centered on x_i allows to identify z_i with a $n_i - 1$-jet of a holomorphic map from the disk to X *with non zero differential.* This space of jets being denoted by $\mathcal{J}_{n_i-1}$, we see that the curvilinear schemes of length n_i supported at one point can be identified with

$$W_{n_i} := \mathcal{J}_{n_i-1} / \operatorname{Aut} \Delta_{n_i},$$

where $\Delta_{n_i} = \operatorname{Spec} \mathbb{C}[t]/t^{n_i}$. It is easy to see that the action is free.

In the general case, we will define similarly the pseudoholomorphic curvilinear subschemes of length n_i supported at one point as

$$W_{n_i} := \mathcal{J}^{ph}_{n_i-1} / \operatorname{Aut} \Delta_{n_i},$$

where $\mathcal{J}^{ph}_{n_i-1}$ is the set of jets of order $n_i - 1$ of pseudoholomorphic map from the disk to X with non zero differential.

Now, in the integrable case, these spaces of jets are constructed inductively as follows: $\mathcal{J}_0 = X$, $\mathcal{J}_1 = T_X -$ *0-section.* In general a jet of order k is a tangent lifting of the corresponding jet of order $k-1$, i.e.

$$\mathcal{J}_k \subset T_{\mathcal{J}_{k-1}},$$

the lifting condition being the following:

Let $\pi_{k-1} : \mathcal{J}_{k-1} \to \mathcal{J}_{k-2}$ be induced by the inclusion $\mathcal{J}_{k-1} \subset T_{\mathcal{J}_{k-2}}$, and denote similarly $\pi_k : T_{\mathcal{J}_{k-1}} \to \mathcal{J}_{k-1}$. Then

$$\mathcal{J}_k = \{\phi \in T_{\mathcal{J}_{k-1}} / (\pi_{k-1})_* \phi = \pi_k(\phi) \in \mathcal{J}_{k-1} \subset T_{\mathcal{J}_{k-2}}\}. \tag{2.1}$$

Notice that these spaces of jets $\mathcal{J}_k$ describe as well the jets of real maps from a real segment to X, that we will denote by $\mathcal{J}_k^{\mathbb{R}}$. The complex structure is used to interpret them as jets of holomorphic maps from the disk to X, which allows to construct the $\operatorname{Aut}\Delta_{k+1}$-action on them.

It turns out that we have exactly the same picture in the pseudo-holomorphic setting (*cf.* [6]). Namely, let X be endowed with a $\mathcal{C}^\infty$ almost complex structure J. We want to study the spaces of k-jets of pseudoholomorphic maps from the disk to X with non zero differential. For $k = 1$ they are of course identified with the data of a point of X and of a (real) tangent vector to X at x, with the help of the complex structure on $T_{X,x}$ which identifies real tangent vector to complex tangent vectors of type $(1,0)$. This situation still exists at higher order because of the following lemma

LEMMA 1. — *If (X, J) is an almost complex variety, T_X admits a natural almost complex structure $\tilde{J}$ which is compatible with J in the sense that the structural map $\pi : T_X \to X$ has a complex linear differential.*

Proof. — The formula for $\tilde{J}$ is the following. Let $x = (x_i)$ be local coordinates on X, and let $(x, \dot{x})$ be the induced coordinates on T_X. The almost complex structure J is described by a matrix $J(x)$. For $(v, \dot{v}) \in T_{X,(x,\dot{x})}$, we define then

$$\tilde{J}(v, \dot{v}) = (Jv, \dot{J}_{\dot{x}} v + J\dot{v}).$$

Here, $\dot{J}_{\dot{x}}$ means the differential of the matrix $J(x)$ with respect to the tangent vector $\dot{x}$. One verifies easily that $\tilde{J}^2 = -1$, and that the definition does not depend on the choice of cordinates. □

We can apply inductively this lemma to conclude that each space of k-jets $\mathcal{J}_k^{\mathbb{R}}(X)$ has an induced almost complex structure. Now we have

LEMMA 2. — *The elements of $\mathcal{J}_k^{\mathbb{R}}(X)$ identify naturally with the space $\mathcal{J}_k^{ph}(X)$ of k-jets of pseudoholomorphic maps from the disk to X.*

Proof. — Let us consider first the spaces $\mathcal{J}_k^{\text{diff}}(X)$ of k-jets of *differentiable* maps from the disk to X. They are built inductively as follows : $\mathcal{J}_k^{\text{diff}}(X)$ is contained in the fibration over $\mathcal{J}_{k-1}^{\text{diff}}(X)$ with fiber

$\mathrm{Hom}(T_\Delta, T_{\mathcal{J}^{\mathrm{diff}}_{k-1}(X)})$. Elements of $\mathcal{J}^{\mathrm{diff}}_k(X)$ have to be tangent liftings of the corresponding element of $\mathcal{J}^{\mathrm{diff}}_{k-1}(X)$, that is have to satisfy the analogue of the compatibility condition (2.1). But they also have to satisfy the following integrability condition: let $j_k \in \mathcal{J}^{\mathrm{diff}}_k(X) \subset \mathrm{Hom}(T_\Delta, T_{\mathcal{J}^{\mathrm{diff}}_{k-1}(X), j_{k-1}})$, where j_{k-1} is the corresponding $k-1$-th order jet. Again view j_{k-1} as an element of $\mathrm{Hom}(T_\Delta, T_{\mathcal{J}^{\mathrm{diff}}_{k-2}(X), j_{k-2}})$. The integrability says:

(*) The composite

$$j_{k-1} \circ j_k : T_\Delta \otimes T_\Delta \to T_{\mathcal{J}^{\mathrm{diff}}_{k-2}(X), j_{k-2}}$$

is symmetric.

Now one uses the almost complex structure on each $\mathcal{J}^{\mathbb{R}}_k(X)$ as follows: assume the lemma has been proved for $k-1$. Hence there is an inclusion

$$\mathcal{J}^{\mathbb{R}}_{k-1}(X) = \mathcal{J}^{ph}_{k-1}(X) \subset \mathcal{J}^{\mathrm{diff}}_{k-1}(X).$$

Let now

$$j_k \in \mathcal{J}^{\mathbb{R}}_k(X) \subset T_{\mathcal{J}^{\mathbb{R}}_{k-1}(X)}.$$

Using the almost complex structure on the space of jets, we can then extend j_k by $\mathbb{C}$-linearity to an element $\tilde{j}_k$ of

$$\mathrm{Hom}(T_\Delta, T_{\mathcal{J}^{\mathbb{R}}_{k-1}(X)}) \subset \mathrm{Hom}(T_\Delta, T_{\mathcal{J}^{\mathrm{diff}}_{k-1}(X)}).$$

Because the almost complex structures on $\mathcal{J}_{k-1}(X)$ and $\mathcal{J}_{k-2}(X)$ are compatible, $\tilde{j}_k$ satisfies the tangent lifting condition. To see that it gives a k-th order pseudoholomorphic jet, one notes that it satisfies the integrability condition since the successive elements of $\mathrm{Hom}(T_\Delta, T_{\mathcal{J}_{\cdot}})$ considered are complex linear, so that the symmetry condition (*) is obviously satisfied. So one gets a k-th order differentiable jet and it is immediate to verify that it is pseudoholomorphic. □

Now that we have identified $\mathcal{J}^{\mathbb{R}}_k(X)$ as the space of pseudoholomorphic k-jets with non zero differential, we have on it the action of $\mathrm{Aut}\,\Delta_{k+1}$, which is free, hence gives a quotient W_{k+1} that we interpret as the set of pseudoholomorphic curvilinear subschemes of length $k+1$ supported at one point. The set of pseudoholomorphic curvilinear subschemes of X of length n will then be defined as a set as the disjoint union over all partitions $n = n_1 + \ldots + n_l$, $n_i \neq 0$ of the sets $W_{n_1,\ldots,n_l}$ parametrizing the data of isomorphism classes of jets j_i of length n_i supported at one point x_i with the x_i's all distinct. Notice that each $W_{n_1,\ldots,n_l}$ has a differentiable structure, being the quotient of the open subset of $\Pi_i W_{n_i}$, $n_1 \geq \ldots \geq n_l$ parametrizing l-uple of jets with disjoint supports, by the subgroup of the symmetric group S_l which permutes factors with equal multiplicities. Note also that one recovers $X_0^{(n)}$ as $W_{1,\dots,1}$. Furthermore the description of the spaces of pseudoholomorphic jets given in Lemma 2 shows that the set of pseudoholomorphic schemes with given support is diffeomorphic to the set of holomorphic schemes with the same support for any integrable complex structure on X defined near this support. This proves the last assertion of Theorem 4.

To conclude the construction of $\mathrm{Hilb}^n_{\mathrm{curv}}(X)$, it now remains to put a differentiable structure on this set theoretic union, and this is the local (instead of finite order) theory of pseudoholomorphic curves which will provide the differentiable charts. Let $z \in \mathrm{Hilb}^n_{\mathrm{curv}}(X)$; then $z = \sqcup_i z_i$, where each $z_i \in \mathrm{Hilb}^{n_i}_{\mathrm{curv}}(X)$ is supported at one point x_i, and $\sum_i n_i = n$. Clearly $\mathrm{Hilb}^n_{\mathrm{curv}}(X)$ identifies naturally to $\Pi_i \,\mathrm{Hilb}^{n_i}_{\mathrm{curv}}$ near z, so that it suffices to define the local charts for $\mathrm{Hilb}^{n_i}_{\mathrm{curv}}(X)$ near z_i and to take the product charts for $\mathrm{Hilb}^n_{\mathrm{curv}}(X)$ near z. In the sequel we then put $n = n_i$ and $z = z_i$ with $x = x_i$ supporting z.

Recall that the space W_n of pseudoholomorphic jets of order $n-1$ modulo $\mathrm{Aut}\,\Delta_n$ is of complex dimension $(N-1)n+1$, $N = \dim_{\mathbb{C}} X$. Let $0 \in W$ be a differentiable ball of complex dimension $(N-1)n$ parametrizing a family of pseudoholomorphic curves

$$\psi : W \times \Delta_\epsilon \to X$$

satisfying:

i) The map

$$\Psi_n : W \times \Delta_\epsilon \to W_n, \tag{2.2}$$

which to (w,x) associates the isomorphism class of the $n-1$-jet of ψ_w at x is a local diffeomorphism near $(0,0)$.

ii) The isomorphism class of the $n-1$-jet of ψ_0 at 0 is the point $z \in \mathrm{Hilb}^n_{\mathrm{curv}}(X)$.

That such families exist for sufficiently small ϵ follows from the local theory of pseudoholomorphic curves (see [13]). The finite order analysis showed already that there are no finite order obstruction to its existence. Some supplementary analysis is needed here to ensure the existence of actual curves instead of formal jets. The charts we will use for $\mathrm{Hilb}^n_{\mathrm{curv}}(X)$ near z are then simply given by the maps

$$\begin{aligned}\Psi^{(n)} : W \times \Delta_\epsilon^{(n)} &\to \mathrm{Hilb}^n_{\mathrm{curv}}(X),\\ (w,z) &\mapsto \psi_w(z).\end{aligned} \tag{2.3}$$

From now on we will restrict to the case $N = 2$, while presumably the result remains true in any dimension. The proof that these maps provide indeed charts for a differentiable structure on $\mathrm{Hilb}_{\mathrm{curv}}(X)$, compatible with the differentiable structures on each stratum, follows then from the two next propositions.

