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MARK KISIN

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PRIME TO p FUNDAMENTAL GROUPS, AND TAME GALOIS ACTIONS

by Mark KISIN

Introduction.

If $\mathcal{U}$ is a variety over a field F, then the geometric étale fundamental group $\pi_{1,\mathrm{geom}}(\mathcal{U})$ gives rise to a map

$$\rho : \mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{Out}(\pi_{1,\mathrm{geom}}(\mathcal{U}))$$

from the absolute Galois group of F to the outer automorphism group of $\pi_{1,\mathrm{geom}}(\mathcal{U})$, defined by pulling back étale coverings of $\mathcal{U} \otimes_F F^{\mathrm{sep}}$ by Galois automorphisms.

Grothendieck conjectured that when F is a number field, ρ should be a strong invariant of $\mathcal{U}$, and should determine $\mathcal{U}$, if the group $\pi_{1,\mathrm{geom}}(\mathcal{U})$ was sufficiently far from being abelian.

The point of this paper is to study the analogous map

$$\rho : \mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{Out}(\pi_{1,\mathrm{geom}}^{(p')}(\mathcal{U})),$$

where $\pi_{1,\mathrm{geom}}(\mathcal{U})$ is replaced by its maximal prime to p quotient $\pi_{1,\mathrm{geom}}^{(p')}(\mathcal{U})$, and the field F is assumed to be complete with respect to a discrete valuation, and have residue characteristic p.

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Our main results (Corollary 1.16) show for example, that when $\mathcal{U}$ is smooth, and has a smooth compactification $\mathcal{X}$, with complement $\mathcal{Z} = \mathcal{X} - \mathcal{U}$ a normal crossings divisor in $\mathcal{X}$, and the pair $(\mathcal{X}, \mathcal{Z})$ has a sufficiently nice integral model then ρ factors through a tame quotient, and in fact can be completely characterized by the reduction of $\mathcal{X}$ and $\mathcal{Z}$ modulo π, together with a certain auxiliary structure (log. structure) on this reduction. For example, if $(\mathcal{X}, \mathcal{Z})$ extends to a semi-stable pair (X, Z) in the sense of [de J1, 6.3] then this auxiliary structure depends only on the reduction of (X, Z) modulo π^2, so that ρ can be characterized by the reduction of (X, Z) modulo π^2.

The tameness result can be thought of as a non-abelian version of the tameness of vanishing cycles for a semi-stable scheme [RZ, 2.23]. Our proof is quite different however, since we do not have the machinery of abelian cohomology available. In fact the author was unable to make the proof of [RZ] go through, even though we are dealing only with the non-abelian analogue of H^1.

As applications of our results, we show that for *any* variety $\mathcal{U}$ over F, the wild inertia has finite image under ρ. Note that unlike the situation of Galois representations on abelian cohomology, this does not seem to be forced by purely group theoretic considerations.

Our results also cast new light on our previous work [Ki]. There we proved that if $f : \mathcal{X} \longrightarrow \mathcal{S}$ is a smooth proper map of F-schemes, with geometrically connected fibres, $\mathcal{Z} \subset \mathcal{X}$ a normal crossings divisor relative to $\mathcal{S}$, and $\mathcal{U} = \mathcal{X} - \mathcal{Z}$, then the maps

$$\rho_s : \mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{Out}(\pi_{1,\mathrm{geom}}^{(p')}(\mathcal{U}_s)),$$

defined for each F rational point s of $\mathcal{S}$, are equal for F rational points which are sufficiently close π-adically. The methods of [Ki1] were entirely rigid analytic. Here we obtain a weaker version of this result (Corollary 2.9), however the proof is entirely different, and is deduced, after some technicalities, from the fact, mentioned above, that under suitable circumstances, the map ρ only depends on the reduction of $\mathcal{U}$ modulo a suitable power of π. Unfortunately the proof depends on the existence of a very nice integral model for f, so that it gives the result unconditionally only in the case of curves.

We use heavily the notion of scheme equipped with logarithmic structure for which [Ka1] is a general reference. In particular, if (X, Z) is a semi-stable pair of $\mathcal{O}_F$ schemes, and $\mathcal{U}$ is the generic fibre of $X - Z$, then a key point is to characterise the group $\pi_{1,\mathrm{geom}}^{(p')}(\mathcal{U})$ in terms of the log. structure on X. Here $\mathcal{O}_F$ is the valuation ring of F. We show that $\pi_{1,\mathrm{geom}}^{(p')}(\mathcal{U})$ is equal to a certain *logarithmic étale fundamental group* of X. Observing that the log. structure on X can be recovered from the geometry of the the pair (X, Z) modulo a power of π, we obtain that $\pi_{1,\mathrm{geom}}^{(p')}(\mathcal{U})$ together with the Galois action on it can be recovered from the reduction of (X, Z) modulo the same power of π.

In the course of relating $\pi_{1,\mathrm{geom}}^{(p')}(\mathcal{U})$ to logarithmic structures, we need an unpublished result of Fujiwara and Kato [FK], which is the logarithmic analogue of the Zariski-Nagata purity theorem. Since we understand that [FK] is still in preparation, and seems unlikely to appear for some time, we include a proof here (which basically follows [FK]), for the convenience of the reader.

Part of the proof of these results (Proposition 1.7) involves giving an étale local characterisation of log. étale coverings for certain nice log. schemes over $\mathcal{O}_F$. If X is a semi-stable $\mathcal{O}_F$ scheme, this characterisation can be used to relate the vanishing cycles functor to the total direct image functor of the projection from the log. étale site to the étale site of the closed fibre of X. The details will appear in a paper in preparation [Ki3]. Compare also [Na].

As a consequence one can show, that if $f : \mathcal{X} \longrightarrow \mathcal{S}$ is a proper smooth map of F-schemes, which extends to a flat map of proper flat $\mathcal{O}_F$ schemes $X \longrightarrow S$, and if the fibre over some $\mathcal{O}_F$ valued point $\tilde{s}$ of S is semi-stable, then the Galois representations on the l-adic ($l \neq p$) cohomology of the fibres $\mathcal{X}_t$ of f are isomorphic for all F-valued points t sufficiently close to the generic point of $\tilde{s}$. This gives a different approach to the results of [Ki2] although, as in the non-abelian case, the results are much weaker since they require the existence of the nice integral models X and S.

Other, possibly more interesting, applications are the subject of [Ki3]. There we use the realization of the vanishing cycles functor as a direct image functor to prove results about endomorphisms of log. smooth curves, and in particular to prove a Lefschetz type Trace Formula, for certain types of endomorphisms. This formula seems to suggest the existence of a theory of cycles for log. schemes.

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1. Specialisation and tame fundamental group.

1.1. Let F be a field which is complete with respect to a discrete valuation, which has residue field k of characteristic p, ring of integers $\mathcal{O}_F$, and a uniformiser for π. We give $\mathrm{Spec}(\mathcal{O}_F)$ the structure of a fine log. scheme, induced by the map of monoids $\mathbb{N} \longrightarrow \mathcal{O}_F, 1 \mapsto \pi$. In the following we make the convention that all monoids are commutative. We will sometimes use additive and sometimes multiplicative notation for the monoid law.

1.2. We are going to consider log. schemes (X, M), where M is a fine, saturated log. structure in the sense of [Ka1]. We refer to such a log. scheme as an fs log. scheme. Unless otherwise stated, we always assume that the schemes underlying our log. schemes are locally noetherian.

An important example is given by the class of log. schemes which are log. smooth over $\mathcal{O}_F$ with its canonical log. structure, explained above. By [Ka1, 3.5], this means that étale locally on X, there is an étale map $\phi : X \longrightarrow \mathrm{Spec}(\mathcal{O}_F[P]/(x - \pi))$, where P is a finitely generated, integral saturated monoid, $x \in P$, the log. structure on X is induced by $\phi^{-1}(P) \longrightarrow \mathcal{O}_X$, and the torsion subgroup of $P^{\mathrm{gp}}/(x)$ has prime to p order. Such an (X, M) is *log regular* by [Ka2, 8.2], and the underlying scheme X is normal [Ka2, 4.1].

1.3. For a scheme Y we denote by $\mathbf{Et}^{(p')}(Y)$ the category of étale coverings $Y' \longrightarrow Y$ whose connected components are Galois of prime to p order, and quotient coverings of these.

If Y is an F-scheme, then we set $\mathbf{Et}^{(p')}_{\mathrm{geom}}(Y) = \varprojlim_{F'} \mathbf{Et}^{(p')}(Y \otimes_F F')$, where F' runs through finite separable extensions of F. Note that the group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts on $\mathbf{Et}^{(p')}_{\mathrm{geom}}(Y)$ in a natural way.

If (Y, M) is a fs log. scheme denote by $U \subset Y$ the open subscheme where the log. structure on Y is trivial. We call a finite log. étale map

$(Y', M') \longrightarrow (Y, M)$ *prime to* p if its restriction to U is in $\mathbf{Et}^{(p')}(U)$. We denote by $\mathbf{Et}^{(p')}_{\log.}(Y, M)$ the category of fs log. schemes $(Y', M') \longrightarrow (Y, M)$ over (Y, M) which are finite (on the underlying schemes) log. étale, and prime to p.

Now suppose that (Y, M) has the structure of an fs log. scheme over $\mathcal{O}_F$. We set $\mathbf{Et}^{(p')}_{\log.\mathrm{geom}}(Y, M) = \varprojlim_{F'} \mathbf{Et}^{(p')}_{\log.}((Y, M) \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(\mathcal{O}_{F'}))$, where F' runs over finite, separable, tame extensions of F. Here $\mathcal{O}_{F'}$ has its canonical log. structure, generated by a uniformiser, and $(Y, M) \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(\mathcal{O}_{F'})$ denotes the fibre product in the category of fs log. schemes. As above, there is a natural Galois action on this category.

For (Y, M) as above, we denote by (Y_s, M_s) the special fibre of (Y, M). The main point of this section is to prove the following theorem.

