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ON THE GHOST CENTRE OF LIE SUPERALGEBRAS

by Maria GORELIK

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1. Introduction.

1.1. — Let $\mathfrak{g}$ be a complex finite dimensional Lie algebra and $\mathcal{Z}(\mathfrak{g})$ be the centre of its universal enveloping algebra. Then $\mathcal{Z}(\mathfrak{g})$ acts on a simple $\mathfrak{g}$-module by an infinitesimal character. If $\mathfrak{g}$ is semisimple, Duflo proved in [D], that the annihilator of a Verma module is generated by the kernel of the corresponding infinitesimal character.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a complex finite dimensional Lie superalgebra and $\mathcal{Z}(\mathfrak{g})$ be the (super)centre of its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. All $\mathfrak{g}$-modules considered below are assumed to be $\mathbb{Z}_2$-graded and "$\mathfrak{g}$-simple module" means simple as graded module. The centre $\mathcal{Z}(\mathfrak{g})$ acts on a simple $\mathfrak{g}$-module by an infinitesimal character, but, even in the "nice" case $\mathfrak{g} = \mathrm{osp}(1, 2l)$, the annihilator of a Verma module is not always generated by the kernel of the corresponding infinitesimal character. In [GL] we described, for the case $\mathfrak{g} = \mathrm{osp}(1, 2l)$, a polynomial subalgebra $\tilde{\mathcal{Z}}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ which acts on a simple module by "supercharacter". The annihilator of a Verma module is generated by the kernel of the corresponding supercharacter.

In this paper we introduce a notion of *ghost centre* $\widetilde{\mathcal{Z}}(\mathfrak{g})$ (see 2.1.2). This is a subalgebra of $\mathcal{U}(\mathfrak{g})$ which contains both $\mathcal{Z}(\mathfrak{g})$ and the centre of

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$\mathcal{U}(\mathfrak{g})$ considered as an associative algebra. The algebra $\widetilde{\mathcal{Z}}(\mathfrak{g})$ acts on a simple module by "supercharacter".

By definition, $\widetilde{\mathcal{Z}}(\mathfrak{g})$ is the sum of $\mathcal{Z}(\mathfrak{g})$ and of the so-called *anticentre* $\mathcal{A}(\mathfrak{g})$. The last one is the set of invariants of $\mathcal{U}(\mathfrak{g})$ with respect to a "nonstandard adjoint action" $\mathrm{ad}'\mathfrak{g}$ introduced in [ABF]. The anticentre is a $\mathbb{Z}_2$-graded subspace of $\mathcal{U}(\mathfrak{g})$. The even part of the anticentre consists of the even elements which anticommute with the odd elements of $\mathcal{U}(\mathfrak{g})$ and commute with the even ones. The odd part of the anticentre consists of the odd elements which commute with all elements of $\mathcal{U}(\mathfrak{g})$. Thus the product of two elements from the anticentre belongs to the centre and the product of an element from the centre and an element from the anticentre belongs to the anticentre. Moreover, in the case when any non-zero central element is a non-zero divisor, $\mathcal{Z}(\mathfrak{g}) \cap \mathcal{A}(\mathfrak{g}) = \{0\}$ and so $\widetilde{\mathcal{Z}}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g}) \oplus \mathcal{A}(\mathfrak{g})$.

As well as $\mathcal{Z}(\mathfrak{g})$ itself, $\widetilde{\mathcal{Z}}(\mathfrak{g})$ is not easy to describe and, in general, it is not a noetherian algebra. However the anticentre can be described easily. First of all, it is trivial if the dimension of $\mathfrak{g}_1$ is infinite and it is pure even (resp. odd) if the dimension of $\mathfrak{g}_1$ is even (resp. odd) (see Corollary 3.1.3). Moreover $\mathcal{A}(\mathfrak{g})$ itself as well as its image in the symmetric algebra can be described in terms of the adjoint action of $\mathfrak{g}_0$ on its enveloping algebra $\mathcal{U}(\mathfrak{g}_0)$ (see Theorem 3.3). In particular, in the case when the top external degree $\Lambda^{\mathrm{top}}\mathfrak{g}_1$ of $\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module, this theorem gives a linear isomorphism from the centre of $\mathcal{U}(\mathfrak{g}_0)$ onto $\mathcal{A}(\mathfrak{g})$. The above condition on $\Lambda^{\mathrm{top}}\mathfrak{g}_1$ holds for the simple finite-dimensional Lie superalgebras apart from the $W(n)$ type. For the simple Lie superalgebras of type $W(n)$ the anticentre is zero.

The existence of non-zero anticentral elements implies two "negative" results. The first one is that the direct generalization of the Gelfand-Kirillov conjecture does not hold for a Lie superalgebra $\mathfrak{g}$ if $\dim \mathfrak{g}_1$ is a non-zero even integer and $\Lambda^{\mathrm{top}}\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module (see 3.5.2). In particular, it does not hold for $\mathfrak{g} = \mathrm{osp}(1, 2l)$; for $\mathrm{osp}(1,2)$ this was proven earlier in [Mu2]. The second one is that Separation theorem does not hold for the classical basic Lie superalgebras apart from the simple Lie algebras and the superalgebras $\mathrm{osp}(1, 2l)$ (see 4.5).

1.2. — In the case $\mathfrak{g} = \mathrm{osp}(1, 2l)$ Arnaudon, Bauer, Frappat ([ABF]) and Musson ([Mu1]) constructed a remarkable even element T in the enveloping algebra $\mathcal{U}(\mathfrak{g})$. This element is $\mathrm{ad}'\mathfrak{g}$-invariant and its Harish-Chandra projection is the product of hyperplanes corresponding to the positive odd roots. The element T has been called "Casimir's ghost" in [ABF], since its square belongs to the centre.

In 3.3 we construct such an element $T \in \mathcal{A}(\mathfrak{g})$ for any $\mathfrak{g}$ such that $\dim \mathfrak{g}_1$ is finite and $\Lambda^{\mathrm{top}}\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module. The image of T in the symmetric algebra belongs to $\Lambda^{\mathrm{top}}\mathfrak{g}_1$. In Section 4 we show that in the case when $\mathfrak{g}$ is a basic classical Lie superalgebra, the Harish-Chandra projection of T is also the product of hyperplanes corresponding to the positive odd roots.

In [S2] A. Sergeev described the set of "anti-invariant polynomials" which are the invariants of the dual algebra $\mathcal{U}(\mathfrak{g})^*$ with respect to the nonstandard adjoint action $\mathrm{ad}'\mathfrak{g}$.

1.3. Content of the paper. — In Section 3 we define our main objects: the anticentre $\mathcal{A}(\mathfrak{g})$ and the ghost centre $\widetilde{\mathcal{Z}}(\mathfrak{g})$. We describe the action of $\widetilde{\mathcal{Z}}(\mathfrak{g})$ on the modules of finite length in the case when $\mathfrak{g}$ is finite dimensional.

In Section 3 we show that $\mathcal{A}(\mathfrak{g})$ is equal to zero if $\dim \mathfrak{g}_1$ is infinite. Moreover all elements of $\mathcal{A}(\mathfrak{g})$ are either even (if $\dim \mathfrak{g}_1$ is even) or odd (otherwise). We describe $\mathcal{A}(\mathfrak{g})$ and its image in $\mathcal{S}(\mathfrak{g})$ in Theorem 3.3.

In Section 4 we consider the case when $\mathfrak{g}$ is a complex classical basic Lie superalgebra. In this case, the Harish-Chandra projection of $\mathcal{Z}(\mathfrak{g})$ is described by Kac and Sergeev (see [S1]). In Corollary 4.2.4, we describe the Harish-Chandra projection of $\mathcal{A}(\mathfrak{g})$.

We say that an element $u \in \mathcal{U}(\mathfrak{g})$ acts on a module M by a *superconstant* if it acts by the multiplication by a scalar on each graded component M_i $(i = 0, 1)$. In the case when $\mathfrak{g}$ is finite dimensional and $\dim \mathfrak{g}_1$ is even, any element of $\widetilde{\mathcal{Z}}(\mathfrak{g})$ acts on a simple module M by a superconstant (see 2.2). In Corollary 4.4.4 we show that if $\mathfrak{g}$ is a basic classical Lie superalgebra then any element of $\mathcal{U}(\mathfrak{g})$ acting by a superconstant on each simple finite dimensional module belongs to $\widetilde{\mathcal{Z}}(\mathfrak{g})$. Moreover $\widetilde{\mathcal{Z}}(\mathfrak{g})$ coincides with the centre (and the centralizer) of the even part $\mathcal{U}(\mathfrak{g})_0$ of the universal enveloping algebra. For the case $\mathfrak{g} = \mathrm{osp}(1, 2l)$ the last result was proven in [GL].

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2. Ghost centre.

