ANNALES

DE

L'INSTITUT FOURIER

Jan Hendrik BRUINIER & Eberhard FREITAG

Local Borcherds products

Tome 51, nº 1 (2001), p. 1-26.

<http://aif.cedram.org/item?id=AIF_2001__51_1_1_0>

Ann. Inst. Fourier, Grenoble
51, 1 (2001), 1–27

LOCAL BORCHERDS PRODUCTS

by J. H. BRUINIER and E. FREITAG

1. Introduction.

Let L be an even lattice of signature $(2, l)$. Throughout we assume $l \geqslant 3$ and furthermore that $L \otimes_{\mathbb{Z}} \mathbb{Q}$ contains two orthogonal hyperbolic planes (some results require $l \geqslant 4$). Let Γ be a subgroup of finite index of the orthogonal group of L and let X_Γ be the Baily-Borel compactification of $\mathcal{H}_l/\Gamma$, where $\mathcal{H}_l$ denotes the corresponding Hermitean symmetric domain. The boundary of this compactification is a curve, which usually has many components. We consider in each component of this curve a generic point s. We want to investigate the local divisor class group of X_Γ in s. This is roughly the group of analytic line bundles $\operatorname{Pic}(U_{\mathrm{reg}})$ on the regular locus of a small open neighbourhood U of s. Up to some exceptional cases the boundary of X_Γ consists of singular points. In any case, here we define

$$U_{\mathrm{reg}} = U \cap (\mathcal{H}_l/\Gamma).$$

As a precise definition of the local Picard group we take

$$\operatorname{Pic}(X_\Gamma, s) = \varinjlim \operatorname{Pic}(U_{\mathrm{reg}}), \tag{1.1}$$

where U runs through all open neighbourhoods of s. Our first goal is to compute this local Picard group. To this end one has to determine the local analytic cohomology

$$\varinjlim H^1(U_{\mathrm{reg}}, \mathcal{O}_{X_\Gamma}).$$

Keywords: Automorphic form – Automorphic product – Orthogonal group – Heegner divisor – Local Picard group.
Math. classification: 11F55 – 14L35.

This has been done in a more general context by Ballweg in his Heidelberg thesis [Ba]. Using the result of Ballweg we obtain a satisfactory description of the local Picard group.

The cusp s corresponds to a Γ-conjugacy class of parabolic subgroups $P \subset \mathrm{O}(2,l)$. We choose one such P and denote by Γ_∞ the intersection of P with Γ.

For sufficiently neat Γ this is a two-step nilpotent group, which splits into a semi-direct product

$$0 \longrightarrow \mathfrak{t} \longrightarrow \Gamma_\infty \longrightarrow \Lambda \longrightarrow 0.$$

The main result of the local computation can be expressed by the exact sequence

$$\operatorname{Hom}(\Lambda, R^*) \longrightarrow \operatorname{Pic}(X_\Gamma, s) \longrightarrow H^2(\Gamma_\infty, \mathbb{Z}) \longrightarrow \varinjlim H^2(U_{\mathrm{reg}}, \mathcal{O}_{X_\Gamma}).$$

Here R denotes the ring of convergent power series in one complex variable and R^* its group of units. (This ring occurs as the local ring of s in its boundary component). The kernel of $H^2(\Gamma_\infty, \mathbb{Z}) \to \varinjlim H^2(U_{\mathrm{reg}}, \mathcal{O}_{X_\Gamma})$ turns out to be non-trivial. Let D be a divisor in a small neighbourhood of s. From the above exact sequence we see that there are two obstructions for D to be the divisor of a meromorphic function (in a possibly smaller neighbourhood of s). The first obstruction is a Chern class in $H^2(\Gamma_\infty, \mathbb{Z})$. We will find many examples where this Chern class is not trivial. If it is trivial there is a second obstruction, a (usually non-unitary) character of Λ. So our computation shows that it is a rather restrictive property of a divisor to be principal, i.e., the divisor of a meromorphic function.

