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DAVID MATHIEU

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UNIVERSAL REPARAMETRIZATION OF A FAMILY OF CYCLES: A NEW APPROACH TO MEROMORPHIC EQUIVALENCE RELATIONS

by David MATHIEU [(1)]

Introduction.

Let Z be a reduced analytic space of finite dimension (by 'analytic', we will always mean 'complex-analytic').

Daniel Barlet proved that the set $\mathcal{B}_n(Z)$ of *compact* n-cycles of Z is an analytic space of finite dimension (see [Ba75]). Here we consider the set $\mathcal{C}_n^{\mathrm{loc}}(Z)$ of *all* (closed) n-cycles; we can not hope to have a *finite dimensional* analytic structure on this set, but, roughly speaking, we want to provide some 'nice' subsets of $\mathcal{C}_n^{\mathrm{loc}}(Z)$ with such a structure.

Let us be more precise: a subset of $\mathcal{C}_n^{\mathrm{loc}}(Z)$ rather easy to define and to handle is the set of cycles described by an analytic family of n-cycles $(X_s)_{s\in S}$ parametrized by a weakly normal analytic space S. Let $\chi : S \to \mathcal{C}_n^{\mathrm{loc}}(Z)$, $s \mapsto X_s$ be the map associated with this family. The analytic structure to be defined on $\chi(S)$ should not depend on the parametrizing space S. So the problem we raise can be rewritten as a problem of 'universal reparametrization'; we prove:

THEOREM. — *Let S be a weakly normal analytic space of finite dimension and $(X_s)_{s\in S}$ be a 'semi-proper', 'regular' analytic family of n-*

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cycles of Z. Then there exist a unique weakly normal analytic space Q of finite dimension, and an analytic family of n-cycles $(\widetilde{X}_c)_{c\in Q}$ parametrized by Q, such that for every weakly normal analytic space T, and every analytic family of n-cycles $(Y_t)_{t\in T}$ satisfying:

$$\forall\, t \in T,\ \exists\, s \in S,\ s.t.\ \ Y_t = X_s,$$

then there exists a unique analytic map $\varphi : T \to Q$ such that the family $(Y_t)_{t\in T}$ is the pull-back by φ of the family $(\widetilde{X}_c)_{c\in Q}$, that is: for all $t \in T$, $Y_t = \widetilde{X}_{\varphi(t)}$.

Let us comment on the conditions we put on $(X_s)_{s\in S}$. In the case of compact cycles, the map $\chi : S \to \mathcal{B}_n(Z)$ is analytic with values in a *finite* dimensional space. A sufficient condition, so that its image $\chi(S)$ may be analytic, is then *semi-properness*, according to a theorem of Kuhlmann (see [Ku64], [Ku66]).

Here χ is, at first sight, only continuous. Actually, there are some analytic maps associated with the family $(X_s)_{s\in S}$, but they are defined only locally on S and on Z, with values in a locally analytic subset of an *infinite* dimensional Banach space. So two problems appear:

• we have to generalize Kuhlmann's theorem to the case of semi-proper maps with values in infinite dimensional spaces; with this aim, we use ideas of Barlet and Mazet (see [Ma74]);

• we have to put a *regularity* condition on the analytic family of n-cycles $(X_s)_{s\in S}$, which ensures us that the semi-local behaviour (on a relatively compact open set of Z) of the cycles determines their global behaviour: the cycles should not (set-theoretically and topologically) *escape to infinity.*

We can notice that the natural candidate for the underlying topological space of Q is the quotient space S/R_χ of S by the equivalence relation defined by χ, which already gives us a one-to-one reparametrization of the family $(X_s)_{s\in S}$. So expressing it, we see that our problem is close to the problem of *analytic equivalence relations* studied by Grauert [Gr83].

In a last part, we introduce *meromorphic families of n-cycles* of Z, parametrized by a weakly normal space S, in a way similar to Remmert's definition of meromorphic maps. We especially study the case of a meromorphic family of cycles without (set-theoretic) escape to infinity: its graph in $S \times \mathcal{C}_n^{\mathrm{loc}}(Z)$ is a finite dimensional analytic space.

Then we give criteria, so that the projection on S of an analytic subset of $S \times Z$ defines a meromorphic family of cycles without (set-theoretic) escape to infinity (these criteria are close to the assumptions of Grauert's theorem about *meromorphic equivalence relations*, see [Gr86]); this can be seen as a problem of *geometric flattening* (see [Ba78]). A similar problem has already been studied by Siebert [Si93].

We conclude with a theorem of universal reparametrization for semi-proper regular meromorphic families of cycles.

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1. Preliminaries.

In this paper, all analytic spaces are supposed to be reduced, of finite dimension if the contrary is not explicitly stated, and countable at infinity; moreover, 'analytic subset' means 'closed analytic subset', and 'neighbourhood' means 'open neighbourhood'.

1.1. Semi-properness and quasi-properness.

1.1.1. *Definitions.* — Let X and Y be Hausdorff topological spaces.

DEFINITION 1. — *A continuous map $\varphi : X \to Y$ is semi-proper (see [AS71]) if, for every point y of Y, there exist a neighbourhood V of y in Y and a compact subset K of X such that*

$$\varphi(K) \cap V = \varphi(X) \cap V.$$

We collect below some properties of semi-proper maps:

LEMMA 1. — (i) *If φ is semi-proper, then $\varphi(X)$ is closed in Y and locally compact. Moreover, the map $\varphi : X \to Y$ is semi-proper if and only if $\varphi : X \twoheadrightarrow \varphi(X)$ is semi-proper and $\varphi(X)$ is closed in Y. Eventually, when $\varphi : X \twoheadrightarrow \varphi(X)$ is semi-proper, there exists an open set V of Y, containing $\varphi(X)$, such that $\varphi : X \to V$ is semi-proper.*

(ii) *If $\varphi : X \to Y$ is semi-proper and if $Y' \subset Y$ is such that $\varphi(X) \cap Y'$ is locally compact (for instance, when Y' is open or closed in Y), then the restriction $\varphi_{|\varphi^{-1}(Y')} : \varphi^{-1}(Y') \to Y'$ is semi-proper.*

(iii) *The composition* $\pi \circ \varphi$ *of continuous maps* $\varphi : X \to Y$ *and* $\pi : Y \to Z$ *is semi-proper in the following cases:*

• φ *is semi-proper and* π *is a homeomorphism* (φ *semi-proper,* π *proper suffice when* X, Y *and* Z *are locally compact)*;

• φ *is proper and surjective and* π *is semi-proper. On the other hand, if the composition* $\pi \circ \varphi$ *is semi-proper, with* φ *surjective, then* π *is semi-proper.*

(iv) *When* X *and* Y *are locally compact, Kuhlmann* [Ku66] *introduced the following definition of semi-properness, which is, in that case, equivalent to the previous one : for every compact set* L *of* Y, *there exists a compact set* K *of* X *such that* $\varphi(X) \cap L = \varphi(K)$.

As far as *semi-proper analytic* maps are concerned, we have the following result:

THEOREM 1. — *Let* X *be an analytic space of finite dimension,* Y *be an open set of an infinite dimensional Banach space* $\mathbb{E}$. *Let* $\varphi : X \to Y$ *be a semi-proper analytic map. Then* $\varphi(X)$ *is an analytic subset of finite dimension of* Y.

We refer to [Ma84] for the theory of infinite dimensional spaces. We only insist on the fact that here $\varphi(X)$ is an analytic subset of finite dimension of Y, and thus is an analytic space of finite dimension of the classical theory.

This theorem is in fact a generalization of Remmert's proper mapping theorem, which puts together results of Kuhlmann (semi-proper case, see [Ku64], [Ku66], and [AS71]; actually, our theorem enables us to give a new proof of Kuhlmann's) and results of Mazet and Barlet (infinite dimension case, see [Ma74], [Ma84]).

The proof is long and technical; we refer to [Mt99].

The restriction of a semi-proper analytic map $\varphi : X \to Y$ to an analytic subset is seldom semi-proper itself. We shall obtain better results in this way with a notion stronger than semi-properness.

DEFINITION 2. — *An analytic map (or, at least, a map with analytic fibers)* $\varphi : X \to Y$ *is quasi-proper (see* [AS71]) *if, for every point* y *of* Y, *there exist a neighbourhood* V *of* y *in* Y *and a compact set* K *of* X *such that, for every point* y' *of* $V \cap \varphi(X)$ *and every irreducible component* C *of* $\varphi^{-1}(y')$, *we have* $C \cap K \neq \emptyset$.

Thus, a proper map is quasi-proper, and a quasi-proper map is semi-proper.

Let $\varphi : X \to Y$ be an analytic map. We say that an analytic subset A of X is *quasi-saturated* for φ if for every point $a \in A$, every irreducible component of $\varphi^{-1}\varphi(a) \cap A$ is an irreducible component of $\varphi^{-1}\varphi(a)$. For instance, any connected component of X is quasi-saturated. Moreover:

LEMMA 2. — (i) *Let $\varphi : X \to Y$ be a quasi-proper map, and A an analytic subset of X, quasi-saturated for φ. Then the restriction $\varphi_{|A} : A \to Y$ is quasi-proper.*

(ii) *The degeneracy set of order p of φ is quasi-saturated for φ. Thus, if the map $\varphi : X \to Y$ is quasi-proper, so is $\varphi_{|\mathrm{Deg}^p\varphi} : \mathrm{Deg}^p\varphi \to Y$. So (by Theorem 1), $\varphi(X)$ and $\varphi(\mathrm{Deg}^p\varphi)$ are analytic subsets of Y.*

Let us recall that the (geometric) *corank* of an analytic map is defined by

$$\operatorname{gcork}\varphi := \min_{x\in X} \operatorname{gcork}_x \varphi, \quad \text{where} \quad \operatorname{gcork}_x \varphi := \dim_x \varphi^{-1}\varphi(x)\,;$$

the degeneracy set of order k is the analytic subset $\mathrm{Deg}^p\varphi$ of X defined by $\{x \in X,\ \operatorname{gcork}_x \varphi \geq p\}$; we note (see [Fi76])

$$\mathrm{Deg}\,\varphi = \mathrm{Deg}^{\operatorname{gcork}\varphi+1}\varphi.$$

1.1.2. *Quotient defined by a semi-proper analytic map.* — We recall that a continuous map $\varphi : X \to Y$ defines an equivalence relation R_φ on X by: $x\, R_\varphi\, y$ iff $\varphi(x) = \varphi(y)$. Then we have a canonical decomposition

$$\varphi : X \xrightarrow{q} X/R_\varphi \xrightarrow[\sim]{h} \varphi(X) \xhookrightarrow{i} Y,$$

(h is a continuous bijection, and a homeomorphism when φ is semi-proper).

Moreover, the quotient space X/R_φ is provided with a canonical ringed structure $Q = (X/R_\varphi, \mathcal{O}_Q)$: for every open set V of X/R_φ, $\mathcal{O}_Q(V) := \varphi_{*,R_\varphi}\mathcal{O}_X(V)$ is the ring of functions $q^{-1}(V) \to \mathbb{C}$ which are constant on the classes of R_φ (*i.e.*, on the fibers of φ).

We now state:

THEOREM 2. — *Let X be a weakly normal analytic space of finite dimension, Y be an open set of an infinite dimensional Banach space $\mathbb{E}$. Let $\varphi : X \to Y$ be a semi-proper analytic map. Let R_φ be the equivalence relation defined by φ and $Q := (X/R_\varphi, \mathcal{O}_Q)$ be the ringed quotient space. Then Q is a weakly normal analytic space of finite dimension.*

The proof we give in [Mt99] generalizes Cartan's ideas about proper equivalence relations (see [Ca60]). We precisely show that Q is the weak normalization of $\varphi(X)$.

1.2. Analytic families of cycles.

1.2.1. *Definitions.* — For this section, we refer to [Ba75].

Let $\operatorname{sym}^k \mathbb{C}^p$ be the quotient of $(\mathbb{C}^p)^k$ under the action of the k-th symmetric group of permutations $\mathfrak{S}_k$; it can be properly embedded in $\bigoplus_{h=1,\dots,k} S_h(\mathbb{C}^p)$, where $S_h(\mathbb{C}^p)$ is the h-th component of the symmetric algebra of $\mathbb{C}^p$. If U is a polydisc of $\mathbb{C}^n$, let

- $H(\overline{U}, \bigoplus_{h=1,\dots,k} S_h(\mathbb{C}^p))$ be the Banach space of maps continuous on $\overline{U}$, analytic on U, with values in $\bigoplus_{h=1,\dots,k} S_h(\mathbb{C}^p)$, and

- $H(\overline{U}, \operatorname{sym}^k \mathbb{C}^p)$ be the analytic subset of the previous space, containing the maps with values in $\operatorname{sym}^k \mathbb{C}^p$.

