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APPLICATIONS OF THE p-ADIC NEVANLINNA THEORY TO FUNCTIONAL EQUATIONS

by A. BOUTABAA & A. ESCASSUT

Introduction.

Let K denote an algebraically closed field of characteristic zero, complete for an ultrametric absolute value, let a be a point of K, let R be a positive number and let $d(a, R^-)$ be the disk $\{x|\ |x-a| < R\}$. We denote by $\mathcal{A}(K)$ the K-algebra of entire functions in K and by $\mathcal{M}(K)$ the field of meromorphic functions in K, i.e., the field of fractions of $\mathcal{A}(K)$. In the same way, we denote by $\mathcal{A}(d(a, R^-))$ the K-algebra of analytic functions in $d(a, R^-)$, i.e., the set of power series converging inside $d(a, R^-)$, and by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions in $d(a, R^-)$, i.e., the field of fractions of $\mathcal{A}(d(a, R^-))$. Furthermore, here we consider the K-subalgebra $\mathcal{A}_b(d(a, R^-))$ of bounded analytic functions inside $d(a, R^-)$, and we denote by $\mathcal{M}_b(d(a, R^-))$ its field of fractions.

Given $(m, n) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0)\}$ we denote by (m, n) the greatest common divisor of m and n.

For every $n \in \mathbb{Z}$, $|n|_\infty$ will denote the archimedean absolute value of n.

In [3] we stated an improvement to the p-adic Nevanlinna's second Main Theorem which is exactly what we need for applying this theory to the

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present study devoted to functional equations of the form $gQ(f) = P(f)$ in $\mathcal{M}(K)$ and in $\mathcal{M}(d(a,R^-))$. On the other hand, there does exist a Nevanlinna Theorem for p-adic unbounded analytic functions inside a disk such as $d(a,R^-)$. We will recall and use it in order to obtain similar results for such functions. However, most results are somewhat weaker because the p-adic improvement mentioned above when applying to meromorphic functions in all K, no longer works for functions just defined in $d(a,R^-)$.

Our study will be applied to certain curves and to differential equations.

Parametrization of curves.

We denote by D an infinite bounded set included in a disk $d(a,r)$, for some $r < R$. Theorems 1 and 2 may apply to curves of genus 0 as well as curves of genus ≥ 1.

THEOREM 1. — *Let $P, Q \in K[X]$ be two relatively prime polynomials of degrees s and t respectively, let n be the number of distinct zeros of Q, let $m \in \mathbb{N}^*$, and let $g \in \mathcal{M}(d(a,R^-))$ be a non constant function all poles of which have order $\geq m$. Suppose that there exists a function $f \in \mathcal{M}(d(a,R^-))$ satisfying $g(x)Q(f(x)) = P(f(x))$ for every $x \in D$ which is not a pole of f or g.*

i) *Assume that $f \notin \mathcal{M}_b(d(a,R^-))$. Then $mn \leq t+2m$. Moreover, if $s > t$, then $mn \leq \min(t+2m, s+m)$.*

ii) *Assume $f, g \in \mathcal{M}(K)$. Then $mn < t+2m$. Moreover, if $s > t$, then $mn < \min(t+2m, s+m)$.*

Examples. — 1) Let Γ be the curve of equation $y^4(x-b')(x-b'') = (x-c)^3$ (with b', b'', c all distinct) and let $f,\ g \in \mathcal{M}(d(a,R^-))$ be such that $(f(u), g(u)) \in \Gamma$ for all $u \in D$. Then by Theorem 1 $f,\ g \in \mathcal{M}_b(d(a,R^-))$.

2) Let Γ be the curve of equation $y^3(x-b')(x-b'') = (x-c)^3$ (with b', b'', c all distinct) and let $f,\ g \in \mathcal{M}(K)$ be such that $(f(u), g(u)) \in \Gamma$ for all $u \in D$. Then by Theorem 1 $f,\ g$ are constant.

THEOREM 2. — *Let $P(X) = A\prod_{i=1}^{k}(X-a_i)^{s_i}$, $Q(X) = B\prod_{j=1}^{n}(X-b_j)^{t_j}$ be two relatively prime polynomials of $K[X]$ of respective degrees s and t. Let $m \in \mathbb{N}^*$. Let $f,\ g \in \mathcal{M}(d(a,R^-))$ satisfy $(g(x))^m Q(f(x)) = P(f(x))$ for all $x \in D$.*

α) If $k+n > 1+\frac{1}{m}\Big((m,|s-t|_\infty)+\sum_{i=1}^{k}(m,s_i)+\sum_{j=1}^{n}(m,t_j)\Big)$ then both f and g lie in $\mathcal{M}_b(d(a,R^-))$.

Moreover, if f lies in $\mathcal{A}(d(a,R^-))$, and if $k+n > 1+\frac{1}{m}\Big(\sum_{i=1}^{k}(m,s_i)+\sum_{j=1}^{n}(m,t_j)\Big)$ then $f \in \mathcal{A}_b(d(a,R^-))$ and $g \in \mathcal{M}_b(d(a,R^-))$.

β) If both f,g lie in $\mathcal{M}(K)$, and if $k+n \geq 1+\frac{1}{m}\Big((m,|s-t|_\infty)+\sum_{i=1}^{k}(m,s_i)+\sum_{j=1}^{n}(m,t_j)\Big)$, then both f and g are constant.