In this paper the ground field is $\mathbb{C}$. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra such that $\mathfrak{g}_1 \neq 0$. We consider $\mathfrak{g}_0$ as a Lie subsuperalgebra of $\mathfrak{g}$. All $\mathfrak{g}_0$-modules and $\mathfrak{g}$-modules are assumed to be $\mathbb{Z}_2$-graded. We denote by Π the parity change functor: $\Pi(M)_0 := M_1$, $\Pi(M)_1 := M_0$. Denote by $\mathcal{U}(\mathfrak{g})$ the enveloping superalgebra of $\mathfrak{g}$ and by $\mathcal{Z}(\mathfrak{g})$ the (super)centre of $\mathcal{U}(\mathfrak{g})$.

2.1. — For a homogeneous $u \in \mathcal{U}(\mathfrak{g})$ denote by $d(u)$ its $\mathbb{Z}_2$-degree. For a $\mathcal{U}(\mathfrak{g})$-bimodule M one defines the adjoint action of $\mathfrak{g}$ on M by setting $(\operatorname{ad} g)m := gm - (-1)^{d(g)d(m)}mg$ where $m \in M, g \in \mathfrak{g}$ are homogeneous elements and $d(m)$ denotes the $\mathbb{Z}_2$-degree of m. Define a twisted adjoint action ad' of $\mathfrak{g}$ on M as the adjoint action of $\mathfrak{g}$ on the bimodule $\Pi(M)$. One has

$$(\operatorname{ad}' g)(u) = gm - (-1)^{d(g)(d(m)+1)}mg.$$

Assume that M has a superalgebra structure such that $g(m_1m_2) = g(m_1)m_2$, $m_1(g(m_2)) = ((m_1)g)m_2$ and $(m_1m_2)g = m_1(m_2)g$ for all $g \in \mathfrak{g}$, $m_1, m_2 \in M$. Then for any homogeneous $m_1, m_2 \in M$ and $g \in \mathfrak{g}$ one has

$$\begin{aligned}(\operatorname{ad}' g)(m_1m_2) &= ((\operatorname{ad} g)m_1)m_2 + (-1)^{d(g)d(m_1)}m_1((\operatorname{ad}' g)m_2)\\ &= ((\operatorname{ad}' g)m_1)m_2 + (-1)^{d(g)(d(m_1)+1)}m_1((\operatorname{ad} g)m_2).\end{aligned}$$

Moreover if m is $\operatorname{ad}'\mathfrak{g}$-invariant then

$$(1) \qquad (\operatorname{ad}' g)(m_1m) = ((\operatorname{ad} g)m_1)m, \quad (\operatorname{ad} g)(m_1m) = ((\operatorname{ad}' g)m_1)m.$$

2.1.1. *Example.* — Let N be a $\mathcal{U}(\mathfrak{g})$-module and $\operatorname{End}(N)$ be the ring of its $\mathbb{C}$-linear endomorphisms. Then $\operatorname{End}(N)$ admits a natural structure of graded $\mathcal{U}(\mathfrak{g})$-bimodule. Let θ be the endomorphism of N which is equal to id (resp. $-$id) on the even (resp. odd) component of N. Then θ is an even $\operatorname{ad}'\mathfrak{g}$-invariant homomorphism which commutes with the even elements of $\operatorname{End}(N)$ and anticommutes with the odd elements of $\operatorname{End}(N)$. The formulas (1) imply that the multiplication by θ induces an isomorphism from $\mathrm{End}(N)$ considered as $\mathrm{ad}\,\mathfrak{g}$-module onto $\mathrm{End}(N)$ considered as $\mathrm{ad}'\mathfrak{g}$-module. The similar assertion fails for $\mathcal{U}(\mathfrak{g})$ (the structure of $\mathcal{U}(\mathfrak{g})$ as $\mathrm{ad}'\mathfrak{g}$-module is given in Lemma 3.1.2).

2.1.2. — Let us call *anticentre* $\mathcal{A}(\mathfrak{g})$ the set of elements of $\mathcal{U}(\mathfrak{g})$ which are invariant with respect to ad'. Remark that the anticentre is a $\mathbb{Z}_2$-homogeneous subspace of $\mathcal{U}(\mathfrak{g})$. The even part of the anticentre consists of the even elements which anticommute with the odd elements of $\mathcal{U}(\mathfrak{g})$ and commute with the even ones. The odd part of the anticentre consists of the odd elements which commute with all elements of $\mathcal{U}(\mathfrak{g})$. Clearly the anticentre is a module over the centre and the product of any two elements of the anticentre belongs to the centre. For example, for $\mathfrak{g} = \mathrm{osp}(1, 2l)$, $\mathcal{A}(\mathfrak{g})$ is a free rank one module over $\mathcal{Z}(\mathfrak{g})$ (see [GL], 4.4.1). This is not true for a general Lie superalgebra.

Let us call *ghost centre* $\widetilde{\mathcal{Z}}(\mathfrak{g})$ the sum of $\mathcal{A}(\mathfrak{g})$ and $\mathcal{Z}(\mathfrak{g})$. It is clear that $\widetilde{\mathcal{Z}}(\mathfrak{g})$ is a subalgebra of $\mathcal{U}(\mathfrak{g})$ which contains the centre of $\mathcal{U}(\mathfrak{g})$ considered as an associative algebra. Moreover $\widetilde{\mathcal{Z}}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g}) \oplus \mathcal{A}(\mathfrak{g})$ if any non-zero element of $\mathcal{Z}(\mathfrak{g})$ is a non-zero divisor.

In order to describe the action of $\widetilde{\mathcal{Z}}(\mathfrak{g})$ on simple modules, note that Schur's lemma for Lie superalgebras takes the following form (see [K2], [BZ]).

2.1.3. Lemma. — *Let $\mathfrak{g}$ be a finite or countable dimensional Lie superalgebra and $M = M_0 \oplus M_1$ be a simple $\mathfrak{g}$-module. Then either $\mathrm{End}(M)^{\mathrm{ad}\,\mathfrak{g}} = k\,\mathrm{id}$ or $\mathrm{End}(M)^{\mathrm{ad}\,\mathfrak{g}} = k\,\mathrm{id} \oplus k\sigma$ where the odd element σ provides a $\mathfrak{g}$-isomorphism $M \xrightarrow{\sim} \Pi(M)$ and $\sigma^2 = \mathrm{id}$.*

2.1.4. — Using Example 2.1.1, we conclude that $\mathrm{End}(M)^{\mathrm{ad}'\mathfrak{g}} = \mathrm{End}(M)^{\mathrm{ad}\,\mathfrak{g}}\theta$. This implies the following lemma describing the action of $\widetilde{\mathcal{Z}}(\mathfrak{g})$ on simple modules.

Lemma. — *Let $\mathfrak{g}$ be finite or countable dimensional Lie superalgebra, $M = M_0 \oplus M_1$ be a simple $\mathfrak{g}$-module and z be an element of $\widetilde{\mathcal{Z}}(\mathfrak{g})$. Then the action of z on M is proportional to*

$$\begin{array}{ll} \mathrm{id}, & \text{if } z \in \mathcal{Z}(\mathfrak{g}) \text{ and } z \text{ is even},\\ 0, & \text{if } z \in \mathcal{Z}(\mathfrak{g}) \text{ and } z \text{ is odd},\\ \theta, & \text{if } z \in \mathcal{A}(\mathfrak{g}) \text{ and } z \text{ is even},\\ \sigma\theta, & \text{if } z \in \mathcal{A}(\mathfrak{g}) \text{ and } z \text{ is odd}. \end{array}$$

2.2. Case $\dim \mathfrak{g}_1$ **is even.** — In this case all elements of $\mathcal{A}(\mathfrak{g})$ are even (see 3.1.3). Denote by $\widetilde{\mathbb{C}}$ the algebra spanned by id and θ. Then $\widetilde{\mathbb{C}} = \mathbb{C}[\theta]/(\theta^2 - 1)$. Denote by π the algebra involution of $\widetilde{\mathbb{C}}$ sending θ to $-\theta$.

DEFINITION. — *An algebra homomorphism* $\chi : \widetilde{\mathcal{Z}}(\mathfrak{g}) \to \widetilde{\mathbb{C}}$ *is called a supercharacter if* $\chi(\mathcal{Z}(\mathfrak{g})) = \mathbb{C}$ *and* $\chi(\mathcal{A}(\mathfrak{g})) \subseteq \mathbb{C}\theta$.

By Lemma 2.1.4, $\widetilde{\mathcal{Z}}(\mathfrak{g})$ acts on a simple modules M by a supercharacter χ_M. Moreover $\chi_{\Pi(M)} = \pi\chi_M$.

2.2.1. — The standard consequence of Schur's lemma is the following statement. Any finite length module M has a unique decomposition into a direct sum of submodules M_i such that, for any fixed i, all simple subquotients of M_i have the same infinitesimal character and these characters are pairwise distinct for different i. Similarly, one can deduce from Lemma 2.1.4, that any finite length module M has a unique decomposition into a direct sum of submodules M_j such that, for any fixed j, all simple subquotients of M_j have the same supercharacter and these supercharacters are pairwise distinct for different j. This new decomposition is a refinement of the previous one. For example, let L be a simple module such that $\mathcal{A}(\mathfrak{g})$ does not lie in $\operatorname{Ann} L$. Then L and $\Pi(L)$ have different supercharacters. This, for instance, implies that though they have the same infinitesimal character, there are no non-trivial extensions of L by $\Pi(L)$.