THEOREM 1.4. — *Suppose that (X, M) is log. smooth and proper over $\mathcal{O}_F$, with geometrically connected generic fibre, and denote by U the open set of X where the log. structure M is trivial. There are equivalences of categories,*

$$\mathbf{Et}^{(p')}_{\log.\mathrm{geom}}(X_s, M_s) \xrightarrow{\sim} \mathbf{Et}^{(p')}_{\log.\mathrm{geom}}(X, M) \xrightarrow{\sim} \mathbf{Et}^{(p')}_{\mathrm{geom}}(U).$$

These equivalences are compatible with the Galois action on both sides. In particular the Galois action on the right hand side factors through a tame quotient.

1.5. The proof consists of three parts. The first, contained in Propositions 1.7 and 1.10, shows that there is an equivalence $\mathbf{Et}^{(p')}_{\log.}(X, M) \xrightarrow{\sim} \mathbf{Et}^{(p')}(U)$. The second is Proposition 1.15, which shows that every object of $\mathbf{Et}^{(p')}_{\mathrm{geom}}(U)$ is in fact defined over a tame extension of F. This proves the second equivalence in the theorem. Finally, we show the first equivalence by a lifting, and algebraisation argument.

Theorem 1.4 allows one to show that, for example, if M is trivial away from the special fibre, and X is semi-stable then the category $\mathbf{Et}^{(p')}_{\mathrm{geom}}(U)$ depends only on the first nilpotent neighbourhood of the special fibre X_s of X. That is, $\mathbf{Et}^{(p')}_{\mathrm{geom}}(U)$ depends only on the reduction of X modulo π^2. The reason for this is that the reduction of X modulo π^2 completely determines the log. structure on X_s, (see 2.4), and hence the left most term of 1.4.1.

For applications the semi-stable case is the most important, and some of the proofs simplify in this case, however it still seems necessary to use the machinery of log. structures to obtain our results, even in this simple case.

1.6. If P is a finitely generated, torsion free, saturated monoid, and q is a positive integer, then we denote by $\frac{1}{q}P$ the monoid

$$\frac{1}{q}P = \{x \in P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} : x^q \in P\}.$$

We consider finitely generated, saturated monoids Q, with $P \subset Q \subset \frac{1}{q}P$. Such a Q is automatically integral.

Note that as Q is torsion free, $\mathbb{Z}[Q]$ is normal (for example, because it is log. smooth over $\mathbb{Z}$ with its trivial log. structure, hence log. regular) By [Ka1, 3.5], the map $\mathbb{Z}[P] \longrightarrow \mathbb{Z}[Q]$ is log. étale over points of $\mathrm{Spec}(\mathbb{Z})$ where q is invertible.

A map $f : (Y,N) \longrightarrow (X,M)$ of fs log. schemes is called *Kummer étale* if étale locally on X it has a chart

$$(P_X \longrightarrow M, Q_Y \longrightarrow N, P \longrightarrow Q),$$

with P, Q finitely generated, integral, saturated, and torsion free, $P \subset Q \subset \frac{1}{q}P$ for some integer q, which is invertible on X, and the map $Y \to X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q])$ is étale (in the classical sense). Note that this implies that f is log. étale, and quasi-finite on the underlying schemes.

If the map f above is finite, we say that Y is a *Kummer covering* of X. If $P^\times = \{1\}$, and $Y = X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q])$, then $Y \to X$ is called a *standard Kummer covering*. If f is finite on the underlying schemes, and is Kummer, then étale locally on X, $(Y,N) \longrightarrow (X,M)$ is a disjoint union of standard Kummer coverings. Indeed, the proof of 1.7 below shows that we can always choose P such that $P^\times = \{1\}$, and then the result follows by replacing X by its strict henselisation at a point.

PROPOSITION 1.7. — *Let $f : (Y,N) \longrightarrow (X,M)$ be a quasi-finite, log. étale map of fs log. schemes. Then f is Kummer étale.*

Proof. — If W is a scheme, and S a monoid, we denote by S_W the étale sheaf of monoids with constant value W. Fix a chart (defined étale locally) $X \longrightarrow \mathrm{Spec}(\mathbb{Z}[P])$, with P finitely generated and saturated. Denote by $P^\times$ the group of invertible elements of P. As $P^{\mathrm{gp}}/P^\times$ is a free abelian group, the projection $P^{\mathrm{gp}} \to P^{\mathrm{gp}}/P^\times$ admits a section, which induces a section to $P \to P/P^\times$. Thus we may regard $P/P^\times$ as a submonoid of P, and this induces an isomorphism $P \xrightarrow{\sim} P/P^\times \oplus P^\times$. One sees from the definition of associated log. structure, that $P/P^\times \to P \to M_X$ is a chart. Thus we may assume that $P^\times = \{1\}$.

We apply [Ka1, 3.5] which tells us that we can find a chart

$$(P_X \longrightarrow M, Q_Y \longrightarrow N, P \longrightarrow Q),$$

with Q a finitely generated, saturated monoid, such that $P^{\mathrm{gp}} \longrightarrow Q^{\mathrm{gp}}$ is injective with finite cokernel whose order is invertible on X, and the map $Y \longrightarrow X \times_{\mathrm{Spec}\mathbb{Z}[P]} \mathrm{Spec}\mathbb{Z}[Q]$ is étale. Now the proof in [Ka, 3.12] shows that actually, we may take Q to be a submonoid of $(\mathbb{Z}^r \oplus P^{\mathrm{gp}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ (note however that the roles of P and Q are exchanged in *loc. cit.*). In particular we may assume that Q is torsion free.

Now for some integer q, which is invertible on X, the map $P^{\mathrm{gp}} \longrightarrow \frac{1}{q}P^{\mathrm{gp}}$ lifts to Q^{gp}. Indeed, since $P^{\mathrm{gp}} \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$ lifts to a map $Q^{\mathrm{gp}} \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$. Since Q is torsion free, the map $Q^{\mathrm{gp}} \longrightarrow \frac{1}{q}P^{\mathrm{gp}}$ is injective. As the cokernel of $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ has order invertible on X, we may assume that q is invertible on X.

For a ring R and a subring S we denote by $n(S,R)$ the integral closure of S in R. Let

$$Q_1 = Q \cap \frac{1}{q}P = Q^{\mathrm{gp}} \cap \frac{1}{q}P \subset \frac{1}{q}P^{\mathrm{gp}},$$

where the second equality holds as Q is saturated. Since, $n(\mathbb{Z}[P], \mathbb{Z}[\frac{1}{q}P^{\mathrm{gp}}]) = \mathbb{Z}[\frac{1}{q}P]$, we have

$$n(\mathbb{Z}[P], \mathbb{Z}[Q^{\mathrm{gp}}]) = \mathbb{Z}\left[\frac{1}{q}P\right] \cap \mathbb{Z}[Q^{\mathrm{gp}}] = \mathbb{Z}[Q_1].$$

In particular this shows that $\mathbb{Z}[Q_1]$ is a finite $\mathbb{Z}[P]$ module, so that Q_1 is finitely generated. We remark that if $g \in Q^{\mathrm{gp}}$, then there is an $h \in P$ such that $(gh)^q \in P$, so that $gh \in Q_1$. This shows that $Q^{\mathrm{gp}} = Q_1^{\mathrm{gp}}$.

Put $W = X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q_1])$, $W' = X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q])$. We have a commutative diagram with cartesian squares:

$$\begin{array}{ccccccc}
Y & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{Spec}(\mathbb{Z}[Q]) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}[Q_1]) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}[P]).
\end{array}$$

Since $Y \longrightarrow X$ is quasi-finite, and $Y \longrightarrow W'$ is étale, the map $\mathrm{Spec}(\mathbb{Z}[Q]) \longrightarrow \mathrm{Spec}(\mathbb{Z}[Q_1])$ is quasi-finite, over points in the image of

$Y \longrightarrow \mathrm{Spec}(\mathbb{Z}[P])$. Thus, by Zariski's Main Theorem, [EGAIV, 8.12.6], this map is an open immersion over such a point, since both $\mathbb{Z}[Q]$ and $\mathbb{Z}[Q_1]$ are normal with the same field of fractions.This shows that $Y \longrightarrow W$ is étale.

I claim that the restriction to $\mathrm{Spec}(\mathbb{Z}[Q])$ of the canonical log. structure on $\mathrm{Spec}(\mathbb{Z}[Q_1])$ is equal to the one induced by Q, over points in the image of $Y \longrightarrow \mathrm{Spec}(\mathbb{Z}[P])$.

Assume this result for a moment. Then $P \to Q_1$ is chart for f. We have seen that Q_1 is finitely generated, and it is integral and saturated by construction. Thus we may replace Q by Q_1, and assume that $Q \subset \frac{1}{q}P$. The proposition follows.

It remains to show the claim above. I am grateful to the referee for providing the following short argument, which is more transparent than my original one.

Let $\mathfrak{P} \in \mathrm{Spec}(\mathbb{Z}[Q])$ be a point over which $\mathbb{Z}[Q_1] \to \mathbb{Z}[Q]$ is an open immersion. Let $(p) = \mathbb{Z} \cap \mathfrak{P}$, and $\mathfrak{p}$ the image of $\mathfrak{P}$ in $\mathrm{Spec}(\mathbb{F}_p[Q])$. Denote by $g : \mathbb{F}_p[Q_1] \to \mathbb{F}_p[Q]$ the induced map (if $p = 0$, replace $\mathbb{F}_p$ by $\mathbb{Q}$ here and above). Set $S_{\mathfrak{p}} = \mathbb{F}_p[Q] - \mathfrak{p}$. It is enough to show that the induced map $h : Q_1/(Q_1 \cap S_{\mathfrak{p}}) \to Q/(Q \cap S_{\mathfrak{p}})$ is an isomorphism. h is clearly injective, so it remains to show surjectivity. Let $r \in Q$. Since $\mathbb{F}_p[Q_1]_{(g^{-1}\mathfrak{p})} \xrightarrow{\sim} \mathbb{F}_p[Q]_{(\mathfrak{p})}$, there exists $a \in \mathbb{F}_p[Q_1]$, and $s \in g^{-1}(S_{\mathfrak{p}})$ such that $e_r \cdot s = a \in \mathbb{F}_p[Q]$. Write $a = \sum_{p \in Q_1} n_p e_p$, $s = \sum_{p \in Q_1} n'_p e_p$. Since $g(s) \in S_{\mathfrak{p}}$, we have $n'_{p_0} \neq 0$ for some $p_0 \in Q_1 \cap S_{\mathfrak{p}}$. As

$$\sum_{p \in Q_1} n_p e_p = e_r \Big(\sum_{p \in Q_1} n'_p e_p \Big) = \sum_{p \in Q_1} n'_p e_{p+r}$$

we see that $n_{r+p_0} \neq 0$, and in particular $r + p_0 \in Q_1$. As $h(r + p_0) = r$, the claim follows. □

1.8. We define an *fs formal log. scheme* (Y, N) to be an inductive limit of fs log. schemes $\varinjlim(Y_n, N_n)$, with strict transition maps, such that $\varinjlim Y_n$ is a formal scheme. If $\mathcal{I}$ is a sheaf of ideals of definition of Y, we denote by $(Y, N)/\mathcal{I}$ the scheme $Y/\mathcal{I}$ with fs log. structure induced by N (i.e by N_n for n sufficiently large).