In Section 4 we investigate these obstructions for *Heegner divisors*. Recall that for any β in the discriminant group of L and any negative rational integer m (satisfying a congruence condition modulo 1) the Heegner divisor $H(\beta, m)$ of discriminant (β, m) is an algebraic divisor on X_Γ (see (4.1) for a precise definition). It defines an element of $\operatorname{Pic}(X_\Gamma, s)$ which can be realized as an automorphy factor for Γ_∞ on the inverse image U of a small neighbourhood of s under the canonical map $\mathcal{H}_l \to X_\Gamma$. To construct this automorphy factor explicitly we introduce a certain *local Borcherds product* Ψ. This is a holomorphic function on $\mathcal{H}_l$ defined as an infinite product, whose divisor (Ψ) is invariant under Γ_∞. The restriction of (Ψ) to U equals the pullback of $H(\beta, m)$. The function Ψ can be viewed as a local analogue of the automorphic products discovered by Borcherds [Bo1] or more precisely of the generalized Borcherds products attached to Heegner divisors considered in [Br1]. Needless to mention that our construction is quite trivial compared to the deep theory of Borcherds. However, it seems remarkable that the local products have similar properties and carry non-trivial information on the geometry of X_Γ.

The automorphy factor

$$J(g, Z) = \Psi(gZ)/\Psi(Z) \qquad (g \in \Gamma_\infty,\ Z \in \mathcal{H}_l)$$

is a cocycle, which represents the image of $H(\beta, m)$ in $\operatorname{Pic}(X_\Gamma, s)$. It can be computed explicitly, and as an immediate consequence we may determine its Chern class in $H^2(\Gamma_\infty, \mathbb{Z})$ (Proposition 4.3).

It turns out that these Chern classes can be described by means of certain vector valued theta series of weight $1+l/2$ for the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ (Theorem 4.5 and Proposition 5.1). One may infer that they are related to the global obstructions for the existence of automorphic products, that occur in the theory of Borcherds [Bo1], [Bo2].

In the special case that L is unimodular we use a result due to Waldspurger [Wal] to show that the local obstructions generate the space of global obstructions. Let H be a linear combinations of Heegner divisors. Assume that for every one-dimensional irreducible component B of the boundary of X_Γ and a generic point $s \in B$ the divisor H is a torsion element of $\operatorname{Pic}(X_\Gamma, s)$. Then our Proposition 5.1 combined with Waldspurger's result implies that there exists a Borcherds product for the orthogonal group of L whose divisor equals H (see Theorem 5.4).

As a corollary we find that any meromorphic modular form for the orthogonal group of L, whose divisor is a linear combination of Heegner divisors, is a Borcherds product. This was also proved in greater generality in [Br1], [Br2]. However, in these papers a completely different argument is used, which does not say anything about the local Picard groups of X_Γ.

2. Boundary components.

As in the introduction we consider an even lattice L of rank $l+2$, whose symmetric bilinear form $(\cdot,\cdot)$ has signature $(2,l)$. Then the quadratic form

$$q(x) = \frac{1}{2}(x, x)$$

has integral values. Throughout we assume $l \geqslant 3$ and that $L \otimes_{\mathbb{Z}} \mathbb{Q}$ splits two hyperbolic planes. (This is always true if $l \geqslant 5$.) The dual lattice of L is denoted by L'.

We extend the bilinear form to a $\mathbb{C}$-bilinear form on the complexification $L \otimes_{\mathbb{Z}} \mathbb{C}$. We consider the following chain of subsets of the associated projective space $P(L \otimes_{\mathbb{Z}} \mathbb{C})$:

$$\mathcal{H}_l \subset \mathcal{K} \subset \mathcal{N} \subset P(L \otimes_{\mathbb{Z}} \mathbb{C}).$$

Here $\mathcal{N}$ denotes the zero quadric, i.e., the subset of $P(L \otimes_{\mathbb{Z}} \mathbb{C})$ represented by vectors z of norm zero $(z, z) = 0$. The open subset $\mathcal{K}$ is defined by the condition $(z, \bar{z}) > 0$. It has two connected components. We choose one of them and denote it by $\mathcal{H}_l$. The real orthogonal group $\mathrm{O}(2, l) = \mathrm{O}(L \otimes_{\mathbb{Z}} \mathbb{R})$ acts on $L \otimes_{\mathbb{Z}} \mathbb{C}$, $P(L \otimes_{\mathbb{Z}} \mathbb{C})$, $\mathcal{N}$, and $\mathcal{K}$. A subgroup of index 2 (the spinor kernel) $\mathrm{O}'(2, l)$ acts on $\mathcal{H}_l$.