2.3. Case $\dim \mathfrak{g}_1$ **is odd.** — In this case all elements of $\mathcal{A}(\mathfrak{g})$ are odd (see 3.1.3). Retain notation of 2.2. The algebra spanned by id and $\sigma\theta$ (see Lemma 2.1.3) is isomorphic to $\widetilde{\mathbb{C}}$. However if L is a simple module such that $aL \neq 0$ for some $a \in \mathcal{A}(\mathfrak{g})$, then the product of θ and the image of a in $\operatorname{End} L$ provides an isomorphism $s : L \xrightarrow{\sim} \Pi(L)$. One can choose a such that $s^2 = \mathrm{id}$. There are two possible choices of such s which differ by sign. As a consequence, in this case, it is more natural to define an *odd supercharacter* as a pair of homomorphism $(\chi, \pi\chi)$ where χ satisfies the conditions given in Definition 2.2 and π is the involution of $\widetilde{\mathbb{C}}$ sending $\sigma\theta$ to $-\sigma\theta$. Observe that if $L \not\cong \Pi(L)$ then $\chi = \pi\chi$.

As in 2.2.1, odd supercharacters allow us to construct a decomposition of any module of finite length, but, probably, it always coincides with the decomposition coming from the infinitesimal characters.

2.3.1. *Example.* — Let $\mathfrak{g}_1$ be generated by x and $\mathfrak{g}_0$ be generated by $[x,x]$. Then $\mathcal{U}(\mathfrak{g}) = \mathbb{C}[x]$, $\mathcal{Z}(\mathfrak{g}) = \mathbb{C}[x^2]$ and $\mathcal{A}(\mathfrak{g}) = \mathbb{C}[x^2]x$ is a cyclic $\mathcal{Z}(\mathfrak{g})$-module generated by x. The list of the simple representations of $\mathfrak{g}$ is the following:

a) Two trivial representations (one is even and one is odd). The corresponding odd supercharacter sends $\mathcal{A}(\mathfrak{g})$ to zero.

b) Two-dimensional representations $L(\lambda)$ ($\lambda \in \mathbb{C} \setminus \{0\}$) spanned by v and xv where $x^2v = \lambda v$. The corresponding odd supercharacter sends x to $\pm\sqrt{\lambda}\sigma\theta$. The representations $L(\lambda)$ and $\Pi(L(\lambda))$ are isomorphic.

3. Anticentre $\mathcal{A}(\mathfrak{g})$.

In this section we describe the anticentre $\mathcal{A}(\mathfrak{g})$. The anticentre is trivial if $\mathfrak{g}_1$ is not finite dimensional (see Corollary 3.1.3), so starting from 3.2 we assume that $\mathfrak{g}_1$ is finite dimensional.

3.1. — Denote by $\mathcal{F}$ the canonical filtration of $\mathcal{U}(\mathfrak{g})$ given by $\mathcal{F}^k := \mathfrak{g}^k$. Recall that this is an $\operatorname{ad}\mathfrak{g}$-invariant filtration and that the associated graded algebra $\operatorname{gr}_{\mathcal{F}}\mathcal{U}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g})$ is supercommutative. For $u \in \mathcal{U}(\mathfrak{g})$ denote its image in $\mathcal{S}(\mathfrak{g})$ by $\operatorname{gr} u$. Remark that $(\operatorname{ad}' x)(u) = 2xu - (\operatorname{ad} x)(u)$ for $x \in \mathfrak{g}_1$ and $u \in \mathcal{U}(\mathfrak{g})$. Therefore

(2) $\operatorname{gr}((\operatorname{ad}' x)(u))=2(\operatorname{gr} x)(\operatorname{gr} u),\ \forall u\in\mathcal{U}(\mathfrak{g}),\ x\in\mathfrak{g}_1$ s.t. $\operatorname{gr}(xu)=(\operatorname{gr} x)(\operatorname{gr} u)$.

3.1.1. — Let L be an even vector space endowed by a structure of $\mathfrak{g}_0$-module. Denote by $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ the supervector space $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} L$ (here $\mathcal{U}(\mathfrak{g})$ is considered as a right $\mathcal{U}(\mathfrak{g}_0)$-module) equipped with the natural left $\mathcal{U}(\mathfrak{g})$-module structure.

Let L be a submodule of $\mathcal{U}(\mathfrak{g}_0)$ with respect to $\operatorname{ad}\mathfrak{g}_0$-action. Denote by $(\operatorname{ad}'\mathfrak{g})(L)$ the $\operatorname{ad}'\mathfrak{g}$-submodule of $\mathcal{U}(\mathfrak{g})$ generated by L. Note that there is a natural surjective map from $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ to $(\operatorname{ad}'\mathfrak{g})(L)$ given by $u\otimes m \mapsto (\operatorname{ad}' u)m$ for $u \in \mathcal{U}(\mathfrak{g}), m \in L$.

3.1.2. Lemma. — *Let L be a submodule of $\mathcal{U}(\mathfrak{g}_0)$ with respect to $\operatorname{ad}\mathfrak{g}_0$-action. The natural map $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L \to (\operatorname{ad}'\mathcal{U}(\mathfrak{g}))(L)$ is an isomorphism. Moreover $\mathcal{U}(\mathfrak{g}) = (\operatorname{ad}'\mathcal{U}(\mathfrak{g}))\mathcal{U}(\mathfrak{g}_0)$ and thus as $\operatorname{ad}'\mathfrak{g}$-module $\mathcal{U}(\mathfrak{g})$ is isomorphic to $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}\mathcal{U}(\mathfrak{g}_0)$.*

Proof. — Let $\{x_i\}_{i\in I}$ be an ordered basis of $\mathfrak{g}_1$. For any finite subset $J \subseteq I$ set $x_J := \prod_{i\in J} x_i$, where the product is taken with respect to the given order. Then the elements $\{\operatorname{gr} x_J\}_{J\subseteq I}$ form a basis of $\Lambda\mathfrak{g}_1 \subset \mathcal{S}(\mathfrak{g})$. Choose a basis $\{u_j\}_{j\in S}$ in L such that $\{\operatorname{gr} u_j\}_{j\in S}$ are linearly independent in $\operatorname{gr}_{\mathcal{F}}\mathcal{U}(\mathfrak{g}_0)$. Since $\operatorname{gr}_{\mathcal{F}}\mathcal{U}(\mathfrak{g}_0) = \Lambda\mathfrak{g}_1 \operatorname{gr}_{\mathcal{F}}\mathcal{U}(\mathfrak{g}_0)$ one has $\operatorname{gr}(x_J u_j) = (\operatorname{gr} x_J)(\operatorname{gr} u_j)$ for any finite subset $J \subseteq I$ and $j \in S$. Using (2) one concludes that $\operatorname{gr}(\operatorname{ad}' x_J)u_j = 2^{|J|}(\operatorname{gr} x_J)(\operatorname{gr} u_j)$ for any finite subset $J \subseteq I$ and $j \in S$. Therefore the elements $\{(\operatorname{ad}' x_J)u_j\}_{J\subseteq I, j\in S}$ are linearly independent. This proves the first assertion.

For the second assertion, note that $\operatorname{gr}\mathcal{U}(\mathfrak{g})$ is spanned by the elements of the form $(\operatorname{gr} x_J)(\operatorname{gr} u)$ with $u \in \mathcal{U}(\mathfrak{g}_0)$. Now $(\operatorname{gr} x_J)(\operatorname{gr} u) = \operatorname{gr}((\operatorname{ad}' x_J)u)/2^{|J|}$ and so $\operatorname{gr}(\operatorname{ad}'\mathcal{U}(\mathfrak{g}))\mathcal{U}(\mathfrak{g}_0) = \operatorname{gr}\mathcal{U}(\mathfrak{g})$. Therefore $\mathcal{U}(\mathfrak{g}) = (\operatorname{ad}'\mathcal{U}(\mathfrak{g}))\mathcal{U}(\mathfrak{g}_0)$ as required. □

The isomorphism $\mathcal{U}(\mathfrak{g}) \cong \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathcal{U}(\mathfrak{g}_0)$ is proven in [S2], 3.2.