Let (Y, N) be as above and $(X, M) = \varinjlim(X_r, M_r)$ another fs formal log scheme. A map of fs formal log. schemes $\phi : (X, M) \to (Y, N)$ is called Kummer étale, if

(1) If $\mathcal{I}$ is a sheaf of ideals of definition on (Y,N), then $\phi^{-1}(\mathcal{I})$ is a sheaf of ideals of definition on (X,M).

(2) For some $\mathcal{I}$ as in (1), the map $(X,M)/\phi^{-1}(\mathcal{I}) \to (Y,N)/\mathcal{I}$ is Kummer étale.

An easy argument using the lifting property of log. étale maps, shows that if the condition in (2) is satisfied for one choice of $\mathcal{I}$ then it is satisfied for all such choices.

We remark that we will occasionally need to use a variant of 1.7, where Y and X are formal schemes with log. structure rather than schemes. The proof of this variant is easily reduced to the case of log. schemes, by reducing modulo an ideal of definition, and applying 1.7.

Similarly the following result 1.9 is stated for schemes, but also holds for formal schemes, with the same proof.

PROPOSITION 1.9.— *Let (X,M) be a log. regular and $X \longrightarrow \mathrm{Spec}(\mathbb{Z}[P])$, a chart. Let q be a positive integer, which is invertible on X. Take a monoid Q with $P \subset Q \subset \frac{1}{q}P$ which is finitely generated, and saturated. Write $X_Q = X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q])$, where the product is take in the category of fs log. schemes. Then*

(1) $\mathrm{Hom}_X(X_Q, X_Q) = \mathrm{Aut}_X(X_Q)$, *(morphisms as classical schemes) and every element of $\mathrm{Aut}_X(X_Q)$ is induced by multiplying homogeneous components of $\mathbb{Z}[Q]$ by q^{th} roots of unity. In particular $\mathrm{Aut}_X(X_Q)$ is an abelian group.*

(2) *If W is a normal scheme, which is finite over X, and there is a factorisation $X_Q \xrightarrow{h} W \to X$, with h surjective, then there is a unique finitely generated, saturated monoid Q' with $P \subset Q' \subset Q$, such that*

$$W \xrightarrow{\sim} X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q']),$$

and the map $X_Q \to W$ is induced by the inclusion $Q' \subset Q$.

(3) *étale locally we have $Q' = Q^H$, with $H = \mathrm{Aut}_W(X_Q)$ (automorphisms as classical schemes).*

Proof.— (1) The claim that endomorphisms of X_Q over X, are induced by multiplying homogeneous components of $\mathbb{Z}[Q]$ by roots of unity implies that $\mathrm{Hom}_X(X_Q, X_Q) = \mathrm{Aut}_X(X_Q)$. To see that all the endomorphisms have the required form take $x \in Q$, and $\sigma \in \mathrm{Hom}_X(X_Q, X_Q)$. By [Ka2, 11.6], $P \subset \mathcal{O}_X$ and $Q \subset \mathcal{O}_{X_Q}$, so we may view $\sigma(x)x^{-1}$ as a section of $\mathcal{O}_{X_Q|U}$, where U denotes the dense open set where the log. structure on X is trivial. As σ is a map over X, and $x^q \in P$, $(\sigma(x)x^{-1})^q = 1$, whence $\sigma(x)x^{-1}$ is a section of $\mathcal{O}_{X_Q}$, as X_Q is normal, and it is evidently a root of unity.

(2) The uniqueness of Q' makes its existence étale local, thus we may assume that $X = \mathrm{Spec}(A)$ is a domain, and contains μ_q, the group of q^{th} roots of unity. Write $X_Q = \mathrm{Spec}(A_Q)$. As A contains μ_q, the extension of rings of fractions $\mathrm{Fr}(A) \subset \mathrm{Fr}(A_Q)$ is Galois, and abelian, with group of automorphisms G. In fact (1) gives an embedding $G \subset (\mu_q)^d$ for some integer d. As $W = \mathrm{Spec}(B)$ is normal, we have $B = (A_Q)^H$, for some subgroup $H \subset G$.

By (1), there is a pairing $Q^{\mathrm{gp}}/P^{\mathrm{gp}} \times G \to \mu_q$. In fact this is a perfect pairing, since

$$|Q^{\mathrm{gp}}/P^{\mathrm{gp}}| = |G| = \deg(\mathrm{Fr}(A_Q)/\mathrm{Fr}(A)),$$

and if $g \in G$ with $(q,g) = 1$, for every $q \in Q^{\mathrm{gp}}$, then g is the identity by (1). Set

$$Q' = \{q \in Q : (q,h) = 1, h \in H\} = \frac{1}{q} P \cap \{q \in Q^{\mathrm{gp}} : (q,h) = 1, h \in H\}.$$

One sees immediately that Q' is integral and saturated. As $\mathbb{Z}[P] \subset \mathbb{Z}[Q'] \subset \mathbb{Z}[Q]$, $\mathbb{Z}[Q']$ is a finite $\mathbb{Z}[P]$ module, so Q' is finitely generated.

Clearly, the normalisation $A_{Q'}$ of $A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q']$ is contained in B, and we have to show that $A_{Q'} = B$. Using the above perfect pairing, we see that $\mathrm{Aut}_{A_{Q'}}(A_Q) = H$, so by Galois theory $A_{Q'} = (A_Q)^H = B$.

Finally, for the uniqueness of Q', note that any other monoid Q'' with $A_{Q''} = B$, must satisfy $Q'' \subset Q'$, since B is fixed by H. If Q'' is saturated, then since $Q'' \subset Q' \subset \frac{1}{q}P$, to show $Q'' = Q'$, it is enough to show that $(Q'')^{\mathrm{gp}} = (Q')^{\mathrm{gp}}$. Replacing G by H, and P by Q'' we may assume that $P = Q''$. In this case $(Q')^{\mathrm{gp}}$ is fixed by G, and the perfect pairing discussed above shows that $P^{\mathrm{gp}} = (Q')^{\mathrm{gp}}$.

The last claim has already been shown above. □

Proposition 1.10. — *Let (X, M) be a log. regular log. scheme, and denote by U the open subset of X where the log. structure becomes trivial. Let p be a prime, and suppose that every integer which is coprime to p is invertible on X. There is an equivalence of categories $\Phi_F : \mathbf{Et}^{(p')}_{\log.}(X, M) \xrightarrow{\sim} \mathbf{Et}^{(p')}(U)$, given by restriction of coverings to U.*

Proof. — By descent, the full faithfulness is local in the étale topology of X, so we may assume that, in our previous notation, there exists a chart $X \longrightarrow \operatorname{Spec}(\mathbb{Z}[P])$, with P finitely generated, saturated, and torsion free. Let q be a positive integer, which is invertible on X. Set $X_q = X \times_{\operatorname{Spec}(\mathbb{Z}[P])} \operatorname{Spec}(\mathbb{Z}[\frac{1}{q}P])$ with the obvious log. structure.

First we claim

$$\operatorname{Hom}_{(X,M)}(X_q, X_q) \xrightarrow{\sim} \operatorname{Hom}_X(X_q, X_q) \xrightarrow{\sim} \operatorname{Hom}_U(X_q|U, X_q|U).$$

Here the first term means maps as log. schemes. The second equality follows immediately because X_q is normal, and the first from the explicit description of the automorphisms (as a classical scheme) of X_q over X, given in 1.9: As these are all induced by multiplying homogeneous components of $\mathbb{Z}[\frac{1}{q}P]$ by suitable roots of unity, they automatically respect log. structures. On the other hand a morphism of log. schemes $X_q \to X_q$ over X, is determined by its map of underlying schemes, because $M_{X_q} \subset \mathcal{O}_{X_q}$, as noted above.

Now let $(Y_1, N_1), (Y_2, N_2)$ be objects of $\mathbf{Et}^{(p')}_{\text{log.}}(X, M)$, and suppose we have a map $h_U : Y_1|U \longrightarrow Y_2|U$ over U. By the remark preceding 1.7, we may assume that $Y_1 = X_{Q_1}$, $Y_2 = X_{Q_2}$ with $Q_1, Q_2 \subset \frac{1}{q}P$ finitely generated, integral, and saturated, the notation being that of 1.9. h is covered by a map $X_q|U \longrightarrow X_q|U$, which extends to map of log. schemes $X_q \longrightarrow X_q$. On the other hand, since Y_1 and Y_2 are normal, h_U extends to a map of schemes $h : Y_1 \longrightarrow Y_2$. Thus, we have a commutative diagram of schemes:

$$\begin{array}{ccc} X_q & \longrightarrow & X_q \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y_2. \end{array}$$

The vertical maps are induced by the inclusions $Q_1, Q_2 \subset \frac{1}{q}P$, hence they come from maps of log. schemes, by construction, and the top horizontal map does also, by the discussion above. It follows, by 1.9(3), that h comes from a map of log. schemes, which proves the full faithfulness.

For the essential surjectivity we need the following result of Fujiwara - Kato.

PROPOSITION 1.11 (Fujiwara-Kato). — *Let (X, M) be log. regular, and $V \subset X$ open, such that $X - V$ has codimension at least 2 at every point of X. Denote by M_V the restriction of M to V. If $(W, N) \longrightarrow (V, M_V)$*

is a Kummer covering, then (W, N) *extends uniquely to a Kummer covering of* (X, M).

