ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 50, nº 3 (2000), p. 833-863

<http://www.numdam.org/item?id=AIF_2000__50_3_833_0>

Ann. Inst. Fourier, Grenoble
50, 3 (2000), 833–863

EXPLICIT KAZHDAN CONSTANTS FOR REPRESENTATIONS OF SEMISIMPLE AND ARITHMETIC GROUPS

by Yehuda SHALOM

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1. Introduction and discussion of the main results.

We begin by establishing some notation, while recalling the notion of property (T), originally introduced by Kazhdan in [Kaz] (see [HV] for an account):

DEFINITION 1.1.

1. *Let G be a topological group, and $\mathcal{F}$ a family of continuous unitary G-representations. The family $\mathcal{F}$ is said to be* isolated from the trivial representation, *if there is a compact subset $Q \subseteq G$ and $\epsilon > 0$, satisfying the following: For every $(\pi, \mathcal{H}) \in \mathcal{F}$ there is no vector $v \in \mathcal{H}$ which is (Q, ϵ)-invariant, namely, satisfies: $||\pi(g)v - v|| < \epsilon ||v||$, $\forall g \in Q$. Such Q is referred to as a* Kazhdan set, *and ϵ as a* Kazhdan constant *(for Q), for the family $\mathcal{F}$.*

2. *If there are compact set Q and $\epsilon > 0$, which constitute Kazhdan set and constant for the family of all continuous unitary G-representations which do not contain a non-zero G-invariant vector, then G is said to have (Kazhdan's) property (T).*

3. *If there is no $\epsilon' > \epsilon$ satisfying the condition in (2), then ϵ is said to be the* best *Kazhdan constant for the set Q.*

Keywords : Semisimple groups – Arithmetic groups – Lattices – Property (T) – Kazhdan constants.
Math classification: 22D10 – 22D30 – 22E35 – 22E46 – 43A15.

As is well known by now, the group of k-rational points of every semisimple, almost k-simple algebraic group, defined over a locally compact non-discrete field k, has property (T), if its k-rank is at least 2. When $k = \mathbb{R}$ (and only in that case), there exist also simple k-groups of k-rank one with this property, namely, $Sp(n,1)$ $(n \geqslant 2)$ and $\mathbb{F}_{4(-20)}$. Furthermore, any lattice (i.e., a discrete subgroup of finite co-volume) in a group possessing property (T), has this property as well.

A natural problem raised by Serre and by de la Harpe and Valette (cf. [Bur] and [HV, p. 133]), is to compute explicit Kazhdan (sets and) constants for the algebraic groups with property (T), and their lattices. This question has been addressed for some lattices in $PGL_3(k)$ (for certain totally disconnected k) in [CMS], where an interesting family of groups with property (T) was constructed, and the best Kazhdan constant for a natural choice of generators was computed. More recently, new remarkable examples of Kazhdan groups were discovered by A. Zuk [Zu1], which include the constructions in [CMS] (see also [BaSw]). Explicit Kazhdan constants for them are computed in [Zu2]. We note that many of the groups discussed in [CMS], [Zu1], [Zu2] and [BaSw] seem to be non linear, to which the results of the present paper do not apply. For the group $SL_3(\mathbb{Z})$, explicit Kazhdan constants for the family of all the finite dimensional representations and those of the form $\ell^2(SL_3(\mathbb{Z})/\Lambda)$, have been computed by M. Burger in [Bur]. We note also that R. Howe and E.C. Tan [HT, Ch.V 4.1.1] obtained Kazhdan sets and constants for certain semisimple Lie groups. These were defined in terms of the level sets of a function analogous to the Harish-Chandra Ξ-function. Other recent related papers are [BCJ] and [BM].

The problem of computing Kazhdan constants for groups of linear type over (commutative) rings, was studied in detail in [Sh2], where a new approach to property (T) is presented using the notion of bounded generation. Nevertheless, [Sh2] is in many aspects complementary to the present paper, both in the methods employed, as well as in the results obtained. The approach in [Sh2] is more algebraic, and in the cases it may be applied, gives a rather sharp information that does not seem available using the methods of the present paper (such as the behaviour of the Kazhdan constant for the set of unit elementary matrices in $SL_n(\mathbb{Z})$, when n is varied). However, in studying Kazhdan constants for general algebraic groups, and particularly for *uniform* lattices in such groups, the methods of [Sh2] seem inadequate, and the more analytic approach(s) of the current paper will turn out to be fruitful. We shall describe explicit Kazhdan constants for every group of rational points of a semisimple, almost k-simple algebraic group with property (T), over any locally compact non-discrete field k, and consequently, for all lattices in such groups as well. Moreover, for the algebraic groups, we will show that the Kazhdan constants obtained are best possible.

THEOREM A. — *Let k be any locally compact non-discrete field, and let G denote the k-points of a simply-connected, semisimple, almost k-simple linear algebraic group defined over k, which is k-isotropic (i.e., G is not compact). Assume that G has property (T) of Kazhdan. Then for every $2 \leqslant m \in \mathbb{N}$ one can find (explicitly) in G Kazhdan sets of* m elements, *whose* best *Kazhdan constant is:*

$$\epsilon = \sqrt{2 - 2(\sqrt{2m-1}/m)} \qquad (\approx 0.51 \quad \textit{for} \quad m = 2).$$

Example. — For $G = SL_n(\mathbb{R})$, $n \geqslant 3$, we show that the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ embedded in the upper 2×2 left corner of $SL_n(\mathbb{R})$, form a Kazhdan set as in Theorem A (with $m = 2$).

A natural question suggested by Theorem A is whether the same conclusion holds in the anisotropic case, namely, when G is a compact (simple) algebraic group. Every such group obviously has property (T), but it seems even unknown in general if it has a *finite* Kazhdan set (when k is connected this is known to be the case, and there are also some results for totally disconnected k –see [Be1] and also [Sh1, §5] where this stronger property (T) is studied). Actually, it is the norm estimate from which the above Kazhdan constant is derived (see Theorem 3.1 below), which seems more natural and interesting, rather than the constant itself. The work of Lubotzky, Phillips and Sarnak [LPS], although not formulated exactly in this language, may be viewed as supplying an affirmative answer to this question when $G = SO(3, \mathbb{R})$. Such a result, for additional compact groups, would lead to extensions of the results of [LPS] on uniform distribution of points when k is connected (see also [Lub1, Problem 10.9.3]), and to new constructions of “Ramanujan graphs” in the totally disconnected case (cf. [Lub1, Ch. 4] for details).

Although Theorem A deals uniformly with the higher rank groups on one hand, and the rank one (Kazhdan-) groups on the other, its proof for the two families is based on rather different ideas. In both cases the question of Kazhdan constants for all representations is subsequently reduced to that of the regular representation of a suitable copy of SL_2, which is dealt with using Kesten's theorem for free groups. However, for higher rank groups the reduction relies on Kazhdan's original argument [Kaz], while for the rank one groups it is based essentially on the classification of the class-one unitary dual of these groups, together with estimates of the exponential rate of decay of their matrix coefficients. For the explicit construction of the sets guaranteed in Theorem A see the proof of the theorem in Section 3 (the higher rank case) and Section 5 (rank one case). The proof that the above mentioned constant is indeed optimal is presented at the second part of Section 3.

Theorem A yields explicit Kazhdan sets with best constants, but these sets are of a rather special form. In Section 4 we shall use a method different from the ones in the proof of Theorem A, which enables one to derive explicit Kazhdan constants for many other sets of elements. These estimates, however, are not tight in general. We will show in Section 5 how this method, which is based on L^p-integrability of matrix coefficients, can be used to obtain Kazhdan constants also for simple Lie groups without property (T), for any family of representations which admits a spectral gap (in a sense explained there).

Let us address the question of Kazhdan constants for lattices. There is no known general method to construct generating sets for (nonuniform) lattices, and in a sharp contrast to the situation with simple algebraic groups (as in the example proceeding Theorem A), a Kazhdan set for a discrete group *must generate* the whole group (see the proof of Corollary 6.2 in Section 6). The Kazhdan sets we shall determine are of the form: $(*)\quad \{\gamma \in \Gamma : ||\gamma|| \leqslant M\}$, where $\|\cdot\|$ is the distance function on G given by $||g|| = d(g\cdot e, e)$, and d is a G-invariant metric on the associated symmetric space (or Bruhat-Tits building, when k is totally disconnected). Assume that the G-invariant measure m on $\Gamma\setminus G$ is normalized to have total mass one. Making quantitative the standard argument that a lattice in a Kazhdan group is Kazhdan as well, we show in Section 6:

THEOREM B.— *Let G be as in Theorem A, and let $\Gamma < G$ be a lattice. Assume that (Q,ϵ) form Kazhdan constants for G. Fix some $0 \leqslant \delta < \frac{\epsilon^2}{8}$, and choose $M < \infty$ satisfying $m(\Gamma\cdot B_M) \geqslant 1-\delta$, where $B_M = \{g : ||g|| \leqslant M\}$. Denote $R = 2M + \max\{\| g \| \,|g \in Q\}$, and let Γ_R be the finite set $\Gamma_R = \Gamma \cap B_R$. Then Γ_R is a Kazhdan set for Γ, with Kazhdan constant $\left(\frac{\epsilon^2-8\delta}{1-2\delta}\right)^{1/2}$.*

Ever since its appearance, property (T) of Kazhdan has proven useful and influential in various areas of research. The computation of explicit Kazhdan constants yields quantitative versions for many of the consequences derived from property (T), and we refer to [HV] for numerous applications of this property, which will not be considered here. We only note that Kazhdan's original result [Kaz] about finite generation of lattices with property (T), can now be made quantitative. Indeed, a set of generators of the form $(*)$ above for the lattices in Theorem B can be determined, where the dependence of M on Γ is explicit. In fact, a consequence of the uniformity in Theorem A over the various groups, is that this geometric description of generating sets for the lattices with property (T) hardly depends on the ambient group (see Corollary 6.2).