3.1.3. Corollary. — *If $\mathfrak{g}_1$ has infinite dimension then $\mathcal{A}(\mathfrak{g}) = 0$. If $\dim \mathfrak{g}_1$ is even, all elements of $\mathcal{A}(\mathfrak{g})$ are even and if $\dim \mathfrak{g}_1$ is odd, all elements of $\mathcal{A}(\mathfrak{g})$ are odd.*

Proof. — Retain notation of Lemma 3.1.2. Any element of $\mathcal{U}(\mathfrak{g})$ can be written in a form $u = \sum_J (\operatorname{ad}' x_J)u_J$ where $u_J \in \mathcal{U}(\mathfrak{g}_0)$. Take $u \neq 0$ and set $m = \max\{|J| \mid u_J \neq 0\}$. Assume that $m < \dim \mathfrak{g}_1$. Take J such that $|J| = m$ and $u_J \neq 0$; take $i \in I \setminus J$. Modulo $\sum_{|J'|<m+1}(\operatorname{ad}' x_{J'})\mathcal{U}(\mathfrak{g}_0)$ one has

$$(\operatorname{ad}' x_i)u = \sum_{|J'|=m} (\operatorname{ad}' x_i x_{J'})u_{J'} \neq 0.$$

Thus if $u \in \mathcal{A}(\mathfrak{g})$ then $m = \dim \mathfrak{g}_1$. Since $\mathcal{A}(\mathfrak{g})$ is a $\mathbb{Z}_2$-graded subspace of $\mathcal{U}(\mathfrak{g})$, the assertion follows. □

In the rest of the paper $\mathfrak{g}_1$ is assumed to be finite dimensional.

3.2. Ind **and** Coind. — Consider $\mathfrak{g}_0$ as a (pure even) Lie superalgebra. Let L be a (graded) $\mathfrak{g}_0$-module. Denote by $\operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ the supervector space $\operatorname{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), L)$ (here $\mathcal{U}(\mathfrak{g})$ is considered as a left $\mathcal{U}(\mathfrak{g}_0)$-module) equipped with the following left $\mathcal{U}(\mathfrak{g})$-module structure: $(uf)(u') := f(u'u)$ for any $f \in \operatorname{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), L),\ u, u' \in \mathcal{U}(\mathfrak{g})$.

The module $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ is isomorphic to the module $\operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} {}_*L$ where $\mathfrak{g}_0$-module ${}_*L$ is obtained from L by a certain twist (see [BF], [Ch]). In this subsection we give an explicit construction of this isomorphism.

3.2.1. — Retain notation of Lemma 3.1.2. For $k \in \mathbb{N}$ set

$$\mathcal{F}_o^k := \sum_{J \subseteq I, |J| \leqslant k} \mathcal{U}(\mathfrak{g}_0) x_J.$$

One has $x_J x_{J'} = \pm x_{J \cup J'}$ modulo $\mathcal{F}_o^{|J|+|J'|-1}$. This implies that $\mathcal{F}_o^p \mathcal{F}_o^q \subseteq \mathcal{F}_o^{p+q}$ and thus $\mathcal{F}_o$ is a filtration of $\mathcal{U}(\mathfrak{g})$. In particular, $\mathcal{F}_o^k$ are $\mathcal{U}(\mathfrak{g}_0)$-bimodules and the filtration does not depend from the choice of $\{x_i\}_{i \in I}$.

Consider $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g}_0)$ as left $\mathcal{U}(\mathfrak{g}_0)$-modules through the left multiplication. Denote by ι a $\mathfrak{g}_0$-homomorphism from $\mathcal{U}(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{g}_0)$ such that $\ker \iota = \mathcal{F}_o^{|I|-1}$ and $\iota(x_I) = 1$. Recall that $\ker \iota$ does not depend from the choice of basis in $\mathfrak{g}_1$. Modulo $\mathcal{F}_o^{|I|-1}$ for any $g \in \mathfrak{g}_0$ one has $g x_I = x_I g + c(g) x_I$ where $c(g)$ stands for the eigenvalue of g in the one-dimensional $\mathfrak{g}_0$-module $\Lambda^{\mathrm{top}} \mathfrak{g}_1$. Thus

$$(3) \qquad \iota(ug) = \iota(u)(g - c(g)), \quad \iota(gu) = g\iota(u), \quad \forall g \in \mathfrak{g}_0, u \in \mathcal{U}(\mathfrak{g}).$$

Define a map $(.|.)$ from $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} \mathcal{U}(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{g}_0)$ by setting $(u|u') = \iota(uu')$. For any subsets J, J' of I set $\delta_{J,J'} = 1$ if $J = J'$ and $\delta_{J,J'} = 0$ otherwise.

3.2.2. Lemma. — *For any $J \subseteq I$ there exist $u_J, v_J \in \mathcal{U}(\mathfrak{g})$ such that $(u_J|x_{J'}) = (x_{J'}|v_J) = \delta_{J,J'}$.*

Proof. — We prove the existence of v_J by induction on $r = |I \setminus J|$. For $r = 0$, $J = I$ and $v_I = 1$ satisfies the conditions.

Fix $J \subseteq I$. For any $J' \subseteq I$ such that $|J'| \leqslant |J|$, one has $x_{J'} x_{I \setminus J} = \pm x_{I \setminus J \cup J'}$ modulo $\ker \iota = \mathcal{F}_o^{|I|-1}$. Thus $(x_{J'} \mid x_{I \setminus J}) = 0$ for $J \neq J'$ and $(x_J \mid x_{I \setminus J}) = \pm 1$. Set

$$v := x_{I \setminus J} - \sum_{|J'| > |J|} v_{J'} (x_{J'} \mid x_{I \setminus J}).$$

Then $(x_{J'}|v) = 0$ for any $J' \subseteq I, J' \neq J$ and $(x_J|v) = \pm 1$. This proves the assertion.

The existence of u_J can be shown similarly. □

3.2.3. — Consider $\mathfrak{g}_1$ as odd vector space endowed by the structure of $\mathfrak{g}_0$-module. Then $\Lambda^{\mathrm{top}} \mathfrak{g}_1$ is a one-dimensional $\mathfrak{g}_0$-module which is even iff $\dim \mathfrak{g}_1$ is even. For a (graded) $\mathfrak{g}_0$-module L denote by ${}_*L$ the graded $\mathfrak{g}_0$-module $L \otimes \Lambda^{\mathrm{top}} \mathfrak{g}_1$. We will consider ${}_*L$ as the vector space L (if $\dim \mathfrak{g}_1$

is even) or $\Pi(L)$ (otherwise) equipped by a new structure of $\mathfrak{g}_0$-module given by $g * m := (g + c(g))m$ for any $g \in \mathfrak{g}_0, m \in L$.

3.2.4. PROPOSITION. — *For any $\mathfrak{g}_0$-module L the linear map Ψ defined by*

$$\Psi(u' \otimes m)(u) := (u|u') * m, \ \forall m \in L, \ u, u' \in \mathcal{U}(\mathfrak{g})$$

provides an isomorphism $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L \xrightarrow{\sim} \operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} (L \otimes \Lambda^{\mathrm{top}} \mathfrak{g}_1)$.

Proof. — If $\dim \mathfrak{g}_1$ is even then $(u|u') = 0$ provided that u, u' are graded elements of distinct parity in $\mathcal{U}(\mathfrak{g})$. Similarly, if $\dim \mathfrak{g}_1$ is odd then $(u|u') = 0$ provided that u, u' are graded elements of the same parity in $\mathcal{U}(\mathfrak{g})$. This shows that the map Ψ respects the $\mathbb{Z}_2$-grading.

For any $g \in \mathfrak{g}_0$ and $m \in L$ one has, by (3)

$$(u|u'g) * m = \iota(uu'g) * m = \big(\iota(uu')(g - c(g))\big) * m = (u|u') * (gm)$$

and thus $\Psi(u'g \otimes m) = \Psi(u' \otimes gm)$. Moreover $\Psi(u' \otimes m)$ is a $\mathfrak{g}_0$-linear map since

$$\Psi(u' \otimes m)(gu) = (gu|u')m = g(u|u')m = g\Psi(u' \otimes m)(u).$$

Hence Ψ is a well-defined map from $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} L$ to $\operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} {}_*L$.

For any $s \in \mathcal{U}$ one has

$$\Psi(su' \otimes m)(u) = (u|su')m = (us|u')m = \Psi(u' \otimes m)(us) = (s\Psi(u' \otimes m))(u)$$

and so Ψ is a homomorphism of left $\mathcal{U}(\mathfrak{g})$-modules.

Any element of $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ can be written in the form $\sum_{J \subseteq I} x_J \otimes m_J$ where $m_J \in L$. Fix $J' \subseteq I$ and choose $u_{J'} \in \mathcal{U}(\mathfrak{g})$ as in Lemma 3.2.2. Then $\Psi(\sum_{J \subseteq I} x_J \otimes m_J)(u_{J'}) = m_{J'}$. This implies that $\ker \Psi = 0$.

Fix $J \subseteq I$ and choose $v_J \in \mathcal{U}(\mathfrak{g})$ as in Lemma 3.2.2. Then for any $m \in L$ one has $\Psi(v_J \otimes m)(x_{J'}) = \delta_{J,J'} m$. This implies the surjectivity of Ψ and completes the proof. □

3.3. — Retain notation of Lemma 3.2.2.