Given this proposition we may prove the essential surjectivity of Φ_F as follows. Suppose that $U' \longrightarrow U$ is étale, Galois of order prime to p. Since X is normal, it is regular at all its codimension 1 points, and U' extends to a Kummer covering over these points. Hence, by the proposition it extends to a Kummer covering of (X, M). □

It remains to prove 1.11. We remark that although the proof in the general case is a little delicate, the case of interest in the applications will be that of semi-stable schemes, and their base changes by extensions of valuation rings. In this case the proof is much easier, since locally such schemes are quotients of regular ones by a finite group action, and 1.11 can be proved by pulling coverings back to these regular coverings, and applying the usual Zariski-Nagata purity theorem [SGA2, X Thm 3.4].

Proof of 1.11. — We may assume that X is the spectrum of a strictly henselian local ring A, with closed point x. We may assume that W is connected. Fix a chart $X \longrightarrow \operatorname{Spec}(\mathbb{Z}[P])$ with $P \xrightarrow{\sim} (M/\mathcal{O}_X^*)_x$ (compare with the proof of (1.7)). We proceed in several steps.

Step 1. — If Q is a finitely generated, integral, saturated monoid with $P \subset Q \subset \frac{1}{q}P$, for some positive integer q (which we do not assume invertible on X) then we may replace X by $X_Q = X \times_{\operatorname{Spec}(\mathbb{Z}[P])} \operatorname{Spec}(\mathbb{Z}[Q])$ and W by the normalisation W_Q of $W \times_{\operatorname{Spec}(\mathbb{Z}[P])} \operatorname{Spec}(\mathbb{Z}[Q])$. In particular, we may suppose that the map $W \longrightarrow V$ is finite étale.

Indeed, suppose that W_Q extends to a Kummer covering Y_Q of X_Q. Then after replacing Q by a larger monoid having all the same properties, we may assume that Y_Q is étale over X_Q. In fact the components of Y_Q are of the form X_{Q_i} for various monoids $Q_i \subset P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and it is enough to replace Q by the saturation of Q in $\sum Q_i^{\mathrm{gp}} \subset P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$.

As X is strictly henselian this implies that Y_Q is totally split over X_Q. Let $\tilde{W}$ be the normalisation of X in W. Since W is connected, so is $\tilde{W}$, whence it must be a quotient of X_Q. Let p be a prime factor of q, which is not invertible on X, and write

$$Q' = \{x \in Q : x^n \in P, (n, p) = 1\}.$$

Consider the factorisation $X_Q \longrightarrow X_{Q'} \longrightarrow X$. Write $X_Q = \operatorname{Spec}(A_Q)$, $X_{Q'} = \operatorname{Spec}(A_{Q'})$, and $\tilde{W} = \operatorname{Spec}(B)$. Consider a height 1 prime $\mathfrak{p}$ of X over p. The extension $(A_{Q'})_{\mathfrak{p}}/\mathfrak{p} \to (A_Q)_{\mathfrak{p}}/\mathfrak{p}$ is purely inseparable, so that

$(A_{Q'})_{\mathfrak{p}}$ contains the maximal étale $A_{\mathfrak{p}}$ subalgebra of $(A_Q)_{\mathfrak{p}}$. As $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is étale this implies that $B_{\mathfrak{p}} \subset (A_{Q'})_{\mathfrak{p}}$. Taking the normalisation of A in both sides, we get $B \subset A_{Q'}$, so that $\tilde{W}$ is a quotient of $X_{Q'}$. Replacing Q by Q', we may assume that $Q \subset \frac{1}{q}P$, with q invertible on X. Hence by (1.9(2)), $\tilde{W}$ is a Kummer covering.

Step 2.— $\dim X = 2$. Consider any finitely generated integral monoid P such that $P^{\times} = \{1\}$ and such that P^{gp} has rank 2. If $e_1, e_2, e_3 \in P$ are part of a set of g generators for P and generate P^{gp} as a group, then we have $e_3^c = e_1^a e_2^b$ for suitable integers a, b, c. After relabelling e_1, e_2, e_3 we may assume that a, b, c are non-negative. Since $P^{\times} = \{1\}$ $c > 0$. Thus if we denote by P' by the monoid generated in $P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ by P, $e_1^{1/c}$ and $e_2^{1/c}$, then P' is generated by $g-1$ elements. If P' is generated by two elements, then $P' \xrightarrow{\sim} \mathbb{N}^2$. Thus repeating the above construction, we arrive at a map of monoids $P \longrightarrow P' \longrightarrow P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$, with $P' \xrightarrow{\sim} \mathbb{N}^2$, and $(P')^n \subset P$ for some integer n.

We apply this to the case $\dim X = 2$, and the chart $X \longrightarrow \mathrm{Spec}(\mathbb{Z}[P])$. By Step 1 we may assume that $W \longrightarrow V$ is étale.

By [Ka2, 3.2] the completion $\hat{\mathcal{O}}_{X,x}$ is isomorphic to $R[\![P]\!]/(\theta)$, where R is a regular local ring, and $\theta \in R[\![P]\!]$ has constant term in $m_R \backslash m_R^2$ (m_R is the radical of R). It follows that if P is a free monoid, then X is regular.

If the rank of P^{gp} is less than 2, then $P \xrightarrow{\sim} \mathbb{N}$ and X is regular at x. If the rank of P^{gp} is 2 we apply the preceding discussion. By step 1, we may replace X by $X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[P'])$, and W by $W \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[P'])$. Thus we may assume that $P \xrightarrow{\sim} \mathbb{N}^2$. Hence, we may assume that X is regular. However in this case the result follows by the Zariski-Nagata purity theorem.

Step 3.— $\dim X \geqslant 3$. By step 1, we may assume that W is étale over V.

Let $\mathfrak{p}$ be a prime ideal of M_X of codimension 1, and consider $\bar{X} = \mathrm{Spec}(\mathcal{O}_{X,x}/\mathfrak{p})$ with its canonical log. regular structure $M_{\bar{X}}$ induced by pulling back $M_X \backslash \mathfrak{p}$ ([Ka2, 7.2]). Set $\bar{V} = V \times_X \bar{X}$, and similarly for $\bar{W}$. By the induction hypothesis, $\bar{W}$ extends to a Kummer covering of $\bar{X}$. Thus, if we replace P by $\frac{1}{q}P$, X by $X_{\frac{1}{q}P}$, W by $W_{\frac{1}{q}P}$ and $\mathfrak{p}$ by a prime ideal of $M_{X_{\frac{1}{q}P}}$ lying over $\mathfrak{p}$ (notation as in step 1), for a suitable q which is invertible on X, we may assume that $\bar{W}$ extends to a finite étale cover $\bar{Y}$ of $\bar{X}$.

Since $\bar{X}$ is the spectrum of a strictly henselian ring, $\bar{Y}$ is a disjoint union of trivial coverings. If $\hat{X}$, $\hat{V}$ and $\hat{W}$ denote the completions of X, V and W respectively along $\mathfrak{p}$, then this shows that $\hat{W}$ is a disjoint union of trivial coverings of $\hat{V}$. Thus W is a disjoint union of trivial coverings of V by [SGA2, X 2.2,2.3], and thus trivially extends to an étale covering of X. $\square$

1.12. We will need the following formal version of 1.10. Let (X, M) be log. smooth over $\mathcal{O}_F$, and denote by $\mathfrak{X}$ the π-adic completion of X, and by U^{an} the π-adic analytic space associated to U. We equip $\mathfrak{X}$ with a log. structure $M_{\mathfrak{X}}$, by pulling back the one on X via the map of locally ringed spaces $\mathfrak{X} \to X$. We define the categories $\mathbf{Et}^{(p')}_{\mathrm{log.}}(\mathfrak{X}, M_{\mathfrak{X}})$ and $\mathbf{Et}^{(p')}(U^{\mathrm{an}})$, as for schemes. Namely $\mathbf{Et}^{(p')}(U^{an})$ consists of analytic spaces V, equipped with a finite étale map to U^{an}, such that each connected component of V is a quotient of a Galois covering of U^{an} of prime to p order, and $\mathbf{Et}^{(p')}_{\mathrm{log.}}(\mathfrak{X}, M_{\mathfrak{X}})$ consists of formal schemes $(\mathfrak{Y}, N)$ equipped with a fs log. structure and a finite, log. étale map to $(\mathfrak{X}, M_{\mathfrak{X}})$, whose restriction to U^{an} is an object of $\mathbf{Et}^{(p')}(U^{\mathrm{an}})$.

Then the functor

$$\mathbf{Et}^{(p')}_{\mathrm{log.}}(\mathfrak{X}, M_{\mathfrak{X}}) \to \mathbf{Et}^{(p')}(U)$$

is fully faithful. The proof is identical to that of the full faithfulness in 1.10, using Remark 1.8. We ignore the question of whether this functor is an equivalence, as it will not be needed.

We now turn to the second part of the proof of Theorem 1.4, as outlined above. Namely, showing that for smooth F-schemes, with log. smooth $\mathcal{O}_F$ compactifications, étale prime to p covers are all defined over tame extensions. The following easy lemma will be useful

LEMMA 1.13. — *Let U be a scheme of essentially finite type over F. Let F'/F be a Galois, totally ramified extension of p power order. If $U' = U \otimes_F F'$, is connected then the functor*

$$\mathbf{Et}^{(p')}(U) \overset{V \mapsto V \otimes_F F'}{\longrightarrow} \mathbf{Et}^{(p')}(U')$$

is fully faithful.

Proof. — We may assume that F'/F has degree p. For an object V of $\mathbf{Et}^{(p')}(U)$ denote $V \otimes_F F'$ by V'. If V, W are in $\mathbf{Et}^{(p')}(U)$, then we have

$$\mathrm{Hom}_U(V, W) = \mathrm{Hom}_V(V, V \times_U W),$$

and if V is connected, then the latter set is simply the number of connected components of $V \times_U W$ which have degree 1 over V. Thus the lemma will follow if we can show that any connected object in $\mathbf{Et}^{(p')}(U)$ remains connected after the base change $\otimes_F F'$.