If B is a polydisc of $\mathbb{C}^p$, $\operatorname{sym}^k B$ is the image of B^k in $\operatorname{sym}^k \mathbb{C}^p$.

Then we have: *there exists a natural bijection between the set of ramified covers of degree k over U included in $U \times B$ and the set of analytic maps $U \to \operatorname{sym}^k B$.*

Let Z be an analytic space.

A *n-cycle* of Z is a locally finite (formal) linear combination $X := \sum_{i\in I} m_i X_i$ of irreducible analytic subsets of dimension n X_i of Z (pairwise distincts), together with non-negative integers m_i (called *multiplicities*). The *support* of the cycle X is the analytic subset $|X| := \bigcup_{i\in I} X_i$ of Z.

A *scale* $E = (U, B, j)$ *on* Z is the data of an open set V of Z, two polydiscs U and B of $\mathbb{C}^n$ and $\mathbb{C}^p$ respectively, and an analytic isomorphism j of V onto an analytic subset A of a neighbourhood of $\overline{U} \times \overline{B}$ in $\mathbb{C}^{n+p}$. The *domain* of the scale is the open set $W_E := j^{-1}((U \times B) \cap A)$ of Z.

If X is a n-cycle of Z, the scale $E = (U, B, j)$ is said to be *adapted* to X if $j^{-1}(\overline{U} \times \partial B) \cap |X| = \emptyset$. In this case, the analytic subset $j(|X|)$ can be seen, thanks to the projection $U \times B \to U$, as a ramified cover X_E over U, every irreducible component of $j(|X|)$ carrying the multiplicity of the corresponding irreducible component of X. We denote by $k := \deg_E X$ the degree of the ramified cover X_E, and by $F_E(X) : U \to \operatorname{sym}^k B$ the analytic map associated with X_E (according to the previous paragraph); moreover, the restriction $F_E(X)_{|\overline{U}'}$ belongs to $H(\overline{U}', \operatorname{sym}^k B)$, where U' is a relatively compact polydisc in U (in the sequel, we shall always assume that such a polydisc is fixed when we consider a scale).

Let S be a weakly normal analytic space. Let $(X_s)_{s\in S}$ be a family of cycles of Z parametrized by S. This family is said to be *analytic*, if, for every $s_0 \in S$, for any scale $E = (U, B, j)$ on Z adapted to X_{s_0}, there exists a neighbourhood S_E of s_0 in S such that conditions (a) and (b) hold:

(a) for every point s of S_E, E is adapted to X_s and $\deg_E X_s = \deg_E X_{s_0}$;

(b) the map $\mathfrak{f}_E : S_E \times U \to \operatorname{sym}^k B$, $(s,t) \mapsto F_E(X_s)(t)$ is analytic.

Furthermore, when b) holds, we have

(c) the map $f_E : S_E \to H(\bar{U}', \operatorname{sym}^k B)$, $s \mapsto F_E(X_s)_{|\bar{U}'}$ is analytic.

The *graph* of an analytic family of cycles $(X_s)_{s\in S}$ is the analytic subset of $S \times Z$ defined by $G := \{(s,z) \in S \times Z$ such that $z \in |X_s|\}$. Conversely, here is a useful construction of an analytic family of cycles (see [Ba75], Theorem 1):

PROPOSITION 1. — *Let Z and S be analytic spaces, S being normal, G be an analytic subset of $S \times Z$, $\pi : G \to S$ and $p : G \to Z$ the restrictions to G of the projections on S and on Z. Suppose that, for all irreducible component G_i of G, the projection $\pi_i : G_i \to S$ is surjective and of constant corank n. Then the irreducible components Γ_s^ν of the fibres $\pi^{-1}(s) = \{s\} \times p(\pi^{-1}(s))$ of π can be provided with multiplicities m_s^ν, generically equal to* 1, *such that the cycles $(X_s := (p(\Gamma_s^\nu), m_s^\nu))_{s\in S}$ define an analytic family of n-cycles of Z parametrized by S.*

1.2.2. *Topology of the set of n-cycles.* — Let $\mathcal{C}_n^{\mathrm{loc}}(Z)$ be the set of n-cycles of Z, provided with the topology generated by finite intersections of the following sets:

$$\Omega^k(E) := \big\{X \in \mathcal{C}_n^{\mathrm{loc}}(Z) \text{ such that } E \text{ is adapted to } X \text{ with } \deg_E(X) = k\big\},$$

defined for all scales E on Z and all integers k. This topology is Hausdorff. Besides, every cycle has a countable fundamental system of neighbourhoods.

With every family of cycles $(X_s)_{s\in S}$, we associate a map

$$\chi : S \longrightarrow \mathcal{C}_n^{\mathrm{loc}}(Z), \quad s \longmapsto X_s .$$

We shall say that the family $(X_s)_{s\in S}$ is *continuous* if the map χ is continuous.

With these notations, the condition (a) above can be written:

(a) $S_E \subset \chi^{-1}(\Omega^{\deg_E X_{s_0}}(E))$,

and implies that χ is continuous in s_0.

Actually, the neighbourhood $\chi^{-1}(\Omega^{\deg_E X_{s_0}}(E))$ verifies clearly (a), and (b) as well, and we shall always set $S_E = \chi^{-1}(\Omega^{\deg_E X_{s_0}}(E))$.

We shall often have to work with *finite families of scales*; we introduce here some notations.

Let $(X_s)_{s\in S}$ be a continuous family of n-cycles of Z parametrized by S, and $\chi : S \to \mathcal{C}_n^{\mathrm{loc}}(Z)$ the associated mapping. Let s_0 be a point of S, and $\mathcal{E}$ be a finite family of scales $(E_i)_{i\in I}$ adapted to X_{s_0} — with, say, $E_i = (U_i, B_i, j_i)$, and $k_i := \deg_{E_i} X_{s_0}$, $U_i' \subset\subset U_i$, W_i the domain of E_i, and $S_{E_i} := \chi^{-1}(\Omega^{k_i}(E_i))$. Set

$$\Omega_{\mathcal{E}} := \bigcap_{i\in I} \Omega^{k_i}(E_i), \quad S_{\mathcal{E}} := \bigcap_{i\in I} S_{E_i} = \chi^{-1}(\Omega_{\mathcal{E}}), \quad W_{\mathcal{E}} := \bigcup_{i\in I} W_i\,;$$

we call $W_{\mathcal{E}}$ the domain of the family $\mathcal{E}$, and we shall say that the family of scales $\mathcal{E}$ *covers* a subset A of Z if $A \subset W_{\mathcal{E}}$.

When the family of cycles is moreover analytic, we have an analytic map

$$f_{\mathcal{E}} := \prod_i f_{E_i|S_{\mathcal{E}}} : S_{\mathcal{E}} \longrightarrow \prod_i H(\overline{U}_i', \operatorname{sym}^{k_i} B_i).$$

1.2.3. *Comparison of the topologies of $\mathcal{C}_n^{\mathrm{loc}}(Z)$ and $H(\overline{U}, \operatorname{sym}^k B)$.* We provide $H(\overline{U}, \operatorname{sym}^k B)$ with the topology induced by the following metric:

$$D(X,Y) := \sup_{t\in\overline{U}} \Big(\inf_{\sigma\in\mathfrak{S}_k} \sum_{i=1}^{k} \big|x_i(t) - y_{\sigma(i)}(t)\big| \Big),$$

where X and Y belong to $H(\overline{U}, \operatorname{sym}^k B)$, with $X(t) := [x_1(t), \ldots, x_k(t)]$ and $Y(t) := [y_1(t), \ldots, y_k(t)]$ for t in $\overline{U}$, the points $x_i(t)$ and $y_i(t)$ lying in B.

The major result is the following one (see [Ba75], Theorem 2):

PROPOSITION 2. — *Let $E := (U, B, j)$ be a scale on Z, let X_0 be a cycle belonging to $\Omega^k(E)$ (so to $H(\overline{U}, \operatorname{sym}^k B)$). Let $F := (V, C, h)$ be a scale on $U \times B$, such that X_0 belongs to $\Omega^\ell(F)$. Thus there exists a neighbourhood $\mathcal{V}$ of X_0 in $H(\overline{U}, \operatorname{sym}^k B)$ which is included in $\Omega^\ell(F)$.*

This proposition implies that the map

$$H(\overline{U}, \operatorname{sym}^k B) \longrightarrow \mathcal{C}_n^{\mathrm{loc}}(U \times B),$$

which maps a ramified cover X_E over $\overline{U}$ to the cycle $X \cap (U \times B)$ of $U \times B$, is continuous.

In order to generalize this proposition to global cycles of Z, we denote

- by $\mathcal{C}_n^{\mathrm{loc}}(W, Z)$ the set of cycles of an open set W of Z which are intersections with W of cycles of Z, provided with the topology of $\mathcal{C}_n^{\mathrm{loc}}(W)$.
- by $\prod_{i \in I}^{\mathcal{C}} H(\overline{U}_i, \operatorname{sym}^{k_i} B_i)$ the set of I-uples of ramified covers associated with the same (global) cycle of Z.

Now, we can state:

PROPOSITION 3. — *Let $\mathcal{E}$ be a finite family of scales, with the above notations. The following map is continuous:*

$$\begin{aligned}\prod\nolimits_{i \in I}^{\mathcal{C}} H(\overline{U}_i, \operatorname{sym}^{k_i} B_i) &\longrightarrow \mathcal{C}_n^{\mathrm{loc}}(W_{\mathcal{E}}, Z),\\ X &\longmapsto X \cap W_{\mathcal{E}}.\end{aligned}$$

The proof follows easily from the previous proposition. It is a bit technical, but no more complicated, to prove this last proposition:

PROPOSITION 4. — *Let $E = (U, B, j)$ be a scale. The following map is continuous:*

$$\begin{aligned}(\mathcal{C}_n^{\mathrm{loc}}(Z) \supset) \quad \Omega^k(E) &\longrightarrow H(\overline{U}, \operatorname{sym}^k B),\\ X &\longmapsto X_E.\end{aligned}$$

1.2.4. *Regular analytic families of cycles.* — In Theorem 3 below, we consider the quotient S/R_χ, where $\chi : S \to \mathcal{C}_n^{\mathrm{loc}}(Z)$ is the map associated with an analytic family of n-cycles $(X_s)_{s \in S}$. We work with a finite family $\mathcal{E}$ of scales, and want to deduce the analyticity of the 'global' quotient S/R_χ from the analyticity of the 'local' quotient $S_{\mathcal{E}}/R_{f_{\mathcal{E}}}$. To be sure that the latter quotient is an open set of the former, or, as we explain in the Introduction, to be sure that the local behaviour (on the domain $W_{\mathcal{E}}$ of $\mathcal{E}$) of $(X_s)_{s \in S}$ determines its global behaviour, we need to put some conditions on the analytic family of cycles.

First of all, we say that a finite family of scales $\mathcal{E}$ *well determines* the continuous family of cycles $(X_s)_{s\in S}$ around X_{s_0} if the map

$$\begin{aligned}(\mathcal{C}_n^{\mathrm{loc}}(Z)\supset)\quad \chi(S_{\mathcal{E}}) &\longrightarrow \mathcal{C}_n^{\mathrm{loc}}(W_{\mathcal{E}},Z),\\ X_s &\longmapsto X_s\cap W_{\mathcal{E}}\end{aligned}$$

is injective; that is to say, two cycles X_s and $X_{s'}$, for $s,\ s'\in S_{\mathcal{E}}$, coincide as soon as their intersections with $W_{\mathcal{E}}$ coincide.

Such a phenomenon can occur: set $D_a := \{z\in\mathbb{C},\ \text{such that } |z|<a\}$; then $(X_s := \{z\in D_1,\ \text{such that } z(z-1-s)=0\})_{s\in D_{1/2}}$ is an analytic family of 0-cycles of D_1 parametrized by $D_{1/2}$; X_s equals either $\{0\}+\{1+s\}$ or $\{0\}$; for $\nu\in\mathbb{N}^*$, set $s_\nu=-1/\nu$, then, for any finite family $\mathcal{E}$ of scales adapted to $X_0=\{0\}$ (its domain $W_{\mathcal{E}}$ is then a relatively compact open set of D_1), and for ν large enough, $X_{s_\nu}\cap W_{\mathcal{E}}=X_0\cap W_{\mathcal{E}}=\{0\}$ but $X_{s_\nu}\neq X_0$: thus the family is not well determined by $\mathcal{E}$ around X_0. The pathology comes from the fact that an irreducible component of X_{s_ν}, namely $\{1-1/\nu\}$, converges to the point 1 lying in the boundary ∂D_1.