Finally, if $f,\ g \in \mathcal{M}(K)\setminus K$ and if $k+n \geq 1+\frac{1}{m}\Big(1+\sum_{i=1}^{k}(m,s_i)+\sum_{j=1}^{n}(m,t_j)\Big)$, then f admits at least one pole of order $< m$.

Examples. — 3) Let $c',c'' \in K$ (with $c' \neq c''$) and let Γ be the curve of equation $y^3 = (x-c')^2(x-c'')$. Let $f,g \in \mathcal{M}(K)$ be such that $(f(u),g(u)) \in \Gamma$ for all $u \in D$. If f and g lie in $\mathcal{A}(K)$, they are constant. If f and g are not constant, then f admits at least one pole of order 1 or 2. Here the genus is clearly 0, therefore there exist $f,g \in K(u)$ satisfying $g^3 = (f-c')^2(f-c'')$.

4) In the same way, let Γ be the curve of equation $y^3(x-b)^2 = (x-c)$ (with $b \neq c$) and let $f,g \in \mathcal{M}(K)$ be such that $(f(u),g(u)) \in \Gamma$ for all $u \in D$. If f and g lie in $\mathcal{A}(K)$, they are constant. If f and g are not constant, then f admits at least one pole of order 1 or 2.

5) Let Γ be the curve of equation $y^3(x-b)^2 = (x-c')^2(x-c'')$ (with b,c',c'' all distinct) and let $f,g \in \mathcal{M}(d(a,R^-))$ be such that $(f(u),g(u)) \in \Gamma$ for all $u \in D$. Then by Theorem 2 $f,g \in \mathcal{M}_b(d(a,R^-))$.

6) Let Γ be the curve of equation $y^2(x-b')(x-b'') = (x-c)$ (with b',b'',c all distinct) and let $f \in \mathcal{A}(d(a,R^-))$ and let $g \in \mathcal{M}(d(a,R^-))$ be such that $(f(u),g(u)) \in \Gamma$ for all $u \in D$. Then by Theorem 2 $f \in \mathcal{A}_b(d(a,R^-))$ and $g \in \mathcal{M}_b(d(a,R^-))$.

Remark. — The conclusion appears without determining the genus of curves of equation $y^mQ(x) = P(x)$. However, Theorem 2 lets us easily obtain Picard Berkovich's Theorem [1] as far as curves of genus 1 and 2 are concerned. Actually, Corollaries A) and B) are even more general because they also apply to analytic functions in $d(a,R^-)$.

COROLLARY A. — *Let Γ be an algebraic curve on K of genus 1 or 2 and let $f,g \in \mathcal{M}(K)$ be such that $(f(u),g(u)) \in \Gamma$ $\forall u \in D$. Then f and*

g are constant.

COROLLARY B. — *Let Γ be a non degenerate elliptic curve on K and let $f, g \in \mathcal{A}(d(a, R^-))$ be such that $(f(u), g(u)) \in \Gamma$ $\forall u \in D$. Then f and g are bounded.*

COROLLARY C. — *Let Γ be an algebraic curve on K of genus 2 and let f, $g \in \mathcal{M}(K)$ (resp. $f, g \in \mathcal{M}(d(a, R^-))$) be such that $(f(u), g(u)) \in \Gamma$ $\forall u \in D$. Then both f and g are constant (resp. lie in $\mathcal{M}_b(d(a, R^-))$).*

Indeed, every algebraic curve of genus 1 (resp. 2) is birationally equivalent to a smooth elliptic (resp. hyperelliptic) curve [14]. So, we can apply Theorem 2 with $m = 2$, $t = 0$, $s = n = 3$ in Corollary A), and $s \geq 4$, $n \geq 4$ in Corollary C). Corollary B) is obvious.

Here we take this opportunity to recall that there exists no parametrization of conics with a center, by entire functions, on the field K. Such a result cannot be extended to bounded analytic functions as show the functions sin and cos defined in $d(0, (p^{-\frac{1}{p-1}})^-)$ when the residue characteristic of K is p (resp. $d(0, 1^-)$ when the residue characteristic of K is 0).

PROPOSITION A. — *Let Γ be a non degenerate conic with a center in K, and let $f, g \in \mathcal{A}(d(a, R^-))$ be such that $\big(f(u), g(u)\big) \in \Gamma$ $\forall u \in D$. Then f and g are bounded in $d(a, R^-)$. Moreover, if both f and g lie in $\mathcal{A}(K)$, then they are constant.*

Remark. — There exists no generalization of the p-adic Nevanlinna Theorem to bounded analytic functions.

Equality $f^m + g^n = 1$.

In [4] it was proven that the equation $f^m + g^n = 1$ in $\mathcal{M}(K)$ leads to $f, g \in K$ as soon as the least commun multiple q of m and n satisfies: $\frac{1}{m} + \frac{1}{n} + \frac{1}{q} \geq 1$ and that in $\mathcal{A}(K)$ it leads to $f, g \in K$ as soon as $\min(m, n) \geq 2$. Here we are now able to generalize these conclusions.