Suppose V is a connected object. $\mathrm{Gal}(F'/F)$ acts transitively on the connected components of V', so if V' is not connected, it must split into p components each isomorphic to V. This gives us a map $V \longrightarrow \mathrm{Spec}(F')$, whence a map of U schemes

$$V \longrightarrow U \otimes_F F' = U'.$$

As the latter is connected, this map must be a surjection, as it is étale. However that is impossible, as the degree of V over U is prime to p. □

LEMMA 1.14. — *Let (Y, M_Y) be a log. smooth $\mathcal{O}_F$ scheme, and denote by (X, M_X) the spectrum of the strict henselisation of Y at an étale point x over a closed point of the special fibre of Y, with the induced log. structure M_X. Let F' be a finite, Galois, totally ramified extension of F, of p power order. We give $\mathcal{O}_{F'}$ its canonical log. structure. For a Kummer covering $(Z, M_Z) \longrightarrow (X, M_X)$ write $(Z', M_{Z'}) = (Z, M_Z) \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(\mathcal{O}'_F)$, where the product is taken in the category of fs log. schemes.*

The functor which assigns $(Z', M_{Z'})$ to (Z, M_Z) induces an equivalence of categories between Kummer coverings of (X, M_X) and Kummer coverings of $(X', M_{X'})$.

Proof. — First we show that the functor is fully faithful. Denote by U and U' the dense open subschemes of X and X' respectively over which the log. structures are trivial. Consider the functor which takes a finite étale, prime to p covering $W \longrightarrow U$ to the covering $W' = W \otimes_F F' \longrightarrow U'$. By the full faithfulness in 1.10, it is enough to show that this functor is fully faithful. Hence by Lemma 1.13, it is enough to show that U' is connected.

We may assume that F' has degree p over F. Then, as in 1.13, if U' is not connected, it splits into a disjoint union of p copies of U, and we obtain a map $U \longrightarrow \mathrm{Spec}(F')$. As the generic fibre X_η of X is normal, this induces a map $X_\eta \longrightarrow \mathrm{Spec}(F')$, whence a map $X \rightarrow \mathrm{Spec}(\mathcal{O}_{F'})$, as X is normal. It will suffice to show that no such map exists.

Let $S = \mathrm{Spec}(\mathcal{O}_F)$, $S' = \mathrm{Spec}(\mathcal{O}_{F'})$ endowed with the canonical log. structure, and $S^{\mathrm{triv}} = \mathrm{Spec}(\mathcal{O}_F)$ endowed with its trivial log. structure. Finally, denote by π' a uniformiser of $\mathcal{O}_{F'}$. Let $f(X) \in \mathcal{O}_F[X]$ be a minimal polynomial for π'. We have

$$(\pi' f'(\pi'))d\pi'/\pi' = f'(\pi')d\pi' = df(\pi') = 0 \in \omega^1_{X/S}.$$

Since f is the minimal polynomial for π', and $\mathcal{O}_X$ is a domain, we have $\pi' f(\pi') \neq 0 \in \mathcal{O}_X$, so $d\pi'/\pi' = 0 \in \omega^1_{X/S}$, because $\omega^1_{X/S}$ is a free $\mathcal{O}_X$ module. It follows that there exists $a \in \mathcal{O}_X$, such that $d\pi'/\pi' = ad\pi/\pi \in \omega_{X/S^{\mathrm{triv}}}$.

On the other hand we must have $\pi = \pi'^p u$ for some $u \in \mathcal{O}^*_{F'}$. Writing $u = \sum_{i=0}^\infty a_i \pi'^i$, where $a_i \in \mathcal{O}_F$, we see that $du = bd\pi' \in \omega^1_{S'/S^{\mathrm{triv}}}$. Thus we have

$$\begin{aligned} d\pi/\pi = pd\pi'/\pi' + du/u &= (p + u^{-1}b\pi')d\pi'/\pi' \\ &= (ap + au^{-1}b\pi')d\pi/\pi \in \omega^1_{X/S^{\mathrm{triv}}}. \end{aligned}$$

Now $\omega^1_{X/S^{\mathrm{triv}}}$ is a finitely generated $\mathcal{O}_X$ module, hence is π-adically separated. As $(ap + au^{-1}b\pi')^p$ is divisible by π, we conclude that $d\pi/\pi = 0 \in \omega^1_{X/S^{triv}}$. But X is log. smooth and faithfully flat over S, so $\omega^1_{S/S^{triv}} \subset \omega^1_{X/S^{triv}}$ by [Ka1, 3.12]. It follows that $d\pi/\pi = 0 \in \omega^1_{S/S^{\mathrm{triv}}}$, a contradiction.

Next we prove the essential surjectivity. We keep the above notation. Consider a Kummer covering

$$(Z', M_{Z'}) \longrightarrow (X', M_{X'}).$$

We have to show that Z' is defined over (X, M). By 1.9, and the full faithfulness above, it is enough to show that $(Z', M_{Z'})$ is dominated by a Kummer covering defined over (X, M). By 1.10 it is enough to show that the étale covering $Z'|_{U'} \to U'$ is dominated by an étale covering defined over U.

Now let $P \subset \mathcal{O}_X$ be a chart for X, with $\pi \in P$. The submonoid $P' \subset \mathcal{O}_{X'}$ generated (as a fs. monoid) by P and π' is a chart for X'. By 1.7 $\mathcal{O}_{Z'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{U'}$ is contained in an $\mathcal{O}_{U'}$ algebra obtained by adjoining q^{th} roots of elements of P', for some integer q which is invertible on X. Thus it is enough to show that $\pi'^{1/q}$ is contained in the $\mathcal{O}_{U'}$ algebra generated by adjoining $\pi^{1/q}$.

Choose positive integers a, b such that $ap - bq = 1$. We have

$$\pi'^{1/q} = \pi'^{ap/q-b} = \pi^{a/q} u^{-a/q} \pi'^{-b}.$$

Since $\mathcal{O}_{X'}$ is strictly henselian, we have $u^{1/q} \in \mathcal{O}_{X'}$, and the result follows. ☐

PROPOSITION 1.15.— *Let (X, M_X) be a proper log. smooth $\mathcal{O}_F$ scheme, with geometrically connected generic fibre, and denote by $U \subset X$ the dense open subset, where the log. structure is trivial. If F' is a finite Galois, totally ramified extension of F of p power order, then the base change functor $\mathbf{Et}^{(p')}(U) \longrightarrow \mathbf{Et}^{(p')}(U) \otimes_F F'$ is an equivalence of categories.*

Proof. — The full faithfulness follows from 1.13, as U is geometrically connected.

Write $(X', M_{X'}) = (X, M) \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(\mathcal{O}_{F'})$, where as usual the product is taken in the category of fine saturated log. schemes. By 1.10 any object of $\mathbf{Et}^{(p')}(U \otimes_F F')$ extends uniquely to a log. étale covering of $(X', M_{X'})$. Let $(Z', M_{Z'})$ be such a covering. By Lemma 1.14 for each étale point $x \in X$, over the special fibre, $(Z', M_{Z'})$ descends to a log. covering of (X, M_X) in an étale neighbourhood of x. By the full faithfulness of 1.14 and properness of X, these descended objects glue into a finite log. étale covering of (X, M_X), and it clearly has prime to p order if the original log. covering does. □

Finally we come to

Proof of Theorem 1.4. — Write $\mathbf{Et}^{(p')}_{\mathrm{tame}}(U) = \varprojlim_{F'} \mathbf{Et}^{(p')}(U) \otimes_F F'$, where F' runs through tame Galois extensions of F. We have equivalences of categories

$$\mathbf{Et}^{(p')}_{\mathrm{log.geom}}(X, M) \xrightarrow{\sim} \mathbf{Et}^{(p')}_{\mathrm{tame}}(U) \xrightarrow{\sim} \mathbf{Et}^{(p')}_{\mathrm{geom}}(U).$$

Indeed the first equivalence follows from 1.10, and the second from 1.15.

It remains to show that the functor

$$\mathbf{Et}^{(p')}_{\mathrm{log.geom}}(X, M) \to \mathbf{Et}^{(p')}_{\mathrm{log.geom}}(X_s, M_s)$$

induced by restriction to the special fibre is an equivalence. Let $\mathfrak{X}$ be the π-adic completion of X. We give $\mathfrak{X}$ a log. structure $M_{\mathfrak{X}}$ by pulling back the log. structure on X via the map of locally ringed spaces $\mathfrak{X} \longrightarrow X$.

We claim that the pullback functor $\mathbf{Et}^{(p')}_{\mathrm{log.}}(X, M) \longrightarrow \mathbf{Et}^{(p')}_{\mathrm{log.}}(\mathfrak{X}, M_{\mathfrak{X}})$ is an equivalence of categories. Using the full faithfulness of 1.10, and its formal analogue 1.12, we see that to check that our functor is fully faithful, it is enough to check that the functor $\mathbf{Et}^{(p')}(U) \to \mathbf{Et}^{(p')}(U^{an})$ which takes a covering of U to its associated rigid analytic space is fully faithful. This is well known [Lut].

We check essential surjectivity. A log. covering of $(\mathfrak{X}, M_{\mathfrak{X}})$ induces a finite étale, Galois, prime to p map of rigid analytic spaces $W^{\mathrm{an}} \longrightarrow U^{\mathrm{an}}$, where U^{an} denotes the rigid analytic space associated to U. By [Lut] W^{an} comes from an algebraic covering W, which extends to a log. étale covering of (X, M) by 1.10. We have to check that the π-adic completion of this log. covering is isomorphic to the original log. covering of $\mathfrak{X}$. However this follows from the full faithfulness discussed in 1.12, since the two coverings give the same étale covering of U^{an}, by construction.

By [Ka1, 3.14] the functor

$$\mathsf{Et}^{(p')}_{\mathrm{log.}}(\mathfrak{X}, M_{\mathfrak{X}}) \to \mathsf{Et}^{(p')}_{\mathrm{log.}}(X_s, M_s)$$

induced by reduction modulo π, is an equivalence of categories.