Precisely: a subset $\mathfrak{X}$ of $\mathcal{C}_n^{\mathrm{loc}}(Z)$ *escapes to infinity* in a cycle $X_0\in\mathfrak{X}$ if there exist a sequence $(X_j)_{j\in\mathbb{N}}$ of cycles in $\mathfrak{X}$ and irreducible components Γ_j of $|X_j|$ such that

- $(X_j)_{j\in\mathbb{N}}$ converges to X_0 in $\mathcal{C}_n^{\mathrm{loc}}(Z)$,
- for every compact set L of Z, we have $\Gamma_j\cap L=\emptyset$ for j large enough.

We say that a continuous family $(X_s)_{s\in S}$ of n-cycles of Z escapes to infinity in X_{s_0} if $\chi(S)$ does.

We give now a property equivalent to *non*-escape to infinity:

PROPOSITION 5. — *Let $\mathfrak{X}$ be a subset of $\mathcal{C}_n^{\mathrm{loc}}(Z)$, and set $X_0\in\mathfrak{X}$. The following properties are equivalent:*

(i) *$\mathfrak{X}$ does not escape to infinity in X_0;*

(ii) *there exist a compact set L_0 of Z and a neighbourhood $\mathcal{V}_0$ of X_0 in $\mathcal{C}_n^{\mathrm{loc}}(Z)$ such that every irreducible component of every cycle in $\mathcal{V}_0\cap\mathfrak{X}$ intersects L_0.*

Furthermore, when these conditions are satisfied for the subset $\chi(S)$ associated with an analytic family of n-cycles $(X_s)_{s\in S}$, there exists a finite family of scales adapted to X_{s_0} which well determines $(X_s)_{s\in S}$ around X_{s_0}.

Proof. — The equivalence is quite clear. To prove the last statement, it suffices to cover L_0 with a finite family of scales adapted to X_{s_0}. □

Besides, when a family of cycles $(X_s)_{s\in S}$ doesn't escape to infinity in any cycle, the projection $\pi : G \to S$ of its graph onto S is quasi-proper: for $s_0 \in S$, set L_0 and $\mathcal{V}_0$ given by property (ii), let S_0 be a neighbourhood of s_0, relatively compact in the open set $\chi^{-1}(\mathcal{V}_0)$; then it is easy to check that the condition of Definition 2 holds with S_0 and the compact set $(\overline{S}_0 \times L_0) \cap G$ of G.

Consider now the 1-cycles of $\mathbb{C}^2$ defined by

$$X_\nu := \{(x,y) \in \mathbb{C}^2, \text{ s.t. } y = x^\nu\},$$

with $\nu \in \mathbb{N}^*$, and $X_0 := \{y = 0\}$; the scale $E := \{|x| < \frac{1}{2},\ |y| < \frac{3}{4}\}$ is adapted to every cycle X_ν and well determines the set of cycles $(X_\nu)_{\nu\in\mathbb{N}}$. However, it is clear that $(X_\nu \cap W_E)_\nu$ converges to $X_0 \cap W_E$ in $\mathcal{C}_n^{\mathrm{loc}}(W_E, Z)$ but $(X_\nu)_\nu$ does *not* converge to X_0 in $\mathcal{C}_n^{\mathrm{loc}}(Z)$.

To avoid also this kind of pathology, we give a last definition, which includes non-escape to infinity (that is, a set-theoretic control) plus a *topological* control (which will enable us to deduce the semi-properness of a map $f_{\mathcal{E}}$ from the semi-properness of χ).

DEFINITION 3. — *The subset $\mathfrak{X}$ of $\mathcal{C}_n^{\mathrm{loc}}(Z)$ is said to be regular if the following condition holds for every $X_0 \in \mathfrak{X}$: there exist a compact set L_0 of Z and a neighbourhood $\mathcal{V}_0$ of X_0 in $\mathcal{C}_n^{\mathrm{loc}}(Z)$ such that*

- *every irreducible component of every cycle of $\mathcal{V}_0 \cap \mathfrak{X}$ intersects L_0;*

- *$\mathfrak{X}$ does not topologically escape to infinity in X_0, that is: there exists a relatively compact neighbourhood W_0 of L_0 in Z, such that, if $(X_\nu)_{\nu\in\mathbb{N}}$ and X are cycles in $\mathcal{V}_0 \cap \mathfrak{X}$ such that $(X_\nu \cap W_0)_{\nu\in\mathbb{N}}$ converges to $X \cap W_0$ in $\mathcal{C}_n^{\mathrm{loc}}(W_0, Z)$, then $(X_\nu)_{\nu\in\mathbb{N}}$ converges to X in $\mathcal{C}_n^{\mathrm{loc}}(Z)$.*

A continuous family $(X_s)_{s\in S}$ of n-cycles is regular if $\chi(S)$ is.

Remark. — If $\mathfrak{X}$ is a subset of $\mathcal{C}_n^{\mathrm{loc}}(Z)$, we define its graph in $\mathcal{C}_n^{\mathrm{loc}}(Z) \times Z$ as follows:

$$\mathcal{G} := \{(X, z) \text{ s.t. } z \in |X|\} \subset \mathfrak{X} \times Z \subset \mathcal{C}_n^{\mathrm{loc}}(Z) \times Z.$$

The study of this graph yields equivalent conditions to set-theoretic and topological non-escape to infinity:

LEMMA 3. — *The following assertions are equivalent:*

(i) *$\rho : \mathcal{G} \twoheadrightarrow \mathfrak{X}$ is quasi-proper;*

(ii) *$\mathfrak{X}$ does not escape to infinity and is locally compact.*

Assume that $\mathfrak{X}$ does not escape to infinity and is locally compact. For a given $X_0 \in \mathfrak{X}$, set L_0, $\mathcal{V}_0$ as in Proposition 5; let W_0 be a relatively compact neighbourhood of L_0 in Z.

The map $h : \mathcal{V}_0 \cap \mathfrak{X} \to \mathcal{C}_n^{\mathrm{loc}}(W_0, Z)$, $X \mapsto X \cap W_0$, is continuous and injective. Set $\mathfrak{Y} := h(\mathcal{V}_0 \cap \mathfrak{X})$; thus, $\mathfrak{Y}$ is a subset of $\mathcal{C}_n^{\mathrm{loc}}(W_0, Z)$, which is locally compact and does not escape to infinity. Let $\mathcal{H}$ denote the graph of $\mathfrak{Y}$ in $\mathcal{C}_n^{\mathrm{loc}}(W_0) \times W_0$; set

$$\mathcal{G}_{W_0} := \mathcal{G} \cap (\mathfrak{X} \times W_0) \quad \text{and} \quad H : \mathcal{G}_{W_0} \to \mathcal{H},\ (X, z) \mapsto (X \cap W_0, z).$$

LEMMA 4. — *With these notations, the following assertions are equivalent:*

(i) *$\mathfrak{X}$ does not topologically escape to infinity in X_0;*

(ii) *h is a homeomorphism;*

(iii) *H is semi-proper.*

2. Regular semi-proper analytic families of cycles.

From now on, by 'cycle' we shall always mean a n-cycle of a reduced, finite dimensional analytic space Z.

2.1. Analyticity of the quotient.

If $(X_s)_{s \in S}$ is a family of cycles, we define an equivalence relation in S as follows:

$$s \sim s' \iff X_s = X_{s'}.$$

In fact, this relation is exactly the equivalence relation R_χ defined by the map $\chi : S \to \mathcal{C}_n^{\mathrm{loc}}(Z)$ associated with $(X_s)_{s \in S}$.

If the family $(X_s)_{s \in S}$ is continuous, the quotient S/R_χ is a Hausdorff topological space. Moreover, we have a canonical ringed structure $\mathcal{O}_{S/R_\chi}$ on S/R_χ.

This relation is not necessarily an *analytic equivalence relation* in the sense of [Gr83]; nevertheless, we have:

LEMMA 5. — *Let $(X_s)_{s \in S}$ be an analytic family of cycles, such that, for every $s_0 \in S$, there exists a finite family $\mathcal{E}$ of scales well determining the family of cycles around X_{s_0}. Then the equivalence classes of the relation defined by $(X_s)_{s \in S}$ locally coincide with the fibers of the analytic* *map* $f_{\mathcal{E}} : S_{\mathcal{E}} \to \prod_i^{\mathcal{C}} H(\overline{U}'_i, \mathrm{sym}^{k_i} B_i)$. *Especially, these classes are analytic subsets of* S.

Proof. — Set s, $s' \in S_{\mathcal{E}}$. First, $f_{\mathcal{E}}(s) = f_{\mathcal{E}}(s')$ implies the equality of the ramified covers defined by X_s and $X_{s'}$ over $\overline{U}'_i$, so over $\overline{U}_i$ too, and this, for all i, thus $X_s \cap W_{\mathcal{E}} = X_{s'} \cap W_{\mathcal{E}}$; finally, we obtain $X_s = X_{s'}$ since the family of scales $\mathcal{E}$ well determines $(X_s)_{s \in S}$ around X_{s_0}; conversely, $X_s = X_{s'}$ clearly implies that $f_{\mathcal{E}}(s) = f_{\mathcal{E}}(s')$. □

We can state now the following theorem:

THEOREM 3. — *Let Z and S be finite dimensional analytic spaces, S being weakly normal. Let R_χ be the equivalence relation defined by an analytic family $(X_s)_{s \in S}$ of n-cycles of Z. Assume that*

$$\text{(H)} \quad \begin{cases} \text{for every point } s_0 \text{ of } S, \text{ there exists a finite family } \mathcal{E} \text{ of scales which} \\ \text{well determines the family of cycles around } X_{s_0}, \text{ and such that the} \\ \text{analytic map } f_{\mathcal{E}} = \prod_i f_{E_i|S_{\mathcal{E}}} : S_{\mathcal{E}} \twoheadrightarrow f_{\mathcal{E}}(S_{\mathcal{E}}) \subset \prod_i H(\overline{U}'_i, \mathrm{sym}^{k_i} B_i) \\ \text{is semi-proper onto its image.} \end{cases}$$

Then the ringed quotient space $(S/R_\chi, \mathcal{O}_{S/R_\chi})$ is a weakly normal, finite dimensional analytic space.

Proof. — Let $(X_s)_s$ be an analytic family of n-cycles satisfying (H).

Let $q : S \twoheadrightarrow S/R_\chi$ be the canonical surjection. Let $t_0 := q(s_0)$ in S/R_χ. For the family of scales $\mathcal{E}$ given by (H), the map

$$f_{\mathcal{E}} : S_{\mathcal{E}} \longrightarrow \prod_i H(\overline{U}'_i, \mathrm{sym}^{k_i} B_i)$$

is analytic and semi-proper onto its image.

Now, since $\prod_i H(\overline{U}'_i, \mathrm{sym}^{k_i} B_i)$ is included in $\prod_i H(\overline{U}'_i, \oplus_h S_h(\mathbb{C}^p))$, with the induced topology, the map

$$f_{\mathcal{E}} : S_{\mathcal{E}} \longrightarrow \prod_i H(\overline{U}'_i, \oplus_h S_h(\mathbb{C}^p))$$

is still semi-proper onto its image.

By Lemma 1, (i), there exists an open set ω of

$$\prod_i H\Big(\overline{U}'_i, \bigoplus_{h=1,\dots,k} S_h(\mathbb{C}^p)\Big)$$

such that the map $f_{\mathcal{E}} : S_{\mathcal{E}} \to \omega$ is semi-proper: furthermore, this map is defined in a weakly normal space, with values in an open set of a Banach space, thus we can apply Theorem 2.

It follows that the quotient $S_{\mathcal{E}}/R_{f_{\mathcal{E}}}$ is an analytic space (of finite dimension), which topologically coincide with $q(S_{\mathcal{E}})$ (by Lemma 5); the latter is a neighbourhood of t_0 in S/R_{χ}, and we have an isomorphism of ringed spaces: $\mathcal{O}_{S_{\mathcal{E}}/R_{f_{\mathcal{E}}}} = \mathcal{O}_{S/R_{\chi}} | \, q(S_{\mathcal{E}})$.

Therefore, the ringed space $(S/R_{\chi}, \mathcal{O}_{S/R_{\chi}})$ is analytic in a neighbourhood of any of its points: so it is an analytic space, which is moreover weakly normal, as the quotient of a weakly normal space (see [KK83], 72.4). □

2.2. Semi-properness of χ and semi-properness of $f_{\mathcal{E}}$.

DEFINITION 4. — *An analytic family of cycles* $(X_s)_{s \in S}$ *is said to be semi-proper if the map* $\chi : S \twoheadrightarrow \chi(S)$ *is semi-proper.*

PROPOSITION 6. — *The condition* (H) *holds in particular when the analytic family of cycles is regular and semi-proper.*

Proof. — Let X_{s_0} be a cycle in $\chi(S)$. Let L_0, $\mathcal{V}_0$, and W_0 be respectively the compact set of Z, the neighbourhood of X_{s_0} in $\mathcal{C}_n^{\mathrm{loc}}(Z)$, and the relatively compact neighbourhood W_0 of L_0 in Z given by the regularity of the family in X_{s_0} (see Definition 3).