Having settled the general question of explicit Kazhdan constants, we return in Section 7 to consider a different application of the method applied in Section 4. The aim of this section is the following result:

THEOREM C. — *Let $G = \Pi G_i$ be a semisimple Lie group with finite center, $(\pi, \mathcal{H})$ a unitary G-representation, and μ a probability measure on G. Denote by $\pi(\mu)$ the μ-convolution (or averaging) operator on $\mathcal{H}$, defined by $\langle \pi(\mu)u, v\rangle = \int \langle \pi(g)u, v\rangle d\mu(g)$. Suppose that for every i one has $I \not\prec \pi|_{G_i}$ (hereafter I stands for the trivial representation, and $\prec$ for weak containment of unitary representations - see 2.1 below).*

(1) *If $H < G$ is a closed non-amenable subgroup, then $I \not\prec \pi|_H$.*

(2) *If μ is not supported on a closed amenable subgroup, then the spectral radius of $\pi(\mu)$ satisfies $r_{sp}\pi(\mu) < 1$.*

Furthermore, if we assume regarding π the weaker assumption that only $1_G \not\prec \pi$, then the conclusion of (1) holds true under the assumption that the projection of H to every *simple factor does not have amenable closure, and the conclusion of (2) holds true under the assumption that the projection of μ to* every *simple factor is not supported on a closed amenable subgroup.*

Notice that if G is Kazhdan, Theorem C yields a uniform gap from 1 for $r_{sp}\pi(\mu)$, over *all* the unitary representations π without an invariant vector. It is easy to verify that (2) in the Theorem implies (1) (see the argument at the end of this paragraph). The latter may be regarded as a "weak containment" analogue of Howe-Moore's theorem ([Mo1], [HM]), which replaces "amenable subgroup" by "compact subgroup", and "weak containment" by "proper containment" (of the trivial representation). Notice also that (2) implies that as a Kazhdan set for a simple Kazhdan Lie group, one can take any set of elements which is not contained in a closed amenable subgroup. Indeed, then there exists a *finite* subset with this property, and on it spread, say uniformly, a probability measure μ.

In Section 8 we present another application of our methods in considering the question of existence of a Kazhdan constant valid for *all* the generating sets of a finitely generated group. Very little seems to be known about the problem in general. We discuss its relation with uniform growth rate of groups, and establish a result in this direction for certain hyperbolic groups.

Acknowledgments. — Since the Fall of 95 this paper has been circulated in different forms, and was revised several times. The first version was written in October 95, while visiting the Max Planck Institute at Bonn. We would like to thank the MPI for its hospitality and support during that visit. A second version was completed on July 96, jointly with Amos Nevo, to whom we thank for his contribution and for illuminating discussions on various subjects related to the paper. In writing Sections 4 and 7 we have benefited from [Ne], both in the methods employed, as well as in learning about some relevant literature. The next version was distributed during the Fall of 97, in which, among other improvements, Theorem A was brought to its present unified form, and Theorem C was put. After [Sh2] was written, a fourth version of the paper included the results there (under the current title). The present form of the paper was finalized at the beginning of 99.

2. Preliminaries.

We briefly review some of the definitions and results relevant to our discussion. All the representations considered hereafter are assumed to be unitary and continuous (in the strong operator topology). For any representation π, $\infty \cdot \pi$ denotes a countable direct sum of copies of π. Throughout this section, G denotes a locally compact, second countable group.

Let us first clarify an ambiguity in the notion of weak containment.

DEFINITION 2.1. — *Let ρ, π be two unitary G-representations. We say that π is* weakly contained *in ρ, denoted $\pi \prec \rho$, if either one of the following equivalent conditions holds:*

1. *Every diagonal matrix coefficient of π can be approximated, uniformly on compact sets, by convex combinations of matrix coefficients of ρ.*

2. *When extended to a representation of the group algebra* $L^1(G)$, π *and* ρ *satisfy for all* $f \in L^1(G)$, $\| \pi(f) \| \leqslant \| \rho(f) \|$.

We say that ρ *and* π *are weakly equivalent, denoted* $\rho \sim \pi$, *if* $\rho \prec \pi$ *and* $\pi \prec \rho$. *The representation* π *is said to be weakly contained in a set of representations* $\mathcal{F}$, *if it is weakly contained in the direct sum of the representations in* $\mathcal{F}$.

The equivalence between the two conditions above was proved by Eymard [Ey]. Condition 1 is the original definition due to Fell [Fe]. There are many other characterizations of weak containment (see e.g. [Dix], [Kir], and also [CHH]). On the other hand, some authors, for example [Zi], [Mar], [HV], use the following definition:

$(*)$ π is weakly contained in ρ if every $n \times n$ submatrix of π can be approximated, uniformly on compact sets, by $n \times n$ submatrices of ρ.

We note that $(*)$ is a stronger requirement. Indeed, by Definition 2.1 every unitary representation π satisfies $\pi \sim \infty \cdot \pi$, but this is not the case according to definition $(*)$; for instance when π is finite dimensional. Nevertheless, the connection between the two definitions is simple, as the following shows:

PROPOSITION. — *For any unitary G-representations* τ *and* π, *one has* $\tau \prec \pi$ *in the sense of 2.1 iff* $\tau \prec \infty \cdot \pi$ *in the sense of* $(*)$.

Indeed, if every submatrix of τ can be approximated by a submatrix of $\infty \cdot \pi$, then of course this holds also for 1×1 matrices. For the converse, we recall that if τ is irreducible, the two definitions can be shown to be equivalent (see [Fe, 2.2] and the remark thereafter). For a general representation one then uses a direct integral argument to deduce the proposition.

For some probability measures μ on G, our purpose will be to compute the operator norm $||\pi(\mu)||$ for a general unitary G-representation π, in terms of the operator norm $||\lambda(\mu)||$ for the regular G-representation λ. Recall that the operator $\pi(\mu)$ is defined by $\langle \pi(\mu)v, u\rangle = \int \langle \pi(g)v, u\rangle d\mu(g)$. The following theorem, due to M. Cowling, U. Haagerup and R. Howe [CHH], will enable us to do that.

THEOREM 2.2 (Weak containment in the regular representation). *Let* $(\pi, \mathcal{H})$ *be a unitary G-representation. Assume that there exists a dense subspace* $W \subseteq \mathcal{H}$, *such that for any* $u, v \in W$, $\langle \pi(g)u, v\rangle \in L^{2+\epsilon}(G)$, *for all* $\epsilon > 0$. *Then for every* $f \in L^1(G)$, $\| \pi(f) \| \leqslant \| \lambda(f) \|$, *where* λ *is the regular representation of* G. *Consequently,* $\pi \prec \lambda$.

We remark that it will often suffice to invoke the following classical variant of Theorem 2.2, due to Dixmier [Dix]: Under the stronger assumption $\epsilon = 0$, the representation π may be embedded in a multiple of the regular G-representation (cf. [HT, Ch.V 1.2.4]).

It will be convenient to establish the following notation:

DEFINITION 2.3. — *Let $S \subseteq G$ be a finite set of elements. We say that S is a* 2m-discrete free symmetric set *if $|S| = 2m$, $S = S^{-1}$ and S generates freely a discrete free subgroup of G.*

There are rather few examples of unitary representations of discrete groups, for which explicit Kazhdan constants are known, even for regular representations (see [HRV1], [HRV2], [GH] and the references therein). Fortunately, we will be able to reduce our problem to one regarding the regular representation of the free group. There, using the following well known result of Kesten, one can obtain a (tight) estimate of the Kazhdan constant for a set of free generators.

THEOREM [Ke]. — *Let $S = S^{-1} \subseteq \mathbb{F}_m$ be a set of free generators (and their inverses) for the free group $\mathbb{F}_m$, and let $\mu = \frac{1}{|S|}\sum_{s\in S} s$ be the associated convolution (or averaging) operator. Then $\| \lambda(\mu) \| = \sqrt{2m-1}/m < 1$, where λ denotes the regular representation of $\mathbb{F}_m$.*

Finally, the following observation will be extremely useful to us in the sequel. Recall that if $H < G$ is a closed subgroup, then the restriction of the regular G-representation to H is equivalent to a multiple of the regular H-representation. More precisely, $L^2(G)|_H \cong \dim L^2(G/H)\cdot L^2(H)$. Combining this fact when H is a free group, Definition 2.1, and the above theorem of Kesten together, yields:

PROPOSITION 2.4. — *Let $S \subseteq G$ be a $2m$-discrete free symmetric set and $\mu = \frac{1}{|S|}\sum_{s\in S} s$ be the associated averaging operator. Let π be a representation of G with $\pi \prec \lambda$, where λ is the regular representation of G. Then $\| \pi(\mu) \| \leqslant \sqrt{2m-1}/m$.*

3. Proof of Theorem A for higher-rank groups.

Let k be a locally compact non-discrete field, and let G denote the group of k-points of a simply connected, semisimple, almost k-simple k-group, with k-rank greater than 1. As is well known, G contains (the group of k-points of) a k-subgroup, k-isomorphic to either SL_3 or Sp_4 [Mar, I.1.6.2]. It is easy to verify that $SL_2(k) \ltimes k^2 < SL_3(k)$, and $SL_2(k) \ltimes k^3 < Sp_4(k)$, where $SL_2(k)$ acts on k^2 via the defining representation, and on k^3 via the adjoint representation. Therefore G contains a subgroup, which we denote by H, consisting of the k-points of an algebraic k-group, such that H is isomorphic to either $SL_2(k) \ltimes k^2$ or $SL_2(k) \ltimes k^3$. We can now state the following:

THEOREM 3.1.— *Let G and $H \cong SL_2(k) \ltimes k^n$ ($n = 2,3$) be as above. Suppose $S \subseteq SL_2(k) < H$ is a $2m$-discrete free symmetric set (2.3), and let μ denote the associated averaging operator as in 2.4. Let π be any unitary representation of G with no G-invariant vectors. Then $\| \pi(\mu) \| \leqslant \sqrt{2m-1}/m$.*

Proof.— We argue in the following steps (compare e.g. with [Zi, Ch. 7]):

(1) The standard and the adjoint representation of $SL_2(k)$ on k^n ($n = 2,3$) are algebraic, and the adjoint action of $SL_2(k)$ on the dual group of characters $\widehat{k^n}$ is algebraic as well. Thus every orbit is locally closed in the locally compact topology of $\widehat{k^n}$ (see [BZ, 6.15] for a proof valid for any k). Therefore, by Mackey's theorem [Mac] any irreducible unitary representation π of H, is induced from a unitary representation σ of a subgroup which stabilizes some $\chi \in \widehat{k^n}$. This χ is trivial if and only if $\pi|_{k^n}$ is trivial.