PROPOSITION 1. — *Let $\psi : W \times \Delta_\epsilon \to X$, $\psi' : W' \times \Delta_\epsilon \to X$ be two families of pseudoholomorphic disks satisfying the properties i), ii) above. Then the set*

$$Z \subset W \times \Delta_\epsilon^{(n)} \times W' \times \Delta_\epsilon^{(n)}$$

consisting of couples $((w, z), (w', z'))$ such that $\Psi^{(n)}(w, z) = \Psi'^{(n)}(w', z')$ is a subvariety of $W \times \Delta_\epsilon^{(n)} \times W' \times \Delta_\epsilon^{(n)}$ near $((0, n0), (0, n0))$, which projects (up to shrinking W, W', Δ_ϵ if necessary) isomorphically onto each factor $W \times \Delta_\epsilon^{(n)}$, $W' \times \Delta_\epsilon^{(n)}$.

PROPOSITION 2. — *Let $\psi : W \times \Delta_\epsilon \to X$ be a family of pseudoholomorphic disks satisfying the properties i), ii) above. Then for any point $(w, z') \in W \times \Delta_\epsilon^{(n)}$ close enough to $(0, n0)$, with $z' = \sum_{1\leq j\leq r} m_j x_j$, $x_j \in \Delta_\epsilon$ distinct, the map*

$$W \times \Delta_\epsilon^r \to \prod_{1\leq j\leq r} W_{m_j}(X)$$

$$(w, (y_j)) \mapsto ((\psi_w)_{m_j}(y_j)), \tag{2.4}$$

where the last notation means that we consider the $r-$uple of the isomorphism classes of the $m_j - 1$-th order jets of ψ_w at y_j, is a local diffeomorphism near $(w, (x_j))$.

Proposition 1 shows that the maps $\Psi^{(n)}$ are injective on sufficiently small neighbourhoods of the considered point. It also shows that the subsets of $\mathrm{Hilb}^n_{\mathrm{curv}}(X)$ given as the images of sufficiently small open neighbourhoods of $(0, n0)$ do not depend on the choice of Ψ, and that the differentiable structures induced by $\Psi^{(n)}$ on these sets are independent on the choice of ψ. Proposition 2 shows first of all that the maps $\Psi^{(n)}$ induce local diffeomorphisms on the strata near $(0, n0)$, (z), where the stratification on $W \times \Delta_\epsilon^{(n)}$ is the one given by the multiplicities. It follows then that this differentiable structure is compatible with the one on each stratum. Finally, Proposition 2 shows also that our $\Psi^{(n)}$ is in fact also a product of charts defined similarly at any of its points, so that its compatibility with the product charts at any of its points follows from Proposition 1.

Proof of Proposition 1. — It suffices to do it when ψ and ψ' satisfy the conditions that the jets of ψ_0 and ψ'_0 at 0 coincide to order $n-1$ but are different at order n. Indeed, given ψ and ψ' as in the proposition, there exists a ψ'' satisfying the same property, and the supplementary condition that the n-th order jet of ψ''_0 at 0 is different from the one of ψ_0 and ψ'_0. Then the statement for the couples (ψ, ψ'') and (ψ', ψ'') implies the statement for the couple (ψ, ψ').

Now we do the following: the assumptions on ψ and ψ' are now that the smooth pseudoholomorphic disks $\psi_0(\Delta_\epsilon)$, $\psi'_0(\Delta'_\epsilon)$ have exactly a contact of order n at the point $\psi_0(0) = \psi'_0(0) = x$. This implies in particular (see for example [10]) that the pseudoholomorphic disks $\psi_0(\Delta_\epsilon)$ and $\psi'_0(\Delta_\epsilon)$ have exactly n as local intersection number (in a sufficiently small neighbourhood of x), so that, by stability of this local intersection number, up to shrinking W and W' if necessary, each disk $\Psi_w(\Delta_\epsilon)$ meets the disk $\Psi'_{w'}(\Delta_\epsilon)$ along a cycle of length n_i still contained in the same neighbourhood of x_i, the multiplicities being given as the order of contact $+1$, or equivalently the local intersection number near each intersection point. It follows that one has for each $w' \in W'$ a map

$$\eta_{w'} : W \to \Delta'^{(n)}_\epsilon, \tag{2.5}$$

which to w associates the cycle of intersection $\psi_w(\Delta_\epsilon) \cap \psi'_{w'}(\Delta'_\epsilon)$ or more precisely its inverse image by the map $\psi'^{(n)}_{w'}$.

The following lemma, which follows easily from [13], shows that $\eta_{w'}$ is differentiable.

LEMMA 3. — *Let $C \subset X$ be a smooth pseudoholomorphic curve, and let $\psi : w \times \Delta \to X$ be a differentiable family of pseudoholomorphic embeddings parametrized by w. Then locally near C, there exists a differentiable function $F : W \times X \to \mathbb{C}$, such that each $F_w : X \to \mathbb{C}$ is submersive, gives an equation for $\psi_w(\Delta)$, and has holomorphic restriction to C.*

We apply this lemma to $C = \psi'_{w'}(\Delta'_\epsilon)$, and we use the definition of the differentiable structure on $\Delta'^{(n)}_\epsilon$ to get the differentiability of $\eta_{w'}$.

Now the dimensions of W and $\Delta'^{(n)}_\epsilon$ are the same, and this map has injective differential at $w = 0$ when $w' = 0$, since then the intersection cycle is equal to z, so that a tangent vector anihilated by the differential would provide a deformation of ψ_w still containing the cycle z, in contradiction with the fact that the map Ψ_n of (2.2) is a local isomorphism. Hence these maps are local isomorphisms in a neighbourhood of 0 for all w' close enough to 0. Now consider the set Z defined in proposition 1. It is clearly equal to the graph of the map

$$(2.6)\qquad \begin{gathered} W \times W' \to \Delta_\epsilon^{(n)} \times \Delta'^{(n)}_\epsilon \\ (w, w') \mapsto (\psi_w^{-1}(\psi_w(\Delta_\epsilon) \cap \psi'_{w'}(\Delta'_\epsilon)), \psi'^{-1}_{w'}(\psi_w(\Delta_\epsilon) \cap \psi'_{w'}(\Delta'_\epsilon))). \end{gathered}$$

Now this map is differentiable by Lemma 3 above. Hence Z it is a differentiable variety of the same dimension as $W \times W'$ or $W' \times \Delta_\epsilon^{(n)}$ as well.

Finally consider its projection onto $W' \times \Delta'^{(n)}_{\epsilon'}$. Since the map $\eta_{w'}$ of (2.5) is a local diffeomorphism for all w' close enough to 0, this projection contains $w' \times \Delta'^{(n)}_{\epsilon'}$ for all w' close enough to 0. On the other hand, since Z is a graph, the projection on W' is submersive. Hence we conclude that Z projects submersively onto $W' \times \Delta_\epsilon^{(n)}$, hence is a local isomorphism by dimension reasons. □

Proof of Proposition 2. — Introduce as before a family of pseudo-holomorphic disks

$$\psi' : W' \times \Delta'_\epsilon \to X$$

satisfying properties i) and ii) and the supplementary condition that ψ_0 and ψ'_0 coincide exactly up to order $n-1$. Let now $(w, z') \in W \times \Delta_\epsilon^{(n)}$ be sufficiently close to $(0, n0)$. Then the proof of Proposition 1 shows that there is a pseudoholomorphic disk $\psi'_{w'}(\Delta'_\epsilon)$ which meets $\psi_w(\Delta_\epsilon)$ exactly along $\psi_w(z')$. Furthermore the map

$$\eta_{w'} : W \to \Delta'^{(n)}_\epsilon,\ w \mapsto (\psi_{w'}^{(n)})^{-1}(\psi_w(\Delta_\epsilon) \cap \psi'_{w'}(\Delta'_\epsilon))$$

is a local isomorphism near w.

Consider now the map (2.4) near $(w, (x_j))$, where $z' = \sum_j m_j x_j$. Clearly its differential is injective on the tangent space $T_{\Delta_\epsilon^r, (x_j)}$ since the disk $\Psi_w(\Delta_\epsilon)$ is immersed in X. So if u is a non zero tangent vector anihilated by the differential of this map at $(w, (x_j))$, the projection of u to $T_{W,w}$ provides a non trivial tangent vector to W at w which is clearly anihilated by the differential of $\eta_{w'}$, and this provides a contradiction. So the map (2.4) is immersive at any point sufficiently close to $(0, n0)$, hence a local diffeomorphism for reasons of dimensions. □

2.2. $\mathrm{Hilb}^k_{\mathrm{curv}}(X)'$.

This section is somewhat technical. We have included it because it provides a canonically defined open part of the Hilbert scheme of an almost complex fourfold. The reader who is only interested in the abstract existence Theorem 3 may skip it and go to Section 3.

Recall that $\mathrm{Hilb}^k_{\mathrm{curv}}(S)'$ is the open subset of $\mathrm{Hilb}^k(S)$ made of subschemes of length k which at each point are either curvilinear or of length ≤ 3. We want to prove in this section a result similar to Theorem 4 for $\mathrm{Hilb}^k_{\mathrm{curv}}(S)'$. Recall that an element z of $\mathrm{Hilb}^3(S)$ which is not curvilinear is the first infinitesimal neighbourhood of a point x of S, that is $\mathcal{I}_z = \mathcal{M}_x^2$, and we will denote this by $z = x_2$. Similarly we will denote by S_2 the copy of S naturally contained in $\mathrm{Hilb}^3(S)$. More generally one has

$$\mathrm{Hilb}^k_{\mathrm{curv}}(S)' = \mathrm{Hilb}^k_{\mathrm{curv}}(S) \bigsqcup_l (S_0^{(l)} \times \mathrm{Hilb}^{k-3l}_{\mathrm{curv}}(S))_0,$$

which set theoretically makes sense as well if S is replaced by an almost complex fourfold X, so that we can define

$$\mathrm{Hilb}^k_{\mathrm{curv}}(X)' := \mathrm{Hilb}^k_{\mathrm{curv}}(X) \bigsqcup_l (X_0^{(l)} \times \mathrm{Hilb}^{k-3l}_{\mathrm{curv}}(X))_0.$$

Notice that to put then a differentiable structure on $\mathrm{Hilb}^k_{\mathrm{curv}}(X)'$, it suffices to put a differentiable structure on $\mathrm{Hilb}^3(X)$, since we will then put the product differentiable structure at any point of $\mathrm{Hilb}^k_{\mathrm{curv}}(X)'$. So from now on we will only consider the problem of defining a differentiable structure on $\mathrm{Hilb}^3(X)$ which as a set is the disjoint union of $\mathrm{Hilb}^3_{\mathrm{curv}}(X)$ and of a copy X_2 of X.

The topology is the following: a sequence z_n will converge to x_2 if the support of z_n converges to $3x$ and no subsequence converges to a curvilinear scheme : assuming that $z_n \in X_0^{(3)}$, this means in a local $\mathcal{C}^\infty$ identification $X \cong \mathbb{C}^2$, with complex differential at x, that for any subsequence $z_{n_k} = z^1_{n_k} + z^2_{n_k} + z^3_{n_k}$ and sequence of numbers $h_{n_k} \neq 0$ such that the limits (taken in $\mathbb{C}^2$)

$$\lim_{k\to\infty} \frac{z^i_{n_k} - z^j_{n_k}}{h_{n_k}}$$

exist for any i, j, these limits are not all colinear over $\mathbb{C}$ (intrinsically they are tangent vectors to X at x).