Let us consider now a finite family $\mathcal{E}$ of scales $(E_i = (U_i, B_i, j_i))_{i \in I}$ adapted to X_{s_0} and such that the family $\mathcal{E}'$ of scales $(E_i{}' = (U_i', B_i, j_i))_{i \in I}$ covers W_0: so $L_0 \subset W_0 \subset W_{\mathcal{E}'} \subset W_{\mathcal{E}}$; we note

$$\Omega_{\mathcal{E}} := \bigcap_i \Omega^{k_i}(E_i),$$

where $k_i := \deg_{E_i} X_{s_0}$, and $S_{\mathcal{E}} := \chi^{-1}(\Omega_{\mathcal{E}})$. The open set $\Omega_{\mathcal{E}}$ is a neighbourhood of X_{s_0} in $\mathcal{C}_n^{\mathrm{loc}}(Z)$; if we add a finite number of scales (*i.e.* we restrict $\Omega_{\mathcal{E}}$), we can moreover assume that $\Omega_{\mathcal{E}}$ is included in $\mathcal{V}_0$.

It is then clear that this family of scales $\mathcal{E}$ well determines the family of cycles around X_{s_0}.

Since χ is semi-proper, by Lemma 1, (ii), the restriction

$$\chi_{|S_{\mathcal{E}}} : S_{\mathcal{E}} \longrightarrow\!\!\!\!\!\rightarrow \chi(S_{\mathcal{E}})$$

is semi-proper. In order to prove the semi-properness of the map $f_{\mathcal{E}} : S_{\mathcal{E}} \twoheadrightarrow f_{\mathcal{E}}(S_{\mathcal{E}})$, we shall prove that in the commutative diagram

$$\begin{tikzcd} S_{\mathcal{E}} \arrow[r, two heads, "\chi|S_{\mathcal{E}}"] \arrow[dr, two heads] & \chi(S_{\mathcal{E}}) \subset \mathcal{C}_n^{\mathrm{loc}}(Z) \arrow[d, "h"] \\ & f_{\mathcal{E}}(S_{\mathcal{E}}) \subset \prod_i^{\mathcal{C}} H(\bar{U}'_i, \mathrm{sym}^{k_i} B_i) \end{tikzcd}$$

the mapping $h : \chi(S_{\mathcal{E}}) \to f_{\mathcal{E}}(S_{\mathcal{E}})$ is a homeomorphism. The semi-properness of $f_{\mathcal{E}}$ will then follow from Lemma 1, (iii).

Let us show that h is a homeomorphism:

• $h : \chi(S_{\mathcal{E}}) \to f_{\mathcal{E}}(S_{\mathcal{E}})$ is a bijection: it is clear that h is surjective. Furthermore, if $h(X_s) = h(X_{s'})$, that is, if X_s and $X_{s'}$ coincide in $\prod_i H(\bar{U}'_i, \mathrm{sym}^{k_i} B_i)$, then X_s and $X_{s'}$ coincide on the domain of $\mathcal{E}$ (see the proof of Lemma 5). Since $\mathcal{E}$ well determines the family of scales around X_{s_0}, X_s and $X_{s'}$ coincide in $\mathcal{C}_n^{\mathrm{loc}}(Z)$ and h is injective.

• h is continuous by Proposition 4.

• h^{-1} is continuous: let $(s_\nu)_{\nu\in\mathbb{N}}$ and s be points in $S_{\mathcal{E}}$, such that $(f_{\mathcal{E}}(s_\nu))_{\nu\in\mathbb{N}}$ converges to $f_{\mathcal{E}}(s)$ in $f_{\mathcal{E}}(S_{\mathcal{E}}) \subset \prod_i^{\mathcal{C}} H(\bar{U}'_i, \mathrm{sym}^{k_i} B_i)$. According to Proposition 3 (applied to the family $\mathcal{E}'$), $(X_{s_\nu} \cap W_{\mathcal{E}'})_{\nu\in\mathbb{N}}$ converges to $X_s \cap W_{\mathcal{E}'}$ in $\mathcal{C}_n^{\mathrm{loc}}(W_{\mathcal{E}'}, Z)$. Since the family $(X_s)_{s\in S}$ is regular, with $W_0 \subset W_{\mathcal{E}'}$, it follows that $(X_{s_\nu})_{\nu\in\mathbb{N}}$ converges to X_s in $\mathcal{C}_n^{\mathrm{loc}}(Z)$, so in $\chi(S_{\mathcal{E}})$ — note that the non-topological escape is crucial here. □

2.3. Universal reparametrization.

We give now another version of Theorem 3. This version exhibits a universal property of our construction, and, contrary to Theorem 3, will be generalizable to the meromorphic case. This is the result announced in our Introduction:

THEOREM 4. — *Let $(X_s)_{s\in S}$ be a regular semi-proper analytic family of n-cycles of Z parametrized by a weakly normal space S. Then, there exist a unique weakly normal, finite dimensional analytic space Q, homeomorphic to $\chi(S)$, and an analytic family of n-cycles $(\widetilde{X}_c)_{c\in Q}$, satisfying:*

For every weakly normal analytic space T, and every analytic family of n-cycles $(Y_t)_{t\in T}$ such that

$$\forall t \in T,\ \exists\, s \in S, \text{ such that } Y_t = X_s,$$

there exists a unique analytic map $\varphi : T \to Q$, such that the family $(Y_t)_{t\in T}$ is the pull-back by φ of the family $(\widetilde{X}_c)_{c\in Q}$, that is to say : for all $t \in T$, $Y_t = \widetilde{X}_{\varphi(t)}$.

Proof. — Let $Q := (S/R_\chi, \mathcal{O}_{S/R_\chi})$ be the weakly normal analytic space given by Theorem 3 (that we can use thanks to Proposition 6), and for $c = q(s) \in Q$, set $\widetilde{X}_c = X_s$ (this is independent of the choice of s). It is easy to verify that this family is analytic.

Moreover, if T and $(Y_t)_{t\in T}$ satisfy the given condition, let us define:

$$\begin{aligned} \varphi : T &\longrightarrow Q, \\ t &\longmapsto q(s), \text{ if } s \text{ is such that } Y_t = X_s \, ; \end{aligned}$$

then the desired properties hold for φ. Let us only check that φ is analytic: let $\mathcal{E} := (E_i)_i$ be a finite family of scales adapted to $X_{s_0} = Y_{t_0}$, chosen as in the proof of Theorem 3; set as usually $\Omega_\mathcal{E} := \bigcap_i \Omega^{\deg_{E_i} X_{s_0}}(E_i)$ and $S_\mathcal{E} := \chi^{-1}(\Omega_\mathcal{E})$.

Set also $T_\mathcal{E} := \xi^{-1}(\Omega_\mathcal{E})$, where $\xi : T \to \mathcal{C}_n^{\mathrm{loc}}(Z)$ is the map associated with $(Y_t)_{t\in T}$. Since this family is analytic, the map

$$g_\mathcal{E} : T_\mathcal{E} \longrightarrow \prod_i^{\mathcal{C}} H(\overline{U}_i', \mathrm{sym}^{k_i} B_i)$$

is analytic.

$$\begin{tikzcd}
 & Q & \mathcal{C}_n^{\mathrm{loc}}(Z) & \prod_i^{\mathcal{C}} H(\overline{U}_i', \mathrm{sym}^{k_i} B_i) \\
S_\mathcal{E} \arrow[r, two heads, "q"] \arrow[rrr, bend left, "f_\mathcal{E}"] & q(S_\mathcal{E}) \arrow[u, hook] \arrow[r, two heads, "\tilde{\chi}"] & \chi(S_\mathcal{E}) = \Omega_\mathcal{E} \cap \chi(S) \arrow[u, hook] \arrow[r, two heads, "h"] & f_\mathcal{E}(S_\mathcal{E}) \arrow[u, hook] \\
 & T_\mathcal{E} \arrow[u, dashed, "\varphi"] \arrow[r, two heads, "\xi"] \arrow[rr, bend right, "g_\mathcal{E}"'] & \xi(T_\mathcal{E}) = \Omega_\mathcal{E} \cap \xi(T) \arrow[u, hook] \arrow[r, two heads] & g_\mathcal{E}(T_\mathcal{E}) \arrow[u, hook]
\end{tikzcd}$$

Note that $h \circ \widetilde{\chi} : q(S_\mathcal{E}) \twoheadrightarrow f_\mathcal{E}(S_\mathcal{E})$ is the weak normalization of $f_\mathcal{E}(S_\mathcal{E})$. Now the map $g_\mathcal{E} : T_\mathcal{E} \twoheadrightarrow g_\mathcal{E}(T_\mathcal{E}) \subset f_\mathcal{E}(S_\mathcal{E})$ can be lifted to the weak normalizations (see [Fi76], 2.30); since $T_\mathcal{E}$ is already weakly normal, the analytic map thereby defined is nothing else than $\varphi : T_\mathcal{E} \to q(S_\mathcal{E})$. □

Remark. — We explain in [Mt99] how to use Theorem 3 in the study of analytic equivalence relations $R \subset X \times X$ in the sense of Grauert. Any scale adapted to X_0 obviously well determines the family of cycles around X_0 since these cycles are equivalence classes; then, the semi-properness of f_E is also clear when $S = Z = X$: all cycles adapted to E intersect the closure of the domain W_E of the scale.

Of course, we can only prove a particular case of Grauert's result (Theorem 6 of [Gr83]), precisely the case when all equivalence classes are of constant pure dimension (we build them as n-cycles!). On the other hand, the gain of our point of view is that it enables us to distinguish an ambient space Z and a parameter space S.

By the way, we make the following informal comment: our condition of semi-properness (on the family of cycles) is similar to the one of Grauert (semi-properness of the quotient map). But, to this semi-properness w.r.t. S, we must add a condition of 'quasi-properness w.r.t. Z', namely, the non-escape to infinity.

3. Meromorphic families of cycles.

Theorem 4 provides with an analytic structure the subset of $\mathcal{C}_n^{\mathrm{loc}}(Z)$ described by an analytic family of cycles; to generalize it to wider subsets of $\mathcal{C}_n^{\mathrm{loc}}(Z)$, we will introduce here the notion of *meromorphic family of n-cycles*; we also study this notion for itself.

3.1. Definition.

The definition of a meromorphic family of n-cycles is similar to Remmert's definition of a meromorphic map (see [Re57]): a generically defined and analytic map $\Phi : X \dashrightarrow Y$ is a meromorphic map, if there exist a proper modification $\sigma : \widetilde{X} \twoheadrightarrow X$ and an analytic map $\varphi : \widetilde{X} \to Y$ such that φ generically coincides with Φ. One can described $\widetilde{X}$ as an analytic subset of $X \times Y$: it is precisely the closure of the graph of the generic map $X \to Y$. This set is called the *graph*, $\mathrm{Gr}(\Phi)$, of the map.

Similarly, a meromorphic family of cycles parametrized by a weakly normal space S, is an analytic family parametrized by a dense Zariski open set of S, such that there exists a proper modification $\sigma : \widetilde{S} \twoheadrightarrow S$, such that $\widetilde{S}$ parametrizes an analytic family of cycles, generically equal to the previous one. We shall always associate a *graph in* $S \times \mathcal{C}_n^{\mathrm{loc}}(Z)$ with this family, but we will be able to provide this graph with an analytic structure only with further conditions.

In order to express that definition in an way easier to handle, we shall use a *graph in* $S \times Z$.

Let S and Z be analytic spaces (S weakly normal), G be an analytic subset of $S \times Z$. We note N and m the respective dimensions of G and S, π and p the restrictions to G of the projections on S and on Z; since $\pi^{-1}(s) = \{s\} \times p(\pi^{-1}(s))$, $p(\pi^{-1}(s))$ is an analytic subset of Z, and $\operatorname{gcork}_{(s,z)} \pi = \dim_z p(\pi^{-1}(s))$.

Such a map π is said to be *geometrically flat* (see [Ba78]) if G is the graph

$$\{(s,z) \text{ such that } z \in |X_s|\} \subset S \times Z$$

of an analytic family of n-cycles $(X_s)_{s\in S}$, or, equivalently, if the irreducible components X_s^i of the analytic subsets $p(\pi^{-1}(s))$ can be provided with multiplicities m_s^i such that the family of cycles $(X_s := (X_s^i, m_s^i))_{s\in S}$ is analytic.

We shall say that the map $\pi : G \to S$ *defines an analytic family of n-cycles* if there exists a union G_H of irreducible components of G, such that the restriction $\pi_{|G_H} : G_H \to S$ is geometrically flat.