(2) If π is an irreducible unitary representation of H with $I \not\subset \pi|_{k^n}$, then the χ given in (1) is not trivial, therefore its stabilizer in $SL_2(k)$ is solvable and its stabilizer in H is (solvable and hence) amenable.

(3) For every unitary representation σ of an amenable locally compact group F, one has $\sigma \prec L^2(F)$. Therefore by continuity of induction, (2) and (1) imply that for every irreducible H-representation π, satisfying $I \not\subset \pi|_{k^n}$, one has $\pi \prec L^2(H)$.

(4) If π is any unitary representation of H such that $I \not\subset \pi|_{k^n}$ then in a direct integral decomposition $\pi = \int \pi_x \, d\mu(x)$ with π_x irreducible, for μ-a.e. x one has $I \not\subset \pi_x|_{k^n}$ and thus by (3), $\pi_x \prec L^2(H)$. Therefore integrating gives for such $\pi : \pi \prec \infty \cdot L^2(H) \sim L^2(H)$.

(5) By Howe-Moore's Theorem over local fields and the assumption on π, it follows that $I \not\subset \pi|_{k^n}$ and therefore by (4) $\pi|_H \prec L^2(H)$.

(6) By (5) and Proposition 2.4, $\| \pi(\mu) \| \leqslant \sqrt{2m-1}/m$. □

Proof of Theorem A for higher rank groups. — Given any unitary representation $(\pi, \mathcal{H})$ of G such that $I \not\subset \pi$, and any $v \in \mathcal{H}$ with $\| v \| = 1$, we have, using Theorem 3.1:

$$\frac{1}{|S|} \sum_{s \in S} \mathrm{Re}\langle \pi(s)v, v\rangle \leqslant \left| \frac{1}{|S|} \sum_{s \in S} \langle \pi(s)v, v\rangle \right| \leqslant \| \pi(\mu) \| \leqslant \sqrt{2m-1}/m.$$

Consequently, for at least one $s \in S$, $\mathrm{Re}\,\langle \pi(s)v, v\rangle \leqslant \sqrt{2m-1}/m$. But since $\| \pi(s)v - v \|^2 = 2 - 2\mathrm{Re}\,\langle \pi(s)v, v\rangle$, we conclude that S is a Kazhdan set with a Kazhdan constant ϵ as in the statement of Theorem A. In fact, already a subset which consists of one of every two reciprocal elements in S forms a Kazhdan set, with the same constant ϵ. □

It may be interesting to note that G. A. Margulis, in the course of the proof of the superrigidity theorem, also used discrete embeddings of free groups in semisimple groups to obtain some spectral information (see [Mar, p. 189]).

Proof of the optimality of the Kazhdan constants in Theorem A. — We start by noticing that for the measure μ defined above, the universal bound on the *norm*: $\| \pi(\mu) \| \leqslant \sqrt{2m-1}/m$ is best possible, as it is obtained already when π is the regular representation of G (use the discussion preceding Proposition 2.4). In fact, we claim that for any locally compact group, this is the lowest possible uniform bound on $||\pi(\mu)||$, for an averaging operator $\pi(\mu)$ defined on *any* finite symmetric set S, not necessarily a set which generates a free subgroup. To remove first the freeness assumption on $\langle S\rangle$, recall that for any group Γ generated by a symmetric set of $2m$ elements, S, the norm of the corresponding averaging operator μ on $\ell^2(\Gamma)$ is greater than $\sqrt{2m-1}/m$, unless Γ is free on S (see [Ke], [HRV1], [HRV2]). Furthermore, the discreteness assumption on the subgroup Γ generated by the set S is also redundant. Indeed, it is not true then that the restriction of $L^2(G)$ to Γ is a multiple of $\lambda = \ell^2(\Gamma)$, but instead take π to be the restriction to Γ of the regular G-representation, in the following general inequality proved in [Sh4] (see Lemma 2.3 and the remark thereafter): For every finitely generated group Γ, a probability measure μ on Γ, and a unitary Γ-representation π: $\| \lambda(\mu) \| \leqslant \| \pi \otimes \bar{\pi}(\mu) \|$. Since in our case π is the regular G-representation, $\pi \otimes \bar{\pi}$ is a multiple of π as a G-representation, hence also as Γ-representation, and the claim follows.

Proceeding to the main issue, we need to show that the above Kazhdan constant, for a $2m$-discrete free symmetric set S, is best possible. Again, it suffices to see that it is the best Kazhdan constant for the S-action in the regular representation λ of the free group $\mathbb{F}_m$ (generated by S). To this end, we shall exhibit a unitary representation π of $\mathbb{F}_m$ which has the following two properties: (i) The best Kazhdan constant for the action of S in the representation π, is at least the same $\epsilon = \sqrt{2 - 2(\sqrt{2m-1}/m)}$. (ii) π is weakly contained in λ. Then, denoting for any representation by $\kappa(\cdot, S)$ its best Kazhdan constant (for the set S), we deduce from (i) that $\epsilon \leqslant \kappa(\pi, S)$, and from (ii) that $\kappa(\pi, S) \leqslant \kappa(\lambda, S)$, thereby proving our claim.

The representation π of $\mathbb{F}_m$ that we shall take is the one coming from the action on its so called "Poisson μ-boundary" B (we refer to [Fu, §4.1] for definitions, details and proofs of the facts presented below). The space B consists of all infinite (one sided, say to the right) sequences of letters in the generators S (adjacent inverses cancelled), equipped with the product, hence compact, topology. The free group $\mathbb{F}_m$ acts naturally and continuously on B by "stringing" from the left, and B supports a unique μ-stationary probability measure ν (i.e., one which is invariant under convolution by μ). Therefore, ν is quasi-invariant under the action of $\mathbb{F}_m$, and this induces a quasi-regular representation π of $\mathbb{F}_m$ on $L^2(B, \nu)$. The measure ν is in fact simple to describe explicitly, and it can easily be verified that for the constant unit function $w = 1 \in L^2(B, \nu)$, one has $\langle \pi(s)w, w\rangle = \sqrt{2m-1}/m$ for *every* free generator $s \in S$. A simple calculation now shows that any Kazhdan constant for the set S in this representation, must then be at least $\epsilon = \sqrt{2 - 2(\sqrt{2m-1}/m)}$.

Thus it only remains to establish (ii). This is surely well known, but we sketch a proof for completeness. The action of $\mathbb{F}_m$ on B arises naturally from its automorphism action on the $2m$-regular tree, when identified with the Cayley graph of $\mathbb{F}_m$ with respect to S. The action on B can be extended to a (transitive) action of the whole automorphism group of that tree, G, and the stabilizer P of one (and hence every) point is *amenable*. It is also easy to see that ν is quasi-invariant under the whole G-action, hence we may identify π with $L^2(G/P)$. However, inducing $I \prec L^2(P)$ from P to G yields $L^2(G/P) \prec L^2(G)$, and restricting back to $\mathbb{F}_m$ (which is a discrete subgroup of G) completes the proof. □

4. L^p-integrability of matrix coefficients and Kazhdan constants.

We begin by recalling the following:

Definition 4.1.— *Let G be locally compact, second countable group, and $(\pi, \mathcal{H})$ be a unitary G-representation. For $1 \leqslant p < \infty$, we say that π is* strongly L^p, *if there exists a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$, such that for all $u, v \in \mathcal{H}_0$ the matrix coefficient $g \to \langle \pi(g)u, v\rangle$ lies in $L^{p+\epsilon}(G)$, for all $\epsilon > 0$.*

When G is a (semi-)simple Lie group, one considers the subspace of K-finite vectors, where K is a maximal compact subgroup. Given an irreducible non-trivial unitary G-representation π, the matrix coefficients associated with K-finite vectors satisfy well known (from the work of Harish-Chandra) exponential decay estimates. These estimates imply that the (K-finite) matrix coefficients are in $L^p(G)$ for some $p = p(\pi) < \infty$, and that the set of q for which $\langle \pi(g)u, v\rangle \in L^q$ is always an open interval (p, ∞).

A fundamental result in the representation theory of simple groups with property (T), which is due to M. Cowling, asserts that the exponential decay of matrix coefficients of (K-finite) vectors in all non-trivial irreducible unitary representations, is faster than a fixed rate depending only on G. In particular, the matrix coefficients lie in $L^p(G)$ for some fixed $p = p(G) < \infty$ (see e.g. [Co], [Ho] and [HT, Ch.V 3.3.13]). This fact can be formulated more generally, applying to any family of irreducible representations of a simple Lie group, which is isolated from the trivial one.

Theorem 4.2 (L^p-integrability of matrix coefficients [Co], [KM1], [Mo2]).— *Let G be a simple Lie group with finite center. Then for every neighborhood U of the trivial representation $I \in \hat{G}$, there exists $p = p(U) < \infty$ such that every (irreducible) unitary representation of G outside U is strongly L^p. In particular, if G has property (T), then there exists $p < \infty$ such that the above holds for* every *non trivial irreducible representation of G.*

For simple Kazhdan Lie groups, a value of such p, and even the best p (denoted $p(G)$), have been computed explicitly. The first results in this direction are due to Howe [Ho], and they were extended by Li [Li] and Zhu [LZ]. In [Li] a table for $p(G)$ was computed for all but five of the classical Lie groups. See also [Oh] for a further comprehensive study of this issue.