We want to put a differentiable structure on $\mathrm{Hilb}^3(X)$. Let us first analyse the case of an integrable complex structure. We have the following

LEMMA 4. — *Let X be a complex surface, and let ϕ be a holomorphic function defined in a neighbourhood U of a point $x \in X$, such that $d\phi_x \neq 0$. Then the set*

$$Y_\phi := \{z \in \mathrm{Hilb}^3(U) / \exists t \in \mathbb{C},\ l(z \cap X_t) \geq 2\}$$

is a complex hypersurface of $\mathrm{Hilb}^3(X)$ *which contains x_2 and is smooth at x_2. In particular Y_ϕ contains a neighbourhood of x_2 in X_2.*

Here X_t is the curve $\phi^{-1}(t)$ which will be smooth near x for t close to $\phi(x)$.

Proof. — It is easy to see by a dimension count that Y_ϕ is a hypersurface. Since x_2 corresponds to the ideal $\mathcal{M}_x^2$, the subscheme $x_2 \cap X_t$ of X_t, for $t = \phi(x)$ is defined in X_t by $\mathcal{M}_x^2$ hence has length 2, so that x_2 belongs to Y_ϕ. It remains finally to compute the Zariski tangent space of Y_ϕ at x_2. Since $l(x_2 \cap X_t)$ is equal to 2, we can identify schematically near x_2, via the projection on $\mathrm{Hilb}^3(X)$, the hypersurface Y_ϕ with the subset

$$Y'_\phi = \{(z, w, u) \in \mathrm{Hilb}^3(X) \times \mathrm{Hilb}^2(X) \times \mathbb{C} / w \subset z,\ w \subset X_u\}.$$

The Zariski tangent space of Y_ϕ at x_2 is the projection to $T_{\mathrm{Hilb}^3(X), x_2}$ of the Zariski tangent space of Y'_ϕ at (x_2, w, t), with $w = x_2 \cap X_t$, $t = \phi(x)$. The last one is computed as follows:

– The Zariski tangent space $T_{\mathrm{Hilb}^3(X), x_2}$ is equal to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_{x_2}, \mathcal{O}_{x_2})$ and this is easily seen to be equal to $\mathrm{Hom}_{\mathbb{C}}(\mathcal{M}_x^2/\mathcal{M}_x^3, \mathcal{M}_x/\mathcal{M}_x^2)$.

– Similarly, the tangent space $T_{\mathrm{Hilb}^2(X), w}$ is equal to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_w, \mathcal{O}_w)$ that is to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_w/\mathcal{I}_w^2, \mathcal{O}_w)$.

– Finally the tangent space to $\mathbb{C}$ at t sends naturally via a map which we will denote by ρ to a subspace of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_{X_t}, \mathcal{O}_{X_t})$, by the Kodaira-Spencer map associated to the family of curves X_u.

Consider now the following diagramm:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_{x_2}, \mathcal{O}_{x_2}) & \xrightarrow{\delta} & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_{x_2}, \mathcal{O}_w) \\
 & & \uparrow \gamma \\
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_{X_t}, \mathcal{O}_w) & \xleftarrow{\beta} & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_w, \mathcal{O}_w) \\
\alpha \uparrow & & \\
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_{X_t}, \mathcal{O}_{X_t}) & \xleftarrow{\rho} & \mathbb{C}
\end{array}$$

where all the maps are the obvious restriction maps. Retracing through the identifications

$$T_{\mathrm{Hilb},y} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{I}_y, \mathcal{O}_y)$$

used, it is immediate to see that if (u, v, ϵ) is tangent to $\mathrm{Hilb}^3(X) \times \mathrm{Hilb}^2(X) \times \mathbb{C}$ at (z, w, t) the condition $\delta(u) = \gamma(v)$ is then the infinitesimal condition on the deformations of the pair (z, w) for w to remain contained in z, while the condition $\beta(v) = \alpha \circ \rho(\epsilon)$ is the infinitesimal condition on the pair (w, t) for w to remain contained in the curve X_t. In conclusion, we have

$$T_{Y_\phi, x_2} = \{u \in T_{\mathrm{Hilb}^3(X), z} / \exists v \in T_{\mathrm{Hilb}^2(X), w}, \delta(u) = \gamma(v) \,\mathrm{and}\, \beta(v) \in \mathrm{Im}\, \alpha \circ \rho\}.$$

Now, choose holomorphic coordinates u_1, u_2 centered at x, so that $\phi(u_1, u_2) = u_2 + a$, $a = \phi(x)$. Then $\mathcal{O}_w$ admits for basis over $\mathbb{C}$ the elements $1, u_1$, and $\mathcal{I}_w / \mathcal{I}_w^2$ admits for basis over $\mathcal{O}_w$ the elements u_1^2, u_2. Also $\mathcal{M}_x^2 / \mathcal{M}_x^3$ admits for basis over $\mathbb{C}$ the elements $u_1^2, u_2^2, u_1 u_2$, while $\mathcal{O}_{x_2}$ admits for basis over $\mathbb{C}$ the elements $1, u_1, u_2$. In these bases, the map γ associates to

$$h \in \mathrm{Hom}_{\mathbb{C}}(< u_1^2, u_2 >, < 1, u_2 >) \simeq \mathrm{Hom}_{\mathcal{O}_w}(\mathcal{I}_w / \mathcal{I}_w^2, \mathcal{O}_w)$$

the homomorphism $h' \in \mathrm{Hom}_{\mathbb{C}}\,(< u_1^2, u_2^2, u_1 u_2 >, < 1, u_1 >)$ given by

$$h'(u_1^2) = h(u_1^2),\ h'(u_1 u_2) = u_1 h(u_2), h'(u_2^2) = 0.$$

This follows indeed from the fact that γ is the restriction map and that it is $\mathcal{O}_X$-linear. Next one checks easily, using the fact that the curves X_t are given by the equations $u_2 = t$, that $h \in \beta^{-1}(\mathrm{Im}\ \alpha \circ \rho)$ if and only if $h(u_2)$ is proportional to $1 \in \mathcal{O}_w$. It follows that T_{Y_ϕ, x_2} identifies to the set

(2.7)

$$\{H \in \mathrm{Hom}\,(<u_1^2, u_2^2, u_1 u_2>, <u_1, u_2>),\ \exists h \in \mathrm{Hom}\,(<u_1^2, u_2>, <1, u_1>),$$
$$h(u_2) \in 1,\ H(u_1^2) = h(u_1^2),\ H(u_1 u_2) = u_1 h(u_2) \,\mathrm{and}\, H(u_2^2) = 0 \bmod u_2\}.$$

It is obvious that this is the (proper) hyperplane of the tangent space

$$T_{\mathrm{Hilb}^3(X), x_2} = \mathrm{Hom}\,(< u_1^2, u_2^2, u_1 u_2 >, < u_1, u_2 >),$$

described as

$$\{h \in \mathrm{Hom}\,(< u_1^2, u_2^2, u_1 u_2 >, < u_1, u_2 >) /\ h(u_2^2) = 0 \bmod u_2\}, \tag{2.8}$$

so that Y_ϕ is smooth at x_2. □

This computation shows in fact more

LEMMA 5. — *The tangent space to Y_ϕ at x_2 depends only on the tangent space to the curve X_t, $t = \phi(x)$ at x. Furthermore the map*

$$f : \mathbb{P}^1 = \mathbb{P}(\Omega_{X,x}) \to \mathbb{P}^5 = \mathbb{P}(\Omega_{\mathrm{Hilb}^3(X), x_2})$$

which to the hyperplane $T_{X_t,x}$ associates the tangent space to Y_ϕ at x_2, is given by the complete linear system of cubics on $\mathbb{P}^1$.

Proof. — We have seen that in local holomorphic coordinates u_1, u_2 such that $\phi(u_1,u_2) = u_2 + a$, we have

$$T_{Y_\phi,x_2} = \{h \in \mathrm{Hom}_{\mathbb{C}}(< u_1^2, u_2^2, u_1u_2 >, < u_1, u_2 >)/\, h(u_2^2) \text{ proportional to } u_2\}.$$

It is immediate to see that this hyperplane remains unchanged under a change of coordinates such that $u_2' = \alpha u_2 \bmod \mathcal{M}_x^2$, which proves the first statement.

Next, the above formula says that the map $f : \mathbb{P}(\Omega_{X,x}) \to \mathbb{P}(\mathrm{Hom}(S^2\Omega_{X,x}, \Omega_{X,x}))$ sends η to the hyperplane

$$H_\eta = \{h \in \mathrm{Hom}(S^2\Omega_{X,x}, \Omega_{X,x}))/\, h(\eta^2) \text{ proportional to } \eta\}.$$

Writing this map explicitely (or arguing by $PGl(2)$-invariance) shows that it is given by the complete linear system of cubics on $\mathbb{P}^1$. □

COROLLARY 1. — *The locally defined hypersurfaces Y_ϕ cut out schematically (and locally) the smooth subvariety $X_2 \subset \mathrm{Hilb}^3(X)$.*

Indeed, by Lemma 5, the intersection of the tangent spaces to the hypersurfaces Y_ϕ is of codimension at least 4 in $T_{\mathrm{Hilb}^3(X),x_2}$ and this is the codimension of T_{X_2,x_2}. □

It turns out that the analogue of the hypersurfaces Y_ϕ (near each point $x_2 \in \mathrm{Hilb}^3(X)$) can be constructed as differentiable varieties in the almost complex case, so that their intersections with $\mathrm{Hilb}^3_{\mathrm{curv}}(X)$ are smooth codimension 2 real subvarieties. This is done as follows : choose in a neighbourhood U of $x \in X$ a submersive map

$$\phi : X \to \mathbb{C}\,,\ \phi(x) = 0,$$

such that each fiber X_t of ϕ is a pseudoholomorphic curve w.r.t. J. Then consider

$$Y_\phi := \{z \in \mathrm{Hilb}^3(U)/\exists t \in \mathbb{C}\,,\ l(z \cap X_t) = 2\}.$$

Here the intersection $z \cap X_t$ is a subscheme of X_t defined in the natural way : if z is not curvilinear and supported in $y \in X_t$, $z \cap X_t$ is the first infinitesimal neighbourhood of y in X_t. Otherwise writing $z = \sqcup z_i$ with z_i curvilinear supported at y_i the intersection $z \cap X_t$ is the union for $y_i \in X_t$ of the subschemes of X_t supported at y_i and whose length is equal to 1+

the order of coincidence of the equivalence class of the jets of X_t and z_i. It is easy to see that Y_ϕ is a locally closed subset of $\mathrm{Hilb}^3(X)$ which contains U_2. We have now another description of Y_ϕ. Let $\psi : U \to \mathbb{C}$ be a submersive map satisfying the conditions that its fibers are pseudoholomorphic curves and that for any $y \in U$, the curves $X_y^\phi := \phi^{-1}(\phi(y))$ and $X_y^\psi := \psi^{-1}(\psi(y))$ meet transversely at exactly one point (namely y). Define then

$$Y_{\phi,\psi} = \Big(\bigcup_{y\in U} \widetilde{(X_y^\phi)^{(2)} \times X_y^\psi} \Big)^0,$$

where the symbol $\widetilde{\ }$ means the blow-up along the subvariety $(y + X_y^\phi) \times \{y\}$ and the upperscript 0 means that we consider the complementary set of the proper transform of the surface $(X_y^\phi)^{(2)} \times \{y\}$ (this surface is made of subschemes which are contained in $(X_y^\phi)^{(3)}$). Clearly $Y_{\phi,\psi}$ has a natural differentiable structure. Furthermore, if $E = \cup_{y\in U} E_y$ is the exceptional divisor, there is a natural map

$$f : Y_{\phi,\psi} - E \to \mathrm{Hilb}^3_{\mathrm{curv}}(X),$$

which is easily seen to be an immersion whose image is contained in Y_ϕ. We have now

PROPOSITION 3. — *The map f extends to a continuous homeomorphism*

$$Y_{\phi,\psi} \to Y_\phi \subset \mathrm{Hilb}^3(X)$$

which restricts to a differentiable immersion

$$Y_{\phi,\psi} - U_\infty \hookrightarrow \mathrm{Hilb}^3(U) - U_2.$$

In particular, Y_ϕ has a natural differentiable structure.