THEOREM 3. — *Let $f, g \in \mathcal{M}(d(a, R^-))$ satisfy $g^m + f^n = 1$, with $\min(m, n) \geq 3$, $\max(m, n) \geq 4$. Then both f and g lie in $\mathcal{M}_b(d(a, r))$. Moreover, if $f, g \in \mathcal{M}(K)$, and if $\min(m, n) \geq 2$, $\max(m, n) \geq 3$ then f, g are constant.*

THEOREM 4. — *Let $f, g \in \mathcal{A}(d(a, R^-))$ satisfy $g^m + f^n = 1$, with $\min(m, n) \geq 2$. Then f and g are bounded in $d(a, R^-)$.*

On the other hand, in order to complete results in this domain, we have the following proposition:

PROPOSITION B. — *Let $f, g \in \mathcal{A}(d(a, R^-))$ and let $h \in \mathcal{A}_b(d(a, R^-))$ satisfy $f^2 + g^2 = h$. Then f and g are bounded inside $d(a, R^-)$.*

Remark. — The p-adic functions sin and cos are bounded inside $d\big(0, \big(p^{-\frac{1}{p-1}}\big)^-\big)$ when the residue characteristic is p (resp. inside $d(0, 1^-)$ when the residue characteristic is 0) and satisfy $\sin^2 x + \cos^2 x = 1$.

Applications to certain differential equations.

As a corollary of Theorem 1, we have Theorem 5:

THEOREM 5. — *Let $P, Q \in K[X]$ be relatively prime, let $q \in \mathbb{N}^*$, let $F \in K[X_0, X_1, ...X_q] \setminus K[X_0, X_1, ...X_{q-1}]$, let n be the number of distinct zeros of Q, $s = \deg(P)$ and $t = \deg(Q)$.*

i) *Let $f \in \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$. Suppose that the equation $F(y, y', y'', ...y^{(q)})Q(f) = P(f)$ admits a solution in $\mathcal{M}(d(a, R^-))$. Then $(q+1)n \leq t + 2q + 2$. Moreover, if $s > t$, then we have $(q+1)n \leq \min(t + 2q + 2, s + q + 1)$.*

ii) *Let $f \in \mathcal{M}(K) \setminus K$. Suppose that the equation $F(y, y', y'', ..., y^{(q)})$ $Q(f) = P(f)$ admits a solution in $\mathcal{M}(K)$. Then $(q+1)n < t + 2q + 2$. Moreover, if $s > t$, then we have $(q+1)n < \min(t + 2q + 2, s + q + 1)$.*

In [2], it was shown that if the p-adic Yoshida equation $(\mathcal{E})$ $(y')^m = F(x, y)$ (with $F(x, y) \in K(x, y)$) admits solutions in $\mathcal{M}(K) \setminus K(x)$, then $F \in K(x)[y]$, and $\deg_y(F) \leq 2m$. Moreover, in [4], it was shown that if $F \in K[y]$, then any solution of the equation lying in $\mathcal{A}(K)$ is actually a polynomial.

Here we extend this last result by studying all meromorphic solutions of the equation when $F \in K(y)$.

THEOREM 6. — *Let $F(y) \in K(y)$ and suppose that there exists a non constant solution $f \in \mathcal{M}(K)$ of the differential equation $(\mathcal{E})$ $(y')^m = F(y)$.*

Then F is a polynomial $A(y-b)^d$ $(0 \leq d \leq 2m)$ such that $m-d$ divides m. In that case the solutions $f \in \mathcal{M}(K) \setminus K$ of $(\mathcal{E})$ are the functions of the form $f(x) = b + \lambda(x-\alpha)^{\frac{m}{m-d}}$, where λ satisfies $\lambda^{m-d}\left(\frac{m}{m-d}\right)^m = A$.

In particular, when $m = 1$, we obtain the solutions of Malmquist's equation when $F \in K(y)$:

COROLLARY D. — *Let $F \in K(y)$ be such that the equation $y' = F(y)$ admits a non constant solution in $\mathcal{M}(K)$. Then either $F(y)$ is of the form $A(y-b)^2$ and then the solutions in $\mathcal{M}(K)$ are the functions of the form $-\frac{1}{A(x-\alpha)} + b$, or F is a constant A and then the solutions are the functions $y = Ax + c$, with $c \in K$.*

Now, what is true in the field K for meromorphic functions, (and in particular for rational functions), also holds in any algebraically closed field of characteristic zero, as far as rational functions are concerned, because we have an obvious isomorphism from a finite extension E of $\mathbb{Q}$ containing all coefficients of all rational functions involved in $\mathcal{E}$, into K.

COROLLARY E. — *Let L be an algebraically closed field of caracteristic zero. Let $F \in L(y)$. If the equation $(y')^m = F(y)$ admits a non constant solution $f \in L(x)$, then F is a polynomial of the form $A(y-b)^d$ $(0 \leq d \leq 2m)$ such that $m-d$ divides m. In that case the solutions $f \in L(x) \setminus L$ of $(\mathcal{E})$ are the functions of the form $f(x) = b + \lambda(x-\alpha)^{\frac{m}{m-d}}$, where λ satisfies $\lambda^{m-d}\left(\frac{m}{m-d}\right)^m = A$.*

Remarks. — 1) In [12] (§31), two similar examples of Corollary E) are considered, when $m = 2$.

2) Generalizing Theorem 6 to analytic functions in $d(a, R^-)$ does not seem easy because there is no reason to think that F is a polynomial. Yet, should it be a polynomial, when applying the Nevanlinna inequality to such analytic functions, we no longer get a contradiction.