For all these groups, one can readily determine explicit Kazhdan constants using the following general result (see also the discussion after Theorem 5.3 below for an application to rank-1 groups).

THEOREM 4.3. — *Let G be a locally compact second countable group, and $\mathcal{F}$ a family of unitary representations which are all strongly L^p for some (uniform) $p < \infty$. Denote by $\mathcal{F}_{\prec}$ the set of all G-representations which are weakly contained in $\mathcal{F}$. Let $n = 2k$ be an even integer with $n \geqslant p/2$, and $S \subseteq G$ a $2m$-discrete free symmetric set (2.3). Then for any $(\pi, \mathcal{H}) \in \mathcal{F}_{\prec}$ and vectors $\| v \| \leqslant 1$, $\| w \| \leqslant 1$, one has*

$$\frac{1}{|S|} \sum_{s \in S} |\langle \pi(s)v, w \rangle| \leqslant (\sqrt{2m-1}/m)^{1/n}.$$

In particular, there is $s \in S$ such that $|\langle \pi(s)v, v \rangle| \leqslant (\sqrt{2m-1}/m)^{1/n}$. Consequently, $\epsilon = \sqrt{2 - 2(\sqrt{2m-1}/m)^{1/n}}$ is a Kazhdan constant for S, for the family $\mathcal{F}_{\prec}$.

Remark. — Obviously, if $\mathcal{F}$ is the family of all non trivial irreducible representations, then $\mathcal{F}_{\prec}$ contains all the unitary representations without invariant vectors.

Proof. — Assume first that π is actually in $\mathcal{F}$ and let $\bar{\pi}$ be its contragredient representation. Notice that $\bar{\pi}$ is strongly L^p for the same p, and if $v \mapsto \bar{v}$ is the anti-isomorphism between $(\pi, \mathcal{H})$ and $(\bar{\pi}, \bar{\mathcal{H}})$, then by definition $\langle \pi \otimes \bar{\pi}(g)(v \otimes \bar{v}), w \otimes \bar{w} \rangle = |\langle \pi(g)v, w \rangle|^2$. Consider the representation $\tau = \pi^{\otimes k} \otimes \bar{\pi}^{\otimes k} \cong (\pi \otimes \bar{\pi})^{\otimes k}$ (k-fold tensor product). By Hölder inequality, all tensor products (and so their linear combinations) of the assumed dense set of vectors give matrix coefficients in $L^{\frac{p+\epsilon}{n}}$, and therefore in $L^{2+\epsilon}(G)$. It follows from Theorem 2.2 and Proposition 2.4 that

$$\tau \prec L^2(G) \qquad \| \tau(\mu) \| \leqslant \sqrt{2m-1}/m.$$

If we denote for any $u \in \mathcal{H}$ the vector $u_\tau = u \otimes \ldots \otimes u \otimes \bar{u} \otimes \ldots \otimes \bar{u}$ in the representation space of τ, we get for any $v, w \in \mathcal{H}$, $\| v \| \leqslant 1$, $\| w \| \leqslant 1$:

$$\frac{1}{|S|} \sum_{s \in S} |\langle \pi(s)v, w \rangle|^n = \langle \tau(\mu)v_\tau, w_\tau \rangle \leqslant \sqrt{2m-1}/m$$

and therefore

$$\frac{1}{|S|} \sum_{s \in S} |\langle \pi(s)v, w \rangle| \leqslant \Big(\frac{1}{|S|} \sum_{s \in S} |\langle \pi(s)v, w \rangle|^n\Big)^{1/n} \leqslant (\sqrt{2m-1}/m)^{1/n}$$

as required. Moving on to the general case, notice that we have used the L^p integrability only in showing that an appropriate tensor product embeds weakly in the regular representation. As weak containment is a transitive relation (and preserved under taking tensor products), the same conclusion holds with the weaker assumption $\pi \prec \mathcal{F}$. $\square$

Remark. — The idea of using L^p-integrability of matrix coefficients for the purpose of spectral estimates is taken from [Ne]. The argument itself has previously been used in [Co] and [Mo2]. For other applications of this method, in a different flavour, see e.g. [CS] and [Be2].

Evidently, Theorem 4.3 does not yield in general the best Kazhdan constants for the sets described there, as Theorem A shows. Nevertheless, it gives explicit constants for a large family of sets, and yields readily explicit Kazhdan constants also for the rank one Kazhdan groups $Sp(n,1)$ and $\mathbb{F}_{4(-20)}$. For instance, the classification of the unitary dual of $Sp(n,1)$ [Bal] shows that $p(Sp(n,1)) = 2n+1$. (Using the embedding of $Sp(2,1)$ in $\mathbb{F}_{4(-20)}$ this gives also Kazhdan constants for the latter, see Section 5 below). However, the whole classification of the unitary dual is an overkill, and in the next section we will see how the classification of the class-one dual alone (which is considerably easier), enables one to get, using a different idea, some Kazhdan sets with optimal constants.

5. Kazhdan constants for rank one groups.

In this section we discuss rank one simple Lie groups. In its first part we consider such a group, which does not necessarily have property (T), and using the results of Section 4 show how to compute explicitly Kazhdan constants for any family of representations which admits a spectral gap. In the second part we introduce a different approach, which will enable us to construct explicitly Kazhdan sets for the groups $Sp(n,1)$ and $\mathbb{F}_{4(-20)}$, with optimal Kzhdan constants, thereby completing the proof of Theorem A.

For the time being, let G be any simple Lie group with finite center, and $K < G$ a maximal compact subgroup. Recall that a class-one irreducible representation of G, is one that contains a non zero vector invariant under K. We denote by $\mathcal{I}^1(G)$ the set of irreducible class one representations of G, and by $\mathcal{I}^0(G)$ the set of all the unitary representations without a K-invariant vector. Clearly, the set $\mathcal{I}^0(G)$ is isolated from the trivial representation already as representations of K. In fact, this observation leads directly to the following simple result (see [DV] for this and more):

LEMMA 5.1. — *The compact subgroup K forms a Kazhdan set for the family $\mathcal{I}^0(G)$, with the Kazhdan constant $\sqrt{2}$.*

Thus, the problem of finding explicit Kazhdan constants is reduced to the family $\mathcal{I}^1(G)$. To deal with those, we begin by establishing some notation. Let $\pi \in \mathcal{I}^1(G)$, and let v denote the K-invariant unit vector in $\mathcal{H}_\pi$ (unique up to scalar). Let $\varphi_\pi(g) = \langle \pi(g)v, v\rangle$ denote the associated spherical function. Let $G = KA^+K$ be a Cartan decomposition of the simple Lie group G, $\mathfrak{a} =$ Lie A and $\mathfrak{a}^*$, $\mathfrak{a}_c^*$ its dual and complexified dual. Let $\Delta^+ \subseteq \Delta$ be the set of positive roots in a root system Δ, $\phi =$ half the sum of positive roots and W the Weyl group. Let KAN be the Iwasawa decomposition and $P = MAN$ a minimal parabolic subgroup (M = centralizer of A in K). Then to every $\lambda \in \mathfrak{a}_c^*$ (actually $\lambda \in \mathfrak{a}_c^*/W$), one can form the character of $P : man \mapsto e^{\lambda(\log a)}$, and inducing this representation to G yields a representation with a K-invariant vector v_λ (which is unitary if $\lambda \in i\mathfrak{a}$). The function $\psi_\lambda(g) = \langle \pi_\lambda(g)v_\lambda, v_\lambda\rangle$ is a spherical function. By a classical result of Harish-Chandra, for every class one representation $\pi \in \mathcal{I}^1(G)$, the (positive definite) spherical function φ_π is of the form ψ_λ, for some π_λ as above.

Let us focus our attention hereafter on the rank one case, i.e., $\dim \mathfrak{a} = 1$. Then it was shown by Kostant [Ko] that the set of λ's for which the above construction yields a unitary representation $\pi \in \mathcal{I}^1(G)$ is of the form $i\mathfrak{a}^* \cup [-\phi_0, \phi_0] \subseteq \mathfrak{a}_c^*$, where $\phi_0 \leqslant \phi$. The values of the two parameters for the various groups are:

LEMMA 5.2 [Ko].

(1) For $SO(n,1)$, $\phi_0 = \phi = \frac{n-1}{2}$.

(2) For $SU(n,1)$, $\phi_0 = \phi = n$.

(3) For $Sp(n,1)$, $n \geqslant 2$, $\phi_0 = 2n-1 < \phi = 2n+1$.

(4) For $\mathbb{F}_{4(-20)}$, $\phi_0 = 5 < \phi = 11$.

Restrict λ to the region given by $i\mathfrak{a}_+^* \cup [0, \phi_0]$. By [Ko], one gets a bijection $\lambda \to \pi_\lambda$ between the region and $\mathcal{I}^1(G)$ in the first two cases, and a bijection between the region and $\mathcal{I}^1(G) \setminus \{1\}$, in the last two cases.

As is well known, the spherical function $\psi_\lambda(g) = \langle \pi_\lambda(g)v_\lambda, v_\lambda\rangle$ can be estimated along A^+ by (see e.g. [Kn, 8.47] or [GV, 5.1]):

$$\psi_\lambda(a) \sim e^{(\lambda-\phi)\log a}. \tag{1}$$

In fact, this decay holds for all K-finite vectors, not only v_λ. Recall that the Haar measure of G in terms of the polar decomposition is given by $dg = J(a)dk\,da\,dk$, where the volume density is the bi-K-invariant function given along A^+ by (see e.g. [GV, p. 73]):

$$J(a) = \prod_{\alpha\in\Delta^+} \left(e^{\alpha(\log a)} - e^{-\alpha(\log a)}\right) \sim e^{2\phi(\log a)}. \tag{2}$$

From (1) and (2) one then sees readily that if p satisfies

$$p(\phi - Re\,\lambda) \geqslant 2\phi \tag{3}$$

then the spherical function ψ_λ (as well as all K-finite matrix coefficients in π_λ) lie in $L^{p+\epsilon}(G)$, for every $\epsilon > 0$.