THEOREM. — *Assume that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra such that $\mathfrak{g}_1$ is finite dimensional. Then the map $\phi : z \mapsto (\mathrm{ad}' v_\emptyset) z$ provides a linear isomorphism from the $\mathfrak{g}_0$-invariants of $\mathcal{U}(\mathfrak{g}_0) \otimes \Lambda^{\mathrm{top}} \mathfrak{g}_1$ onto the anticentre $\mathcal{A}(\mathfrak{g})$. Moreover one has $\operatorname{gr} \phi(z) = x \operatorname{gr} z$ where x is an element of $\Lambda^{\mathrm{top}}(\mathfrak{g}_1)$.*

Proof. — The proof follows from Lemma 3.1.2 and Proposition 3.2.4. We give the full details below.

Set ${}_*\mathcal{U}(\mathfrak{g}_0) := \mathcal{U}(\mathfrak{g}_0) \otimes \Lambda^{\text{top}}\mathfrak{g}_1$. Using notation of Proposition 3.2.4 one has

$$(\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathcal{U}(\mathfrak{g}_0))^{\mathfrak{g}} = \Psi^{-1}\big((\mathrm{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}}\, {}_*\mathcal{U}(\mathfrak{g}_0))^{\mathfrak{g}}\big).$$

For $z \in ({}_*\mathcal{U}(\mathfrak{g}_0))^{\mathfrak{g}_0}$ denote by f_z the linear map $\mathcal{U}(\mathfrak{g}) \to \mathbb{C}$ such that $f_z(1) = z$ and $f_z(\mathfrak{g}\mathcal{U}(\mathfrak{g})) = 0$. The map $z \mapsto f_z$ provides a linear isomorphism $({}_*\mathcal{U}(\mathfrak{g}_0))^{\mathfrak{g}_0} \xrightarrow{\sim} (\mathrm{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}}\, {}_*\mathcal{U}(\mathfrak{g}_0))^{\mathfrak{g}}$. Therefore the map $z \mapsto \Psi^{-1} f_z$ provides a linear isomorphism $({}_*\mathcal{U}(\mathfrak{g}_0))^{\mathfrak{g}_0} \xrightarrow{\sim} (\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathcal{U}(\mathfrak{g}_0))^{\mathfrak{g}}$. Moreover $\Psi^{-1}(f_z) = v_\emptyset \otimes z$ since $(u|v_\emptyset)z = 0$ for $u \in \mathcal{U}(\mathfrak{g})\mathfrak{g}$ and $(1|v_\emptyset)z = z$. Using Lemma 3.1.2, one concludes that $\phi : z \mapsto (\mathrm{ad}'\, v_\emptyset)z$ is a linear isomorphism $({}_*\mathcal{U}(\mathfrak{g}_0))^{\mathfrak{g}_0} \xrightarrow{\sim} \mathcal{A}(\mathfrak{g})$.

The proof of Lemma 3.2.2 shows that $v_\emptyset = \pm x_I + \sum_{J \neq \emptyset} x_{I \setminus J} d_J$ where d_J are certain elements of $\mathcal{U}(\mathfrak{g}_0)$. Therefore

$$\phi(z) := (\mathrm{ad}'\, v_\emptyset)(z) = \Big(\mathrm{ad}'\Big(x_I + \sum_{J \subsetneq I} c_J x_J\Big)\Big) z \tag{4}$$

where c_J are scalars. By the formula (2), $\mathrm{gr}\,\phi(z) = x\, \mathrm{gr}\, z$ for $x := \mathrm{gr}\, x_I \in \Lambda^{\text{top}}(\mathfrak{g}_1)$. This completes the proof. □

3.3.1. *Remark.* — If $\Lambda^{\text{top}}\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module, the map ϕ of Theorem 3.3 provides a linear isomorphism $\mathcal{Z}(\mathfrak{g}_0) \xrightarrow{\sim} \mathcal{A}(\mathfrak{g})$. In particular, $\mathcal{A}(\mathfrak{g}) \neq 0$ in this case, because $\mathcal{Z}(\mathfrak{g}_0)$ contains the base field.

3.4. — A classification theorem of Kac (see [K1], 4.2.1) states that any complex simple finite dimensional Lie superalgebra is isomorphic either to one of the classical Lie superalgebra or to one of the Cartan Lie superalgebras $W(n), S(n), \widetilde{S}(n), H(n)$.

Evidently $\Lambda^{\text{top}}\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module if $\mathfrak{g}_0$ is a semisimple Lie algebra or if $\mathfrak{g}_1 \cong \mathfrak{g}_1^*$ as $\mathfrak{g}_0$-module. In particular, $\Lambda^{\text{top}}\mathfrak{g}_1$ is trivial for all simple classical Lie superalgebras. It is easy to check that it is trivial also for the Cartan Lie superalgebras $S(n), \widetilde{S}(n), H(n)$.

On the other hand, if $\mathfrak{g}_0$ is reductive and $\Lambda^{\text{top}}\mathfrak{g}_1$ is not a trivial $\mathfrak{g}_0$-module, then $(\mathcal{U}(\mathfrak{g}_0) \otimes \Lambda^{\text{top}}\mathfrak{g}_1)^{\mathfrak{g}_0} = 0$ and so $\mathcal{A}(\mathfrak{g}) = 0$. In particular, for the "strange" non-simple Lie superalgebras $p(n)$ one has $\mathcal{A}(\mathfrak{g}) = 0$ (remark that $\mathcal{Z}(\mathfrak{g}) = \mathbb{C}$, see [Sch]).

3.4.1. *Example.* — Consider a Cartan type Lie superalgebra $\mathfrak{g} := W(n)$ $(n > 2)$. Let us show that $\mathcal{A}(\mathfrak{g}) = 0$. Recall that $W(n)$ is a $\mathbb{Z}$-graded

Lie superalgebra $\mathfrak{g} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}^{(k)}$ so that $\mathfrak{g}_0 = \oplus\mathfrak{g}^{(2k)}$, $\mathfrak{g}_1 = \oplus\mathfrak{g}^{(2k+1)}$ and $\dim\mathfrak{g}^{(k)} = n\binom{n}{k+1}$. The Lie algebra $\mathfrak{g}^{(0)}$ is isomorphic to $\mathfrak{gl}(n)$. Denote by e the image of the identity matrix in $\mathfrak{g}^{(0)}$. One has $[e,u] = ku$ for $u \in \mathfrak{g}^{(k)}$. In particular, e acts on $\Lambda^{\text{top}}\mathfrak{g}_1$ by the multiplication on a positive integer. Since the action of e on $\mathcal{U}(\mathfrak{g}_0)$ is semisimple and all eigenvalues are non-negative integers, $[e,m] \neq 0$ for any non-zero $m \in \mathcal{U}(\mathfrak{g}_0) \otimes \Lambda^{\text{top}}\mathfrak{g}_1$. Hence $\mathcal{A}(\mathfrak{g}) = 0$ by Theorem 3.3.

3.4.2. — Let $\Lambda^{\text{top}}\mathfrak{g}_1$ be a trivial $\mathfrak{g}_0$-module.

DEFINITION. — *Denote by T a non-zero $\text{ad}'\mathfrak{g}$-invariant element belonging to* $(\text{ad}'\mathcal{U}(\mathfrak{g}))(1)$.

The element T is defined up to a non-zero scalar and it is even iff $\dim\mathfrak{g}_1$ is even. Observe that, up to a scalar, T is the unique element of the anticentre whose image in $\mathcal{S}(\mathfrak{g})$ belongs to $\Lambda^{\text{top}}(\mathfrak{g}_1)$.

3.5. Remarks.

3.5.1. — Consider $\mathcal{U}(\mathfrak{g})$ as an associative algebra and denote its centre by Z. Evidently $Z \cap \mathcal{U}(\mathfrak{g})_0 = \mathcal{Z}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})_0$ and $Z \cap \mathcal{U}(\mathfrak{g})_1 = \mathcal{A}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})_1$. Using Corollary 3.1.3 one concludes that

$$\begin{array}{ll} Z = \mathcal{Z}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})_0 & \text{if } \dim\mathfrak{g}_1 \text{ is even or } \dim\mathfrak{g}_1 = \infty, \\ Z = (\mathcal{Z}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})_0) \oplus \mathcal{A}(\mathfrak{g}) & \text{if } \dim\mathfrak{g}_1 \text{ is odd.} \end{array}$$

3.5.2. — In most of the cases $\mathcal{U}(\mathfrak{g})$ is not a domain (see [AL]). However, even if $\mathcal{U}(\mathfrak{g})$ is a domain (for example $\mathfrak{g} = \text{osp}(1,2l)$) the direct generalization of the Gelfand-Kirillov conjecture does not hold for Lie superalgebras.

In fact, let k be a field of characteristic zero and $A_n(k)$ be a Weyl algebra over k. Recall that the centre of a Weyl skew field $W_n(k)$ coincides with k and that $A_n(\overline{k}) = A_n(k) \otimes_k \overline{k}$ where $\overline{k}$ stands for the algebraic closure of k. Therefore a Weyl skew field does not contain non-central elements whose squares are central.