The proofs.

DEFINITIONS AND NOTATION. — *Let* log *be the real logarithm function of base $p > 0$. In K, the valuation v is defined as $v(x) = -\log|x|$.*

Given an interval I, and functions f, g from I to $\mathbb{R}$, we will write $f(r) \leq g(r) + O(1)$ $\forall r \in I$ if there exists a constant $A > 0$ such that $f(r) \leq g(r) + A$ $\forall r \in I$.

Given $f \in \mathcal{A}(d(a, R^-))$ (resp. $f \in \mathcal{A}(K)$), f' is the derivative of f.

Given $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}(d(0, R^-))$, (resp. $\in \mathcal{A}(K)$), we denote by $v(f, \mu)$ the valuation function defined by $v(f, \mu) = \inf_{n \in \mathbb{N}} v(a_n) + n\mu$ $\mu > -\log R$ (resp. $\mu \in \mathbb{R}$). This valuation can be extended to $\mathcal{M}(d(0, R^-))$ (resp. to $\mathcal{M}(K)$) by setting $v(\frac{f}{g}, \mu) = v(f, \mu) - v(g, \mu)$.

LEMMA 1. — *Let $f \in \mathcal{A}(d(0, R^-))$. Then $f \in \mathcal{A}_b(d(0, R^-))$ if and only if $v(f, \mu)$ admits a lower bound when μ tends to $-\log R$.*

Let $\alpha \in K$ and $h = \frac{f}{g} \in \mathcal{M}(d(a, R^-))$ (with $f, g \in \mathcal{A}(d(a, R^-))$). If h has a zero (resp. a pole) of order q at α, we put $\omega_\alpha(h) = q$ (resp. $\omega_\alpha(h) = -q$). If $h(\alpha) \neq 0$ and ∞, we put $\omega_\alpha(h) = 0$.

Let $f = \frac{g}{h} \in \mathcal{M}(d(0, R^-))$ such that $\omega_0(f) = 0$, $h(0) = 1$, g and h having no common zeros. Let $r \in]0, R[$. Then $Z(r, f)$ will denote the counting function of zeroes of f in the disk $d(0, r)$, which is also equal to $-v(g, -\log r) + v(g(0))$, and $N(r, f)$ will denote the counting function of poles of f in the disk $d(0, r)$, which is also equal to $-v(h, -\log r)$.

For each integer $k \geq 1$, $Z_{[k]}(r, f)$ (resp. $N_{[k]}(r, f)$) will denote the counting function of zeros (resp. poles) of f in the disk $d(0, r)$, where such zeros (resp. poles) are counted with the same order as in f when this order is $\leq k$, and with order k when it exceeds k in f. In particular, $Z_{[1]}(r, f)$ (resp. $N_{[1]}(r, f)$) is just denoted $\overline{Z}(r, f)$ (resp. $\overline{N}(r, f)$).

Finally, in a p-adic field, as noticed in [7], the Nevanlinna's function $T(r, f)$ previously defined in [3], is also equal to

$$T\left(r, \frac{g}{h}\right) = \max(-v(g, -\log r), -v(h, -\log r)),$$

when $g(0) = h(0) = 1$.

LEMMA 2. — *Let $f \in \mathcal{M}(d(0, R^-))$ (resp. $f \in \mathcal{M}(K)$, with $f(0) \neq 0$ and ∞, and let $I =]0, R[$, (resp. $I =]0, +\infty[$). Then $T(r, f - k) \leq T(r, f) + O(1)$ $(r \in I)$, $\forall k \in K$, $Z(r, f - k) \leq T(r, f) + O(1)$ $(r \in I)$ $\forall k \in K$, $N(r, f - k) \leq T(r, f) + O(1)$ $(r \in I)$ $\forall k \in K$.*

Let S be a finite subset of K , and let $h \in \mathcal{M}(d(0, R^-))$. We will denote by $Z_0^S(r, h')$ the counting function of zeros of h', excluding those which are zeros of $h - c$ for any $c \in S$.

In [3] we gave this improved p-adic Nevanlinna Theorem for meromorphic functions in K. The main improvement holds in the term $-\log r$. It comes from the intervention of the valuation function of a logarithmic derivative in the classic proof of the theorem. Moreover, in order to obtain the most powerful form of the theorem, one has to keep $\overline{N}(r,f)$ inside the inequality, rather than replacing it prematurely by $T(r,f)$. If we now consider meromorphic functions inside $f \in \mathcal{M}(d(0,R^-))$, We saw in [6] that the same inequality holds and we obtain this more general theorem:

THEOREM N. — *Let $a_1,...,a_q \in K$, and let $S = \{a_1,...,a_q\}$. Let $f \in \mathcal{M}(K)$ (resp. $f \in \mathcal{M}(d(0,R^-))$) be non constant, such that $f(0) \notin S \cup \{0,\infty\}$. Let $I =]0,+\infty[$ (resp. $I =]0,R[$. Then*

$$(q-1)T(r,f) \leq \overline{N}(r,f)+\sum_{i=1}^{q}\overline{Z}(r,f-a_i)-Z_0^S(r,f')-\log r+O(1)\ (r \in I).$$

However, if $f \in \mathcal{M}_b(d(0,R^-))$, then $T(r,f)$ is bounded in I, so the inequality is trivial. And if $f \in \mathcal{M}(d(0,R^-))\setminus\mathcal{M}_b(d(0,R^-))$, the inequality is not trivial but the term $-\log r$ is no longer efficient when applying the theorem because r is now bounded.