Here U_∞ is a copy of U contained in the exceptional divisor of $Y_{\phi,\psi}$ and is defined as follows: for each y, consider the exceptional divisor E_y which is a $\mathbb{P}^1$-bundle over X_y^ϕ. Its fiber over $y \in X_y^\phi$ is isomorphic to $\mathbb{P}(N_y)$ where N_y is the normal bundle of $(y+X_y^\phi)\times\{y\}$ in $(X_y^\phi)^{(2)}\times X_y^\psi$ at (y, y, y). Then $\mathbb{P}(N_y)$ contains the hyperplane tangent to the surface $(X_y^\phi + y) \times X_y^\psi$ at y, and this defines the point y_∞.

Proof. — Indeed, for each $y \in X$, the singular curve $X_y := X_y^\phi \cup_y X_y^\psi$ has a natural holomorphic structure, for which the inclusion i_y into X is pseudoholomorphic. Hence we have the Hilbert scheme $\mathrm{Hilb}^3_{\mathrm{curv}}(X_y)$, $\mathrm{Hilb}^3(X_y)$ and inside it the open sets of certain irreducible components

$$\mathrm{Hilb}^{2,1}_{\mathrm{curv}}(X_y),\ \mathrm{Hilb}^{2,1}(X_y)$$

which are the sets of subschemes $z \subset X_y$ such that $l(X_y^\phi \cap z) = 2$. Now it is immediate to check that

$$\mathrm{Hilb}^{2,1}_{\mathrm{curv}}(X_y) \cong (\widetilde{(X_y^\phi)^{(2)} \times X_y^\psi})^0 - y_\infty, \mathrm{Hilb}^{2,1}(X_y) \cong (\widetilde{(X_y^\phi)^{(2)} \times X_y^\psi})^0.$$

Now, since i_y is a pseudoholomorphic immersion, we have a differentiable immersion

$$i_y : \mathrm{Hilb}^3_{\mathrm{curv}}(X_y) \to \mathrm{Hilb}^3_{\mathrm{curv}}(X)$$

which extends the map f. It is indeed defined using the definition of $\mathrm{Hilb}_{\mathrm{curv}}(X_y)$ using jets, and the differentiability is proved using the description of $\mathrm{Hilb}_{\mathrm{curv}}(X)$ given in the previous section.

This extension varies differentiably with the parameter y, hence provides the desired extension of f to $Y_{\phi,\psi} - U_\infty$. Looking at the topologies near the non curvilinear points, one shows that this extension extends continuously to a map

$$\tilde{f} : Y_{\phi,\psi} \to \mathrm{Hilb}^3(X).$$

It is obvious that $\tilde{f}$ takes value in Y_ϕ. That $\tilde{f}$ is a homeomorphism onto Y_ϕ follows from the fact that one can invert it: an element $z \in Y_\phi$ meets some curve X_t^ϕ along a subscheme w of length 2, and t is unique since $X_t \cap X_{t'} = \emptyset$, $t \neq t'$. On the other hand, there is a well defined residual point $x = z - w$ and z is contained in the singular curve $X_t^\phi \cup X_x^\psi$ which is a curve X_y for some y. But then $z \in \mathrm{Hilb}^{2,1}(X_y)$ hence determines a point of $Y_{\phi,\psi}$. The fact that $\tilde{f}$ is an immersion along the curvilinear part $Y_{\phi,\psi} - U_\infty$ is not difficult to prove using the description of the differentiable structure of $\mathrm{Hilb}^3_{\mathrm{curv}}(X)$ given in the previous section. □

The hypersurfaces Y_ϕ will be used to construct a differentiable structure on $\mathrm{Hilb}^3(U)$ near U_∞ as follows. First of all, one constructs a map

$$\Phi : U \times \mathbb{P}^1 \to S$$

over $\mathbb{P}^1$, where $\mathbb{P}^1 = \mathbb{P}(\Omega^{1,0}_{X,x})$ and $S \to \mathbb{P}^1$ is the total space of the bundle $\mathcal{O}(1)$, satisfying the following properties:

i) Each $\Phi_t : U \to \mathbb{C}$, $t \in \mathbb{P}^1$, is submersive, takes value 0 at x and has pseudoholomorphic fibers.

ii) The map $\mathbb{P}^1 \to \mathbb{P}(T^{1,0}_{X,x})$, which to t associates $T_{X_{\Phi_t^{-1}(0)},x} \subset T^{1,0}_{X,x}$ is the natural isomorphism $\mathbb{P}(T^{1,0}_{X,x}) \cong \mathbb{P}(\Omega^{1,0}_{X,x})$, $u \mapsto u^\perp$.

Notice that property ii) will then also imply that the map which to t associates $T_{X_{\Phi_t^{-1}(\Phi_t(y))},y} \subset T^{1,0}_{X,y}$ is an isomorphism for y close to x. Hence we may assume this property to be true in U.

Then for each $t \in \mathbb{P}^1$ we have the "hypersurface" $Y_t := Y_{\Phi_t}$ of $\mathrm{Hilb}^3(X)$ which contains U_2, has a differentiable structure and is immersed in $\mathrm{Hilb}^3(X)$ away from U_∞. Now let $y \in U$, $t \in \mathbb{P}^1$ and denote by $K_{y,t}$ the complex threefold $(\widetilde{(X_y^{\Phi_t})^{(2)}} \times X_y^{\psi})^0$ which is contained in Y_t by the previous lemma. Here $\psi : U \to \mathbb{C}$ is an auxiliary map which is submersive, and has pseudoholomorphic fibers transverse to the fibers of Φ_t. We can make explicit infinitesimal computations in $K_{y,t} = \mathrm{Hilb}^{2,1}(X_y^{\Phi_t} \cup_y X_y^{\psi})$ exactly as in the proof of Lemmas 4, 5, which gives the following result; introduce first the surface $S_{y,t} \subset K_{y,t}$, parametrizing length 3 subschemes of $X_y = X_y^{\phi} \cup X_y^{\psi}$ which are the union of a point of X_y^{ϕ} and of a length 2 subscheme of X_y supported on y. Notice that there is a one dimensional family of such length 2 subschemes, since the Zariski tangent space of X_y at y has rank 2. We have

LEMMA 6. — *The subsets $Y_{t'} \cap K_{y,t}$ for $t \neq t'$ are codimension two real subvarieties of $K_{y,t}$ containing the point y_∞ and smooth at y_∞. Furthermore their tangent space at y_∞ is a complex hyperplane of $T_{K_{y,t},y_\infty}$. This remains true for $t = t'$ if one defines $Y_t \cap K_{y,t}$ near the point y_∞ as the surface $S_{y,t}$. Furthermore the map*

$$\mathbb{P}^1 \to T_{K_{y,t},y_\infty}$$

which to $t' \in \mathbb{P}^1$ associates the tangent complex hyperplane to $Y_{t'} \cap K_{y,t}$ at y_∞ is given by the complete linear system of quadrics on $\mathbb{P}^1$.

The proof is exactly the same as for Lemmas 4, 5. The only point to note here is the smooth convergence of the hypersurfaces $Y_{t'} \cap K_{y,t}$ near y_∞ to the surface $S_{y,t}$, which is easy. □

COROLLARY 2. — *The dimension 4 real subvarieties*

$$K_{y,t,t',t''} := K_{y,t} \cap Y_{t'} \cap Y_{t''}$$

of $K_{y,t}$ are smooth at y_∞ and the natural differentiable map

$$\bigsqcup_{t',t''} K_{y,t,t',t''} \to K_{y,t}$$

identifies differentiably the left hand side to the blow-up of $K_{y,t}$ at y_∞.

Now we can let y move, and since everything varies differentiably with y, we conclude

COROLLARY 3. — *$\mathrm{Hilb}^3(U)$ contains for each triple $\{t, t', t''\} \in (\mathbb{P}^1)^{(3)}$ the smooth real 6-dimensional variety $Y_t \cap Y_{t'} \cap Y_{t''}$, which contains*

U_2. This variety varies differentiably with the parameter $\{t,t',t''\} \in (\mathbb{P}^1)^{(3)}$. Furthermore the natural map

$$\bigsqcup_{\{t',t''\}\in\mathbb{P}^1} Y_t \cap Y_{t'} \cap Y_{t''} \to Y_t$$

identifies the left hand side to the blow-up of Y_t along X_2.

This blow-up indeed makes sense in the differentiable category, since it is clear that the normal bundle of X_2 in Y_t has a natural complex structure : it is indeed isomorphic at y_2 to the tangent space to $K_{y,t}$ at y_∞. □

Now we are almost done. From the last corollary, we get a 12-dimensional differentiable variety

$$\mathcal{Y} = \bigsqcup_{\{t,t',t''\}\in(\mathbb{P}^1)^{(3)}} Y_t \cap Y_{t'} \cap Y_{t''}$$

and a map

$$g : \mathcal{Y} \to \mathrm{Hilb}^3(U)$$

which is continuous and clearly differentiable where this makes sense, i.e in $g^{-1}(\mathrm{Hilb}^3_{\mathrm{curv}}(X))$. We show now

LEMMA 7. — *The map g induces a diffeomorphism from $\mathcal{Y} - F$ onto $V - X_2$, where $V \subset \mathrm{Hilb}^3(U)$ is the neighbourhood of U_2 consisting of those subschemes not contained in any fiber of one of the maps Φ_t, and $F := g^{-1}(U_2)$.*

Proof. — The inverse map is obtained by constructing a differentiable map

$$\chi : \mathrm{Hilb}^3_{\mathrm{curv}}(X) \to (\mathbb{P}^1)^{(3)}$$

defined in $V - X_2$. This map is obtained by noting that for any pseudo-holomorphic subscheme of length 2 supported in U there exists exactly one $t \in \mathbb{P}^1$ such that w is contained in one fiber of the map Φ_t. Hence we get a differentiable map

$$\mu : \mathrm{Hilb}^2(X) \to \mathbb{P}^1.$$

Next one can use the incidence variety

$$\mathrm{Hilb}^{2,3}_{\mathrm{curv}}(X) = \{(w,z) \in \mathrm{Hilb}^2(X) \times \mathrm{Hilb}^3_{\mathrm{curv}}(X) \mid w \subset z\}.$$

Using the description of $\mathrm{Hilb}_{\mathrm{curv}}(X)$ given in the previous section, it is easy to show that this is a smooth variety and that the natural map

$$pr_2 : \mathrm{Hilb}^{2,3}_{\mathrm{curv}}(X) \to \mathrm{Hilb}^3_{\mathrm{curv}}(X)$$

is finite of degree 3 (i.e. every fiber is a set of 3 points counted with positive multiplicities). It follows that we have a composite (differentiable) map

$$\nu : \mathrm{Hilb}^3_{\mathrm{curv}}(X) \to (\mathrm{Hilb}^2_{\mathrm{curv}}(X))^{(3)} \xrightarrow{\mu^{(3)}} (\mathbb{P}^1)^{(3)}.$$

By construction and by definition of V, $\nu^{-1}(\{t,t',t''\}) \cap V$ is contained in $Y_t \cap Y_{t'} \cap Y_{t''} \subset \mathcal{Y}$, which gives the inverse map g^{-1}. □