This situation presents no problem since it suffices to 'forget' some irreducible components in order to get a geometrically flat morphism. This is no longer the case in the following example (however elementary): $\pi : G \twoheadrightarrow S$ is the blow-up of the origin in $S := \mathbb{C}^2$: in that case, the fibre $\pi^{-1}(0) = \{0_{\mathbb{C}^2}\} \times \mathbb{P}_1(\mathbb{C})$ contains an infinity of 'limit' cycles, which depend on the sequence $(s_n)_{n\in\mathbb{N}}$ converging to 0.

We give now the following definition:

Definition 5. — *Let $\mathcal{M}(\pi)$ denote the set of all couples $(\widetilde{S}, \sigma)$, where $\widetilde{S}$ is a weakly normal space and $\sigma : \widetilde{S} \twoheadrightarrow S$ a proper modification such that the strict transform of π defines an analytic family of n-cycles.*

We say that the map $\pi : G \to S$ defines a meromorphic family of n-cycles of Z if $\mathcal{M}(\pi)$ is non-empty.

Before studying this definition — and especially constructing this meromorphic family—, we describe a basic construction and recall some elementary facts about proper modifications and strict transforms.

3.1.1. *A basic construction.* — Let $\pi : G \to S$ be a map (with notations as above) and A be an analytic subset of S, such that the restriction of π to

$G \setminus \pi^{-1}(A) = \pi^{-1}(S \setminus A)$ onto $S \setminus A$ is geometrically flat: that is to say, we have an analytic family of n-cycles $(X_s)_{s\in S\setminus A}$ of cycles of Z parametrized by $S \setminus A$, with supports $|X_s| := p(\pi^{-1}(s))$.

We show here how to build a continuous family of cycles from this one.

Let $\chi : S \setminus A \to \mathcal{C}_n^{\mathrm{loc}}(Z)$ be the map associated with the family $(X_s)_{s\in S\setminus A}$. This map has a graph

$$\Gamma := \big\{(s, X_s),\ s \in S \setminus A\big\} \subset S \times \mathcal{C}_n^{\mathrm{loc}}(Z).$$

Let $\overline{\Gamma}$ be the closure of Γ in $S \times \mathcal{C}_n^{\mathrm{loc}}(Z)$, p_S and $p_{\mathcal{C}}$ be the restrictions to $\overline{\Gamma}$ of the two projections, and

$$\mathfrak{X} := p_{\mathcal{C}}(\overline{\Gamma}).$$

We call $\mathfrak{X}$ the *set of cycles defined by* π. Note that these are not the limits of the cycles $(X_s)_{s\in S\setminus A}$: we take the limits of the *couples* $(s, X_s)_{s\in S\setminus A}$.

Since the map $p_{\mathcal{C}} : \overline{\Gamma} \twoheadrightarrow \mathfrak{X} \subset \mathcal{C}_n^{\mathrm{loc}}(Z)$, $(s, X) \mapsto X$ is continuous, $(X)_{(s,X)\in\overline{\Gamma}}$ is a continuous family of cycles in $\mathcal{C}_n^{\mathrm{loc}}(Z)$ parametrized by $\overline{\Gamma}$ and going all over $\mathfrak{X}$.

$$\begin{tikzcd}
S \times Z \supset G \arrow[r, "p"] \arrow[d, two heads, "\pi"] & Z & \\
S \supset S \setminus A \arrow[rr, two heads, "\chi"] & & \chi(S \setminus A) \arrow[d, hook] \\
S \times \mathcal{C}_n^{\mathrm{loc}}(Z) \supset \overline{\Gamma} \arrow[u, "p_S"] \arrow[rr, two heads, "p_{\mathcal{C}}"] & & \mathfrak{X} \subset \mathcal{C}_n^{\mathrm{loc}}(Z)
\end{tikzcd}$$

We eventually check that

LEMMA 6. — *If* (s, X) *belongs to* $\overline{\Gamma}$, *then* $|X|$ *is included in* $p(\pi^{-1}(s))$.

3.1.2. *Proper modifications and strict transforms.* — First, we note that if S is weakly normal and $\sigma : \widetilde{S} \twoheadrightarrow S$ is a proper modification, then we can assume, as we shall always do afterwards, that $\widetilde{S}$ is weakly normal too: it suffices to compose σ with the weak normalization of $\widetilde{S}$, which is a proper modification. Let

- G be an analytic subset of $S \times Z$;

- Λ be a nowhere dense analytic subset of S;
- $\sigma : \widetilde{S} \twoheadrightarrow S$ be the proper modification of S with center Λ (so, $\widetilde{\Lambda} := \sigma^{-1}(\Lambda)$ is a nowhere dense analytic subset of $\widetilde{S}$);
- $\widetilde{G}$ be the strict transform of G by σ, namely, the smallest analytic subset of $\widetilde{S} \times Z$ containing $(\sigma \times \mathrm{Id}_Z)^{-1}(\pi^{-1}(S \setminus \Lambda))$.

We note $\widetilde{\pi}$, $\tau := \sigma \times \mathrm{Id}_Z$ and $\widetilde{p} := p \circ \tau$ the projections of $\widetilde{G}$ on $\widetilde{S}$, G and Z respectively.

Then we have

A) τ is proper;

B) for all $\tilde{s} \in \widetilde{S}$, $\widetilde{p}\big(\widetilde{\pi}^{-1}(\tilde{s})\big) \subset p\big(\pi^{-1}(\sigma(\tilde{s}))\big)$ (hence, $\mathrm{gcork}_{(\tilde{s},z)}\, \widetilde{\pi} \leq \mathrm{gcork}_{(\sigma(\tilde{s}),z)}\, \pi$), with equality for $\tilde{s} \notin \widetilde{\Lambda}$.

We come back now to meromorphic families of cycles.

3.1.3. *Meromorphic families of cycles.* — Let $\pi : G \to S$ be a map defining a meromorphic family of n-cycles: thus $\mathcal{M}(\pi) \neq \emptyset$, and let $\sigma : \widetilde{S} \twoheadrightarrow S$ be the proper modification of S with center Λ, and $(\widetilde{X}_{\tilde{s}})_{\tilde{s} \in \widetilde{S}}$ the analytic family of n-cycles given by Definition 5. Let $\widetilde{\chi} : \widetilde{S} \to \mathcal{C}_n^{\mathrm{loc}}(Z)$ be the map associated with $(\widetilde{X}_{\tilde{s}})_{\tilde{s}}$.

According to the point B) above, we have

$$|\widetilde{X}_{\tilde{s}}| = \widetilde{p}\big(\widetilde{\pi}^{-1}(\tilde{s})\big) = p\big(\pi^{-1}(\sigma(\tilde{s}))\big), \quad \text{if } \tilde{s} \notin \widetilde{\Lambda}.$$

and if we note, for every $s \in S \setminus \Lambda$

$$X_s := \widetilde{X}_{\tilde{s}}\,,$$

where $\tilde{s}$ is the single pre-image of s, it is easy to verify, using the isomorphism $\sigma_{|\widetilde{S} \setminus \widetilde{\Lambda}} : \widetilde{S} \setminus \widetilde{\Lambda} \to S \setminus \Lambda$, that $(X_s)_{s \in S \setminus \Lambda}$ is an analytic family of n-cycles of Z parametrized by $S \setminus \Lambda$.

Thus, we can make the construction described above; let $\chi : S \setminus \Lambda \to \mathcal{C}_n^{\mathrm{loc}}(Z)$ denote the map associated with this family, Γ the graph of χ in $(S \setminus \Lambda) \times \mathcal{C}_n^{\mathrm{loc}}(Z)$, p_S and $p_{\mathcal{C}}$ the restrictions to $\overline{\Gamma}$ of the projections, and $\mathfrak{X}(\pi) := p_{\mathcal{C}}(\overline{\Gamma})$.

LEMMA 7. — *With these notations, we have $\overline{\Gamma} := (\sigma \times \widetilde{\chi})(\widetilde{S})$, and $\mathfrak{X}(\pi) = \widetilde{\chi}(\widetilde{S})$. Moreover, $\overline{\Gamma}$ and $\mathfrak{X}(\pi)$ do not depend on the choice of $(\widetilde{S}, \sigma) \in \mathcal{M}(\pi)$.*

Proof. — We easily prove that the map

$$\begin{aligned} \sigma \times \widetilde{\chi} : \widetilde{S} &\longrightarrow \overline{\Gamma}, \\ \tilde{s} &\longmapsto (\sigma(\tilde{s}), \widetilde{X}_{\tilde{s}}) \end{aligned}$$

is well-defined, continuous, proper and surjective.

It follows that $\overline{\Gamma} := \sigma \times \widetilde{\chi}(\widetilde{S})$ and $\widetilde{\chi}(\widetilde{S}) = \mathfrak{X}(\pi)$.

$$\begin{tikzcd}
\widetilde{G} \arrow[r, "\tau"] \arrow[d, "\tilde{\pi}"'] & G \arrow[r, "p"] \arrow[d, "\pi"'] & Z & & \\
\widetilde{S} \arrow[r, two heads, "\sigma"] \arrow[drrr, two heads, "\sigma\times\tilde{\chi}"'] \arrow[ddrrrr, bend right, "\tilde{\chi}"'] & S & \supset & S\setminus\Lambda \arrow[r, two heads, "\chi"] & \chi(S\setminus\Lambda) \arrow[d, hook] \\
 & & & \overline{\Gamma} \arrow[ull, "ps"'] \arrow[r, two heads, "pc"] & \mathfrak{X}(\pi) \arrow[d, equal] \\
 & & & & \widetilde{\chi}(\widetilde{S})
\end{tikzcd}$$

Now, if $(\widetilde{S}', \sigma')$ is another couple in $\mathcal{M}(\pi)$, it is easy to check that we have $(\sigma' \times \widetilde{\chi}')(\widetilde{S}') = (\sigma \times \widetilde{\chi})(\widetilde{S}) = \overline{\Gamma}$; this yields the last assertion. □

This lemma enables us to give this definition:

DEFINITION 6. — *The continuous family of n-cycles $(X)_{(s,X)\in\overline{\Gamma}}$ is called (the) meromorphic family of n-cycles defined by $\pi : G \to S$, and $\overline{\Gamma}$ is called (the) graph of this meromorphic family of n-cycles.*

3.2. Meromorphic families of cycles without escape to infinity.

Suppose that $\mathcal{M}(\pi) \neq \emptyset$. Let $\mathcal{MNE}(\pi)$ denote the set of $(\widetilde{S}, \sigma)$ in $\mathcal{M}(\pi)$ such that the analytic family $(\widetilde{X}_{\tilde{s}})_{\tilde{s}\in\widetilde{S}}$ does *not* escape to infinity, that is to say: the set of cycles $\widetilde{\chi}(\widetilde{S})$ does not escape to infinity. Since $\widetilde{\chi}(\widetilde{S}) = \mathfrak{X}(\pi)$ for every $(\widetilde{S}, \sigma)$ in $\mathcal{M}(\pi)$, we have

$$\mathcal{MNE}(\pi) \neq \emptyset \iff \mathcal{MNE}(\pi) = \mathcal{M}(\pi).$$

The next proposition provides $\overline{\Gamma}$ with an analytic structure, together with a universal property:

PROPOSITION 7. — *Let $\pi : G \to S$ be a map such that $\mathcal{MNE}(\pi) \neq \emptyset$. Then there exists a unique couple $(A, \alpha) \in \mathcal{M}(\pi)$ such that for every couple $(S', \sigma') \in \mathcal{M}(\pi)$, there exists a unique analytic map $\beta : S' \to A$ such that $\sigma' = \alpha \circ \beta$.*

Proof. — Set $(\widetilde{S}, \sigma) \in \mathcal{M}(\pi)$.

We will apply to the map $\sigma \times \widetilde{\chi} : \widetilde{S} \to S \times \mathcal{C}_n^{\mathrm{loc}}(Z)$ the proof of Theorem 3: it will yield that the quotient $\widetilde{S}/R_{\sigma\times\widetilde{\chi}}$ is a weakly normal, finite dimensional analytic space A, homeomorphic to $\overline{\Gamma}$.

Since the analytic family $(\widetilde{X}_{\tilde{s}})_{\tilde{s}\in\widetilde{S}}$ does not escape to infinity, there exists, for every $\tilde{s}_0 \in \widetilde{S}$, a finite family of scales which well determines it around $\widetilde{X}_{\tilde{s}_0}$ (Proposition 5). Thus we have an analytic map $\widetilde{f}_{\mathcal{E}} = \prod_i \widetilde{f}_{E_i|\widetilde{S}_{\mathcal{E}}} : \widetilde{S}_{\mathcal{E}} \twoheadrightarrow \widetilde{f}_{\mathcal{E}}(\widetilde{S}_{\mathcal{E}})$, whose fibers coincide in $S_{\mathcal{E}}$ with the fibers of $\widetilde{\chi}$ (Lemma 5).