Lemma 3 is immediate and comes from Lemma 1:

LEMMA 3. — *Let $f \in \mathcal{M}(d(0,R^-)) \setminus \mathcal{M}_b(d(0,R^-))$ be such that $f(0) \notin S \cup \{0,\infty\}$. Then $T(r,f)$ is unbounded when r tends to R.*

Remark. — When applying Theorem N to functions $h(t)$, we will be able to make a change on the variable t of h so that the condition "$h \in \mathcal{M}(K)$ non constant, such that $h(0) \notin S\cup\{0,\infty\}$" be satisfied without loss of generality.

We will also use the following classical lemmas.

LEMMA 4. — *Let $a \in K$ and let $f \in \mathcal{M}(K)$ be such that $f(x) \neq a$ $\forall x \in K$. Then there exists $h \in \mathcal{A}(K)$ such that $f = \frac{1}{h} + a$.*

LEMMA 5. — *If $a,b,m,s \in \mathbb{N}^*$ satisfy $sa = mb$, then $a \geq \frac{m}{(s,m)}$.*

General notation in the proofs of all theorems. — When f,g are supposed to belong to $\mathcal{M}(d(a,R^-))$ but are not supposed to belong to $\mathcal{M}(K)$, we put $I =]0,R[$, and when f,g are supposed to belong to $\mathcal{M}(K)$, then we put $I =]0,+\infty[$.

In Theorems 1 and 2, functional equalities true whenever $x \in D$ obviously hold for all $x \in d(a, R^-)$ (resp. for all $x \in K$ when $f, g \in \mathcal{M}(K)$) provided x is not a pole for f or g. Besides, without loss of generality, we will assume $a = 0$.

Proof of Theorem 1. — If $n < 2$, the inequality $mn < t + 2m$ is trivial. So we suppose $n \geq 2$. Let $Q(X) = \prod_{j=1}^{n}(X - b_j)^{t_j}$. Since P and Q have no common zeros, each zero α of $Q(f(t))$ is a pole of $g(t)$, and therefore it is a zero of order at least m of $Q(f(t))$. As a consequence, for each zero α of $f - b_j$ we have $t_j \omega_\alpha(f - b_j) \geq m$ hence

$$\overline{Z}(r, f - b_j) \leq \frac{t_j}{m} Z(r, f - b_j). \tag{1}$$

But since $Z(r, f - b_j) \leq T(r, f) + O(1)$ $(r \in I)$, by (1) we have

$$\sum_{j=1}^{n} \overline{Z}(r, f - b_j) \leq \frac{t}{m} T(r, f) + O(1) \quad (r \in I). \tag{2}$$

Then, by Theorem N, we obtain

$$(n-1)T(r, f) \leq \frac{t}{m} T(r, f) + \overline{N}(r, f) - \log r + O(1) \quad (r \in I). \tag{3}$$

In particular, this implies

$$(n-2)T(r, f) \leq \frac{t}{m} T(r, f) - \log r + O(1) \quad (r \in I) \tag{4}$$

and therefore, by Lemma 3, if f does not lie in $\mathcal{M}_b(d(0, R^-))$, we have $m(n-2) \leq t$. And if f, g lies in $\mathcal{M}(K)$, then when $-\log r$ tends to $+\infty$ it is seen that the inequality (4) implies $(n-2) < \frac{t}{m}$.

Now, suppose that $s > t$. Then each pole α of f is a pole of g and therefore satisfies $\omega_\alpha(f)(s - t) = \omega_\alpha(g)$, hence $\overline{N}(r, f) \leq N(r, f)(\frac{s-t}{m})$. Consequently, Relation (3) becomes

$$(n-1)T(r, f) \leq \frac{t}{m} T(r, f) + \Big(\frac{s-t}{m}\Big) N(r, f) - \log r + O(1) \quad (r \in I), \tag{5}$$

and so we have $(n-1) \leq \frac{t}{m} + \frac{s-t}{m}$, thereby $mn \leq \min(s + m, t + 2m)$. Finally, if in addition , f and g lie in $\mathcal{M}(K)$, as r tends to $+\infty$, inequality (5) becomes strict and therefore $mn < \min(s + m, t + 2m)$, which finishes the proof.

Proof of Theorem 2. — It is clear that if f is constant so is g. Suppose that f is not constant. Then we have $s = \sum_{i=1}^{k} s_i$, $t = \sum_{j=1}^{n} t_j$. For each