Finally, over each $y_2 \in X_2$ the fiber $g^{-1}(y_2)$ clearly identifies to $(\mathbb{P}^1)^{(3)}$ since all the varieties $Y_t \cap Y_{t'} \cap Y_{t''}$ contain U_2. But $(\mathbb{P}^1)^{(3)}$ is also isomorphic to $\mathbb{P}^3$ so that the subvariety (of real codimension 2) F of $\mathcal{Y}$ is a $\mathbb{P}^3$-bundle over U_2. To get a differentiable structure on $\mathrm{Hilb}^3(U)$ it suffices now to show that F can be contracted in $\mathcal{Y}$ onto U_2 in the differentiable category. For this it suffices to show that the Euler class of the (real oriented rank two) normal bundle of F in $\mathcal{Y}$ restricts on each $\mathbb{P}^3_{y_2}$ (fiber of g over y_2) to the Chern class $c_1(\mathcal{O}_{\mathbb{P}^3}(-1))$. But this follows from Corollary 3, since it says that for each t, the subvariety

$$\tilde{Y}_t := U_{\{t',t''\}\in\mathbb{P}^1} Y_t \cap Y_{t'} \cap Y_{t''}$$

is the blow-up of Y_t along X_2, so that the exceptional variety $F_t = \tilde{Y}_t \cap F$ has a normal bundle in $\tilde{Y}_t$ which restricts on each $\mathbb{P}^2_{y_2}$ (fiber of $g_{|\tilde{Y}_t}$ over y_2) to the Chern class $c_1(\mathcal{O}_{\mathbb{P}^2}(-1))$. On the other hand $N_{F_t/\tilde{Y}_t}$ is the restriction of $N_{F/\mathcal{Y}}$ to F_t and the last $\mathbb{P}^2$-fibers are hyperplanes sections $\mathbb{P}^3_y \cap \tilde{Y}_t$. Hence we have

$$e(N_{F/\mathcal{Y}})_{|\mathbb{P}^2_{y_2}} = c_1(\mathcal{O}_{\mathbb{P}^2}(-1))$$

which implies

$$e(N_{F/\mathcal{Y}})_{|\mathbb{P}^3_{y_2}} = c_1(\mathcal{O}_{\mathbb{P}^3}(-1)).$$

To be more precise, one can show using the above construction that there exists a unique differentiable structure on the contraction of F to U_2 in $\mathcal{Y}$, for which the map g is differentiable, and which induces the differentiable structure already defined on each of the hypersurfaces

$$Y_\phi \subset \mathrm{Hilb}^3(U).$$

□

To conclude the construction of a differentiable structure on $\mathrm{Hilb}^3(X)$, it would suffice to show the compatibility of these local constructions, which we leave to the reader.

A similar (more complicated) study can be done in the case of $\mathrm{Hilb}^4(X)$, hence more generally for the open set $\mathrm{Hilb}^k_{\mathrm{curv}}(X)''$ parametrizing pseudoholomorphic subschemes which are at each of their points either curvilinear or of length ≤ 4. The point is that all the length 4 finite subschemes of a complex surface X can be described using only the almost complex structure. Namely they are either curvilinear, or the union of the first infinitesimal neighbourhood of a point and of another reduced point, or supported at one point x and defined by a colength 4 ideal containing $\mathcal{M}_x^3$ and contained in $\mathcal{M}_x^2$; such an ideal is determined by an hyperplane in

$$\mathcal{M}_x^2/\mathcal{M}_x^3 \cong S^2\Omega_{X,x}.$$

So this last set that we denote by X_2' is parametrized by the $\mathbb{P}^2$-bundle $\mathbb{P}(S^2\Omega_X)$ over X. Hence we can define similarly in the almost complex case $\mathrm{Hilb}^4(X)$ as the disjoint union of $\mathrm{Hilb}^4_{\mathrm{curv}}(X)$, $X_2 \times X - \mathrm{Diag}$ and $X_2' = \mathbb{P}(S^2\Omega_X^{(1,0)})$. We have a similar set theoretic description of $\mathrm{Hilb}^k_{\mathrm{curv}}(X)''$ as the quotient by the adequate permutation group of an open set of a union of products of the above with smaller dimensional Hilbert schemes.

Then in the case of $\mathrm{Hilb}^4(X)$ one can show that there is a differentiable structure on this set, which makes it a 16-dimensional compact variety. This gives immediately the analogous result for $\mathrm{Hilb}^k_{\mathrm{curv}}(X)''$.

3. A general construction for the Hilbert scheme.

We will consider in this section a differentiable fourfold X endowed with an almost complex structure J of class $\mathcal{C}^\infty$. Our goal is to construct for each n a manifold $\mathrm{Hilb}^n(X)$ of real dimension $4n$ having the following properties:

i) There is a continuous proper map

$$c : \mathrm{Hilb}^n(X) \to X^{(n)},$$

which is a diffeomorphism over $X_0^{(n)}$. More generally the fiber of c over a cycle $z = \sum_i n_i x_i \in X^{(n)}$ admits an identification to the product over i of the singular varieties $\mathrm{Hilb}^{n_i}(\mathbb{C}^2)_0$ parametrizing subschemes of length n_i supported at 0. (This is exactly the description of the fibers of the Hilbert-Chow map for the Hilbert scheme associated with an integrable complex structure.)

ii) When the almost complex structure J is integrable, $\mathrm{Hilb}^n(X)$ is diffeomorphic to the Hilbert scheme relative to the complex structure I.

iii) The construction behaves well under deformation. In particular, the manifold $\mathrm{Hilb}^n(X)$ depends only on the deformation class of J.

iv) $\mathrm{Hilb}^n(X)$ has a stable almost complex structure, which is in the same cobordism class as the one given by the complex structure when J is a deformation of an integrable complex structure.

v) The manifold $\mathrm{Hilb}^n(X)$ is well defined up to diffeomorphisms isotopic to the identity.

Remark 1. — The symmetric product $X^{(n)}$ can (and will in the sequel) be considered as a singular differentiable variety (its set of differentiable functions is given by the differentiable functions on X^n that are invariant under the symmetric group S_n). However the map c is never differentiable for this differentiable structure.

The construction is as follows: Consider $Z \subset X^{(n)} \times X$, the incidence subset

$$Z = \{(z, x) \in X^{(n)} \times X / \, x \in z\}.$$

We will show

PROPOSITION 4. — *There exists a neighbourhood $W \subset X^{(n)} \times X$ of Z, and a relative integrable complex structure I on $W/X^{(n)}$, which varies differentiably with the parameter $z \in X^{(n)}$.*

Concretely, this means that for each $z \in X^{(n)}$, there is a neighbourhood $W_z \subset X$ of $\mathrm{Sup}\, z$, and a (integrable) complex structure I_z on W_z, which varies differentiably with z. Notice that such an object implies conversely that X has an almost complex structure. Indeed it suffices to restrict I to the subset $W_n = \mathrm{pr}_1^{-1}(X_n^{(n)})$ of W, where $X_n^{(n)} \cong X$ is the set of cycles supported at one point. Then W_n is a neighbourhood of the diagonal Δ in $X \times X$ and the relative complex structure I provides a complex structure on the relative (with respect to pr_1) tangent bundle $T_{X\times X/X}{}_{|\Delta}$, which is isomorphic to T_X.

The proof of the proposition will in fact exhibit a family of such relative integrable complex structures, parametrized by a contractible basis, so that the construction below gives a family of manifolds (compact when X is) parametrized by a contractible basis, hence, by Ehresman fibration theorem, a manifold well defined up to diffeomorphisms isotopic to the identity.

We use this integrable complex structure I as follows: first of all we perform the relative construction of the Hilbert scheme for the family of complex structures I_z on W_z. This gives a singular differentiable variety

$$\mathrm{Hilb}_I^n(W/X^{(n)}) \xrightarrow{\pi} X^{(n)},$$

which is relatively smooth by [5]. Next consider the relative Hilbert-Chow map

$$c_{\mathrm{rel}} : \mathrm{Hilb}^n_I(W/X^{(n)}) \to W^{(n)/X^{(n)}}.$$

Since W is open in $X^{(n)} \times X$ and the map $W \to X^{(n)}$ is the first projection, we have also an open inclusion

$$i : W^{(n)/X^{(n)}} \to X^{(n)} \times X^{(n)}.$$

Defining

$$C_{\mathrm{rel}} := i \circ c_{\mathrm{rel}} : \mathrm{Hilb}^n_I(W/X^{(n)}) \to X^{(n)} \times X^{(n)},$$

we have $\pi = \mathrm{pr}_1 \circ C_{\mathrm{rel}}$.The Hilbert scheme $\mathrm{Hilb}^n(X)$ will then be defined as

$$\mathrm{Hilb}^n(X) = C_{\mathrm{rel}}^{-1}(\mathrm{Diag}), \tag{3.9}$$

where now $\mathrm{Diag} \subset X^{(n)} \times X^{(n)}$ is the diagonal. We will then define $c : \mathrm{Hilb}^n(X) \to X^{(n)}$ as $\mathrm{pr}_1 \circ C_{\mathrm{rel}} = \mathrm{pr}_2 \circ C_{\mathrm{rel}}$. By definition, the fiber of c over $z \in X^{(n)}$ is equal to the fiber of $c_z : \mathrm{Hilb}^n_{I_z}(W_z) \to W_z^{(n)}$, which proves assertion i).

Note that the construction above defines $\mathrm{Hilb}^n(X)$ only as a topological space, since the map C_{rel} is continuous but not differentiable. We have now

THEOREM 5. — *For an adequate (family of) choice of the relative integrable complex structure I (which will be explicitely described in Proposition 5),* $\mathrm{Hilb}^n(X)$ *can be naturally endowed with the structure of a smooth manifold of dimension $4n$.*

Notice that by properness of the (relative) Hilbert-Chow map, $\mathrm{Hilb}^n(X)$ is always compact, when X is compact. Finally, we will show assertion iv)

THEOREM 6. — *The variety* $\mathrm{Hilb}^n(X)$ *defined above has a stable almost complex structure (i.e. there exists a complex structure on $T_{\mathrm{Hilb}^n(X)} \oplus T$, for some trivial bundle T on* $\mathrm{Hilb}^n(X)$*), whose complex cobordism class depends only on the deformation class of I.*

We now turn to the proofs of these statements. Let us first introduce the following notation: for each partition $S = \{S_1, \ldots, S_r\}$ of $\{1, \ldots, n\}$ into disjoint subsets, let $\Delta_S \subset X^n$ be the corresponding diagonal

$$\Delta_S = \{(x_1, \ldots, x_n) \in X^n /\, x_i = x_j \ if\ i,\, j \in S_l \ \text{ for some } l\}.$$

If S' is a refinement of S, one has $\Delta_S \subset \Delta_{S'}$. The number $\mid S \mid = r$ is also the number of distinct points in the n-uple corresponding to the general point of Δ_S. The symmetric product S_n acts in an obvious way on the set of partitions of $\{1, \ldots, n\}$ and we have $\Delta_{\sigma(S)} = \sigma(\Delta_S)$, where on the right S_n acts on X^n.