Now the theorem follows by making the same argument for $(X, M) \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(\mathcal{O}_{F'})$, for each finite separable extension F' of F. $\square$

COROLLARY 1.16. — *If (X, M_X) and U is as above, denote by $\pi_1^{(p')}(U)$ the maximal, prime to p quotient of the geometric étale fundamental group of U. Then the canonical map $\mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{Out}(\pi_1^{(p')}(U))$, kills the wild inertia subgroup. This map is determined by the reduction of the log. scheme (X, M_X) modulo π.*

Proof. — Immediate from 1.4. $\square$

2. Applications.

THEOREM 2.1. — *Let $\mathcal{U}$ be a variety over F. Under the map*

$$\mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{Out}(\pi^{(p')}_{1,\mathrm{geom}}(\mathcal{U})),$$

the image of the wild inertia is finite.

Proof. — By Nagata's compactification theorem, we may embed $\mathcal{U}$ as an dense open subvariety of a proper variety $\mathcal{X}$. Put $\mathcal{Z} = \mathcal{X} - \mathcal{U}$.

By [deJ1, 6.5], after replacing F, by a finite extension, we may assume that there exists a smooth, proper variety $\mathcal{X}'$, and a proper dominant map $f : \mathcal{X}' \longrightarrow \mathcal{X}$ such that f is generically finite, $\mathcal{Z}' = f^{-1}(\mathcal{Z})$ is a normal crossings divisor in $\mathcal{X}'$, and the pair $(\mathcal{X}', \mathcal{Z}')$ extends to a semi-stable pair of $\mathcal{O}_F$ schemes (X', Z'). That is, X' is regular, and if X'_s denotes the special fibre of X', then $X'_s \cup Z$ is a normal crossings divisor in X'.

Let $\mathcal{W}$ be a geometrically connected object of $\mathbf{Et}^{(p')}_{\text{geom}}(\mathcal{U})$, which is also Galois. By (1.4) $f^*(\mathcal{W})$ is defined over a tame extension of F. After replacing F by a finite separable extension, we may assume that the connected components of $\mathcal{U}'' = \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}'|_{\mathcal{U}}$, are geometrically connected. Having made this extension, we claim that $\mathcal{W}$ is actually defined over a tame extension of F, which will prove the theorem. (Note that $\mathcal{X}'$ may no longer have semi-stable reduction after replacing F by the above extension, however in the following we only use the fact that $f^*(\mathcal{W})$ is defined over a tame extension, which remains true.)

Since $f^*(\mathcal{W})$ is defined over a tame extension of F, after replacing F by a tame extension, we may assume that there is an étale, prime to p Galois covering $\mathcal{V}$ of $\mathcal{U}' = \mathcal{X}' - \mathcal{Z}'$ which becomes isomorphic to $f^*(\mathcal{W})$ after a finite extension of F.

Denote by p_1 and p_2 the two projections of $\mathcal{U}''$ to $\mathcal{U}'$. $p_1^*(\mathcal{V})$ and $p_2^*(\mathcal{V})$ become isomorphic after a finite extension of scalars, and this isomorphism satisfies the usual cocycle condition. As the connected components of $\mathcal{U}''$ are geometrically connected, this isomorphism is actually defined after a tame extension by 1.13. Thus, replacing F by a tame extension, we may assume that $p_1^*(\mathcal{V}) \xrightarrow{\sim} p_2^*(\mathcal{V})$, satisfying the usual cocycle condition.

Using [SGA1, IX 4.12] we may descend $\mathcal{V}$ to a finite étale covering $\mathcal{W}^+$ of $\mathcal{U}$. By construction there is a finite separable extension E/F such that $\mathcal{W}$ is defined over E, and $\mathcal{W}^+ \otimes_F E \xrightarrow{\sim} \mathcal{W}$.

Thus $\mathcal{W}^+$ is a finite, connected étale covering of $\mathcal{U}$, of prime to p order. By descent, the Galois automorphisms of $\mathcal{V}$ descend to $\mathcal{W}^+$ if and only if their pullbacks via p_1^* and p_2^* are equal. However this condition must be satisfied, as it is after a finite extension of scalars, so $\mathcal{W}^+$ is Galois over $\mathcal{U}$. □

2.2. As a second application of the results of §1, we recover some of the results of [Ki1]. Let $f : \mathcal{X} \longrightarrow \mathcal{S}$ be a smooth proper map of finite type F schemes, with geometrically connected fibres. Let $\mathcal{Z} \subset \mathcal{X}$ be a normal crossings divisor relative to $\mathcal{S}$. Put $\mathcal{U} = \mathcal{X} - \mathcal{Z}$.

In [Ki] we showed that under quite general hypotheses the maps

$$\rho_s : \mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{Out}(\pi^{(p')}_{1,\mathrm{geom}}(\mathcal{U}_s))$$

were equal for sufficiently nearby F rational points s of $\mathcal{S}$. Here we give a different proof of this sort of result in certain cases.

The idea of the argument is as follows. Suppose that $f : \mathcal{X} \longrightarrow \mathcal{S}$ extended to a proper map $\bar{f} : X \longrightarrow S$ of $\mathcal{O}_F$ schemes, and put Z equal to the closure of $\mathcal{Z}$ in X. We assume that Z is a normal crossings divisor in X relative to S. This means that for each point $z \in Z$, there is an étale neighbourhood of X, $h : W \to X$, and an étale map $g : W \to \mathbb{A}^n_S$ such that $h^{-1}(Z)$ is the preimage under g of a normal crossing divisor in $\mathbb{A}^n_S$ cut out by a product of co-ordinate functions.

Suppose that for each $\mathcal{O}_F$ valued point s of S, the pair (X_s, Z_s) is semi-stable. By 1.16 the map $\mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{Out}(\pi^{(p')}_{1,\mathrm{geom}}(\mathcal{U}_s))$ depends only on the special fibre of X_s equipped with its log. structure. It turns out that this log. structure depends only on the reduction of X_s modulo π^2. Thus $\mathcal{O}_F$ valued points which are sufficiently close π- adically will give rise to the same maps.

We will prove the result under the weaker hypothesis that we have the situation described above only after a finite extension of scalars. Unfortunately even this is only known if the fibres of f are one dimensional.

In a positive direction we show that if $\bar{f}$ is flat, and the pair (X_s, Z_s) is semi-stable for some $\mathcal{O}_F$ valued point s, then (X_t, Z_t) is semi-stable for all $\mathcal{O}_F$ valued points t sufficiently close to s.

2.3. For technical reasons we deal with a slightly more general class of schemes than semi-stable ones. Fix a positive integer n, and denote by P the monoid

$$P = \langle e_1, \dots e_{r+s}, u; e_{r+1} \dots e_{r+s} = u^n \rangle.$$

Then $\mathcal{O}_F[P]/(u - \pi)$ is log. smooth over $\mathcal{O}_F$. We call an $\mathcal{O}_F$ log. scheme (X, M_X) n-semi-stable if étale locally, it admits a smooth map $X \longrightarrow \mathrm{Spec}(\mathcal{O}_F[P]/(u - \pi))$, which is induced by a strict map of log. schemes.

A technical advantage of n-semi-stable log. schemes, is that if F' is a finite extension of F, then the underlying scheme of the product $(X, M_X) \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(\mathcal{O}_{F'})$ coincides with the ordinary scheme theoretic product $X \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(\mathcal{O}_{F'})$ This is certainly false for general log. schemes which are log. smooth over $\mathcal{O}_F$, since in general one has to take the normalisation of the scheme theoretic product.

We can express the definition of n-semi-stable log. schemes more scheme theoretically as follows. Let X be an $\mathcal{O}_F$ scheme, and $Z \subset X$ a closed subscheme. We call the pair (X, Z) an n-semi-stable pair if étale locally, there exists a smooth map $\phi : X \longrightarrow \mathrm{Spec}(\mathcal{O}_F[P]/(u - \pi))$ such that $Z = \phi^{-1}(V(e_1 \dots e_r))$. Pulling back the log. structure induced by P, we obtain étale locally on X an n-semi-stable log. structure. By [Ka 2, 11.6] any log. regular structure on X is determined by the underlying scheme X, together with the open subset $U \subset X$, where the log. structure is trivial. In our situation, U is given by the complement of Z and the special fibre of X. Hence the log. structures we have obtained, are independent of the chosen map ϕ, and therefore glue into a log. structure on X, determined by the pair (X, Z). Conversely, given an n-semi-stable log. scheme, we get an n-semi-stable pair, by taking Z to be the closed subscheme of X locally cut out by the product $e_1 \dots e_r$. Alternatively, we can define Z globally, as the complement of the open subset of X, where the structural morphism $(X, M) \to (\mathrm{Spec}(\mathcal{O}_F), \mathbb{N})$ is *vertical* [Na2, 7.3].

For n-semi-stable schemes, we can show that the reduction of the log. scheme (X, M_X) modulo π depends only on the reduction modulo a finite, explicitly computable, power of π of the underlying schemes (X, Z). This should be true more generally for log. smooth log. schemes. See [Ki3], Prop 2.3 for the case of vertical log. structure.

Let us denote by $(X^{(n)}, Z^{(n)})$, and $(X^{(n)}, M_X^{(n)})$ the reductions modulo π^{n+1} of (X, Z) and (X, M_X) respectively.

PROPOSITION 2.4.— *Let (X, Z) be an n-semi-stable pair, and (X, M_X) the corresponding log. scheme. Then the reduction of (X, M_X) modulo π is completely determined by $(X^{(n)}, Z^{(n)})$.*

More precisely, suppose that (X_1, Z_1) (X_2, Z_2) are n-semi-stable pairs, and denote by (X_1, M_{X_1}) and (X_2, M_{X_2}) the corresponding log. schemes. If

$$\phi : (X_1^{(n)}, Z_1^{(n)}) \xrightarrow{\sim} (X_2^{(n)}, Z_2^{(n)})$$

is an isomorphism of pairs of underlying schemes (i.e an isomorphism of $X_1^{(n)}$ and $X_2^{(n)}$ taking $Z_1^{(n)}$ onto $Z_2^{(n)}$) then there exists a unique isomorphism of log schemes

$$\tilde{\phi}_0 : (X_1^{(0)}, M_{X_1}^{(0)}) \xrightarrow{\sim} (X_2^{(0)}, M_{X_2}^{(0)})$$

such that $\tilde{\phi}_0$ agrees with the reduction of ϕ modulo π on the underlying schemes.