$i=1,...,k$, every zero α of $f-a_i$ is a zero of g, and therefore is a zero of order $\omega_\alpha(f-a_i)=\frac{m\omega_\alpha(g)}{s_i}$ But by Lemma 5, we have $\omega_\alpha(f-a_i)\geq\frac{m}{(m,s_i)}$. In the same way, for each $j=1,...,n$, every zero β of $f-b_j$ is a pole of g, and therefore is a zero of order $\omega_\beta(f-b_j)=\frac{m\omega_\beta(g)}{t_j}$, and by Lemma 5 we have $\frac{m\omega_\beta(g)}{t_j}\geq\frac{m}{(m,t_j)}$. So, we have

$$\overline{Z}(r,f-a_i)\leq\frac{(m,s_i)}{m}T(r,f)+O(1)\quad(r\in I),\tag{6}$$

$$\overline{Z}(r,f-b_j)\leq\frac{(m,t_j)}{m}T(r,f)+O(1)\quad(r\in I).\tag{7}$$

Then, applying Theorem N to f at the points $a_1,...,a_k$, $b_1,...b_n$ and using (6) and (7) we obtain

$$\begin{aligned}(8)\ &(k+n-1)T(r,f)\\ &\leq\frac{1}{m}\Big(\sum_{i=1}^{k}(m,s_i)+\sum_{j=1}^{n}(m,t_j)\Big)T(r,f)+\overline{N}(r,f)-\log r+O(1)\ (r\in I).\end{aligned}$$

We will now find an upper bound of $\overline{N}(r,f)$. Let γ be a pole of f we have

$$(s-t)\omega_\gamma(f)=m\omega_\gamma(g).\tag{9}$$

In the same way, from (9) we obtain $|\omega_\gamma(f)|_\infty\geq\frac{m}{(m,|s-t|_\infty)}$ and therefore

$$\overline{N}(r,f)\leq\frac{(m,|s-t|_\infty)}{m}N(r,f)\leq\frac{(m,|s-t|_\infty)}{m}T(r,f).\tag{10}$$

We will prove (11):

$$\begin{aligned}&(k+n-1)T(r,f)\leq\frac{1}{m}\Big((m,|s-t|_\infty)\\ (11)\qquad &+\sum_{i=1}^{k}(m,s_i)+\sum_{j=1}^{n}(m,t_j)\Big)T(r,f)-\log r+O(1)\ (r\in I).\end{aligned}$$

If $s=t$ we just have the inequality $\overline{N}(r,f)\leq N(r,f)\leq T(r,f)$. Then from (8) we obtain

$$\begin{aligned}&(k+n-1)T(r,f)\\ &\leq\frac{1}{m}\Big(m+\sum_{i=1}^{k}(m,s_i)+\sum_{j=1}^{n}(m,t_j)\Big)T(r,f)-\log r+O(1)\quad(r\in I).\end{aligned}$$

But in this case we have $(m, |s-t|_\infty) = m$. So, we obtain (11). And if $s \neq t$, by (8), we obtain relation (11) again.

Thus, from relation (11), using Lemma 3, if f does not lie in $\mathcal{M}_b(d(0,R^-))$ we can get

$$(12) \qquad (k+n-1) \leq \frac{1}{m}\Big((m,|s-t|_\infty) + \sum_{i=1}^{k}(m,s_i) + \sum_{j=1}^{n}(m,t_j)\Big),$$

a contradiction with the hypothesis. Hence f must lie in $\mathcal{M}_b(d(0,r^-))$ and so must g.

Now, suppose $f, g \in \mathcal{M}(K)$. When r tends to $+\infty$ we can see that the inequality (12) becomes strict. Therefore, if the inequality is not satisfied, f and g are constant. Conversely, suppose that f is not constant, and that all poles of f have order $\geq m$. By (11) we have

$$\begin{aligned}&(k+n-1)T(r,f)\\ &\leq \frac{1}{m}\Big(\sum_{i=1}^{k}(m,s_i) + \sum_{j=1}^{n}(m,t_j)\Big)T(r,f) + \frac{1}{m}T(r,f) - \log r + O(1)\end{aligned}$$

$(r \in I)$, which implies $k+n-1 < \frac{1}{m}\Big(1 + \sum_{i=1}^{k}(m,s_i) + \sum_{j=1}^{n}(m,t_j)\Big)$. Hence, if this inequality is not true, f must admit at least one pole of order $< m$.

Finally, suppose $f \in \mathcal{A}(d(0,R^-))$. Then (8) becomes

$$(13) \qquad (k+n-1)T(r,f) \leq \frac{1}{m}\Big(\sum_{i=1}^{k}(m,s_i)+\sum_{j=1}^{n}(m,t_j)\Big)T(r,f)-\log r+O(1) \ (r \in I),$$

and then, if f does not lie in $\mathcal{A}_b(d(0,R^-))$, by Lemma 3 we obtain $k+n-1 \leq \frac{1}{m}\Big(\sum_{i=1}^{k}(m,s_i) + \sum_{j=1}^{n}(m,t_j)\Big)$, hence $k+n-1 \leq \frac{1}{m}(s+t)$, a contradiction. This finishes the proof.

Proof of Proposition B. — It is easily seen that f^2+g^2 factorizes in the form $(f+ig)(f-ig)$ with $i^2=-1$. Suppose that $|f(x)|+|g(x)|$ is not bounded in $d(a,R^-)$. At least one of the functions $f+ig$, $f-ig$ is not bounded. But, by classic results on analytic functions [9], $\mathcal{A}(d(a,R^-)) \setminus \mathcal{A}_b(d(a,R^-)$ is stable for the multiplication. Therefore, f^2+g^2 is not bounded.