As is well known (and is easy to verify), the sets $U_\lambda = \{\pi_{\lambda_1} | \lambda < \lambda_1 \leqslant \phi\}$ form a base for the neighborhoods of the trivial representation in the Fell topology. The following result presents explicit Kazhdan constants for any family of representations, in terms of a bound on the spectral parameter λ.

THEOREM 5.3. — *Let G, K and ϕ be as above, and fix some $0 \leqslant \lambda < \phi$. Denote by $\mathcal{F}_\lambda$ the set of* all *the G-representations which do not contain weakly any representation $\pi \in U_\lambda$. Let n be any even integer with $n \geqslant \phi/\phi - \lambda$, and S be a $2m$-discrete free symmetric set. Then $S \cup K$ forms a Kazhdan set for $\mathcal{F}_\lambda$, with the Kazhdan constant $\epsilon = \frac{1}{\sqrt{2}} \min\{\sqrt{2}, \sqrt{2 - 2(\sqrt{2m-1}/m)^{\frac{1}{n}}}\} = \sqrt{1 - (\sqrt{2m-1}/m)^{\frac{1}{n}}}$.*

Proof. — Denote $\mathcal{F}^1_\lambda = \mathcal{F}_\lambda \cap \mathcal{I}^1(G)$. Then by (3) (and the remark proceeding it), together with Theorem 4.3, we see that $\sqrt{2 - 2(\sqrt{2m-1}/m)^{\frac{1}{n}}}$ is a Kazhdan constant for the set S, for the family of all the representations which are weakly contained in $\mathcal{F}^1_\lambda$. Now, given any representation $\pi \in \mathcal{F}_\lambda$, we have a (unique) decomposition $\pi = \pi_1 \oplus \pi_2$, where π_1 is weakly contained in $\mathcal{F}^1_\lambda$, and $\pi_2 \in \mathcal{I}^0(G)$. Given any unit vector v, write the corresponding (orthogonal) decomposition $v = v_1 + v_2$. Then, for i=1 or 2 we have $||v_i|| \geqslant 1/\sqrt{2}$. We can now use the computation above in case $i = 1$, or Lemma 5.1 in case $i = 2$, to obtain easily the required estimate. $\square$

We note that one often has a good deal of information about the set of representations π_λ which can occur (weakly) in a given family of representations. For example, a well known result of Harish-Chandra implies that the π_λ which occur (weakly) in the regular representation of G on $L^2(G/\Gamma)$, where Γ is any discrete subgroup, correspond through the action of the Casimir operator, to the L^2-spectrum of the Laplacian on the locally symmetric space $K\backslash G/\Gamma$. From [Sh4, Proposition 2.4] it follows that a value p for which $L^2(G/\Gamma)$ is strongly L^p is determined already by its value in the class one spectrum, which in turn, depends (using (3) above) only on the bottom of the Laplacian spectrum, λ_0, by the formula $p = 2\phi/(\phi - \sqrt{\phi^2 - \lambda_0})$ (see 5.2 above for the values of ϕ in the different groups). Substituting this p in Theorem 4.3 yields explicit Kazhdan constants for the representation $L^2(G/\Gamma)$, in terms of a bound from 0 on λ_0 (which exists iff $L^2(G/\Gamma)$ does not contain weakly the trivial representation). If Γ is a lattice, a lower bound on the positive spectrum yields Kazhdan constants for the G-representation on the subspace of zero mean functions in $L^2(G/\Gamma)$, but here [Sh4, Proposition 2.4] does not apply, and Theorem 5.3 should be used. Finally, we should remark that adding K to the Kazhdan set in Theorem 5.3 will not affect the calculation of Kazhdan constants for lattices, as will be seen in the next section. Also notice that Theorem 5.3, together with Lemma 5.2, yields explicit Kazhdan constants for the groups $Sp(n,1)$ and $\mathbb{F}_{4(-20)}$. However, these are not optimal. Our next purpose is to describe a different approach which gives the constants guaranteed in Theorem A (thereby completing the proof of that theorem).

Proof of Theorem A for the rank one Lie groups with property (T).— Although our method works equally well for all the groups $Sp(n,1)$, $\mathbb{F}_{4(-20)}$, it will be more convenient to make the following reduction: We claim that to complete the proof of Theorem A for any of the above remaining rank one groups, say G, it suffices to prove it for $Sp(2,1)$. Indeed, every $Sp(n,1)$ $(n>1)$ clearly contains a copy of $Sp(2,1)$, and it is also known that the same holds for $\mathbb{F}_{4(-20)}$ (cf. [BB, p. 41]). By restricting any G-representation to this copy of $Sp(2,1)$, together with Howe-Moore's theorem, it is easy to see that the same Kazhdan constants of $Sp(2,1)$ apply to G. (Incidentally, notice that this argument shows that to establish property (T) for all the rank one groups, it suffices to consider only $Sp(2,1)$.)

Thus, we shall deal henceforth only with $G = Sp(2,1)$. Notice that G contains a natural copy of $SO(2,1)$, by viewing $\mathbb{R}$ as a subfield of the quaternionic ring $\mathbb{H}$. Therefore, the above discussion and the following result complete the proof of Theorem A.

THEOREM 5.4.— *Let $SO(2,1) < G = Sp(2,1)$ be the inclusion of groups as above. Suppose that π is a non trivial irreducible unitary* *representation of G. Then $\pi|_{SO(2,1)}$ is strongly L^1. Hence, by the remark proceeding Theorem 2.2, for every unitary G-representation π with no invariant vectors one has $\pi|_{SO(2,1)} \subset \infty \cdot L^2(SO(2,1))$. Consequently, as in the proof of Theorem A for the higher rank groups (see Section 3), if $S \subset SO(2,1)$ is a $2m$-discrete free symmetric set, then $\epsilon = \sqrt{2-2(\sqrt{2m-1}/m)}$ is a (best) Kazhdan constant for the set S (and actually, for any subset consisting of one of every two reciprocal elements in S).*

Proof. — Notice first that for our embedding of $SO(2,1)$ in G, we have also inclusions of maximal compact subgroups, and the two groups share a mutual Cartan subgroup. Let γ denote the positive root of $SO(2,1)$. Then, as is well known, the positive roots of G are γ (with multiplicity 4) and 2γ (with multiplicity 3). Thus, with the notations at the beginning of this section, we have $\phi = \gamma/2$ for $SO(2,1)$, and $\phi = 5\gamma$ for G.

Let now τ be *any* unitary representation of G with no invariant vectors, and assume that v is a K-invariant vector (where $K < G$ denotes a maximal compact subgroup). Then v lies in a subrepresentation $\sigma \subset \tau$ whose spectral decomposition $\sigma = \int \sigma_x$ consists of class-one representations only, and $v = \int v_x$, where $v_x \in \mathcal{H}_{\sigma_x}$ is a K-invariant vector for (almost) every x. Since the estimate (1) of the decay of the spherical functions holds uniformly over all the class-one representations, it follows from the computation in (3) (and a simple direct-integral calculation), that the matrix coefficient $\langle \tau(g)v, v\rangle$ lies in $L^{10/(5-3)+\epsilon}(G) = L^{5+\epsilon}(G)$ for all $\epsilon > 0$. From [Co, 2.2.6] it then follows that for every non trivial irreducible G-representation, all matrix coefficients associated with the K-finite vectors are in $L^{2\cdot 5+\epsilon}(G) = L^{10+\epsilon}(G)$. Therefore, every such matrix coefficient decays exponentially along the (positive Weyl chamber of the) Cartan subalgebra $a \in \mathfrak{a}_+$ as $\exp(5\gamma(a))^{-2/10+\epsilon} = \exp(\gamma(a)/2)^{-2/1+\epsilon}$, which implies that the restriction of this matrix coefficient to $SO(2,1)$ is in $L^{1+\epsilon}$, as required. (We have used here twice [Co, 2.2.4], once in each direction, together with the preliminaries at the beginning of the proof.) □

6. Kazhdan constants for lattices: Proof of Theorem B.

Let G denote the group of k-points of a connected semi-simple k-group. If k is connected, let d be the G-invariant Riemannian metric on the symmetric space G/K induced by the Killing form (K a maximal compact group). Otherwise, the metric on the associated Bruhat-Tits building induces a G-invariant metric on G/K, which is again denoted by d. We denote by e the identity coset in G/K and define $||g|| = d(ge, e)$. $|| \, ||$ is an additive semi-norm on G, i.e., $||g^{-1}|| = ||g||$ and $||gh|| \leqslant ||g|| + ||h||$. Furthermore, $||g|| = 0 \Leftrightarrow g \in K$ and for $h \in K$, one has $||gh|| = ||hg|| = ||g||$. For every $M < \infty$ the ball $B_M = \{g \in G | ||g|| \leqslant M\}$ is compact. Given a lattice $\Gamma < G$, we normalize the G-invariant measure m on $\Gamma \setminus G$ to be a probability measure. With these preliminaries we can present:

Proof of Theorem B. — The proof may be viewed as a quantitative version of the standard argument showing that a lattice in a group with property (T), has this property as well (see also [HV, Lemma 3.3]).