Assume that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra such that $\dim\mathfrak{g}_1$ is even and non-zero, $\Lambda^{\text{top}}\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module and $\mathcal{U}(\mathfrak{g})$ is a domain. Then $\mathcal{A}(\mathfrak{g}) \neq 0$ by Theorem 3.3 and $\mathcal{A}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})_0$ by Corollary 3.1.3. Take any non-zero $a \in \mathcal{A}(\mathfrak{g})$. Since a is a non-zero divisor and $ax + xa = 0$ for an odd element x, $a \notin Z$. However $a^2 \in Z$. This implies that a Weyl skew field and a skew field of fractions of $\mathcal{U}(\mathfrak{g})$ are not isomorphic if $\dim \mathfrak{g}_1$ is a non-zero even integer and $\Lambda^{\mathrm{top}}\mathfrak{g}_1$ is a trivial $\mathfrak{g}$-module.

4. The case of basic classical Lie superalgebras.

In this section $\mathfrak{g}$ denotes a basic classical Lie superalgebra (see [K2] and 4.1 below) such that $\mathfrak{g}_1 \neq 0$. In this case the dimension of $\mathfrak{g}_1$ is even and so all elements of $\mathcal{A}(\mathfrak{g})$ are even. In particular, they anticommute with the odd elements of $\mathcal{U}(\mathfrak{g})$ and commute with the even ones.

In this section we show that the restriction of the Harish-Chandra projection $\mathcal{P}$ on $\mathcal{A}(\mathfrak{g})$ is an injection and describe its image. We also prove that $\widetilde{\mathcal{Z}}(\mathfrak{g})$ coincides with the centralizer of $\mathcal{U}(\mathfrak{g})_0$ and with the set of the elements of $\mathcal{U}(\mathfrak{g})$ acting by superconstants on each simple finite dimensional module.

4.1. Notation. — A finite dimensional simple Lie superalgebra $\mathfrak{g}$ is called basic classical if $\mathfrak{g}_0$ is reductive and $\mathfrak{g}$ admits a non-degenerate invariant bilinear form. The list of basic classical Lie superalgebras is the following as determined by Kac (see [K2]):

a) simple Lie agebras

b) $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1,\alpha)$, $F(4)$, $G(3)$.

Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}_0$ and a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. For a $\mathcal{U}(\mathfrak{g})$-module M and an element $\mu \in \mathfrak{h}^*$ set $M|_\mu = \{m \in M \mid hm = \mu(h)m, \ \forall h \in \mathfrak{h}\}$. When we use the notation $\mathcal{U}(\mathfrak{g})|_\mu$, the action of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$ is assumed to be the adjoint action.

The Harish-Chandra projection $\mathcal{P} : \mathcal{U}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{h})$ is the projection with respect to the following triangular decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (\mathcal{U}(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^-\mathcal{U}(\mathfrak{g}))$ (we identify $\mathcal{S}(\mathfrak{h})$ and $\mathcal{U}(\mathfrak{h})$). An element a of $\mathcal{U}(\mathfrak{g})|_0$ acts on a primitive vector of weight μ ($\mu \in \mathfrak{h}^*$) by multiplication by the scalar $\mathcal{P}(a)(\mu)$. The restriction of $\mathcal{P}$ on $\mathcal{U}(\mathfrak{g})|_0 = \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ is an algebra homomorphism from $\mathcal{U}(\mathfrak{g})_0$ to $\mathcal{S}(\mathfrak{h})$ if $\mathfrak{g} \neq A(n,n)$.

Denote by Δ_0 the set of non-zero even roots of $\mathfrak{g}$. Denote by Δ_1 the set of odd roots of $\mathfrak{g}$. Set $\Delta = \Delta_0 \cup \Delta_1$. Set

$$\overline{\Delta}_0 := \{\alpha \in \Delta_0 \mid \alpha/2 \notin \Delta_1\}, \quad \overline{\Delta}_1 := \{\beta \in \Delta_1 \mid 2\beta \notin \Delta_0\}.$$

Note that $\overline{\Delta}_1$ is the set of isotropic roots. Denote by Δ^+ the set of positive roots and define $\Delta_0^+, \Delta_1^+, \overline{\Delta}_0^+$ as usual.

Denote by W the Weyl group of Δ_0. For any $\alpha \in \Delta_0$ let $s_\alpha \in W$ be the corresponding reflection. Let W' be the subgroup of W generated by the reflections $s_\alpha, \alpha \in \overline{\Delta}_0$. Note that $W = W'$ iff all odd roots are isotropic. Otherwise (if $\mathfrak{g}$ is of the type $B(m,n)$ or $G(3)$) W' is a subgroup of index two.

Set

$$\rho_0 := \frac{1}{2}\sum_{\alpha\in\Delta_0^+}\alpha, \quad \rho_1 := \frac{1}{2}\sum_{\alpha\in\Delta_1^+}\alpha, \quad \rho := \rho_0 - \rho_1.$$

Define the translated action of W on $\mathfrak{h}^*$ by the formula

$$w.\lambda = w(\lambda+\rho) - \rho, \quad \forall\lambda \in \mathfrak{h}^*, w \in W.$$

Define the left translated action of W on $\mathcal{S}(\mathfrak{h})$ by setting $w.f(\lambda) = f(w^{-1}.\lambda)$ for any $\lambda \in \mathfrak{h}^*$.

Fix a non-degenerate $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$ and denote by $(-,-)$ the induced bilinear form on $\mathfrak{h}^*$. This form is non-degenerate and W-invariant.

4.1.1. — For $\lambda \in \mathfrak{h}^*$ denote a graded $\mathfrak{g}$-Verma module of the highest weight λ by $\widetilde{M}(\lambda)$ where the grading is fixed in such a way that a highest weight vector has degree zero. By [LM], an element of $\mathcal{U}(\mathfrak{g})$ annihilating the modules $\widetilde{M}(\lambda)$ for λ running through a Zariski dense subset of $\mathfrak{h}^*$, is equal to zero.

The following result is due to Kac (see, for example, [Ja]).

4.1.2. Lemma. — *Assume that a pair (n,α) belongs to the following set:*

$$(\mathbb{N}^+ \times \overline{\Delta_0^+}) \cup ((1+2\mathbb{N}) \times (\Delta_1^+ \setminus \overline{\Delta_1^+}))$$

and $\lambda \in \mathfrak{h}^*$ is such that $2(\lambda+\rho,\alpha) = n(\alpha,\alpha)$. Then $\widetilde{M}(\lambda)$ contains a primitive vector of the weight $\lambda - n\alpha$. Moreover this vector is even iff α is even. If $\alpha \in \overline{\Delta_1^+}$ and $(\lambda+\rho,\alpha) = 0$ then $\widetilde{M}(\lambda)$ contains an odd primitive vector of the weight $\lambda - \alpha$.

4.2. — For $\beta \in \mathfrak{h}^*$ denote by h_β the element of $\mathfrak{h}$ such that $\mu(h_\beta) = (\mu,\beta)$ for any $\mu \in \mathfrak{h}^*$. Set

$$t := \prod_{\alpha\in\Delta_1^+}(h_\alpha + (\rho,\alpha)).$$

4.2.1. LEMMA. — *The restriction of the Harish-Chandra projection $\mathcal{P}$ provides a linear injective map $\mathcal{A}(\mathfrak{g}) \to t\mathcal{S}(\mathfrak{h})^{W.}$.*

Proof. — Recall that $a \in \mathcal{U}(\mathfrak{g})|_0$ acts on a primitive vector of weight μ ($\mu \in \mathfrak{h}^*$) by multiplication by the scalar $\mathcal{P}(a)(\mu)$. Fix a non-zero $a \in \mathcal{A}(\mathfrak{g})$. Since $\mathcal{A}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})|_0$, a acts on the even component of $\widetilde{M}(\lambda)$ by multiplication by the scalar $\mathcal{P}(a)(\lambda)$ and on the odd component of $\widetilde{M}(\lambda)$ by multiplication by $(-\mathcal{P}(a)(\lambda))$. The intersection of the annihilators of all Verma modules is zero (see 4.1.1) and so $\mathcal{P}(a)$ is a non-zero polynomial in $\mathcal{S}(\mathfrak{h})$.