Proof of Proposition A. — If the equation of Γ is of the form $ax^2+by^2=U$, with $U \in K$, proof is provided by Proposition B. Now, we consider the general case. Since Γ has a center, by a suitable change on variable of the form $u = x - \alpha$, $w = y - \beta$, we may assume that the equation is of the form $au^2 + bw^2 + 2cuw = M$. Moreover, since Γ is non degenerate, we know that $M \neq 0$. We first notice that if $a = b = 0$, then f and g are invertible in $\mathcal{A}(K)$, and therefore are constant. Thus, since a and b play the same role, we may assume, for instance $b \neq 0$. In this way, g is a solution (in $\mathcal{A}(K)$) of the equation $bY^2 + (2cf)Y + (af^2 - M) = 0$. Thereby, the reduced discriminant $f^2(c^2 - ab) + bM$ must be equal to $(bg - cf)^2$. Consequently, putting $h = bg - cf$, we check that $h^2 = f^2(c^2 - ab) + bM$. Thus, f and h satisfy an equation of the form $\lambda f^2 + \mu h^2 = \nu$. But since $bM \neq 0$, and since h lies in $\mathcal{A}(K)$, we come back to the first case, proving that f and h are constant and therefore so is g. This ends the proof.

Lemma 6 will be useful in the proofs of Theorems 3 and 4.

LEMMA 6. — *Let* $m, n \in \mathbb{N}^*$ *satisfy* $\max(m,n) \geq 3$ *and* $\min(m,n) \geq 2$. *Then we have*

$$mn \geq m + n + (m,n) \tag{14}$$

$$mn > m + n. \tag{15}$$

Moreover, if $\min(m,n) \geq 3$ *and* $\max(m,n) \geq 4$, *then*

$$mn > m + n + (m,nn). \tag{16}$$

Proof. — Without loss of generality, we can assume $m \leq n$. If $m \geq 3$, then $mn \geq m + n + (m,n)$. Now, suppose $m = 2$, $n \geq 4$. Then, $mn = 2n \geq n + 2 + (n,2)$. Thus, (14) holds when $m \geq 3$, and when $m = 2$, $n \geq 4$. It also holds when $m = 2$, $n = 3$, and so (14) is proven and (15) is an obvious consequence. Now, suppose $m \geq 3$, $n \geq 4$. If $m \geq 4$, then $mn > 3n \geq m + n + (m,n)$, and if $m = 3$, $n \geq 4$, then $mn = 3n \geq 3 + n + (3,n)$, and this finishes the proof of (16).

Proof of Theorems 3 and 4. — Without loss of generality we may obviously replace n by k, and g^m by $-(g)^m$. Thus we assume that f and g satisfy $g^m = f^k - 1$. We will apply Theorem 2 in the case when $Q(X) = 1$, $P(X) = X^k - 1$. We first place ouselves in the hypothetis of Theorem 3. By Lemma 6, (16) is satisfied, hence we have $k > 1 + \frac{1}{m}((m,k) + k)$, and then by Theorem 2, α), both f, g lie in $\mathcal{M}_b(d(a, R^-))$. Now suppose f, g lie in $\mathcal{M}(K)$. If $\max(m,k) \geq 3$, $\min(m,k) \geq 3$, $\min(m,k) \geq 2$, then by Lemma 6, (14) is satisfied, so we have $k \geq 1 + \frac{1}{m}((m,k) + k)$, and therefore by Theorem 2, β), f and g are constant. This finishes proving Theorem 3.

We now assume the hypotheses of Theorem 4. We notice that if $m = k = 2$, then by Proposition B f and g are bounded. We now suppose $\max(m,k) \geq 3$. By Lemma 6, (15) is satisfied, hence we have $k > 1 + \frac{k}{m}$ and then by Theorem 2, α), f and g are bounded. This finishes the proof of Theorem 4.

Proof of Theorem 6. — Let $f \in \mathcal{M}(K) \setminus K$ be a solution of equation $(\mathcal{E})$. Let b be a pole of $F(X)$ and β a zero of $f - b$. Then β is a pole of $F \circ f$, which contradicts the fact that f is a solution of the equation $(\mathcal{E})$. As a consequence, f must avoid poles of $F(X)$. Therefore $F(X)$ admits at most one pole b and then, is of the form $R(X) = \frac{Q(X)}{(X-b)^\nu}$, with $Q(X) \in K[X]$, $Q(b) \neq 0$ and $\nu \in \mathbb{N}^*$. In this case, by Lemma 4, f is of the form $f = b + \frac{1}{h}$ with $f \in \mathcal{A}(K) \setminus K$. From $(\mathcal{E})$ we deduce that h satisfies $(h')^m = (-1)^m h^{2m+\nu} Q(b + \frac{1}{h})$. Since $h \in \mathcal{A}(K)$, and is not a constant, h admits at least one zero γ, and then we can find $M \in \mathbb{R}$ such that $M \leq v(\gamma)$, and $v(h,\mu) < -v(b)$ $\forall \mu \leq M$. Then, $v(b + \frac{1}{h}, \mu) = v(b)$ $\forall \mu \leq M$, hence $v(Q(b + \frac{1}{h})) = v(Q(b))$ $\forall \mu \leq M$. As a consequence we have $mv(h', \mu) = (2m + \nu)v(h,\mu) + v(Q(b))$ $\forall \mu \leq M$. But it is known that $v(h',\mu) \geq v(h,\mu) - \mu$ $\forall \mu \in \mathbb{R}$, (for instance th. 13.5 in [9]), hence $m\big(v(h,\mu) - \mu)\big) \leq (2m + \nu)v(h,\mu) + v(Q(b))$, and therefore

$$0 \leq (m + \nu)(v(h,\mu) + \mu + v(Q(b)). \tag{17}$$

Now, since $M \leq v(\gamma)$, by classical results (see for example Th. 23.18 in [9]) the function $m(v(h,\mu) + \mu) + v(Q(b))$ is strictly increasing and tends to $-\infty$ with μ, a contradiction to Relation (17). Thus $F(X)$ has no poles and is a polynomial $P(X) = a_0 + a_1 X + ... + a_d X^d$. Then using the inequalities $T(r, f') \leq 2T(r, f)$ and $T(P \circ f, r) = dT(f,r) + O(1)$, $(r \in I)$ (given in [2]), we deduce from $(\mathcal{E})$ that $d \leq 2m$.