Everything will then follow from the following

PROPOSITION 5. — *There exists a relative complex structure $\tilde{I}$ defined on a neighbourhood W of the incidence set $\tilde{Z} \subset X^n \times X$ and for each partition S as above a differentiable retraction $R_S : X^n \to \Delta_S$ defined in a neighbourhood of Δ_S, satisfying the following conditions:*

1. *Everything is equivariant under the action of S_n. Hence we have $\sigma^* \tilde{I} = \tilde{I}$ and $R_{\sigma(S)} = \sigma \circ R_S \circ \sigma^{-1}$ for $\sigma \in S_n$.*

2. *The retractions R_S are compatible in the sense that if S' refines S, so that $\Delta_S \subset \Delta_{S'}$, one has $R_S = R_S \circ R_{S'}$ near Δ_S.*

3. *One has $\tilde{I} = (R_S)^*(I_S)$ near Δ_S, where $I_S := \tilde{I}_{|\Delta_S}$.*

4. *For $x \in \Delta_S$ the fiber $R_S^{-1}(x)$ is a holomorphic subvariety of a neighbourhood of x in X^n for the complex structure induced by I_x.*

(More precisely if $x = (x_1, \ldots, x_n)$, $\tilde{I}_x$ is a complex structure on $V_{x_1} \cup \ldots \cup V_{x_n}$, where the V_{x_i} are neighbourhoods of x_i in X, and we consider the induced complex structure on $V_{x_1} \times \ldots V_{x_n}$.)

We postpone the proof of this proposition until the end of the section. From now on we will denote by I the relative complex structure on $W \subset X^{(n)} \times X$ obtained by passage to the quotient from a $\tilde{I}$ satisfying the properties of Proposition 5 and we perform the construction of $\mathrm{Hilb}^n(X)$ as explained above. We show that for this I, $\mathrm{Hilb}^n(X)$ satisfies the conclusions of Theorems 5, 6.

Proof of Theorem 5. — We have to show that for $\tilde{I}$ as in the Proposition 5, $\mathrm{Hilb}^n(X) = C_{\mathrm{rel}}^{-1}(\mathrm{Diag})$ has a natural differentiable structure. Indeed, let $z \in X^{(n)}$, $z = \sum_{1 \le i \le r} n_i x_i$ and let $\tilde{z} \in X^n$ be a lift of z. Let Δ_S be the minimal diagonal in which $\tilde{z}$ lies, and let U, V be neighbourhoods of $\tilde{z}$ in Δ_S and in X^n respectively, in which the retraction map $R_S : V \to U$ is defined. If V is sufficiently small, the quotient V/S_S is naturally an open neighbourhood V' of z in $X^{(n)}$ and by Property 1, R_S induces a map

$$R'_S : V' \to U.$$

Here S_S is the subgroup of S_n leaving Δ_S pointwise invariant.

By Property 3, the complex structure I_z, $z \in V'$ satisfies the property that there is a relative integrable complex structure I_S on an open subset W' of $U \times X$, whose pull-back by R'_S is equal to I. Hence we get the following alternative definition of $\mathrm{Hilb}^n(X)$ over V': Consider the relative Hilbert scheme

$$\pi : \mathrm{Hilb}^n_{I_S}(W'/U) \to U.$$

It is smooth since U is smooth, and the Hilbert scheme is relatively smooth over U. Then we have the relative Hilbert-Chow map

$$c'_{\mathrm{rel}} : \mathrm{Hilb}^n_{I_S}(W'/U) \to W'^{(n)/V}$$

which combined with the open inclusion

$$W'^{(n)/U} \subset U \times X^{(n)}$$

gives a continuous map

$$C'_{\mathrm{rel}} : \mathrm{Hilb}^n_{I_S}(W'/U) \to U \times X^{(n)}.$$

Then using the Cartesian diagram

$$\begin{array}{cccc} & \mathrm{Hilb}^n_I(W_{V'}/V') & \longrightarrow & \mathrm{Hilb}^n_{I_S}(W'/U) \\ & \downarrow & & \downarrow \\ R'_S : & V' & \longrightarrow & U \end{array}$$

which is obtained using the fact that $I = R'^*_S.I_S$, we get a natural identification, as topological spaces

$$\mathrm{Hilb}^n(X) \cap c^{-1}(V') = C'^{-1}_{\mathrm{rel}}(\Gamma_R)$$

where $\Gamma_R \subset U \times V'$ is the graph of R'_S.

But this can be also translated as follows: let

$$\rho : \mathrm{Hilb}^n(X) \cap c^{-1}(V') \to U$$

denote the restriction of π to $\mathrm{Hilb}^n(X) \subset \mathrm{Hilb}^n(W'/U)$; then the fiber $\rho^{-1}(y) \subset \mathrm{Hilb}^n(W'_y)$ is equal to

$$(R'_S \circ c_y)^{-1}(y) = c_y^{-1}({R'_S}^{-1}(y)).$$

But by Property 4 in Proposition 5, we know that ${R'_S}^{-1}(y)$ is an analytic subset of $X^{(n)}$. More precisely it is the image in $V' = V/S_S$ of the S_S-invariant holomorphic subvariety $R_S^{-1}(\tilde{z})$ of X^n, which passes through $\tilde{z}$ and is transversal to Δ_S at $\tilde{z}$. It is then easy to see that the fiber $\rho^{-1}(y) = c_y^{-1}({R'_S}^{-1}(y))$ is a smooth holomorphic subvariety of $\mathrm{Hilb}^n(W'_y)$, varying differentiably with y, as $R_S^{-1}(\tilde{z}) \subset X^n$ varies differentiably with y. Hence

$$\mathrm{Hilb}^n(X) \cap c^{-1}(V') \subset \mathrm{Hilb}^n(W'/U)$$

is a smooth submanifold.

To conclude that there is a differentiable structure on $\mathrm{Hilb}^n(X)$, it suffices now to verify the compatibility of the differentiable structures defined above on different open sets of $\mathrm{Hilb}^n(X)$. But this follows immediately from the compatibility Properties 2 and 3 satisfied by the R_S, $\tilde{I}$. □

Proof of Theorem 6. — The existence of a stable almost complex structure on $\mathrm{Hilb}^n(X)$ is proved as follows. If one forgets the singularities of $X^{(n)}$, one has just to note the two exact sequences of real "vector bundles" given by the relative tangent bundles sequence of $\pi : \mathrm{Hilb}^n_I(W/X^{(n)}) \to X^{(n)}$ and the normal bundle sequence of $\mathrm{Hilb}^n(X)$ in $\mathrm{Hilb}^n_I(W/X^{(n)})$:

$$(3.10) \qquad 0 \to T_{\mathrm{Hilb}^n_I(W/X^{(n)})/X^{(n)}} \to T_{\mathrm{Hilb}^n_I(W/X^{(n)})} \xrightarrow{\pi_*} \pi^* T_{X^{(n)}} \to 0,$$

defining the relative tangent bundle of π, and

$$(3.11) \quad 0 \to T_{\mathrm{Hilb}^n(X)} \to T_{\mathrm{Hilb}^n_I(W/X^{(n)})}{}_{|\mathrm{Hilb}^n(X)} \to \pi^* T_{X^{(n)}}{}_{|\mathrm{Hilb}^n(X)} \to 0,$$

where $\pi^* T_{X^{(n)}}{}_{|\mathrm{Hilb}^n(X)}$ has been identified to the "normal bundle" of $\mathrm{Hilb}^n(X)$ in $\mathrm{Hilb}^n_I(W/X^{(n)})$, since $T_{X^{(n)}}$ is canonically identified to the normal bundle of Diag in $X^{(n)} \times X^{(n)}$.

Restricting now (3.10) to $\mathrm{Hilb}^n(X)$ and choosing a vector bundle K of even rank such that $\pi^* T_{X^{(n)}}{}_{|\mathrm{Hilb}^n(X)} \oplus K$ is a trivial vector bundle T on $\mathrm{Hilb}^n(X)$, we deduce from (3.10) and (3.11) isomorphisms

$$(3.12) \qquad \begin{aligned} T_{\mathrm{Hilb}^n_I(W/X^{(n)})}{}_{|\mathrm{Hilb}^n(X)} \oplus K &\cong T_{\mathrm{Hilb}^n_I(W/X^{(n)})/X^{(n)}}{}_{|\mathrm{Hilb}^n(X)} \oplus T \\ T_{\mathrm{Hilb}^n_I(W/X^{(n)})}{}_{|\mathrm{Hilb}^n(X)} \oplus K &\cong T_{\mathrm{Hilb}^n(X)} \oplus T. \end{aligned}$$

But the trivial bundle T is of even rank, hence has a complex structure and $T_{\mathrm{Hilb}^n_I(W/X^{(n)})/X^{(n)}}$ has a complex structure induced by the integrable complex structure on each fiber of π. Hence we conclude, combining these two isomorphisms that

$$T_{\mathrm{Hilb}^n(X)} \oplus T \cong T_{\mathrm{Hilb}^n_I(W/X^{(n)})/X^{(n)}}{}_{|\mathrm{Hilb}^n(X)} \oplus T$$

has a complex structure.

In order to make this argument correct, that is to take into account the singularities of $X^{(n)}$, we do the following. Notice that by the construction of the differentiable structure on $\mathrm{Hilb}^n(X)$, the locally defined maps

$$R'_S \circ c : \mathrm{Hilb}^n(X) \to \Delta_S$$

introduced in the Proof of Theorem 5 are differentiable.

Now we will prove

LEMMA 8. — *There exists a differentiable map*

$$\tilde{\phi} : X^n \to X^n$$

which is arbitrarily close to Id, commutes with the action of the symmetric group S_n and satisfies the following properties:

– Near Δ_S, $\tilde{\phi}$ takes value in Δ_S and factors through the retraction R_S.

– The relative complex structure $\tilde{I}$ satisfies $\tilde{\phi}^*\tilde{I} = \tilde{I}$.

Assuming the lemma, let $\phi : X^{(n)} \to X^{(n)}$ be map induced by $\tilde{\phi}$. We can choose then a differentiable embedding

$$i : X^{(n)} \hookrightarrow M$$

into a smooth manifold and extend the relative complex structure I to a relative complex structure I_M on $W_M \to M$, a neighbourhood of Z in $M \times X$.

We then have the relative Hilbert-Chow map

$$C_{\mathrm{rel}} : \mathrm{Hilb}^n(W_M/M) \xrightarrow{c_{\mathrm{rel}}} W_M^{(n)/M} \hookrightarrow M \times X^{(n)}$$

and there is a natural homeomorphism

$$\chi : \mathrm{Hilb}^n(X) \cong C_{\mathrm{rel}}^{-1}(\Gamma_\phi) \subset \mathrm{Hilb}^n(W_M/M),$$

where $\Gamma_\phi \subset M \times X^{(n)}$ is the graph of $i \circ \phi$. Here χ is obtained as the composite of the inclusion

$$\mathrm{Hilb}^n(X) \subset \mathrm{Hilb}^n(W/X^{(n)})$$

and of the natural map induced by $i \circ \phi$

$$\mathrm{Hilb}^n(W/X^{(n)}) \to \mathrm{Hilb}^n(W_M/M),$$

using the fact that $(i \circ \phi)^* I_M = I$.

We show now

LEMMA 9. — *The continuous map χ is a differentiable immersion. Furthermore the normal bundle of* $\mathrm{Hilb}^n(X) \cong \chi(\mathrm{Hilb}^n(X))$ *in* $\mathrm{Hilb}^n(W_M/M)$ *is naturally isomorphic to $(\phi \circ c)^* T_M$.*

Proof. — Let $z \in X^{(n)}$ and let $\tilde{z}$ be a lifting of z in X^n. Let Δ_S be the smallest diagonal in which $\tilde{z}$ lies, and let U, V be small neighbourhoods of

$\tilde{z}$ in Δ_S and X^n respectively so that the retraction R_S induces a retraction $R_S : V \to U$. Then the notations being as in the proof of Theorem 5, we have an open set $c^{-1}(V')$ of $\mathrm{Hilb}^n(X)$ containing $c^{-1}(z)$, which is a differentiable submanifold of $\mathrm{Hilb}^n(W'/U)$. We will give the proof when ϕ is equal to R'_S near z. (In general, one can by assumption write locally $\phi = \phi' \circ R'_S$ with ϕ' a differentiable map from U to Δ_S preserving the relative complex structure I_S and the result follows in the same way.)