To prove the proposition, we will need the following lemma.

LEMMA 2.5.— *With the above notation, set $U = X - Z$. Let x be an étale point of X with image in the closed fibre of X, and denote by*

M_x and $\bar{M}_x$ the induced log. structure on $\mathrm{Spec}(\mathcal{O}_{X,x})$ *and* $\mathrm{Spec}(\mathcal{O}_{X,x}/\pi)$ *respectively.*

(1) *If* $\phi \in \mathrm{Aut}_{\mathcal{O}_F}(\mathcal{O}_{X,x}/\pi^{n+1})$, *leaves* $\mathrm{Spec}(\mathcal{O}_{X,x}/\pi^{n+1})|_U$ *stable, then there exists*

$$\tilde{\phi} \in \mathrm{Aut}_{(\mathrm{Spec}(\mathcal{O}_F),\mathbb{N})}((\mathrm{Spec}(\mathcal{O}_{X,x}), M_x)),$$

such that $\phi = \tilde{\phi}$ *modulo* π, *on the underlying scheme* X.

(2) *If*

$$\tilde{\phi}_0 \in \mathrm{Aut}_{(\mathrm{Spec}(\mathcal{O}_F),\mathbb{N})}((\mathrm{Spec}(\mathcal{O}_{X,x}/\pi), \bar{M}_x)),$$

induces the identity on the underlying scheme $\mathrm{Spec}(\mathcal{O}_{X,x}/\pi)$ *then* $\tilde{\phi}_0$ *is equal to the identity as a map of log. schemes.*

Proof. — We may assume that $X = \mathrm{Spec}(A)$, where A is a strict henselisation of $\mathcal{O}_F[P]/(u-\pi)[\mathbb{Z}^d]$ for a suitable d. Denote by $e_{r+s+1}, \dots e_{r+s+d}$ a basis for $\mathbb{Z}^d$.

Choose lifts $e'_1, \dots e'_{r+s+d}$ of $\phi(e_1) \dots \phi(e_{r+s+d})$, where we again write e_i for the image of e_i in A/π^{n+1}. As ϕ leaves $\mathrm{Spec}(\mathcal{O}_{X,x}/\pi^{n+1})|_U$ stable, $\phi(e_i)$ is in the ideal corresponding to $\mathrm{Spec}(\mathcal{O}_{X,x}/\pi^{n+1})|_Z$, for $i = 1, \dots, r$ so we may assume that e'_i is in the ideal corresponding to Z for $i = 1, \dots, r$.

Also, after modifying e'_{r+s} by a unit congruent to 1 modulo π, we may assume that $e'_{r+1} \dots e'_{r+s} = \pi^n$, and define a map of schemes

$$\tilde{\phi}' : \mathrm{Spec}(A) \to \mathrm{Spec}(\mathcal{O}_F[P]/(u-\pi)[\mathbb{Z}])$$

by sending e_i to e'_i. As A is strictly henselian, and by construction $\tilde{\phi}'$ maps x to x, there exists a unique extension $\tilde{\phi} : \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ of $\tilde{\phi}'$, lifting $\tilde{\phi}_0$.

We claim that $\tilde{\phi}$ extends uniquely to a map of log. schemes. To see this, denote by V the open subset of $\mathrm{Spec}(\mathcal{O}_{X,x})$ where the log. structure is trivial, and write $j : V \hookrightarrow \mathrm{Spec}(\mathcal{O}_{X,x})$ for the natural inclusion. By [Ka2, 11.6] we have $M_x = j_* \mathcal{O}_V^* \cap \mathcal{O}_{X,x}$, and the choice of the lifts e'_i implies that $\tilde{\phi}$ leaves V stable. Hence $\tilde{\phi}$ leaves M_x stable, and $\tilde{\phi}$ extends to a map of log. schemes. This proves (1).

(2) Denote by $L : \bar{M}_x \longrightarrow \mathcal{O}_{X,x}/\pi$ the map defining the log. structure. The conditions imply that for $1 \leqslant i \leqslant r+s+d$, $\phi(e_i) = e_i w_i$, where $w_i \in \bar{M}_x^* = (\mathcal{O}_{X,x}/\pi)^*$ is an invertible element, such that $L(e_i) = L(e_i w_i)$.

Let $\mathfrak{P}$ be a generic point of $\mathrm{Spec}(\mathcal{O}_{X,x}/\pi)$. The induced log. structure $\bar{M}_{\mathfrak{P}}$ on $\mathrm{Spec}(\mathcal{O}_{X,x}/\pi)_{\mathfrak{P}}$ is generated by the image of u. As ϕ is a morphism over $(\mathrm{Spec}\mathcal{O}_F, \mathbb{N})$, whose reduction modulo π is the identity on underlying schemes, it follows that ϕ induces the identity on $(\mathrm{Spec}(\mathcal{O}_{X,x}/\pi_{\mathfrak{P}}), \bar{M}_{\mathfrak{P}})$. Thus the image of w_i in $\bar{M}_{\mathfrak{P}}$ is 1, as $\bar{M}_{\mathfrak{P}}$ is integral. As the natural map

$$\mathcal{O}_{X,x}/\pi \hookrightarrow \bigoplus_{\mathfrak{P}} (\mathcal{O}_{X,x}/\pi)_{\mathfrak{P}}$$

is an injection, we see that the image of w_i in $(\mathcal{O}_{X,x}/\pi)^*$ is equal to 1. □

Proof of 2.4.— Because the morphism $\tilde{\phi}_0$ will be unique we may apply étale descent, and work étale locally on X_1, and X_2. Then, by a standard limit argument, we may choose an étale point at x in the special fibre of X_1, and replace X_1 and X_2 by their strict localisations in x and $\phi(x)$ respectively. With the notation of 2.5, (X_1, Z_1) and (X_2, Z_2) are both abstractly isomorphic to $(\mathrm{Spec}(A), \mathrm{Spec}(A/e_1 \dots e_r))$ for suitable integers r, s, d, because these integers are even determined by the $X_1^{(0)} \xrightarrow{\sim} X_2^{(0)}$.

Fixing such isomorphisms we may identify X_1, X_2 with $\mathrm{Spec}(A)$ in such a way that Z_1 coincides with Z_2. Under these identifications, ϕ corresponds to an automorphism of $\mathrm{Spec}(A/\pi^{n+1})$, of the sort considered in 2.5, and 2.5(1) shows that there exists an isomorphism $\tilde{\phi} : (X_1, M_{X_1}) \xrightarrow{\sim} (X_2, M_{X_2})$, whose reduction modulo π coincides with that of ϕ on underlying schemes. We take $\tilde{\phi}_0$ to be the reduction of ϕ modulo π, which shows existence. The uniqueness follows from 2.5(2): two choices of $\tilde{\phi}_0$ differ by an automorphism of $\mathrm{Spec}(A/\pi)$ equipped with its log. structure, which is the identity on the underlying scheme. □

The following lemma shows that when X is proper, n semi-stability can be detected on the reduction of (X, Z) modulo π^{n+1}.

LEMMA 2.6.— *Let X be a proper flat $\mathcal{O}_F$ scheme, and $Z \subset X$ a normal crossings divisor over $\mathcal{O}_F$. (So in particular, X is smooth at points of Z.) (X, Z) is n-semi-stable if and only if there exists a n-semi-stable pair (X', Z') which is isomorphic to (X, Z) modulo π^{n+1}. That is n-semi-stability depends only on the reduction of (X, Z) modulo π^{n+1}.*

Proof.— The "only if" direction is obvious, so we prove the "if" direction. Let $\mathrm{Spec}A$ be an open affine in X. Replacing A by its strict henselisation at a point in the special fibre, we may assume that A/π^{n+1} is isomorphic to a strict henselisation of $(\mathcal{O}_F[P]/(u-\pi)[\mathbb{Z}^d])/\pi^{n+1}$ for a suitable integer d, with $Z \cap \mathrm{Spec}A/\pi^{n+1}$ cut out by $e_1 \dots e_r$. As in the proof of Lemma 2.5, we denote by $e_{r+s+1}, \dots, e_{r+s+d}$ a basis for $\mathbb{Z}^d$.

Lift the e_i, to $E_i \in A$. Since Z is a normal crossings divisor relative to Spec$\mathcal{O}_F$, we may assume that for $i = 1$ to r, E_i has been chosen so that the product $E_1 \dots E_r \in A$, cuts out Z.

We have $E_{r+1} \dots E_{r+s} = w\pi^n$, with w a unit in A congruent to 1 modulo π. Replacing E_{r+1} by $E_{r+1}w^{-1}$, we may assume that $E_{r+1} \dots E_{r+s} = \pi^n$, though E_{r+1} now lifts e_{r+1} only modulo π.

Put $B = \mathcal{O}_F[P][\mathbb{Z}^d]/(u-\pi)$ and define a map $\phi : B \longrightarrow A$ by $e_i \mapsto E_i$ for $1 \leqslant i \leqslant r+s+d$. We claim that if $b \in$ Spec$B$ denotes the image of the closed point of A, then ϕ induces an isomorphism $\phi^h : B_b^h \longrightarrow A$, where B_b^h denotes the strict henselisation of B at b. Since A was the strict henselisation of X at an arbitrary point of the special fibre, this claim implies the lemma, by the properness of X.

ϕ^h is an isomorphism modulo π. Since A and B have no π-torsion A is B_b^h flat by [Mat, Ex 22.3]. Similarly if we complete A and B_b^h at their respective closed points, then the map $\hat{\phi} : \hat{B}_b^h \longrightarrow \hat{A}$ makes $\hat{A}$ flat over $\hat{B}_b^h$. By [Mat, 22.5] $\hat{\phi}$ is injective, as it is modulo π. Since $\hat{B}_b^h$ is π-adically complete $\hat{\phi}$ is also surjective, as it is modulo π. Thus $\hat{\phi}$ is an isomorphism. As ϕ^h is quasi-finite, and B_b^h is henselian, ϕ^h is finite, whence an isomorphism, as $\hat{\phi}$ is. □

Next we need the following technical result, on lifting log. structures to a formal neighbourhood. The proof is rather similar to that of the previous lemma.