Choose a pair (n, α) and an element λ satisfying the assumption of Lemma 4.1.2. Then $\mathcal{P}(a)(\lambda) = \mathcal{P}(a)(\lambda - n\alpha)$ for $\alpha \in \overline{\Delta}_0^+$ and $\mathcal{P}(a)(\lambda) = -\mathcal{P}(a)(\lambda - n\alpha)$ for $\alpha \in \Delta_1^+ \setminus \overline{\Delta}_1^+$. Note that $\lambda - n\alpha = s_\alpha.\lambda$ if $\alpha \in \overline{\Delta}_0^+$ and $\lambda - n\alpha = s_{2\alpha}.\lambda$ if $\alpha \in (\Delta_1^+ \setminus \overline{\Delta}_1^+)$. Therefore $\mathcal{P}(a)(\lambda) = \mathcal{P}(a)(s_\alpha.\lambda) = s_\alpha.\mathcal{P}(a)(\lambda)$ for $\alpha \in \overline{\Delta}_0^+$ and $\mathcal{P}(a)(\lambda) = -s_{2\alpha}.\mathcal{P}(a)(\lambda)$ for $\alpha \in (\Delta_1^+ \setminus \overline{\Delta}_1^+)$. For fixed $\alpha \in \overline{\Delta}_0^+ \cup (\Delta_1^+ \setminus \overline{\Delta}_1^+)$ the set of $\lambda \in \mathfrak{h}^*$ satisfying the assumption of Lemma 4.1.2 is a Zariski dense subset of $\mathfrak{h}^*$. Thus $\mathcal{P}(a) \in \mathcal{S}(\mathfrak{h})$ is $W.'$-invariant and $\mathcal{P}(a) = -s_{2\alpha}.\mathcal{P}(a)$ for $\alpha \in (\Delta_1^+ \setminus \overline{\Delta}_1^+)$. The last implies that $P(a)$ is divisible by $(h_\alpha + (\rho, \alpha))$.

Now take $\alpha \in \overline{\Delta}_1^+$. Then α is isotropic. Take $\lambda \in \mathfrak{h}^*$ satisfying $(\lambda + \rho, \alpha) = 0$. Using Lemma 4.1.2, we conclude that $\mathcal{P}(a)(\lambda) = (-1)^n \mathcal{P}(a)(\lambda - n\alpha)$ for any $n \in \mathbb{N}$. Therefore $\mathcal{P}(a)(\lambda) = 0$ if $(\lambda + \rho, \alpha) = 0$. Thus $\mathcal{P}(a)$ is divisible by $(h_\alpha + (\rho, \alpha))$.

Hence $\mathcal{P}(a)$ is divisible by $(h_\alpha + (\rho, \alpha))$ for any $\alpha \in \Delta_1^+$ and so it is divisible by t. It is easy to see that t is $W.'$-invariant and $t = -s_{2\alpha}.t$ for $\alpha \in (\Delta_1^+ \setminus \overline{\Delta}_1^+)$. Thus $\mathcal{P}(a)/t$ is $W.$-invariant as required. □

4.2.2. — Define a filtration on $\mathfrak{g}$ by setting $\mathcal{F}_u^0 = 0$, $\mathcal{F}_u^1 = \mathfrak{g}_1$, $\mathcal{F}_u^2 = \mathfrak{g}$ and extend it canonically to an increasing filtration on $\mathcal{U}(\mathfrak{g})$. Let $z \in \mathcal{Z}(\mathfrak{g}_0)$ have a degree r with respect to the canonical filtration. Then, by (4), $\phi(z) \in \mathcal{F}_u^{\dim \mathfrak{g}_1 + 2r}$ and so $\mathcal{P}(\phi(z))$ is a polynomial of degree less than or equal to $(\dim \mathfrak{g}_1 + 2r)/2 = |\Delta_1^+| + r$. In particular, $\mathcal{P}(\phi(1))$ is a polynomial of degree less than or equal to $|\Delta_1^+|$ and so it is equal to t up to a non-zero scalar. Recall that the map ϕ depends on the choice of basis $\{x_i\}_{i \in I}$; choose a basis $\{x_i\}_{i \in I}$ such that $\mathcal{P}(\phi(1)) = t$.

4.2.3. — Fix $r \in \mathbb{N}$ and set $Z_r := \mathcal{Z}(\mathfrak{g}_0) \cap \mathcal{F}^r$. Denote by S_r

the space of $W.$-invariant polynomials of degree less than or equal to r. Take $z \in Z_r$. Combining Lemma 4.2.1 and 4.2.2, we conclude that $(\mathcal{P}(\phi(z_r))/t) \in S_r$. Recall that $\operatorname{gr}\mathcal{Z}(\mathfrak{g}_0) \xrightarrow{\sim} \mathcal{S}(\mathfrak{h})^W$ as graded algebras and so $\dim Z_r = \dim S_r$. Since ϕ is a linear isomorphism, it follows that $\mathcal{P}(Z_r) = tS_r$.

4.2.4. COROLLARY. — *The restriction of the Harish-Chandra projection $\mathcal{P}$ provides a linear bijective map $\mathcal{A}(\mathfrak{g}) \to t\mathcal{S}(\mathfrak{h})^{W.}$. In particular, $\mathcal{P}(T) = t$.*

4.2.5. LEMMA. — *Any non-zero element $z \in \mathcal{A}(\mathfrak{g})$ is a non-zero divisor in $\mathcal{U}(\mathfrak{g})$.*

Proof. — Assume that $zu = 0$. Recall that z acts by multiplication by $\mathcal{P}(z)(\lambda)$ (resp. $-\mathcal{P}(z)(\lambda)$) on the even (resp. odd) graded component of $\widetilde{M}(\lambda)$. Therefore u annihilates $\widetilde{M}(\lambda)$ when λ is such that $\mathcal{P}(z)(\lambda) \neq 0$. Since $\mathcal{P}(z) \neq 0$, the set $\{\lambda \mid \mathcal{P}(z)(\lambda) \neq 0\}$ is a Zariski dense subset of $\mathfrak{h}^*$. By 4.1.1, it implies that $u = 0$. □

4.2.6. COROLLARY.

$$\mathcal{Z}(\mathfrak{g}) \cap \mathcal{A}(\mathfrak{g}) = 0.$$

Proof. — For any $z \in \mathcal{Z}(\mathfrak{g}) \cap \mathcal{A}(\mathfrak{g})$ and any element $u \in \mathfrak{g}_1$ one has $zu = uz = -zu$ and so $zu = 0$. Hence $z = 0$ by Lemma 4.2.5. □

4.3. The structure of $\widetilde{\mathcal{Z}}(\mathfrak{g})$. — The algebra $\widetilde{\mathcal{Z}}(\mathfrak{g})$ has the following easy realization. Consider the algebra $\widetilde{\mathcal{S}(\mathfrak{h})} := \mathcal{S}(\mathfrak{h})[\xi]/(\xi^2 - 1)$. Define a map $\mathcal{P}' : \widetilde{\mathcal{Z}}(\mathfrak{g}) \to \widetilde{\mathcal{S}(\mathfrak{h})}$ by setting $\mathcal{P}'(z) = \mathcal{P}(z)$ for $z \in \mathcal{Z}(\mathfrak{g})$ and $\mathcal{P}'(z) = \mathcal{P}(z)\xi$ for $z \in \mathcal{A}(\mathfrak{g})$. Since $\widetilde{\mathcal{Z}}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})|_0$, the restriction of $\mathcal{P}$ on $\widetilde{\mathcal{Z}}(\mathfrak{g})$ is an algebra homomorphism. Taking into account Corollary 4.2.4, we conclude that $\mathcal{P}'$ provides an algebra isomorphism from $\widetilde{\mathcal{Z}}(\mathfrak{g})$ onto the subalgebra $(\mathcal{P}(\mathcal{Z}(\mathfrak{g})) \oplus t\mathcal{S}(\mathfrak{h})^{W.}\xi)$ of $\widetilde{\mathcal{S}(\mathfrak{h})}$.

4.3.1. — Assume that $\mathfrak{g}$ is of the type $B(m,n)$ or $G(3)$. Then $W' \neq W$ and so t is not $W.$-invariant. Therefore $\mathcal{P}(\mathcal{A}(\mathfrak{g})) \cap \mathcal{P}(\mathcal{Z}(\mathfrak{g})) = \{0\}$. Then, using Corollary 4.2.4, we conclude that the restriction of the Harish-Chandra projection provides an algebra isomorphism $\widetilde{\mathcal{Z}}(\mathfrak{g}) \cong (\mathcal{P}(\mathcal{Z}(\mathfrak{g})) \oplus t\mathcal{S}(\mathfrak{h})^{W.})$.

In all other cases, $\mathcal{P}(\mathcal{A}(\mathfrak{g})) \subset \mathcal{P}(\mathcal{Z}(\mathfrak{g}))$.

4.3.2. — In the case when $\mathfrak{g} = \mathrm{osp}(1,2l)$, $\mathcal{Z}(\mathfrak{g})$ is a polynomial algebra and $\mathcal{A}(\mathfrak{g})$ is a cyclic $\mathcal{Z}(\mathfrak{g})$ module generated by T. In other cases (when $\mathfrak{g}$ is basic classical Lie superalgebra) this does not hold. However, a similar result hold after a certain localization.