Set a relatively compact neighbourhood V of $\sigma(\tilde{s}_0)$ in S; thus the analytic map $\sigma \times \widetilde{f}_{\mathcal{E}} : \sigma^{-1}(V) \cap \widetilde{S}_{\mathcal{E}} \longrightarrow V \times \widetilde{f}_{\mathcal{E}}(\widetilde{S}_{\mathcal{E}})$ is proper — by routine topological arguments (the properness of σ suffices) — and its fibers coincide with those of $\sigma \times \widetilde{\chi}$. It comes from the proof of Theorem 3 that $A := \widetilde{S}/R_{\sigma\times\widetilde{\chi}}$ is a weakly normal, finite dimensional analytic space. This space A is homeomorphic to $(\sigma \times \widetilde{\chi})(\widetilde{S}) = \overline{\Gamma}$; it is clear now that the projection $\alpha := p_S : \overline{\Gamma} \to S$ is a proper modification $A \twoheadrightarrow S$; the strict transform $\pi_A : G_A \to A$ defines an analytic family of cycles, which is nothing else than the continuous family $(X)_{(s,X)\in\overline{\Gamma}}$ (that we call the 'meromorphic family of n-cycles' defined by π): so (A, α) belongs to $\mathcal{M}(\pi) = \mathcal{MNE}(\pi)$.

The other assertions are easy to check (set $\beta := \sigma' \times \widetilde{\chi}' : S' \to A = \overline{\Gamma}$). □

3.3. Geometric flattening.

Here we put some conditions on a map $\pi : G \to S$, so that its strict transform, after a proper modification, may be geometrically flat.

We recall that the non-normal locus $N(S)$ of S is a nowhere dense analytic subset of S.

THEOREM 5. — *Let G be an analytic subset of $S \times Z$, where S and Z are analytic spaces, S being weakly normal. Assume that there exist a nowhere dense analytic subset Σ of S, containing $N(S)$, and an integer n such that*

1) *the restriction of* π *to every irreducible component* H^i *of the open set* $H := G \setminus \pi^{-1}(\Sigma) = \pi^{-1}(S \setminus \Sigma)$ *is of constant corank* n *and maps* H^i *onto* $S \setminus \Sigma$.

Thanks to Proposition 1, *we have thus an analytic family* $(X_s)_{s \in S \setminus \Sigma}$ *of* n*-cycles of* Z *parametrized by* $S \setminus \Sigma$, *and this enables us to construct* $\overline{\Gamma}$ *and* $\mathfrak{X}$, *using the 'basic construction'.*

If the following condition is moreover satisfied:

2) *for every* $s_0 \in S$, *there exist a compact set* L_0 *of* Z *and a relatively compact neighbourhood* S_0 *of* s_0 *in* S *such that, for every* $s \in S_0$ *and every cycle* $X_1 \in \mathfrak{X}$ *included in* $p(\pi^{-1}(s))$, *there exists a neighbourhood* $\mathcal{V}_1$ *of* X_1 *in* $\mathcal{C}_n^{\mathrm{loc}}(Z)$, *such that every irreducible component of every cycle in* $\mathcal{V}_1 \cap \mathfrak{X}$ *intersects* L_0 ;

then π *defines a meromorphic family of* n*-cycles without escape to infinity.*

Remarks. — Set $G_H = \overline{H} = \overline{\pi^{-1}(S \setminus \Sigma)}$: it is an analytic subset of G, exactly the union of irreducible components of G not included in $\pi^{-1}(\Sigma)$. Actually, the irreducible components of H are exactly the intersections with H of irreducible components of G_H. Note that $\pi^{-1}{}_{|G_H}(\Sigma)$ is nowhere dense in G_H, whereas this is not necessarily true for $\pi^{-1}(\Sigma)$ in G. We will prove below that $\pi_{|G_H} : G_H \to S$ satisfies 1) and 2), and defines the same set $\mathfrak{X}$ of cycles as π.

Condition 2) implies the non-escape to infinity of $\mathfrak{X}$, but asks for more: this non-escape should be defined 'uniformly with respect to S'.

Before going further, we give a special case when 1) holds:

1 bis) G is of pure dimension N, and π is quasi-proper and maps every irreducible component of G onto S.

Since π is quasi-proper, the image $\Delta := \pi(Deg\,\pi)$ of the degenerate locus is an analytic subset of S, thanks to Lemma 2, ii). Moreover, it is nowhere dense. The non-normal locus $N(S)$ of S is a nowhere dense analytic subset of S too, and so is $\Sigma := \Delta \cup N(S)$. Finally, it is easy to check that the projection $\pi : \pi^{-1}(S \setminus \Sigma) \to S \setminus \Sigma$ is of constant corank n, and maps every irreducible component of $\pi^{-1}(S \setminus \Sigma)$ onto $S \setminus \Sigma$.

Remarks.

- Some points in this paragraph have been inspired by 'Lemma (n)' and 'Proposition (n)' of [Gr86], or by the proofs and clarifications given in [Si93].

• The proof of Theorem 5 is partly similar to the one of Proposition 7: we want to explain that $\overline{\Gamma}$ is an analytic space of finite dimension (and thus $p_S : \overline{\Gamma} \to S$ a proper modification of S), and that the family of cycles $(X)_{(s,X)\in\overline{\Gamma}}$ is analytic and without escape to infinity — but here, we have no more the assumption that $\mathcal{MNE}(\pi)$ is non-empty, and the point is precisely to build (at least locally) a couple $(\widetilde{S}, \sigma)$ of this set. There are two main steps in this proof, which we separetely prove below: first, we explain that conditions 1) and 2) move (with the same associated set $\mathfrak{X}$ of cycles) from a map π to its strict transform $\widetilde{\pi}$ induced by a proper modification; then, we show how to build, when one has a quasi-proper map $\pi : G \to S$, a local blow-up of an open set of S such that the *maximal* corank of the local strict transform $\widetilde{\pi}$ is strictly less than the one of π. So, by induction, we can get a map of constant corank, satisfying 1) and 2); first of all, we study such a map.

3.3.1. *Special case*: π *of constant corank.* — We prove here the

PROPOSITION 8. — *Suppose that S is normal. Let $\pi : G \to S$ be a map of constant corank and satisfying conditions* 1) *and* 2). *Then π defines an analytic family of cycles without escape to infinity*; *furthermore we have $\mathfrak{X} = \chi(S)$, if $\chi : S \to \mathcal{C}_n^{\mathrm{loc}}(Z)$ is the map associated with this family.*

Proof. — We shall precisely show that $\pi_{|G_H}$ is geometrically flat. Since S is normal, and π of constant corank, it suffices to prove that every irreducible component G^i of G_H is mapped onto S (Proposition 1). We already know that $\pi(G^i)$ contains $S \setminus \Sigma$.

For $s_0 \in \Sigma$, let L_0 be the compact set of Z and S_0 be the neighbourhood of s_0 given by condition 2). Let $(s_\nu)_{\nu\in\mathbb{N}}$ be a sequence of points of $S_0 \setminus \Sigma$ converging to s_0. For every ν, by assumption, G^i contains at least one point, so, as well, one irreducible component, of $\pi^{-1}(s_\nu)$. This irreducible component can be written $\{s_\nu\} \times X^i_{s_\nu}$, where $X^i_{s_\nu}$ is an irreducible component of $|X_{s_\nu}|$ and then intersects L_0; hence, there exists a point z_ν of $L_0 \cap X^i_{s_\nu}$ such that (s_ν, z_ν) lies in G^i. Since $z_\nu \in L_0$, there exists a sub-sequence $(z_{\nu_k})_{k\in\mathbb{N}}$ converging to a point z of L_0, and $(s_0, z) = \lim_{k\to\infty}(s_{\nu_k}, z_{\nu_k})$ is a point of G^i, *q.e.d.*

Thus, we have an analytic family of n-cycles parametrized by S, whose graph is G_H. We can now see that the graph Γ of the associated map $\chi : S \to \mathcal{C}_n^{\mathrm{loc}}(Z)$ is closed in $S \times \mathcal{C}_n^{\mathrm{loc}}(Z)$ and homeomorphic to S, and that $\mathfrak{X}$ coincides with $\chi(S)$. Finally, condition 2) implies that the family is without escape to infinity. □

3.3.2. *Stability of conditions* 1) *and* 2) *and of* $\mathfrak{X}$ *by strict transformation.* — Let $\pi : G \to S$ be a map satisfying conditions 1) and 2), and $\sigma : \widetilde{S} \twoheadrightarrow S$ be a proper modification of S with centre Λ.

By B), we have, if $s := \sigma(\tilde{s})$:

$$\widetilde{p}\big(\widetilde{\pi}^{-1}(\tilde{s})\big) = p\big(\pi^{-1}(s)\big) \text{ if } \tilde{s} \notin \widetilde{\Lambda}\,;$$

then, in particular, for every point $\tilde{s}$ outside $\widetilde{\Lambda} \cup \sigma^{-1}(\Sigma)$:

$$\widetilde{p}\big(\widetilde{\pi}^{-1}(\tilde{s})\big) = |X_s| \quad \text{if} \quad \tilde{s} \notin \widetilde{M} := \widetilde{\Lambda} \cup \sigma^{-1}(\Sigma)\,;$$

and it is easy to verify that the family of cycles $\big(X_{\sigma(\tilde{s})}\big)_{\tilde{s} \in \widetilde{S} \setminus \widetilde{M}}$ is analytic.

Although $\widetilde{M}$ is a nowhere dense analytic subset of $\widetilde{S}$, this could be no more true for $\widetilde{\pi}^{-1}(\widetilde{M})$ in $\widetilde{G}$. To have this property, we shall work with an analytic subset of $\widetilde{G}$, namely

$$\widehat{G} := \overline{\widetilde{\pi}^{-1}(\widetilde{S} \setminus \widetilde{M})} = \overline{\widetilde{G} \setminus \widetilde{\pi}^{-1}(\widetilde{M})},$$

which is exactly the union of the irreducible components of $\widetilde{G}$ non included in $\widetilde{\pi}^{-1}(\widetilde{M})$.

Let $\widehat{\pi}$ and $\widehat{p}$ denote the restrictions to $\widehat{G}$ of $\widetilde{\pi}$ and $\widetilde{p}$. We have still

$$\widehat{p}\big(\widehat{\pi}^{-1}(\tilde{s})\big) = \widetilde{p}\big(\widetilde{\pi}^{-1}(\tilde{s})\big) = |X_s| \quad \text{if } \tilde{s} \notin \widetilde{M},\ s = \sigma(\tilde{s}),$$

and property B) holds with $\widehat{\pi}$ and $\widehat{p}$ instead of $\widetilde{\pi}$ and $\widetilde{p}$.

We prove now:

PROPOSITION 9. — *The map* $\widehat{\pi} : \widehat{G} \to \widetilde{S}$ *(with* $\widetilde{M}$*) satisfies conditions* 1) *and* 2)*, and defines the same set* $\mathfrak{X}$ *of cycles as* π*. Moreover,* $\widehat{\pi}$ *is quasi-proper.*

Proof. — Remark that $\widetilde{M}$ contains the non-normal locus $N(\widetilde{S})$ of $\widetilde{S}$, and that $\widehat{p}$ is of corank n.

- First, we shall prove that $\widehat{\pi}$ is quasi-proper. With this aim, we begin with a lemma:

LEMMA 8. — *For every point* $\widehat{g}_0 := (\tilde{s}_0, z_0) \in \widehat{G}$, *there exists a cycle* Y_0 *of* $\mathfrak{X}$, *such that*

(i) z_0 *lies in* $|Y_0|$;

(ii) $(\sigma(\tilde{s}_0), Y_0)$ *belongs to* $\overline{\Gamma}$.

Sketch of the proof. — It is enough to prove the lemma for $\widehat{g}_0$ lying in $\widehat{\pi}^{-1}(\widetilde{M})$ since we can set $Y_0 := \widehat{p}(\widehat{\pi}^{-1}(\tilde{s}_0)) = |X_{\sigma(\tilde{s}_0)}|$ for $\tilde{s}_0 \notin \widetilde{M}$.

Since $\widehat{\pi}^{-1}(\widetilde{M})$ is nowhere dense (by construction of $\widehat{G}$), one can build an analytic map $\phi : D \to \widehat{G}$, where $D := \{z \in \mathbb{C},\ |z| < 1\}$, such that $\phi(D)$ is analytic in a neighbourhood of $\widehat{g}_0$ and intersects $\widehat{\pi}^{-1}(\widetilde{M})$ only in $\phi(0) = \widehat{g}_0$. Set $E := \{(t, \widehat{g}) \in D \times \widehat{G}$, such that $\widehat{\pi}(\phi(t)) = \widehat{\pi}(\widehat{g})\}$. Then we can check that the projection $\delta : E \to D$ defines an analytic family of cycles parametrized by $D \setminus \{0\}$, with $\delta^{-1}(t) = \{t\} \times \{\widehat{\pi}(\phi(t))\} \times |X_{\sigma(\widehat{\pi}(\phi(t)))}|$ if $t \neq 0$.