LEMMA 2.7. — *Suppose that $X \longrightarrow S$ is a flat map of flat $\mathcal{O}_F$ schemes, $Z \subset X$ is a normal crossings divisor relative to S, and that the pair $(X_{\tilde{s}}, Z_{\tilde{s}})$ is n- semi-stable for some $\mathcal{O}_F$ valued point $\tilde{s}$ of S. Denote by Y the special fibre of $X_{\tilde{s}}$, and by $\hat{X}$ the completion of X along Y. Denote by $\hat{S}$ the completion of S at the closed point of $\tilde{s}$.*

The usual log. structure on $X_{\tilde{s}}$, lifts canonically to a log. structure on $\hat{X}$, such that for any $\mathcal{O}_F$ valued point t of $\hat{S}$, which is sufficiently close to $\tilde{s}$ the usual log. structure on the π-adic completion of (X_t, Z_t) (which is n-semi-stable by 2.6) coincides with the one induced by that on $\hat{X}$.

If we write $\hat{X} = \varinjlim X_n$, the X_n being nilpotent neighbourhoods of Y in X, then the maps $Y \longrightarrow X_n$ are induced by exact immersions of log. schemes.

Proof. — Working étale locally, we may assume that $X = \operatorname{Spec}(A)$ is affine, and that there is an étale map $\phi : X_{\tilde{s}} \longrightarrow \operatorname{Spec}(\mathcal{O}_F[P]/(u-\pi)[\mathbb{Z}^d])$, which is induced by a strict map of log. schemes, with $Z_{\tilde{s}}$ cut out by the product $e_1 \dots e_r$, the notation being as in the proof of the previous lemma.

Denote by $\mathfrak{p}$ the prime ideal on S corresponding to $\tilde{s}$, and by $\mathfrak{m}$, the closed point of $\tilde{s}$ on S. We may assume that $S = \operatorname{Spec}(E)$ is affine. Set $B = (E[P]/(u-\pi)[\mathbb{Z}^d])\widehat{\ }$, where "$\widehat{\ }$" denotes $\mathfrak{m}$-adic completion. We make $\operatorname{Spf}(B)$ into a formal log. scheme, with log. structure induced by P.

Denote by $\hat{A}$ the $\mathfrak{m}$-adic completion of A, and lift $e_i \in A/\mathfrak{p}$ to $E_i \in \hat{A}$, in such a way that $E_{r+1} \dots E_{r+s} = \pi^n$, assuming however that E_{r+s} only lifts e_{r+s} modulo π. Writing $\hat{Z}$ for the completion of Z along Y, we may assume, as in the proof of (2.6), that the product $E_1 \dots E_r$ generates the ideal corresponding to $\hat{Z}$.

We obtain a map $\phi : B \longrightarrow \hat{A}$ defined by $e_i \mapsto E_i$, $1 \leqslant i \leqslant r+s+d$. As A is $\mathfrak{m}$-adically complete, and flat over E, and ϕ is étale modulo $\mathfrak{m}$, ϕ is formally étale.

Thus, we may define a log. smooth log. structure on $\operatorname{Spec}(\hat{A})$ by pulling back the log. structure on $\operatorname{Spec}(B)$ via the map $\operatorname{Spec}(\hat{A}) \longrightarrow \operatorname{Spec}(B)$. As Z and the special fibre of X, are Cartier divisors in X, one checks easily, that the lifts E_i, are uniquely defined up to multiplication by units which are congruent to 1 modulo the ideal of definition of $\hat{X}$. Thus the log. structure we have defined on $\hat{X}$, is independent of choices.

Let t be an $\mathcal{O}_F$ valued point of S, which is sufficiently close to $\tilde{s}$. As there is at most one log. structure on the π-adic completion of X_t making it log. smooth over $\mathcal{O}_F$ [Ka2, 11.6], the log. structure induced on this π-adic completion by $\hat{X}$, must coincide with the one discussed in 2.3, as both are log. smooth. □

Finally, we come to

PROPOSITION 2.8.— *Let $f : \mathcal{X} \longrightarrow \mathcal{S}$ be a proper smooth map of finite type F schemes, and $\mathcal{Z} \subset \mathcal{X}$ a divisor with normal crossings relative to $\mathcal{S}$. Let $s \in \mathcal{S}$ be an F- rational point, and suppose that for some finite extension F' of F, $\mathcal{X} \otimes_F F' \longrightarrow \mathcal{S} \otimes_F F'$ extends to a proper flat map of flat $\mathcal{O}_{F'}$ schemes $\bar{f} : X \longrightarrow S$, such that s extends to an $\mathcal{O}_{F'}$ valued point $\tilde{s}$, $\mathcal{Z} \otimes_F F'$ extends to a normal crossings divisor $Z \subset X$, relative to S, and the fibre $(X_{\tilde{s}}, Z_{\tilde{s}})$ is n-semi-stable.*

Put $\mathcal{U} = \mathcal{X} - \mathcal{Z}$. For all rational points $t \in \mathcal{S}$, which are sufficiently close to s, there is a natural equivalence of categories $\mathbf{Et}^{(p')}_{\text{geom}}(\mathcal{U}_s) \xrightarrow{\sim} \mathbf{Et}^{(p')}_{\text{geom}}(\mathcal{U}_t)$, compatible with the Galois action on both sides.

Proof. — Suppose first that $F' = F$. If t is an F-rational point of $\mathcal{S}$, which is sufficiently π-adically close to s, then it lifts to an $\mathcal{O}_F$ valued point $\tilde{t}$ of S, and by 2.6 we may assume that the fibre $(X_{\tilde{t}}, Z_{\tilde{t}})$ is also n-semi-stable. Now the result follows from 2.4 and 1.4.

Now suppose $F' \neq F$. By [SGA1, IX 4.10], we may replace F by $F' \cap F^{p^{-\infty}}$. Increasing F' leaves all our hypothesis intact, except that we may have to increase n. Thus, we may assume that F' is Galois over F. We have to show that the equivalence

$$\Phi : \mathbf{Et}^{(p')}_{\mathrm{geom}}(\mathcal{U}_s) = \mathbf{Et}^{(p')}_{\mathrm{geom}}(\mathcal{U}_s \otimes_F F') \xrightarrow{\sim} \mathbf{Et}^{(p')}_{\mathrm{geom}}(\mathcal{U}_t \otimes_F F') = \mathbf{Et}^{(p')}_{\mathrm{geom}}(\mathcal{U}_t)$$

is compatible with the action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ on both sides.

To see this we give an alternate description of Φ. Denote by $\hat{X}$ the completion of X along the closed fibre of $X_{\tilde{s}}$, and by $\hat{S}$ the completion of S at the closed point of $\tilde{s}$. We equip $\hat{X}$ with the log. structure given by the previous lemma.

We have equivalences of categories

$$\mathbf{Et}^{(p')}_{\mathrm{geom}}(\mathcal{U}_s) \xrightarrow{\sim} \mathbf{Et}^{(p')}_{\mathrm{log.geom}}(X_{\tilde{s}}, M_{X_{\tilde{s}}}) \xrightarrow{\sim} \mathbf{Et}^{(p')}_{\mathrm{log.geom}}(\hat{X}, M_{\hat{X}}),$$

the first given by Theorem 1.4, and the second by restricting to the closed fibre of $X_{\tilde{s}}$, and then using [Ka1, 3.14] to lift coverings to $\hat{X}$. Here the $M's$ denote the log. structures on the corresponding schemes and formal schemes. Denote the composite by Γ.

The "generic fibre" of $\hat{X}$ is a rigid analytic space, and similarly for $\hat{S}$, (see [deJ2, 7.2.5]), so that the proper map of formal schemes $\hat{X} \longrightarrow \hat{S}$ gives rise to a map of rigid analytic spaces $\hat{X}_\eta \longrightarrow \hat{S}_\eta$. The fibres of this map are the rigid analytic spaces associated to the fibres $\mathcal{X}_t \otimes_F F'$ of the map f, for points t which are sufficiently close to s. Thus we may restrict a log. étale covering of $(\hat{X}, M_{\hat{X}})$ to a covering of such a fibre, which is étale over $\mathcal{U}_t \otimes_F F'$. Strictly speaking, we have only obtained a covering of the the rigid space associated to $\mathcal{U}_t \otimes_F F'$, however all such coverings are algebraic [Lut].

Thus, for suitable t, we have constructed a functor

$$\Psi : \mathbf{Et}^{(p')}_{\mathrm{log.geom}}(\hat{X}, M_{\hat{X}}) \longrightarrow \mathbf{Et}^{(p')}_{\mathrm{geom}}(\mathcal{U}_t),$$

and one checks easily that $\Phi = \Psi \circ \Gamma$. (The key point is that liftings of log. étale coverings over nilpotent, exact thickenings are unique.)

If $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$, and σ^* denotes pulling back by σ, then we have to check that $\Phi \circ \sigma^* \xrightarrow{\sim} \sigma^* \circ \Phi$. Since Ψ is simply restriction to a certain rigid analytic fibre, we clearly have $\Psi \circ \sigma^* = \sigma^* \circ \Psi$. Similarly, since Γ has a quasi-inverse, which is restriction to a suitable rigid analytic fibre, we obtain $\Gamma \circ \sigma^* \xrightarrow{\sim} \sigma^* \circ \Gamma$. Thus

$$\Phi \circ \sigma^* = \Psi \circ \Gamma \circ \sigma^* \xrightarrow{\sim} \Psi \circ \sigma^* \circ \Gamma \xrightarrow{\sim} \sigma^* \circ \Psi \circ \Gamma = \sigma^* \circ \Phi. \qquad \square$$

Using Proposition 2.8, we obtain immediately

COROLLARY 2.9. — *Under the hypotheses of 2.8, suppose that f has geometrically connected fibres, and for each rational F-rational point t of $\mathcal{S}$ denote by ρ_t the map*

$$\mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{Out}(\pi_{1,\mathrm{geom}}^{(p')}(\mathcal{U}_t)).$$

If s is an F rational point of $\mathcal{S}$, $\rho_s \xrightarrow{\sim} \rho_t$ for all F rational points t which are sufficiently close to s.

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Mark KISIN,
University of Sydney
Department of Mathematics and Statistics
Sydney (Australia).
markh@maths.usyd.edu.au