Using the equality $||gv - v||^2 = 2 - 2\,\mathrm{Re}\langle gv, v\rangle$ (for $||v|| = 1$), it suffices to prove the following: Let π be a representation of Γ with a unit vector v such that

$$(1) \qquad \forall \gamma \in \Gamma_R \quad \mathrm{Re}\,\langle \pi(\gamma)v, v\rangle > 1 - \frac{1}{2}\left(\frac{\epsilon^2 - 8\delta}{1 - 2\delta}\right)^{\frac{1}{2}\cdot 2} = \frac{1 - \frac{1}{2}\epsilon^2 + 2\delta}{1 - 2\delta}.$$

Then for the G-representation σ on $\mathrm{Ind}_\Gamma^G \pi$ there exists a unit vector f such that

$$(2) \qquad \forall s \in Q \quad \mathrm{Re}\,\langle \sigma(s)f, f\rangle > 1 - \frac{1}{2}\epsilon^2.$$

Indeed, since (Q, ϵ) form Kazhdan constants for G, the existence of f as in (2) implies that $I \subseteq \mathrm{Ind}_\Gamma^G \pi$, and therefore $I \subseteq \pi$, as required.

Let then δ, M, R and Γ_R be as in the theorem, and let $F \subseteq G$ be a fundamental domain for Γ in G (i.e. $G = \Gamma \cdot F$) chosen such that $m(B_M \cap F) \geqslant 1 - \delta$. Let $v \in \mathcal{H}$ be a unit vector satisfying (1) above, and define the measurable function $f : G \to \mathcal{H}$ by

$$f(\gamma h) = \pi(\gamma)v \quad \gamma \in \Gamma,\, h \in F.$$

Writing every $g \in G$ uniquely as $g = \gamma_g h_g$, where $\gamma_g \in \Gamma$ and $h_g \in F$, we have, recalling that G operates by right translations,

$$f(\gamma g) = f(\gamma\gamma_g h_g) = f(\gamma\gamma_g) = \pi(\gamma)\pi(\gamma_g)v = \pi(\gamma)f(g) \quad \gamma \in \Gamma,\, g \in G.$$

It follows that f is a unit vector in the representation space of $\mathrm{Ind}_\Gamma^G \pi$. Since $m(B_M \cap F) \geqslant 1 - \delta$, for every $s \in G$ the set of $g \in B_M \cap F$ for which $gs \in \Gamma(B_M \cap F)$ has measure at least $1 - 2\delta$. Moreover, if $s \in Q$ and $g \in B_M \cap F$ satisfy $gs = \gamma g_1$ for $\gamma \in \Gamma$, $g_1 \in B_M \cap F$, then

$$\| \gamma \| = \| gsg_1^{-1} \| \leqslant \| g \| + \| s \| + \| g_1^{-1} \| \leqslant R$$

so $\gamma \in \Gamma_R$. It follows that for every $s \in Q$ there exists a set of $g \in F$ with measure at least $1 - 2\delta$ for which

$$\mathrm{Re}\,\langle f(gs), f(g)\rangle = \mathrm{Re}\,\langle \pi(\gamma)v, v\rangle > \frac{1 - \frac{1}{2}\epsilon^2 + 2\delta}{1 - 2\delta}$$

(using $\gamma \in \Gamma_R$ and (1)). Thus, for all $s \in S$,

$$\begin{aligned}\mathrm{Re}\,\langle \sigma(s)f, f\rangle &= \int_F \mathrm{Re}\,\langle f(gs), f(g)\rangle > \frac{(1 - \frac{1}{2}\epsilon^2 + 2\delta)}{(1 - 2\delta)} \cdot (1 - 2\delta) + (-1) \cdot 2\delta \\ &= 1 - \frac{1}{2}\epsilon^2\end{aligned}$$

and therefore (2) is satisfied, as required. □

We note that the proof of Theorem B yields the following sharper result, which will be used in the sequel.

COROLLARY 6.1. — *Let $\mathcal{F}$ be any family of representations of G which has (Q, ϵ) as Kazhdan constants. Then $(\Gamma_R, \left(\frac{\epsilon^2 - 8\delta}{1 - 2\delta}\right)^{1/2})$ as in Theorem B, form Kazhdan constants for all the Γ-representations π for which $Ind_\Gamma^G \pi$ is weakly contained in $\mathcal{F}$.*

As mentioned in the introduction, the explicit Kazhdan constants for the lattices may be used to point out specific sets of generators.

COROLLARY 6.2. — *Let G be a Kazhdan group as in Theorem A. Suppose that $S = \{a, b\} \subset G$ is a Kazhdan set with the best Kazhdan constant $\epsilon = \sqrt{2 - \sqrt{3}}$, as in Theorem A. Specifically, suppose that a and b generate a discrete free subgroup of G, which in the higher rank case, is contained in a copy of $SL_2(k)$ for which $SL_2(k) \ltimes k^n$ is embedded in G, or, in the rank one case, contained in a copy of $SO(2,1)$ embedded naturally in $Sp(2,1)$, itself contained in $Sp(n,1)$ or $\mathbb{F}_{4(-20)}$ (see Section 5). Normalize the G-invariant measure m on $\Gamma\backslash G$ to have total mass one. Let $U \subseteq G$ be any (bounded) subset with*

$$m(\Gamma \cdot U) > \frac{6 + \sqrt{3}}{8} \quad (\approx 0.966).$$

Then the (finite) set $\{UaU^{-1} \cup UbU^{-1}\} \cap \Gamma$ generates Γ.

Proof. — First notice that if S is a Kazhdan set for a countable group Γ, then S generates Γ. Indeed, if $\Gamma_0 < \Gamma$ is the subgroup generated by S, then in $\ell^2(\Gamma/\Gamma_0)$ there is an S-invariant vector, hence a Γ-invariant vector as well. It follows that Γ_0 must be of finite index. Now, if there are at least two Γ_0-orbits in Γ/Γ_0, then in $\ell_0^2(\Gamma/\Gamma_0)$ there exists a Γ_0-invariant vector, and hence a Γ-invariant vector, which is a impossible. Hence Γ_0 has one orbit, and so $\Gamma_0 = \Gamma$.

To show that $USU^{-1} \cap \Gamma$ is a Kazhdan set, we follow the proof of Theorem B. Note that the argument there shows that the set of elements $B_M Q B_M^{-1} \cap \Gamma$ is a Kazhdan set for Γ (contained in a ball of radius R). We can take Q to be the set S above. Then, by the same argument used in the proof of Theorem B, $USU^{-1} \cap \Gamma$ is a Kazhdan set, provided $m(\Gamma \cdot U) > 1-\delta$, and $\delta < \frac{\epsilon^2}{8}$. To obtain the smallest set U we let δ tend to $\frac{\epsilon^2}{8}$ and take $\epsilon = \sqrt{2-\sqrt{3}}$. The required estimate follows. □

We close this section by presenting another result with a uniform feature, this time for certain *non Kazhdan*, rank one lattices. Let us first describe these arithmetic lattices of $SO(n,1)$, sometimes called "the lattices of simplest type".

Let q a quadratic form defined over a totally real number field, satisfying:

1. q has signature $(n,1)$ over $\mathbb{R}$.
2. q^σ is definite as a real quadratic form, for every Galois automorphism $\sigma \neq \mathrm{id}$.

Let $\mathbf{G}$ be the special orthogonal group of q. Consider $\mathbf{G}$ as a $\mathbb{Q}$ group via restriction of scalars, so that $\mathbf{G}(\mathbb{R}) = SO(n,1) \times \prod SO(n+1)$. The projection of $\mathbf{G}(\mathbb{Z})$ to $SO(n,1)$ is an arithmetic lattice Γ, and we denote by $\Gamma(N) = \{\gamma \in \Gamma | \gamma = I_{modN}\}$ its (principal) congruence subgroups. We remark that these lattices cover all the non-uniform arithmetic lattices, and even *all* the arithmetic ones, when n is even (cf. [Lub2, §3] for details). Actually, a result similar to the one we prove below holds, using the same methods, for all the other arithmetic lattices in $SO(n,1)$ as well (with slightly different, but still uniform, constants).

Finally, for $G = SO(n,1)$ ($n \geqslant 2$), we take a maximal compact subgroup $K = SO(n)$, and $G/K \cong \mathbb{H}^n$ is the n-dimensional real hyperbolic space, with an "origin" $O \in \mathbb{H}^n$ fixed by K. (Actually, above and throughout the proof below we should replace $SO(n,1)$ by its index 2 connected component, but this technicality is insignificant for our purposes.)

THEOREM 6.3. — *Let $\Gamma < G = SO(n,1)$ be an arithmetic lattice as above, and let $M < \infty$ be such that the ball with radius M around the origin $O \in \mathbb{H}^n$ contains at least 0.995 of the measure of a fundamental* *domain for the action of* Γ *on* $\mathbb{H}^n$. *Let* $\Gamma_R \subseteq \Gamma$ *be the (finite) set of elements* $\gamma \in \Gamma$ *for which* $d(\gamma O, O) \leqslant 2M+1$. *Then* $(\Gamma_R, 1/6)$ *form Kazhdan constants for all the representations* $\pi = \ell_0^2(\Gamma/\Gamma_1)$, *where* $\Gamma_1 < \Gamma$ *is any congruence subgroup, and* ℓ_0^2 *is the space of zero mean functions.*

This theorem makes explicit and quantitative, well known geometric constructions of expander graphs (as mentioned in [Lub1, §4]). Notice again the uniformity of the result over different n's. Before turning to the proof, notice that Theorem 6.3 implies that the projection of the set Γ_R must generate every congruence quotient, so this set generates a congruence dense subgroup of Γ. As every congruence dense subgroup must be Zariski dense, we deduce the following independent conclusion (stated only for the more interesting family of non-uniform lattices).