More precisely, if $\mathfrak{g} \neq \mathrm{osp}(1,2l)$ (that is $\mathfrak{g}$ is not of the type $B(0,l)$) then $\mathcal{P}(\mathcal{Z}(\mathfrak{g}))$ is strictly contained in $\mathcal{S}(\mathfrak{h})^{W\cdot}$. However, since the product of two elements from the anticentre belongs to the centre, $\mathcal{P}(\mathcal{Z}(\mathfrak{g}))$ contains $t^2\mathcal{S}(\mathfrak{h})^{W\cdot}$. Set $Q := T^2$, $q := t^2$. Then the localized algebras $\mathcal{Z}(\mathfrak{g})[Q^{-1}]$ and $\mathcal{S}(\mathfrak{h})^{W\cdot}[q^{-1}]$ are isomorphic. Moreover $\widetilde{\mathcal{Z}}(\mathfrak{g})[Q^{-1}] = \mathcal{Z}(\mathfrak{g})[Q^{-1}] \oplus \mathcal{A}(\mathfrak{g})[Q^{-1}]$ and $\mathcal{A}(\mathfrak{g})[Q^{-1}]$ is a cyclic $\mathcal{Z}(\mathfrak{g})[Q^{-1}]$-module generated by T.

4.4. The action of $\widetilde{\mathcal{Z}}(\mathfrak{g})$ on the simple modules. — Let us say that an element $u \in \mathcal{U}(\mathfrak{g})$ acts on a $\mathcal{U}(\mathfrak{g})$-module M by a superconstant if it acts by a multiplication by a scalar on each graded component of M. By 2.2, each element of $\widetilde{\mathcal{Z}}(\mathfrak{g})$ acts by a superconstant on any simple module. In this subsection we shall prove that actually $\widetilde{\mathcal{Z}}(\mathfrak{g})$ coincides with the set of elements of $\mathcal{U}(\mathfrak{g})$ which act by superconstants on each simple finite dimensional module. Moreover $\widetilde{\mathcal{Z}}(\mathfrak{g})$ coincides with the centralizer of the even part $\mathcal{U}(\mathfrak{g})_0$ in $\mathcal{U}(\mathfrak{g})$.

4.4.1. — By definition, $\widetilde{\mathcal{Z}}(\mathfrak{g})$ lies in the centralizer of $\mathcal{U}(\mathfrak{g})_0$ in $\mathcal{U}(\mathfrak{g})$ and even in the centre of $\mathcal{U}(\mathfrak{g})_0$ since all elements of $\widetilde{\mathcal{Z}}(\mathfrak{g})$ are even.

Let A be the centralizer of $\mathcal{U}(\mathfrak{g})_0$ in $\mathcal{U}(\mathfrak{g})$ and a be an element of A. Clearly, a acts by a superconstant on any Verma module. On the even component of $\widetilde{M}(\lambda)$ a acts by $\mathcal{P}(a)(\lambda)$. Let $f(a)$ be the function $\mathfrak{h}^* \to k$ such that a acts by $f(a)(\lambda)$ on the odd component of $\widetilde{M}(\lambda)$.

4.4.2. LEMMA. — *For any $a \in A$ the function $f(a) : \mathfrak{h}^* \to k$ is polynomial.*

Proof. — Choose $y \in \mathfrak{n}_1^-$ and $x \in \mathfrak{n}_1^+$ such that $h := [y,x] \in \mathcal{S}(\mathfrak{h})$ and $h \neq 0$. For each $\mu \in \mathfrak{h}^*$ choose a highest weight vector $v_\mu \in \widetilde{M}(\mu)$. Then yv_μ is odd and so

$$xayv_\mu = f(a)(\mu)xyv_\mu = f(a)(\mu)h(\mu)v_\mu.$$

Since $xay \in \mathcal{U}(\mathfrak{g})|_0$ one has $xayv_\mu = \mathcal{P}(xay)(\mu)v_\mu$. Thus

$$f(a)(\mu)h(\mu) = \mathcal{P}(xay)(\mu). \tag{5}$$

This implies that $\mathcal{P}(xay)(\mu)$ vanishes on the hyperplane $\{\mu \mid h(\mu) = 0\}$. Therefore h divides $\mathcal{P}(xay)(\mu)$ and so $f(a) = \mathcal{P}(xay)/h$ is a polynomial.□

4.4.3. — Lemma 4.4.2 implies that an element $a \in A$ acts by $\mathcal{P}(a)(\lambda)$ on the even component of $\widetilde{M}(\lambda)$ and by $f(\lambda)$ on the odd component of $\widetilde{M}(\lambda)$. Arguing as in 4.2.1, we obtain that $P' := \mathcal{P}(a) - f(a)$ belongs to $t\mathcal{S}(\mathfrak{h})^{W\cdot}$. Similarly, $P := \mathcal{P}(a) + f(a)$ belongs to $\mathcal{S}(\mathfrak{h})^{W\cdot}$ and moreover for any $\alpha \in \overline{\Delta}_1{}^{+}$ one has $P(\lambda - \alpha) = P(\lambda)$ if $(\lambda + \rho, \alpha) = 0$. By [S1] and Corollary 4.2.4, this implies that $P = \mathcal{P}(z)$ for some $z \in \mathcal{Z}(\mathfrak{g})$ and $P' = \mathcal{P}(z')$ for some $z' \in \mathcal{A}(\mathfrak{g})$. Then $a - (z + z')/2$ kills any Verma module and so $a = (z + z')/2$. This proves that $\widetilde{\mathcal{Z}}(\mathfrak{g}) = A$.

The intersection of the annihilators of all simple highest weight modules is equal to zero (see 4.1.1). This implies that the set of elements of $\mathcal{U}(\mathfrak{g})$ acting by superconstants on each graded simple finite dimensional module coincides with A. Hence we obtain

4.4.4. Corollary. — *If $\mathfrak{g}$ is a basic classical Lie superalgebra then the following algebras coincide:*

i) *The algebra of elements of $\mathcal{U}(\mathfrak{g})$ which act by superconstants on each graded simple finite dimensional module.*

ii) *The algebra $\widetilde{\mathcal{Z}}(\mathfrak{g})$.*

iii) *The centre of $\mathcal{U}(\mathfrak{g})_0$.*

iv) *The centralizer of $\mathcal{U}(\mathfrak{g})_0$.*

4.5. A remark concerning the separation theorem. — An important structure theorem of Kostant asserts that for any finite dimensional semisimple Lie algebra there exists an ad $\mathfrak{g}$-submodule $\mathcal{H}$ of $\mathcal{U}(\mathfrak{g})$ such that the multiplication map induces the isomorphism $\mathcal{H} \otimes \mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})$. In [Mu1], I. Musson proved the similar assertion for $\mathfrak{g} = \mathrm{osp}(1, 2l)$. These theorems are called the separation theorems. We shall show that separation theorem does not hold for any basic classical Lie superalgebra apart from finite dimensional simple Lie algebras and $\mathfrak{g} = \mathrm{osp}(1, 2l)$.

Indeed, by (1) the right multiplication by T provides a $\mathfrak{g}$-isomorphism from the ad$'\mathfrak{g}$-module generated by 1 onto the ad $\mathfrak{g}$-module generated by T. Thus $T^2 \in (\mathrm{ad}\,\mathcal{U}(\mathfrak{g}))T \xrightarrow{\sim} \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} V$ where V stands for a trivial $\mathfrak{g}_0$-module. Let $\mathfrak{g}$ be a basic classical Lie superalgebra which is neither simple Lie algebra nor $\mathrm{osp}(1, 2l)$. Then $\mathfrak{g}_1$ contains a non-zero element x such that $[x, x] = 0$. One has $x\,\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} V = \{m \in \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} V \mid xm = 0\}$. Since $(\mathrm{ad}\, x)T^2 = 0$ there exists $u \in (\mathrm{ad}\,\mathcal{U}(\mathfrak{g}))T$ such that $T^2 = (\mathrm{ad}\, x)u$. Thus $T^2 \in ((\mathrm{ad}\, x)\mathcal{U}(\mathfrak{g}) \cap \mathcal{Z}(\mathfrak{g}))$. However $(\mathrm{ad}\, x)\mathcal{U}(\mathfrak{g}) \cap \mathcal{Z}(\mathfrak{g}) \neq 0$ contradicts to the existence of an $\operatorname{ad}\mathfrak{g}$-submodule $\mathcal{H}$ satisfying $\mathcal{H} \otimes \mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})$.

5. Questions.

5.1. — The centralizer of $\mathcal{U}(\mathfrak{g})_0$ contains $\widetilde{\mathcal{Z}}(\mathfrak{g})$. Do they coincide provided that $\Lambda^{\mathrm{top}}\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module?

5.2. — Let C be the set of the elements of $\mathcal{U}(\mathfrak{g})$ which act by a superconstant on each simple module. Clearly, C is a subalgebra of $\mathcal{U}(\mathfrak{g})$. By 2.2, C contains $\widetilde{\mathcal{Z}}(\mathfrak{g})$ if $\dim \mathfrak{g}_1$ is even. Assume that $\dim \mathfrak{g}_1$ is even and that the intersection of all graded primitive ideals of $\mathcal{U}(\mathfrak{g})$ is zero. Does this imply that $C = \widetilde{\mathcal{Z}}(\mathfrak{g})$?

5.3. — In the case when $\mathfrak{g}$ is a basic classical Lie superalgebra both answers are positive (see Corollary 4.4.4).

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