Now, when $\phi = R'_S$, the map χ is simply the composite of the differentiable inclusions $c^{-1}(V') \hookrightarrow \mathrm{Hilb}^n(W'/U)$ (written above) and $\mathrm{Hilb}^n(W'/U) \overset{i}{\hookrightarrow} \mathrm{Hilb}^n(W_M/M)$. This proves the first statement.

It remains to compute the normal bundle of the immersion χ. Even if $\chi(\mathrm{Hilb}^n(X)) = C_{\mathrm{rel}}^{-1}(\Gamma_\phi)$, it is not obvious that its normal bundle is isomorphic to $(\phi \circ c)^* T_M$, that is to the pull-back of the "normal bundle" of the graph of Γ_ϕ, because C_{rel} is not a differentiable map for the product differentiable structure on the open set $W_M^{(n)/M}$ of $M \times X^{(n)}$. However, using the relative complex structure I_M we get a relative analytic structure on $W_M^{(n)/M}$, hence a differentiable structure on it. (The differentiable functions are defined locally as the restrictions of differentiable functions on $\mathbb{C}^N$ for some local differentiable imbedding over M

$$W_M^{(n)/M} \hookrightarrow M \times \mathbb{C}^N$$

which is holomorphic on the fibers.)

Then since c_{rel} is holomorphic on fibers, C_{rel} is clearly differentiable for this differentiable structure. On the other hand, we note that the subsets $M \times z$, $z \in X^{(n)}$ of $M \times X^{(n)}$ are also differentiable subvarieties of $M \times X^{(n)}$ for this differentiable structure, (but they do not vary differentiably with z). Hence there is a natural continuous inclusion

$$\mathrm{pr}_1^* T_M \subset T_{M \times X^{(n)}}$$

where the right hand side is the "Zariski" tangent sheaf for the refined differentiable structure.

Finally we note that Γ_ϕ is a differentiable subvariety of $M \times X^{(n)}$ for this differentiable structure: indeed we have inclusions

$$\Gamma_\phi \subset U \times X^{(n)} \subset M \times X^{(n)},$$

where the second inclusion is an immersion, and the fiber of Γ_ϕ over $z' \in U$ in the first inclusion is equal to ${R'_S}^{-1}(z')$, which is an analytic subset of $X^{(n)}$ for the complex structure $I_{z'}$ by property 4 in Proposition 5.

This description of Γ_ϕ shows also easily that it can be locally defined in $M \times X^{(n)}$ by $\dim_{\mathbb{R}} M$ equations with independent differentials for the refined differentiable structure. Indeed each fiber ${R'_S}^{-1}(z') \subset {W'}^{(n)}_{z'}$ is a local complete intersection of complex codimension equal to dim U and transversal to the singularities of ${W'}^{(n)}_{z'}$. Hence $\Gamma_\phi \subset U \times X^{(n)}$ is clearly a singular differentiable subvariety for the refined differentiable structure. Furthermore Γ_ϕ has a locally free normal bundle and one sees easily that the composite map

$$\mathrm{pr}_1^* T_M \subset T_{M \times X^{(n)}} \rightarrow N_{\Gamma\phi}$$

is an isomorphism. Then the isomorphism

$$(3.13) \qquad (\phi \circ c)^* T_M \cong N_{\chi(\mathrm{Hilb}^n(X))/\,\mathrm{Hilb}^n(W_M/M)}$$

is obtained as the pull-back by C_{rel} of this isomorphism.

Note that the isomorphism 3.13 is continuous but not necessarily differentiable although both sides have the structure of differentiable vector bundles. □

Using Lemma 9, we conclude exactly as before. Indeed we have the relative tangent bundle sequence for $\mathrm{Hilb}^n(W_M/M)$ and the normal bundle sequence of $\mathrm{Hilb}^n(X)$ in $\mathrm{Hilb}^n(W_M/M)$. Since by the above computation the normal bundle is naturally isomorphic to the pull-back of the tangent bundle of the basis, we get a stable isomorphism between $T_{\mathrm{Hilb}^n(X)}$ and the restriction to $\mathrm{Hilb}^n(X)$ of the relative tangent bundle $T_{\mathrm{Hilb}^n(W_M/M)/M}$. Then the stable almost complex structure on $\mathrm{Hilb}^n(X)$ will come as before from the natural complex structure on the vector bundle $T_{\mathrm{Hilb}^n(W_M/M)/M}$.

The fact that the isotopy class of the variety $\mathrm{Hilb}^n(X)$ and the cobordism class of this stable almost complex structure depend only on the deformation class of J will follow from the fact that the data $\tilde{I}, R_S$, which are the supplementary parameters introduced in our construction are well defined up to homotopy preserving the properties stated in Proposition 5. More precisely, assuming they are constructed as in the proof of Proposition 5, one checks easily that they are parametrized by a contractible basis, at least for those which are close enough to J_S^g, R_S^g, a metric g on X being fixed (here the notations are those of Proposition 5). Hence the proof of Theorem 6 is finished, assuming Lemma 8. □

Proof of Lemma 8. — Let Δ_k be the union of the diagonals Δ_S with $| S | \leq k$. We will construct inductively a differentiable map $\tilde{\phi}_k : X^n \rightarrow X^n$, defined in a neighbourhood of Δ_k, taking value in Δ_k and satisfying all the properties stated. Since $\Delta_n = X^n$, we will then put $\tilde{\phi} = \tilde{\phi}_n$.

To start the induction, we define $\tilde{\phi}_1 = R_{S_1}$ where S_1 is the smallest diagonal. By the properties of $\tilde{I}$ stated in Proposition 5, it satisfies the properties needed.

Assume now that $\tilde{\phi}_k$ has been constructed. Let W_k be a neighbourhood of Δ_k in which $\tilde{\phi}_k$ is defined and let W'_k be another neighbourhood such that $\overline{W'}_k \subset W_k$. Let now W_{k+1} be a neighbourhood of Δ_{k+1} such that $W_{k+1} - W'_k$ is a disjoint union of components W^S_{k+1} indexed by the partitions S with $| S |= k+1$. We may assume that R_S is defined in W^S_{k+1}.

We will define $\tilde{\phi}_{k+1}$ to be equal to $\tilde{\phi}_k$ in W'_k and to R_S in W^S_{k+1}. It remains now to construct $\tilde{\phi}_k$ in $W_{k+1} \cap (W_k - W'_k) = \sqcup_S W^S_{k+1} \cap W_k$. To do this we construct on each $W^S_{k+1} \cap W_k$ a homotopy $(H_t)_{t\in[0,1]}$ between $\tilde{\phi}_k$ and R_S; then choosing a function f, which takes value 0 near $\partial W'_k$ and 1 near ∂W_k, we define $\tilde{\phi}_{k+1}$ in $W^S_{k+1} \cap W_k$ by the formula

$$\tilde{\phi}_{k+1} = H_{f(x)}(x). \tag{3.14}$$

It clearly will glue with the previously defined $\tilde{\phi}_{k+1}$ to give a differentiable map defined in W_{k+1}. This differentiable map satisfies the properties stated in the lemma outside $W_{k+1} \cap (W_k - W'_k)$ because $\tilde{\phi}_k$ and the $R'_S s$ do.

It remains to see that we can construct the homotopy and the function f in order to satisfy these properties in $W_{k+1} \cap (W_k - W'_k)$. But we know that $\tilde{\phi}_k$ takes value in Δ_k and factors locally through the retractions $R_{S'}$ on $\Delta_{S'}$, $| S' |\leq k$. Since by Proposition 5, each retraction $R_{S'}$ factors through R_S in each $W^S_{k+1} \cap W_k$ we can write (up to shrinking W_k)

$$\tilde{\phi}_k = \psi_S \circ R_S$$

in $W^S_{k+1} \cap W_k$, where ψ_S is a differentiable map from Δ_k into itself which satisfies the property that $\psi^*_S(\tilde{I}_{|\Delta_S}) = \tilde{I}_{|\Delta_S}$.(Here we use the fact that by choosing W_{k+1} sufficiently small we may assume that in each W^S_{k+1} we have $\tilde{I} = R^*_S I_S$.)

We then simply choose a homotopy $(K_t)_{t\in[0,1]}$ between ψ_S and Id on Δ_k so that $K^*_t(\tilde{I}_{|\Delta_k}) = \tilde{I}_{|\Delta_k}$ and define

$$H_t = K_t \circ R_S. \tag{3.15}$$

Furthermore we ask that the function f factors through R_S which is possible if W_{k+1} is sufficiently small.

It is not difficult to see using the inductive construction of $\tilde{\phi}_k$, that such a homotopy exists and that everything can be choosen to commute with the S_n-action. Then it is immediate to see, using the fact that K_t leaves

$\tilde{I}$ invariant, and the fact that in W^S_{k+1} the relative complex structure $\tilde{I}$ is of the form $R^*_S I_S$, that the differentiable map $\tilde{\phi}_{k+1}$ defined by the formulas (3.14), (3.15) in $W_{k+1} \cap (W_k - W'_k)$ leaves $\tilde{I}$ invariant and factors locally through the retractions R_S. □

To conclude, it remains now only to prove Proposition 5.

Proof of Proposition 5. — Let g be a riemannian metric on X. For each S, let Δ^0_S be the open set of points of Δ_S which do not belong to a smaller diagonal. We first construct a relative complex structure I^g_S defined in a neighbourhood of the incidence set $Z_S \subset \Delta^0_S \times X$, and a retraction $R^g_S : X^n \to \Delta_S$ defined in a neighbourhood of Δ_S, satisfying the property that for each $x \in \Delta^0_S$ the fiber $(R^g_S)^{-1}(x)$ is holomorphic for the complex structure induced by $I^g_{S,x}$.

For this we use the exponential map

$$\exp : N_{\Delta_S/X^n} \to X^n, \tag{3.16}$$

which is a diffeomorphism in the neighbourhood of the 0-section. Then we set $R^g_S = \exp \circ \pi_S \circ \exp^{-1}$, where $\pi_S : N_{\Delta_S/X^n} \to \Delta_S$ is the natural bundle map. Also for $x = (x_1, \ldots, x_n)$ with $x_{i_1}, \ldots, x_{i_k}$ distinct, $k = \mid S \mid$ we define W_x as the disjoint union of exponential balls $W_{x_i} \cong B_{\epsilon_i} T_{X,x_i}$ centered at x_i and we define I^g_x to be $\exp(J_{x_i})$ on W_{x_i}, where J_{x_i} is the constant hence integrable complex structure on T_{X,x_i} given by the almost complex structure J.

It is easy to see that the following properties are satisfied:

– Everything is S_n-equivariant. Namely we have $\sigma \circ R^g_S \circ \sigma^{-1} = R^g_{\sigma(S)}$, for $\sigma \in S_n$, and $\sigma^* I_{\sigma^{-1}(S)^g} = I^g_S$.

– For $x \in \Delta^0_S$ the fiber $(R^g_S)^{-1}(x)$ is a complex subvariety of X^n for the complex structure induced by I^g_x near x.

Unfortunately, the R^g_S, I^g_S do not satisfy the conditions $R^g_S = R^g_S \circ R^g_{S'}$ near Δ_S, for $\Delta_S \subset \Delta_{S'}$ and $I^g_{S'} = (R^g_S)^* I^g_S$ near Δ_S.