COROLLARY 6.4. — *For any* $2 \leqslant n \in \mathbb{N}$ *assume that* $\Gamma < G = SO(n,1)$ *is some non-uniform arithmetic lattice. Let* $\Gamma_R \subset \Gamma$ *be the finite subset defined as in Theorem 6.3. Then* Γ_R *generates a Zariski dense subgroup of* Γ *(and* G*).*

Proof of Theorem 6.3. — We first examine the case $n = 2$. Then $SO(2,1) \cong SL_2(\mathbb{R})$ (locally) and by Selberg's 3/16 theorem, together with the Jacquet-Langlands correspondence, we deduce that all the non trivial representations π_λ which occur (weakly) in some $L^2(SO(2,1)/\Gamma(N))$ satisfy $\lambda \geqslant \sqrt{\frac{1}{4} - \frac{3}{16}} = \frac{1}{4}$ (we retain the notation of Section 5). In the general case, fixing the first $n-2$ variables in q induces a natural inclusion of algebraic groups, and from [BS] it follows that the restriction of $L^2(SO(n,1)/\Gamma(N))$ to $SO(2,1)$ must obey the same spectral restriction on the π_λ's as above. Suppose now that S is any set of two elements in $SO(2,1)$ which generates a discrete free subgroup. From Theorem 5.3 it follows that $\epsilon = \sqrt{1 - (\sqrt{3}/2)^{\frac{1}{2}}} \approx 0.26$ is a Kazhdan constant for the set $S \cup SO(2)$, for all $L_0^2(SO(n,1)/\Gamma(N))$ (orthogonal complement to the constants), as representations of $SO(2,1)$. By Howe-Moore's theorem we deduce that these form actually Kazhdan constants for the whole group $SO(n,1)$. Now, as $\mathrm{Ind}_\Gamma^G \ell^2(\Gamma/\Gamma(N)) = L^2(SO(n,1))/\Gamma(N))$, we wish to invoke Corollary 6.1. Take S to be the set of matrices in the example proceeding Theorem A (under the local isomorphism $SO(2,1) \cong SL_2(\mathbb{R})$). Since $SO(2,1)$ is embedded in $SO(n,1)$ as the stabilizer of a totally geodesic hyperplane, it is enough to evaluate the norm of its elements (for the purpose of Corollary 6.1), in their action on the upper half plane. An easy calculation bounds it from 1. Finally, taking $\delta = 0.005 < \epsilon^2/8 \approx 0.008$ in Corollary 6.1, the required estimate follows. □

7. Spectral radius of convolution operators: Proof of Theorem C.

In this section we shall extend the ideas and results presented in Sections 4, 5, to prove Theorem C (stated in the introduction). We shall need the following claim which essentially appeared in the course of the proof of Theorem 4.3. It will be convenient however, to state and prove it here in the following form:

LEMMA 7.1. — (1) *Suppose that the values $m \in \mathbb{N}$ and $1 \leqslant p < \infty$ satisfy $p \leqslant 2m$. Let $\pi_1, ..., \pi_m$ be representations of the semisimple group $G = \prod_{i=1}^n G_i$, decomposed as (outer) tensor products: $\pi_j = \pi_j^{(1)} \otimes ... \otimes \pi_j^{(n)}$. Suppose further that for every $1 \leqslant j \leqslant m$, $1 \leqslant i \leqslant n$ the representation $\pi_j^{(i)}$ of the group G_i, has all the matrix coefficients of its K_i-finite vectors lying in $L^p(G_i)$. Then $\pi_1 \otimes ... \otimes \pi_m \subset \infty \cdot L^2(G)$.*

(2) *Assume the above L^p condition only for $i = 1$, namely that we are given only that $\pi_j^{(1)}$ are strongly L^p for all $1 \leqslant j \leqslant m$. Then there exists some unitary representation σ of $G' = G_2 \times ... \times G_n$, such that as G-representations, we have $\pi_1 \otimes ... \otimes \pi_m \subset \infty \cdot L^2(G_1) \otimes \sigma$, a tensor product of two G-representations, where G acts in $\infty \cdot L^2(G_1)$ and σ through its projections to G_1 and G', resp.*

Proof. — As in the proof of Theorem 4.3, notice that using $p/2 \leqslant m$ and Holder inequality, the product of any m functions in L^p is in L^q for some $q \leqslant 2$, and therefore also in L^2, if the functions are bounded. Denote for $1 \leqslant i \leqslant n$ the G_i-representation: $\sigma_i = \pi_1^{(i)} \otimes \cdots \otimes \pi_m^{(i)}$. Then for a fixed i, linear combinations of tensor products of K_i-finite vectors in each $\pi_j^{(i)}$ form a dense set of vectors in σ_i, for which the corresponding matrix coefficient is in $L^2(G_i)$. Taking this dense set of vectors, and applying Fubini, we get a dense set of matrix coefficients of $\pi_1 \otimes \cdots \otimes \pi_m = \sigma_1 \otimes \cdots \otimes \sigma_n$, which are in $L^2(G_1 \times \cdots \times G_n)$. The first assertion now follows from [HT, Ch.V 1.2.4] (see Theorem 2.2 and the remark thereafter). The second one is proved using a similar argument, applied only to the first factor (notice that we do not claim here anything about the representation σ). □

We preface the proof of Theorem C with some remarks comparing the two (close, but independent) assertions stated there. As mentioned following the statement of Theorem C, part 1 of the theorem (in each one of the assertions) may be viewed as an analogue of Howe-Moore's theorem ([Mo1], [HM]). However, in contrast to Howe-Moore's theorem, the second assertion in Theorem C does not seem to follow from the first one, and its proof will require further considerations. (Of course, only for non simple groups the two assertions differ.) Before turning to the proof of Theorem C, we would like to recall a natural and important family of representations of semisimple groups, for which it seems at the present that only the weaker assumption, namely that of the *second* assertion, is known in general. These are the regular representations of G on the orthogonal complement to the constant functions in $L^2(G/\Gamma)$, where $\Gamma < G$ is any (irreducible) lattice. The fact that the weaker assumption is always satisfied for these representations, has been established by Bekka [Be1]. Recently, it was shown in [KM2] that when Γ is *non uniform*, actually the stronger assumption (i.e., that of the *first* assertion) of Theorem C holds. It seems very plausible that this is the case also for all the uniform lattices. Such result (also combined with Theorem C), would be useful in applications to ergodic theory (see e.g. [FS1], [KM1], [KM2], [Ne]). In fact, it is the application to [FS1] which arose our interest in part 2 (in each one of the assertions) of Theorem C, rather than just 1.

Proof of Theorem C.— Recall from the introduction that it suffices to prove only part (2) in each one of the assertions. We start with the first, and begin by showing that under the assumptions of the theorem, there exists $m \in \mathbb{N}$ such that the m-th tensor power $\pi^{\otimes m}$ of π is a subrepresentation of $\infty \cdot L^2(G)$. (We remark that the difference between the situation here and that in Sections 4 and 5 is that we do not know at this point that π itself is strongly L^p, as the L^p norms in its irreducible components may not be bounded. However, the argument there can still be used.) Write a direct integral decomposition $\pi = \int \pi_x \, d\nu(x)$ with $\pi_x = \pi_x^{(1)} \otimes \cdots \otimes \pi_x^{(n)}$, where every $\pi_x^{(i)}$ is an irreducible G_i-representation. By the assumption, for each i there exists a neighborhood U_i of I in the Fell topology on $\hat{G}_i$, so that for ν-a.e. x we have $\pi_x^{(i)} \notin U_i$. By 4.2 it then follows that for every i there exists $p_i < \infty$ for which the matrix coefficients of all K_i-finite vectors of $\pi_x^{(i)}$ are in $L^{p_i}(G_i)$. Let $p = \max p_i$ and take any $p/2 \leqslant m \in \mathbb{N}$. Then $\pi^{\otimes m} = \int \pi_{x_1} \otimes \cdots \otimes \pi_{x_m} \, d\nu(x_1) \ldots d\nu(x_m)$, and from Lemma 7.1 it follows that for a.e. $x_1,\ldots,x_m$, we have $\pi_{x_1} \otimes \cdots \otimes \pi_{x_m} \subset \infty \cdot L^2(G)$. This indeed implies that $\pi^{\otimes m} \subset \int \infty \cdot L^2(G) \, d\nu = \infty \cdot L^2(G)$.

Suppose now that under the assumptions of the theorem we have $r_{sp}\pi(\mu) = 1$. Then obviously the same holds for any tensor power of π, and hence, by the above, for the regular G-representation. Let $H < G_1$

be the closed subgroup generated by μ_1 (namely, the smallest closed subgroup supporting μ_1). We may view μ as a measure on H, and since as H-representation, the regular G-representation is a multiple of that of H, denoted λ, we deduce that $r_{sp}\lambda(\mu) = 1$. However this and the nonamenability of H, contradict [DG]. (In the course of the proof of the second assertion below, we shall reproduce the main argument of [DG].)

The second assertion requires a more careful analysis. Again, decompose $\pi = \int \pi_x \, d\nu(x)$ with $\pi_x = \pi_x^{(1)} \otimes ... \otimes \pi_x^{(n)}$. Here the assumption $I \not\prec \pi$ implies only that for every $1 \leqslant i \leqslant n$ there exists a neighborhood U_i of the trivial representation in $\hat{G}_i$, such that for ν a.e. x, there is *at least one* i for which in the decomposition $\pi_x = \pi_x^{(1)} \otimes \cdots \otimes \pi_x^{(n)}$, we have $\pi_x^{(i)} \notin U_i$. Using Theorem 4.2 this implies that there is some $p = p(\pi) < \infty$ with the following property: The spectral measure ν is supported on a (not necessarily disjoint) union of n subsets of $\hat{G}$: $\mathcal{F}_1, ..., \mathcal{F}_n$, where $\mathcal{F}_i$ denotes the set of all the G-representations $\pi = \pi_1 \otimes ... \otimes \pi_n$, for which π_i is a strongly L^p G_i-representation.