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[dd, "\delta"'] & D \times \widehat{G} \arrow[d] & & \\
 & \widehat{G} \arrow[r, "\tau"] \arrow[rr, bend left, "\widehat{p}"] \arrow[d, "\widehat{\pi}"] & G \arrow[r, "p"] \arrow[d, "\pi"] & Z \\
D \arrow[ur, "\phi"] & \widetilde{S} \arrow[r, "\sigma"] & S &
\end{tikzcd}$$

Thus, arguing as in the proof of Proposition 8, we prove that $\delta : E \to D$ generates an analytic family of cycles parametrized by all D. The cycle thereby defined over 0 satisfies (i) and (ii). □

The quasi-properness of $\widehat{\pi}$ follows easily from this lemma: every irreducible component B of every (projection on Z of a) fiber of $\widehat{\pi}$ contains an irreducible component of a cycle lying in $\overline{\Gamma}$. Thanks to 2), this latter irreducible component intersects a compact set L_0 if it lies over a neighbourhood S_0; so B intersects L_0 if it lies over $\widetilde{S}_0 := \sigma^{-1}(S_0)$ (here one sees why it is important, to have a non-escape 'uniformly with respect to S').

• We go on with the proof of Proposition 9. We have an isomorphism outside $\widetilde{M}$ (not only between $\widetilde{S}$ and S but also between $\widehat{G}$ and G), so condition 1) is satisfied by $\widehat{\pi}$ outside $\widetilde{M}$.

We have an analytic family of n-cycles $(\widehat{X}_{\tilde{s}})_{\tilde{s} \in \widetilde{S} \setminus \widetilde{M}}$ parametrized by $\widetilde{S} \setminus \widetilde{M}$, associated with a map $\widehat{\chi} : \widetilde{S} \setminus \widetilde{M} \to \mathcal{C}_n^{\mathrm{loc}}(Z)$, let $\widehat{\Gamma}$ denote the graph of $\widehat{\chi}$ in $(\widetilde{S}\setminus\widetilde{M})\times\mathcal{C}_n^{\mathrm{loc}}(Z)$, $\widehat{p}_{\mathcal{C}}$ and $\widehat{p}_{\widetilde{S}}$ the projections of $\overline{\overline{\widehat{\Gamma}}}$ to $\mathcal{C}_n^{\mathrm{loc}}(Z)$ and $\widetilde{S}$, $\mathrm{Id}_{\mathcal{C}}$ the identity in $\mathcal{C}_n^{\mathrm{loc}}(Z)$ and lastly $\widehat{\mathfrak{X}} := \widehat{p}_{\mathcal{C}}(\overline{\overline{\widehat{\Gamma}}})$ the set of cycles defined by $\widehat{\pi}$.

The map $\sigma\times\mathrm{Id}_{\mathcal{C}}:\overline{\overline{\widehat{\Gamma}}}\longrightarrow\overline{\Gamma}$, $(\tilde{s},X)\mapsto(\sigma(\tilde{s}),X)$ is well-defined, continuous, proper and surjective. Thus, the sets of cycles defined by G and $\widehat{G}$ coincide.

$$\begin{tikzcd}
\widetilde{G} \arrow[r, "\tau"] \arrow[d, "\widehat{\pi}"'] \arrow[rrr, bend left, "\widehat{p}"] & G\ (\subset S\times Z) \arrow[rr, "p"] \arrow[d, "\pi"] & & Z \\
\widetilde{S} \arrow[r, two heads, "\sigma"] & S \arrow[r, phantom, "\supset"] & S\setminus\Sigma \arrow[r, two heads, "\chi"] & \chi(S\setminus\Sigma) \arrow[d, hook] \\
 & & \overline{\Gamma} \arrow[ul, "p_S"'] \arrow[r, two heads, "p_{\mathcal{C}}"] & \mathfrak{X} \arrow[d, equal] \\
 & \overline{\overline{\widehat{\Gamma}}} \arrow[uul, "\widehat{p}_{\widetilde{S}}"] \arrow[ur, "\sigma\times\mathrm{Id}_{\mathcal{C}}"] \arrow[rr, two heads, "\widehat{p}_{\mathcal{C}}"] & & \widehat{\mathfrak{X}}
\end{tikzcd}$$

It remains to show that the continuous family of cycles $(X)_{(\tilde{s},X)\in\overline{\overline{\widehat{\Gamma}}}}$ defined thereby satisfies condition 2): for a point $\tilde{s}_0\in\widetilde{S}$, consider the neighbourhood S_0 of $s_0:=\sigma(\tilde{s}_0)$ and the compact set L_0 of Z given by condition 2) applied to G; then, one can check easily that condition 2) moves from π (with S_0 and L_0) to $\widehat{\pi}$ (with $\widetilde{S}_0:=\sigma^{-1}(S_0)$ and L_0). □

Remark. — Proposition 9 implies some properties for the initial map π:

PROPOSITION 10. — *Let $\pi : G\to S$ be a map satisfying (with a nowhere dense analytic subset Σ of S) conditions* 1) *and* 2); *set $G_H:=\overline{\pi^{-1}(S\setminus\Sigma)}$; then $\pi_{|G_H}$ satisfies* 1) *and* 2), *defines the same set of cycles as π, is quasi-proper and surjective onto S. Moreover, $\pi_{|G_H}(\operatorname{Deg}\pi_{|G_H})$ is a nowhere dense analytic subset of S. Finally, the map $p_S:\overline{\Gamma}\to S$ is surjective.*

Proof. — The above construction can be applied to G, strict transform of G by Id_S; so we have $\Lambda=\emptyset$, $\widetilde{M}=\Sigma$ and $\widehat{G}=\overline{\pi^{-1}(S\setminus\Sigma)}=G_H$. The first assertions follow. The assertion (ii) of Lemma 2 implies that $\pi_{|G_H}(G_H)$

is an analytic subset of S, containing the dense open subset $S \setminus \Sigma$ by 1): so $\pi_{|G_H}(G_H) = S$. The same assertion implies that also $\pi_{|G_H}(\mathrm{Deg}\ \pi_{|G_H})$ is an analytic subset of S; since it is included in Σ, it is nowhere dense.

Finally, for every $s \in S$, there exist $(s, z) \in G_H$, and a cycle X of $\mathfrak{X}$, such that z lies in $|X|$ and (s, X) in $\overline{\Gamma}$: the surjectivity of $p_S : \overline{\Gamma} \to S$ follows. □

3.3.3. *Local lowering of the maximal corank.* — Here we prove the following result:

PROPOSITION 11. — *Let $\pi : G \to S$ be a quasi-proper map between analytic spaces G and S. Let s_0 be a point of $\pi(\mathrm{Deg}\,\pi)$ and q be the dimension of the fiber $\pi^{-1}(s_0)$. Then there exists a (local) blow-up $\sigma : \widetilde{V} \twoheadrightarrow V$ of a neighbourhood V of s_0, such that the strict transform $\widetilde{\pi} : \widetilde{G} \to \widetilde{V}$ is of maximal corank strictly less than q.*

Proof. — We develop here the proof of Corollary 1.2 of [Pa94] (for other results about local geometrical flattening, see [HLT73], Theorem 2, and [Si93], I, Lemma 3.3).

We locally describe (see [Fi76], 3.3) the map $\pi : G \to S$ as the projection p of a proper analytic subset Z of the product $V \times W$ to V, where V is an open set of S, and W an open set of $\mathbb{C}^q$. There exists an analytic function $F : V \times W \to \mathbb{C}$ such that $Z \subset F^{-1}(0)$. We develop

$$\forall (y, t) \in V \times W, \quad F(y, t) = \sum_{\beta \in \mathbb{N}^q} \Theta_\beta(y) t^\beta,$$

with $\Theta_\beta \in \mathcal{O}_S(V)$. Let I be the (proper) ideal of $\mathcal{O}_S(V)$ generated by the Θ_β.

Let now $\sigma : V' \twoheadrightarrow V$ be the local blow-up with center I, and $p' : Z' \to V'$ be the strict transform of $p : Z \to V$ by σ. For every $s'_0 \in V'$, the ideal $(\sigma^\star I)_{s'_0}$ of $\mathcal{O}_{V', s'_0}$ is invertible, then principal, let's say generated by $g := \Theta_{\beta_0} \circ \sigma$. For y' near enough s'_0, for all β, we factorize

$$\Theta_\beta\big(\sigma(y')\big) = g(y') \cdot \theta_\beta(y'),$$

where θ_β is holomorphic in a neighbourhood of s'_0. Thus

$$F(\sigma(y'), t) = g(y') \cdot f(y', t), \quad \text{with} \quad f(y', t) := \sum_{\beta \in \mathbb{N}^q} \theta_\beta(y') t^\beta.$$

We can now check that Z' is included in $f^{-1}(0)$ and that f does not vanish identically on any fiber of p': so

$$\mathrm{gcork}_{y'} p' \leq \dim\big(\{y'\} \times W\big) \cap f^{-1}(0) < q.$$

This yields that the maximal corank of p' over V' is strictly less than q.

Then we globalize 'in G': thanks to the quasi-properness of π, all irreducible components of π-fibers over an open set V_0 of s_0 in S intersect a compact set L_0 of G. We cover this compact set with a finite number of $V_j \times W_j$; for every j, we have as above an ideal I_j in $\mathcal{O}_S(V_j)$. We blow up $V := V_0 \cap (\bigcap_j V_j)$ by a $\sigma : \widetilde{V} \twoheadrightarrow V$ along the ideal $(\prod_j I_j)_{|V}$; every ideal $\sigma^\star I_j$ is still invertible (see [Hi73], Lemme 1.12.3), so the previous result remains valid, and the strict transform $\widetilde{\pi} : \widetilde{G} \to \widetilde{V}$ of $\pi_{|\pi^{-1}(V)} : G_{|\pi^{-1}(V)} \to V$ is of maximal corank strictly less than q. □

3.3.4. *Proof of Theorem 5.* — We will prove Theorem 5 by an induction on the maximal corank of π. Condition 1) implies that the (minimal) corank of π is n.

Suppose first that the maximal corank of π is also n. Then, it suffices to take the normalization $\nu : \widehat{S} \twoheadrightarrow S$; it is a proper modification (along $N(S)$), so, by Proposition 9, the strict transform (after a possible restriction to some irreducible components) is a map satisfying conditions 1) et 2), and of constant corank; and by Proposition 8, it defines an analytic family of cycles without escape to infinity (these cycles going all over $\mathfrak{X}$), and π defines a meromorphic family of cycles without escape to infinity.

Suppose now that the theorem is true when the maximal corank of π is strictly less than q.

Let $\pi : G \twoheadrightarrow S$ be a map satisfying conditions 1) and 2), let q be its maximal corank. We can already work with the map $\pi_{|G_H}$, which also satisfies 1) and 2) and defines the same set of cycles as π (Proposition 10).

We want to explain that $\overline{\Gamma}$ is an analytic space of finite dimension. This is a local problem. Let (s_0, X_0) be a point of $\overline{\Gamma}$; we can suppose that s_0 lies in the nowhere dense analytic subset $\Sigma \cap \pi(\mathrm{Deg}^q \pi)$ (otherwise, the maximal corank of π over a neighbourhood of s_0 is strictly less than q and one can skip the following two paragraphs). So, $q := \dim \pi^{-1}(s_0)$ and $|X_0| \subsetneq p(\pi^{-1}(s_0))$.

According to Proposition 11 (we recall that $\pi_{|G_H}$ is quasi-proper, by Proposition 10), there exist a neighbourhood V of s_0 in S (we note $\overline{\Gamma}_V$

and $\mathfrak{X}_V$ the corresponding subsets of $\overline{\Gamma}$ and $\mathfrak{X}$) and a proper modification $\sigma : \widetilde{V} \twoheadrightarrow V$ (precisely a blow-up), such that the strict transform $\widetilde{\pi} : \widetilde{G} \twoheadrightarrow \widetilde{V}$ is of maximal corank strictly less than q.

Then, thanks to Proposition 9, there exists an analytic subset $\widehat{G}$ of $\widetilde{G}$ such that the projection $\widehat{\pi} : \widehat{G} \twoheadrightarrow \widetilde{V}$ verifies conditions 1) and 2), and defines a set of cycles $\widehat{\mathfrak{X}}$ equal to $\mathfrak{X}_V$, and this map $\widehat{\pi}$ is of maximal corank strictly less than q.