So we will modify the R^g_S, I^g_S near the smaller diagonal $\Delta_{S'}$ contained in Δ_S in order to satisfy these properties. To do this, note that we can define more generally for each pair of diagonals $\Delta_S \subset \Delta_{S'}$ a retraction $R^S_{S'}$ of X^n to $\Delta_{S'}$, defined in a neighbourhood of Δ_S. For this denote by

$$R^{S'}_{S,\mathrm{lin}} : N_{\Delta_S/X^n} \to N_{\Delta_S/\Delta_{S'}}$$

the linear invariant projector associated to the $S_{S'}$-action on N_{Δ_S/X^n}. Then exp being as in (3.16) we put

$$R^S_{S'} = \exp \circ R^{S'}_{S,\mathrm{lin}} \circ \exp^{-1}.$$

Note that for $S = S'$, we have $R^S_{S'} = R^g_S$. The relative complex structure

$$I^S_{S'} := (R^g_S)^*(I^g_S)_{|\Delta_{S'}}$$

and the retraction $R^S_{S'}$ also satisfy the property that for $x \in \Delta_{S'}$, the fiber $(R^S_{S'})^{-1}(x)$ is a complex subvariety of X^n for the complex structure induced by $(I^S_{S'})_x$ near x. Furthermore they also satisfy properties 2 and 3 of Proposition 5. (More precisely property 2 is satisfied by $R^{S''}_S$ and $R^{S''}_{S'}$ near $\Delta_{S''}$ for $\Delta_{S''} \subset \Delta_S \subset \Delta_{S'}$.)

So what we will do now is to modify inductively the $I^g_{S''}$'s and $R^{S''}_{S'}$'s near a smaller diagonal $\Delta_S \subset \Delta_{S''}$ using the $I^S_{S''}$'s and $R^S_{S'}$'s. To start with, let Δ_S be the smallest diagonal, i.e. $\mid S \mid = 1$. We first put $R_S = R^g_S$, $I_S = I^g_S$. Next consider the diagonals $\Delta_{S''}$ with $\mid S'' \mid = 2$: we want to construct a relative complex structure $I_{S''}$ on $\Delta_{S''} \times X$ defined in a neighbourhood of the incidence set in $\Delta_{S''} \times X$, and a system of retractions $R'^{S''}_{S'}$ on $\Delta_{S'}$ defined near $\Delta_{S''}$ for any $\Delta_{S'}$ containing $\Delta_{S''}$. We ask that

1. $I_{S''} = I^S_{S''}$ and $R'^{S''}_{S'} = R^S_{S'}$ in a neighbourhood V of Δ_S.
2. $I_{S''} = I^g_{S''}$ and $R'^{S''}_{S'} = R^{S''}_{S'}$ away from a neighbourhood W containing V of Δ_S.
3. The retractions $R'^{S''}_{S'}$ satisfy the compatibility relations $R'^{S''}_{S'} = R'^{S''}_{S'} \circ R'^{S''}_{S'''}$ for $\Delta_{S''} \subset \Delta_{S'} \subset \Delta_{S'''}$.
4. For $x \in \Delta_{S'}$ close to $\Delta_{S''}$ the fiber $(R'^{S''}_{S'})^{-1}(x)$ is a complex subvariety of X^n for the complex structure on X^n defined near x and induced by the complex structure $I'^{S''}_{S'}(x) := I_{S''}(R'_{S''}(x))$ on X near Sup x.
5. Everything is equivariant with respect to the S_n–action.

(Note that the three last properties are satisfied already in V and outside W). Now we will put $R_{S''} = R'^{S''}_{S''}$, which is a retraction of X^n onto $\Delta_{S''}$ defined in a neighbourhood of $\Delta_{S''}$. At the next step, we will modify the $R^g_{S'''}$, for $\mid S''' \mid = 3$ near the diagonals $\Delta_{S''} \subset \Delta_{S'''}$ using the $R'^{S''}_{S'''}$ and so on.

The relative complex structure $I_{S''}$ so defined satisfies by property 1 the condition 3 of Proposition 5. Also the property 4 for $S' = S''$ will give condition 4 of Proposition 5. Finally the compatibility conditions 3 will give the compatibility conditions 2 of Proposition 5 for the diagonals $\Delta_{S'}$ containing $\Delta_{S''}$, and the fact that $R_{S''} = R'^{S''}_{S''} = R^S_{S''}$ near Δ_S will also imply that $R_S \circ R_{S''} = R_S$ near Δ_S.

So it remains only to explain the construction of the $R'^{S''}_{S'}$'s and $I_{S''}$. For this one shows easily that the relative complex structures $I^g_{S''}$, $I^S_{S''}$, and the systems of compatible retractions $(R^{S''}_{S'})_{\Delta_{S''}\subset\Delta_{S'}}$, $(R^S_{S'})_{\Delta_{S''}\subset\Delta_{S'}}$, defined in $(W-V)\cap\Delta_{S''}$ and $W-V$ respectively have the following common construction.

Let $\mathrm{pr}_i:\Delta_{S''}\to X$ be the composition of the inclusion $\Delta_{S''}\subset X^n$ and the i-th projection. Start with a complex structure K_i on the vector bundle $\mathrm{pr}_i^*(T_X)$ and from a local diffeomorphism over $\Delta_{S''}$

$$\psi_i:\mathrm{pr}_i^*\,T_X\to\Delta_{S''}\times X$$

defined in the neighbourhood of the 0-section, onto a neighbourhood of $(\mathrm{pr}_i,\mathrm{Id})^{-1}(\mathrm{diag})$. We assume that $\psi_{i,x}=\psi_{j,x}$ for $x\in\Delta_{S''}$ such that $x_i=x_j$ and similarly for the K_i's. Then we deduce from the ψ_i's a local diffeomorphism over $\Delta_{S''}$ from a neighbourhood of the 0-section to a neighbourhood of the graph of the inclusion

$$\Psi=(\psi_i):(T_{X^n})_{|\Delta_{S''}}\cong\Delta_{S''}\times X^n.$$

Now on $T_{X^n|\Delta_{S''}}$ we have the linear projector onto the invariant part

$$\pi^{S'}_{\mathrm{lin}}:T_{X^n|\Delta_{S''}}\to T_{\Delta_{S'}|\Delta_{S''}}$$

given by the $S_{S'}$-action. Then $\Psi\circ\pi^{S'}_{\mathrm{lin}}\circ\Psi^{-1}$ gives a differentiable map

$$\chi_{S'}:\Delta_{S''}\times X^n\to\Delta_{S''}\times\Delta_{S'}$$

defined in a neighbourhood of the graph of the inclusion of $\Delta_{S''}$ into X^n, such that for each $x\in\Delta_{S''}$, χ_x is a retraction on $\Delta_{S'}$. Next

$$\chi^{-1}_{S''}(\mathrm{diag})\subset\Delta_{S''}\times X^n$$

is easily seen to be diffeomorphic to a neighbourhood of $\Delta_{S''}$ in X^n by the second projection. We get then a retraction $T_{S'}:X^n\to\Delta_{S'}$ defined in a neighbourhood of $\Delta_{S''}$ by the formula

$$T_{S'}=\mathrm{pr}_2\circ\chi_{S'}{}_{|\chi^{-1}_{S''}(\mathrm{diag})},\ \chi^{-1}_{S''}(\mathrm{diag})\overset{\mathrm{pr}_2}{\cong}X^n.$$

One constructs a relative complex structure $I_{S''}$ using the ψ_i's and K_i's by the formula $I_{S''}(x)=(\psi_i)_*(K_i)$ in a small neighbourhood of x_i, where $K_i(x_i)$ is seen as a complex structure on T_{X,x_i}.

In the case of $R^{S''}_{S'}$, $\psi_i^{S''}$ is the exponential map for g and $K_i^{S''}$ is given by J. In the case of $R^S_{S'}$, we have the following description : let $y=(y_i)_{i=1,\dots,n}\in\Delta_{S''}$ and let $R^g_S(y)=x=(x_i)_{i=1,\dots,n}\in\Delta_S$. Then $y_i=\exp_{x_i}(u_i)$ for some $u_i\in T_{X,x_i}$, and the differential $(\exp_{x_i})_*:T_{X,x_i}\cong$

$T_{T_{X,x_i},u_i} \to T_{X,y_i}$ is an isomorphism. Then $K_i^S(y)$ is the complex structure on T_{X,y_i} induced by J on T_{X,x_i}, and ψ_i^S is the composite $\exp_{x_i,u_i} \circ (\exp_{x_i})_*^{-1}$ where $\exp_{x_i,u_i}(v) := \exp_{x_i}(u_i + v)$.

Note that conversely, any set of retractions $(T_{S'})_{\Delta_{S''} \subset \Delta_{S'}}$ and complex structure $I_{S''}$ defined as above satisfy the properties 3 to 5 above.

Now we note that if W is sufficiently small, the K_i^S and $K_i^{S''}$ are very close, and similarly for the ψ_i^S and $\psi_i^{S''}$. So to construct the $I_{S''}$ and the $R'^{S''}_{S'}$ we just do the following: we choose a homotopy $(K_i^t)_{t\in[0,1]}$ between K_i^S and $K_i^{S''}$ and a homotopy $(\psi_i^t)_{t\in[0,1]}$ between ψ_i^S and $\psi_i^{S''}$. Next we choose a function f on $W - V$ which takes the value 0 near ∂V and the value 1 near ∂W. We then define $\psi_i(x) = \psi_i^{f(x)}(x)$ and $K_i(x) = K_i^{f(x)}(x)$. The retractions $T_{S'}$ constructed using these K_i's and ψ_i's will agree with $R^S_{S'}$ near ∂V and with $R^{S''}_{S'}$ with ∂W, hence together with them will give our $R'^{S''}_{S'}$. Similarly the relative complex structure $I_{S''}$ constructed using these K_i's and ψ_i's will coincide with $R^*_S I_S$ near ∂V and with $I^g_{S''}$ near ∂W. Hence we get our $I_{S''}$.

So the proof of Proposition 5, and hence of Theorem 3 is now finished. □

Remark 2. — It would be interesting to compare the approaches given in Sections 2 and 3, the first one providing only a construction for an open part $\mathrm{Hilb}^k_{\mathrm{curv}}(X)_J$ or $\mathrm{Hilb}^k_{\mathrm{curv}}(X)'_J$ of the Hilbert scheme, depending only on the almost complex structure J on X. The construction provided in Section 3 also provides non-compact manifolds $\mathrm{Hilb}^k_{\mathrm{curv}}(X)_I$ and $\mathrm{Hilb}^k_{\mathrm{curv}}(X)'_I$. Namely inside

$$\mathrm{Hilb}^k(X) := \bigsqcup_{z\in X^{(k)}} \mathrm{Hilb}^k(W_z) \cap c_z^{-1}(z)$$

we can consider the open sets

$$\mathrm{Hilb}^k_{\mathrm{curv}}(X)_I := \bigsqcup_{z\in X^{(k)}} \mathrm{Hilb}^k_{\mathrm{curv}}(W_z) \cap c_z^{-1}(z)$$

or

$$\mathrm{Hilb}^k_{\mathrm{curv}}(X)'_I := \bigsqcup_{z\in X^{(k)}} \mathrm{Hilb}^k_{\mathrm{curv}}(W_z)' \cap c_z^{-1}(z).$$

The question is whether they are diffeomorphic to the corresponding manifolds $\mathrm{Hilb}^k_{\mathrm{curv}}(X)_J$ or $\mathrm{Hilb}^k_{\mathrm{curv}}(X)'_J$.

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