Now, if a is a zero of $P(X)$ of order δ and α is a zero of $f - a$ of order s, we have

$$s = \frac{m}{m - \delta} \quad \text{and} \quad m > \delta. \tag{18}$$

So all zeros of $f - a$ are multiple and have the same multiplicity order. If β is a pole of f of order t, we have

$$t = \frac{m}{d - m} \quad \text{and} \quad m < d. \tag{19}$$

So all poles of f have the same order.

Suppose that P has k, $(k \geq 2)$, distinct zeros a_j of order δ_j, $(1 \leq j \leq k)$. By (18) for each $j = 1, ..., k$, $m - \delta_j$ divides m, hence

$m-\delta_j \leq \frac{m}{2}$, therefore $\delta_j \geq \frac{m}{2}$. Suppose $f \in \mathcal{A}(K)$. By Proposition 1.9. in [4], f belongs to $K[X] \setminus K$. Let $u = \deg(f)$. Then $ud = (u-1)m$, hence $u = \frac{m}{m-d}$, so $m-d$ divides m. But since $d = \sum_{j=1}^{k} \delta_j$ and since $\delta_j \geq \frac{m}{2}$, we have $\delta_1 + \delta_2 \geq m$, and finally $m \leq d$. Thus, either $f \notin \mathcal{A}(K)$, and then (by (19)) $m < d$, or $f \in \mathcal{A}(K)$, and then $m \leq d$. Consequently, since $d-m \geq 0$, by (19) we obtain (20)

$$\overline{N}(r,f) = \frac{d-m}{m} N(r,f) \leq \frac{d-m}{m} T(r,f) + O(1), \quad (r \in I). \tag{20}$$

And by (18), we have

$$\overline{Z}(r, f-a_j) = \frac{m-\delta_j}{m} Z(r, f-a_j). \tag{21}$$

Applying Theorem N, we have

$$(k-1)T(r,f) \leq \sum_{j=1}^{k} \overline{Z}(r, f-a_j) + \overline{N}(r,f) - \log r + O(1), \quad (r \in I).$$

Now by (20) and (21) we obtain

$$(k-1)T(r,f) \leq \sum_{j=1}^{k} \frac{m-\delta_j}{m} Z(r, f-a_j) + \frac{d-m}{m} N(r,f) - \log r + O(1), \ (r \in I).$$

And finally

$$(k-1)T(r,f) \leq (k-1)T(r,f) - \log r + O(1), \quad (r \in I). \tag{22}$$

This contradiction shows that P does not have several distinct zeros. So, P is a polynomial of the form $P(X) = A(X-a)^d$.

Now, since $d = \delta$, by (18) and (19), $f-a$ cannot have simultaneously zeros and poles. As a consequence, either $f \in \mathcal{A}(K) \setminus K$ and then, as it was just said, $f \in K[X] \setminus K$, or $f-a$ has no zeros and therefore f is of the form $a + \frac{1}{h}$ with $h \in \mathcal{A}(K) \setminus K$.

Suppose first that $f \in K[x] \setminus K$. Let $\ell = \deg(f)$ and let α be a zero of $f-a$ of order s. We have: $(\ell-1)m = \ell d$, $(s-1)m = sd$, hence $s = \ell = \frac{m}{m-d}$. Then f is of the form $\lambda(x-\alpha)^\ell + a$, with $\lambda^{m-d} = A\frac{(m-d)^m}{m^m}$.

Conversely, given ℓ and λ satisfying the above relations, one checks that, for every $\alpha \in K$, the polynomial $f_\alpha = \lambda(x-\alpha)^\ell + a$ is a solution of the equation $(y')^m = A(y-a)^d$.

Now suppose that f is of the form $a + \frac{1}{h}$ with $h \in K[x] \setminus K$. Then h satisfies $h'^m = (-1)^m A h^{2m-d}$. Hence by [4] h lies in $K[x]$. We deduce that $h = \mu(x-\beta)^k$ with $\beta, \mu, k \in K$ such that $k = \frac{m}{d-m}$ and $\mu = (-1)^m A \frac{(d-m)^m}{m^m}$.

Reciprocally, given k and μ satisfying the above relations, one checks that, for every $\beta \in K$, the rational function $f_\beta = a + \frac{1}{\mu(x-\beta)^k}$ is a solution of the equation $(y')^m = A(y-a)^d$.

Remark. — If we consider Equation $(\mathcal{E})$ in $\mathcal{A}_u(d(a, R^-))$, Relation (21) no longer leads to a contradiction. This is why we cannot generalize Thoerem 6 to functions in $\mathcal{A}_u(d(a, R^-))$.

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