Thus the induction assumption implies that $\widehat{\pi}$ defines a meromorphic family of cycles without escape to infinity, that is to say, there exists a proper modification $\widetilde{\sigma} : V' \twoheadrightarrow \widetilde{V}$ such that the strict transform $\pi' : G' \twoheadrightarrow V'$ defines an analytic family of cycles $(X'_{s'})_{s' \in V'}$ without escape to infinity. Let $\chi' : V' \to \mathcal{C}_n^{\mathrm{loc}}(Z)$ the map associated with this family. We have $\chi'(V') = \widehat{\mathfrak{X}} = \mathfrak{X}_V$, and, in particular, $X_0 = X'_{s'_0}$ for a point $s'_0 \in (\sigma \circ \widetilde{\sigma})^{-1}(s_0) \subset V'$.

$$
\begin{tikzcd}
G' \arrow[r, two heads] \arrow[d, two heads] & \widehat{G} \arrow[r, two heads] \arrow[d, two heads] & G_{H|\pi^{-1}(V)} \arrow[rrr, "p"] \arrow[d, two heads] & & & Z \\
V' \arrow[r, two heads] \arrow[ddddrrrrr, bend right, "\chi'"'] & \widetilde{V} \arrow[r, two heads] & V & \supset & V \setminus \Sigma \arrow[r, two heads] & \chi(V \setminus \Sigma) \arrow[d, hook] \\
& & & & \overline{\Gamma}_V \arrow[ull, "p_S"'] \arrow[r, two heads, "p_C"] & \mathfrak{X}_V \arrow[d, equal] \\
& & & \overline{\widehat{\Gamma}} \arrow[uull, "\widehat{p}_{\widetilde{V}}"'] \arrow[ur, "\sigma \times \mathrm{Id}_C"] \arrow[rr, two heads, "\widehat{p}_C"] & & \widehat{\mathfrak{X}} \arrow[d, equal] \\
& \overline{\Gamma}' \arrow[uuul, "p'_{V'}"] \arrow[urr, "\widetilde{\sigma} \times \mathrm{Id}_C"] \arrow[rrrr, two heads, "p'_C"] & & & & \mathfrak{X}' \arrow[d, equal] \\
& & & & & \chi'(V')
\end{tikzcd}
$$

Now, we will use again the arguments of the proof of Proposition 7. For every $s' \in V'$, we can choose a finite family $\mathcal{E}$ of scales adapted to the cycle $X'_{s'}$, and well determining the family of cycles around it. Let $f'_{\mathcal{E}} : V'_{\mathcal{E}} \to \prod_{i \in I}^{C} H(\overline{U}'_i, \mathrm{sym}^{k_i} B_i)$ be the associated analytic map, and V_0 be a relatively compact neighbourhood of s_0 in V. Since the composition $\sigma \circ \widetilde{\sigma} : V' \twoheadrightarrow V$ is proper, so is the map

$$(\sigma \circ \widetilde{\sigma}) \times f'_{\mathcal{E}} : (\sigma \circ \widetilde{\sigma})^{-1}(V_0) \cap V'_{\mathcal{E}} \longrightarrow V_0 \times f'_{\mathcal{E}}(V'\mathcal{E}).$$

So we can use Theorem 3.

We obtain that $V'/R_{(\sigma\circ\widetilde{\sigma})\times\chi'}$ is a finite dimensional, weakly normal analytic space, which can be identified to a neighbourhood of (s_0, X_0) in $\overline{\Gamma}$. So this last space is provided with a finite dimensional, weakly normal analytic structure such that the map $p_S : \overline{\Gamma} \twoheadrightarrow S$ is a proper modification and $(X)_{(s,X)\in\overline{\Gamma}}$ an *analytic* family of n-cycles without escape to infinity, defined by the strict transform of π by p_S. All that precisely means that π defines a *meromorphic* family of n-cycles without escape to infinity. □

3.4. Universal reparametrization of a semi-proper regular meromorphic family of cycles.

Here, we want to generalize Theorem 4 to the case of meromorphic families of cycles. So we need to introduce regularity and semi-properness.

DEFINITION 7. — *Let $\mathcal{MRSP}(\pi)$ denote the set of $(\widetilde{S}, \sigma)$ in $\mathcal{M}(\pi)$ such that the analytic family $(\widetilde{X}_{\tilde{s}})_{\tilde{s}\in\widetilde{S}}$ is regular and semi-proper, that is to say: the set $\widetilde{\chi}(\widetilde{S}) \subset \mathcal{C}_n^{\mathrm{loc}}(Z)$ is regular and the map $\widetilde{\chi} : \widetilde{S} \twoheadrightarrow \widetilde{\chi}(\widetilde{S}) = \mathfrak{X}(\pi)$ is semi-proper.*

The meromorphic family of n-cycles defined by a map $\pi : G \to S$ is a semi-proper regular meromorphic family of n-cycles of Z if $\mathcal{MRSP}(\pi) \neq \emptyset$.

First, we extend Theorem 5:

PROPOSITION 12. — *Let $\pi : G \to S$ be a map satisfying condition 1) of Theorem 5. Suppose that the following conditions hold:*

For every $s_0 \in S$, there exist a compact set L_0 of Z and a relatively compact neighbourhood S_0 of s_0 in S such that, for every $s \in S_0$, and for every cycle X_1 of $\mathfrak{X}$ included in $p(\pi^{-1}(s))$, there exists a neighbourhood $\mathcal{V}_1$ of X_1 in $\mathcal{C}_n^{\mathrm{loc}}(Z)$, such that

2) *every irreducible component of every cycle in $\mathcal{V}_1 \cap \mathfrak{X}$ intersects L_0;*

3) *there exists a relatively compact open set W_1 in Z, such that, if for some cycles $(X_\nu)_{\nu\in\mathbb{N}}$ and X in $\mathcal{V}_1 \cap \mathfrak{X}$, $(X_\nu \cap W_1)_{\nu\in\mathbb{N}}$ converges to $X \cap W_1$ in $\mathcal{C}_n^{\mathrm{loc}}(W_1, Z)$, then $(X_\nu)_{\nu\in\mathbb{N}}$ converges to X in $\mathcal{C}_n^{\mathrm{loc}}(Z)$.*

Then the meromorphic family of n-cycles defined by π is regular.

Moreover, if $p_C : \overline{\Gamma} \twoheadrightarrow \mathfrak{X}$ is semi-proper, then this meromorphic family of n-cycles is semi-proper (so $\mathcal{MRSP}(\pi) \neq \emptyset$).

Proof. — We can prove, as for condition 2) in Theorem 5, that condition 3) moves to the strict transforms of π, and implies the topological non-escape — and so the regularity — of the analytic family of n-cycles we finally obtained.

Assume that $p_{\mathcal{C}}:\overline{\Gamma} \twoheadrightarrow \mathfrak{X}$ is semi-proper. Since $\sigma \times \widetilde{\chi}$ is proper and surjective, the semi-properness of $\widetilde{\chi} = p_{\mathcal{C}} \circ (\sigma \times \widetilde{\chi})$ comes from Lemma 1, (iii). $\square$

Now, we can state:

THEOREM 6. — *Let $\pi : G \to S$ be a map such that $\mathcal{MRSP}(\pi) \neq \emptyset$, i.e. defining a semi-proper, regular meromorphic family of n-cycles.*

There exist a weakly normal analytic space Q of finite dimension, and an analytic family of n-cycles $(\widetilde{X}_c)_{c\in Q}$ such that for every weakly normal space T, every map $\rho : H \to T$, where H is an analytic subset of $T \times Z$, such that $\mathcal{M}(\rho) \neq \emptyset$, satisfying

$$\mathfrak{X}(\rho) \subset \mathfrak{X}(\pi),$$

there exists a meromorphic map $\Phi : T \dashrightarrow Q$, such that the family $(Y_t)_t$ is the pull-back of the family $(\widetilde{X}_c)_{c\in Q}$ by the analytic map $\varphi : \mathrm{Gr}(\Phi) \to Q$.

Proof. — By assumption, there exists a $(\widetilde{S}, \sigma)$ such that the induced family $(\widetilde{X}_{\tilde{s}})_{\tilde{s}}$ is regular and semi-proper: so, by Theorem 3, $Q : = \widetilde{S}/R_{\widetilde{\chi}}$ can be provided with a weakly normal, finite dimensional analytic structure.

Let us consider now the couple (A, α) given by Proposition 7 (π is such that $\mathcal{MNE}(\pi) \neq \emptyset$). The weakly normal space A is homeomorphic to $\overline{\Gamma}$; we have an analytic family of cycles $(X)_{(s,X)\in A}$. The map $p_{\mathcal{C}} : A = \overline{\Gamma} \twoheadrightarrow \mathfrak{X}(\pi) \subset \mathcal{C}_n^{\mathrm{loc}}(Z)$ is the map associated with this family; it is semi-proper (thanks to Lemma 1, (iii): indeed, $\widetilde{\chi} = p_{\mathcal{C}} \circ (\sigma \times \widetilde{\chi})$ is semi-proper, with $\sigma \times \widetilde{\chi}$ surjective); since $\mathfrak{X}(\pi)$ is moreover regular, this family $(X)_{(s,X)\in A}$ is regular and semi-proper; so we can apply Theorem 4 to it: there exist a weakly normal analytic space Q, and an analytic family of n-cycles $(\widetilde{X}_c)_{c\in Q}$, such that for every weakly normal analytic space B, and every analytic family of n-cycles $(Y_b)_{b\in B}$ (associated with a map $\xi : B \to \mathcal{C}_n^{\mathrm{loc}}(Z)$), satisfying

$$\xi(B) \subset p_{\mathcal{C}}(A) = \mathfrak{X}(\pi),$$

there exists a unique analytic map $\varphi : B \to Q$, such that the family $(Y_b)_{b\in B}$ is the pull-back by φ of the family $(\widetilde{X}_c)_{c\in Q}$.

Assume now that we have a weakly normal space T, a map $\rho : H \to T$, where H is an analytic subset of $T \times Z$, such that $\mathcal{M}(\rho) \neq \emptyset$, satisfying

$$\mathfrak{X}(\rho) \subset \mathfrak{X}(\pi)\,;$$

that especially yields that $\mathfrak{X}(\rho)$ is regular, so without escape to infinity; by Proposition 7, there exist a couple (B, β) in $\mathcal{MNE}(\rho)$ (where B is a weakly normal space homeomorphic to the graph $\overline{\Delta} \subset T \times \mathcal{C}_n^{\mathrm{loc}}(Z)$), and an analytic family of cycles $(Y)_{(t,Y)\in B}$, associated with a map $\xi : B \to \mathcal{C}_n^{\mathrm{loc}}(Z)$, such that

$$\xi(B) = \mathfrak{X}(\rho).$$

Since $\mathfrak{X}(\rho) \subset \mathfrak{X}(\pi)$, we are in the conditions, above recalled, of Theorem 4; there exists a map $\varphi : B \to Q$, such that the family $(Y)_{(t,Y)\in B}$ is the pull-back by φ of the family.

$$\begin{tikzcd}
 & G \arrow[d, two heads, "\pi"] & & \\
H \arrow[d, two heads, "\rho"'] & S & & \\
T \arrow[rrr, dashed, bend right, "\Phi"] & & \overline{\Gamma} \simeq A \arrow[ul] \arrow[r, two heads] & \mathfrak{X}(\pi) \simeq Q \\
 & & & \cup \\
 & & \overline{\Delta} \simeq B \arrow[uull, bend left, "\beta"] \arrow[uur, "\varphi"'] \arrow[r, two heads, "\xi"] & \mathfrak{X}(\rho)
\end{tikzcd}$$

Finally, since we have a proper modification $\beta : B \to T$, we have a meromorphic map $\Phi : T \dashrightarrow Q$ (with $\mathrm{Gr}(\Phi) = B$), satisfying the desired properties. □

Remark. — We briefly explain in [Mt99] the links between these theorems and the formerly known results about meromorphic equivalence relations (see [Gr86], [Si93]). Here, we only make the following comments:

• the condition of *regularity* that Grauert puts on the *fibration* defined by a meromorphic equivalence relation is close to the condition 2) defined on the set $\mathfrak{X}$ in our Theorem 5;

• the condition of *fibre cycle-separability* defined by Siebert on the *fiber cycle space* generated by a generically open map is similar to our condition of non-escape to infinity;

• in our situation, we must add the assumption '$p_C : \overline{\Gamma} \twoheadrightarrow \mathfrak{X}$ is semi-proper', which is more or less obvious in the case $S = Z = X$ (or in the case of families defined by the fibers of a map).

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David MATHIEU,
Université Nancy I
Institut Élie Cartan (Mathématiques)
BP 239
54506 Vandœuvre–lès–Nancy Cedex (France).