Now, let us assume that $r_{sp}\pi(\mu) = 1$. Then, since there are only finitely many $\mathcal{F}_i$'s, we may, by passing to an appropriate subrepresentation (and re-arranging the order of the simple factors), assume that π is supported on $\mathcal{F}_1$. Again, any tensor power of π has spectral radius 1, and using the second part of Lemma 7.1, together with the argument in the above proof of the first assertion, we deduce that there exists some representation σ of G' such that the spectral radius of the μ-convolution operator, acting in $\infty \cdot L^2(G_1) \otimes \sigma \cong \infty \cdot (L^2(G_1) \otimes \sigma)$, is 1 (the notations here follow those in Lemma 7.1). Obviously, the same conclusion then holds for $L^2(G_1) \otimes \sigma$.

We shall now make use of the well known general fact that σ (as well as any representation) can be embedded as a subrepresentation of $L^2(Y, \theta)$, where (Y, θ) is some probability measure G'-space, on which G' acts by measure preserving transformations (see e.g. [Zi, 5.2.13]). We thus obtain a measurable, measure preserving action of G on the σ-finite measure space $G_1 \times Y$, where G acts on the first coordinate through G_1, and on the second through G'. As $L^2(G_1) \otimes \sigma \subset L^2(G_1) \otimes L^2(Y, \theta) \cong L^2(G_1 \times Y)$, we deduce that the spectral radius of the μ-convolution operator acting in the latter representation is 1 as well. From [FS2] (which extends [DG], see also [Sh3, §2.1]), and our conclusion regarding the spectral radius, it follows that there exists a mean (namely, a positive, normalized linear functional) on $L^\infty(G_1 \times Y)$, which is invariant under μ-a.e. $g \in G$. Projecting this mean to G_1 yields a mean ϕ on $L^\infty(G_1)$, which is invariant under almost every $g \in G_1$, with respect to the projection of μ to G_1, denoted μ_1.

Let $H < G_1$ be the closed subgroup generated by μ_1. By the assumption of the theorem H is not amenable. Recall now (e.g., from the standard proof that a closed subgroup of an amenable group is amenable, cf. [Gre, 2.3.2]) that there exists an H-equivariant embedding of $L^\infty(H)$ in $L^\infty(G_1)$. This embedding induces a projection in the opposite direction between the corresponding spaces of means. Thus, projecting the mean ϕ yields a mean ψ on $L^\infty(H)$, which is invariant under μ_1-a.e. $h \in H$. Finally, restricting ψ to the subspace $UCB(H)$ of H-uniformly continuous functions on H, defines a mean on that space which is again invariant under μ_1-a.e. $h \in H$. However, the H-action on this space of means is *continuous.* Since μ_1 generates H, we conclude that the latter mean is invariant under *all* H. We have therefore constructed an H-invariant mean on the space $UCB(H)$, in contradiction to the non-amenability of H. □

8. Uniform Kazhdan constants and hyperbolic groups.

The natural question of existence of a uniform Kazhdan constant, valid for all generating sets, was raised in [LW] for groups with property (T). We formulate it here in a more general context.

DEFINITION 8.1.— *Let $\mathcal{F}$ be a family of unitary representations of a finitely generated group Γ. We say that $\mathcal{F}$ is* uniformly isolated *from the trivial representation, if there exists $\epsilon > 0$ which forms a Kazhdan constant for* all *generating sets of Γ, for all the representations in $\mathcal{F}$.*

There is no infinite Kazhdan group for which it is known whether the family of all representations with no invariant vectors is uniformly isolated from the trivial representation. In fact, the following seemingly easier question, looks quite intractable in general.

QUESTION 8.2.— *Let Γ be a finitely generated non amenable group. Is the regular representation $\ell^2(\Gamma)$ necessarily uniformly isolated from the trivial representation?*

We will prove that the answer to 8.2 is affirmative when Γ is hyperbolic and residually finite. We note that by a result of T. Delzant (see [GH]), the assumption of residual finiteness can be dispensed with, but we shall not consider this matter here. An affirmative answer to Question 8.2

implies a positive answer to another natural problem considered in [SW], and more fully in [GH], namely, the question of uniform growth rate of non amenable groups. In fact, we have:

PROPOSITION 8.3. — *If for a finitely generated group Γ, the regular representation $\ell^2(\Gamma)$ is uniformly isolated from the trivial representation, then Γ has uniform growth rate, namely, there exists $\alpha > 1$ such that for every generating set $S \subseteq \Gamma$, the number of different elements which can be expressed as words of length at most n in the set S, exceeds α^n for all n large enough.*

Proof. — If S is any generating set for Γ, apply the uniform Kazhdan constant ϵ of the regular representation, to the normalized characteristic functions $1_{S^n}/\sqrt{|S^n|}$. An easy calculation shows that Γ has uniform growth rate which is at least $1+\epsilon$. $\square$

In [SW] Shalen and Wagreich proved that fundamental groups of compact hyperbolic 3-manifolds have uniform growth rate. In fact, this result holds for every torsion-free non-elementary hyperbolic group, a result due to T. Delzant (see [GH]). In this direction we have the following:

THEOREM 8.4. — *Let Γ be a finitely generated, residually finite, word hyperbolic group, which is not elementary (i.e. not virtually cyclic). Then for every m there exists a subgroup $\Gamma_1 < \Gamma$ of finite index, such that the following holds: Every generating set of Γ_1 contains m elements which generate freely a subgroup of Γ_1. Consequently, $\ell^2(\Gamma)$ is uniformly isolated from the trivial representation.*

Proof. — Fix $m \in \mathbb{N}$. By the assumption and [Gro, 5.3 A], one can find $\Gamma_1 < \Gamma$ of finite index with the following property: every m elements in Γ_1 generate a free subgroup (although they may not be free generators for this subgroup). As Γ is non-elementary, we may also assume, taking a group of index large enough, that Γ_1 is not generated by m elements. Let $\{\gamma_1 \dots \gamma_k\}$ be any set of generators for Γ_1 ($k > m$), and consider the subgroup $F_1 = \langle \gamma_1 \dots \gamma_m \rangle$, which is free by the above. Then $F_1 \cong \mathbb{F}_n$-the free group on n generators for some $n \leqslant m$. If $n < m$ denote $F_1 = \langle \beta_1 \dots \beta_n \rangle$ for some (perhaps different) free generators $\beta_i \in \Gamma_1$. Now, join to $\beta_1 \dots \beta_n$ the next $m-n$ elements $\gamma_{m+1} \dots \gamma_{2m-n}$ and look at the subgroup $F_2 = \langle \beta_1 \dots \beta_n, \gamma_{m+1} \dots \gamma_{2m-n} \rangle$. Again, by the choice of Γ_1, F_2 is free on some set with at most m generators, and if this set has less than m elements, join to it the next γ's from $\{\gamma_1 \dots \gamma_k\}$ to obtain a new set of m elements. Repeat the argument, and notice that this process must terminate before the set $\{\gamma_1 \dots \gamma_k\}$ is exhausted, since $\{\gamma_1 \dots \gamma_k\}$ generates Γ_1, which is not generated by m elements. Therefore, collecting all the γ's involved in the process yields a subset $\{\gamma_1 \dots \gamma_l\}$ generating a free subgroup $\mathbb{F}_m$ on m generators.

Next, consider $\mathbb{Z}^m \cong \mathbb{F}_m/[\mathbb{F}_m, \mathbb{F}_m]$ and denote by $\bar{\gamma}$ the image of γ in $\mathbb{Z}^m$. Choose any m elements, say $\{\bar{\gamma}_1 \dots \bar{\gamma}_m\}$, from $\{\bar{\gamma}_1 \dots \bar{\gamma}_l\}$, which generate a subgroup with the maximal rank m (i.e. m elements in $\mathbb{Z}^m$ which form a base for $\mathbb{Q}^m$ over $\mathbb{Q}$). Clearly $\gamma_1 \dots \gamma_m$ are as required, for otherwise, by Schreier's theorem, they generate a free group $\mathbb{F}_{m_1}$ on $m_1 < m$ generators, and so their images in $\mathbb{Z}^m$ generate a subgroup with rank $m_1 < m$. The first assertion is therefore established. To prove the second, notice first that $\ell^2(\Gamma)$ is uniformly isolated from the trivial representation as Γ_1 representation (where Γ_1 is as in the first assertion, say, for $m = 2$). This follows from Kesten's theorem (see §2) and the fact that if $F_2 < \Gamma$ then $\ell^2(\Gamma)|_{F_2} \cong \dim \ell^2(\Gamma/F_2) \cdot \ell^2(F_2)$ as F_2 representations. To complete the proof it therefore clearly suffices to recall the following result:

LEMMA 8.5 (see [SW, Lemma 3.4]). — *Let G be a group with a finite generating set S. Let H be a subgroup of index d in G. Then H has a generating set consisting of elements which can be expressed as words of length at most $2d - 1$ in the generating set S.* □

Using Theorem 8.4 we can deduce the following more general result:

THEOREM 8.6. — *Let G be a simple Lie group with finite center, and $\Gamma < G$ a discrete, finitely generated, non-elementary hyperbolic group. Let $\mathcal{F}$ be any family of unitary G-representations which is isolated from the trivial representation. Then, restricted to Γ, $\mathcal{F}$ is* uniformly *isolated from the trivial representation. Furthermore, for every $\epsilon > 0$ there exists a subgroup $\Gamma_1 < \Gamma$ of finite index, such that for every generating set $S \subseteq \Gamma_1$, every unitary G-representation $(\pi, \mathcal{H}) \in \mathcal{F}$, and every unit vector $v \in \mathcal{H}$, there exists $s \in S$ such that $|\langle \pi(s)v, v\rangle| < \epsilon$.*

Proof. — Every finitely generated linear group is residually finite. Now use Theorems 4.2 and 4..3 (with $v = w$), Theorem 8.4, and Lemma 8.5.

□

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Manuscrit reçu le 5 mars 1999,
accepté le 15 octobre 1999.

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