We now describe the boundary of $\mathcal{H}_l$ in the zero quadric $\mathcal{N}$: Let $F \subset L \otimes_{\mathbb{Z}} \mathbb{R}$ be an isotropic line. It is easy to see that the associated point in $\mathcal{N}$ lies in the boundary of $\mathcal{H}_l$.

Remark and Definition 2.1.

i) *Let $F \subset L \otimes_{\mathbb{Z}} \mathbb{R}$ be an isotropic line. Then F represents a boundary point of $\mathcal{H}_l$. A boundary point of this type is called* special, *otherwise* generic. *A set consisting of one special boundary point is called a zero-dimensional boundary component.*

ii) *Let $F \subset L \otimes_{\mathbb{Z}} \mathbb{R}$ be a two-dimensional isotropic subspace. The set of all generic boundary points, which can be represented by an element of $F \otimes_{\mathbb{R}} \mathbb{C}$ is called the one-dimensional boundary component attached to F.*

This gives a one-to-one correspondence between boundary components and isotropic spaces F of the corresponding dimension. The boundary of $\mathcal{H}_l$ is the disjoint union of the boundary components.

The proof of the last statement follows from the following description: Let $F \subset L \otimes_{\mathbb{Z}} \mathbb{R}$ be a two dimensional isotropic subspace. Then there exists a complementary isotropic space $\tilde{F} \subset L \otimes_{\mathbb{Z}} \mathbb{R}$ such that $F + \tilde{F}$ is the sum of two orthogonal hyperbolic planes. There exist a basis e_1, e_3 of F and a basis e_2, e_4 of $\tilde{F}$ such that

$$(e_1, e_2) = 1, \tag{2.1}$$
$$(e_3, e_4) = 1,$$

$$\begin{aligned}(e_i, e_i) &= 0 \quad (j = 1, \dots, 4),\\ (e_i, e_j) &= 0 \quad (i = 1, 2,\ j = 3, 4).\end{aligned}$$

We briefly write (z_1, z_2, z_3, z_4) instead of $z_1e_1 + \cdots + z_4e_4$. Hence the elements of F are of the form $(z_1, 0, z_3, 0)$. We assume that this is not a multiple of a real point. Then z_1, z_3 both must be different from 0, and we may normalize such that $z_1 = 1$, i.e., the point is of the form $(1, 0, \tau, 0)$. We have to clarify whether it is in the boundary of $\mathcal{H}_l$. This depends on the choice of the component $\mathcal{H}_l \subset \mathcal{K}$. The point $(\mathrm{i}, \mathrm{i}, 1, 1)$ is contained in $\mathcal{K}$. We may replace e_1 by $-e_1$ and e_2 by $-e_2$ and therefore assume that it is contained in $\mathcal{H}_l$. Then

$$\mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}_l, \quad (z_1, z_2) \mapsto (z_1, z_2, -z_1z_2, 1)$$

defines an embedding of the product of two usual complex upper half planes $\mathcal{H}$ into $\mathcal{H}_l$. We have (in the projective space)

$$\lim_{t\to\infty} [\mathrm{i}t, \tau, -\tau\mathrm{i}t, 1] = \lim_{t\to\infty} \left[1, \frac{\tau}{\mathrm{i}t}, -\tau, \frac{1}{\mathrm{i}t}\right] = [1, 0, -\tau, 0].$$

This means that the point $(1, 0, -\tau, 0)$ belongs to the boundary of $\mathcal{H}_l$ if and only if the imaginary part of τ is positive. Thus the one dimensional boundary components can be considered as usual complex upper half planes. It should be mentioned that the set of all boundary points, which can be represented by a point of $F \otimes_{\mathbb{Z}} \mathbb{C}$ (including the special ones), can be identified in the same manner with $\mathcal{H} \cup \mathbb{R} \cup \infty$.

Rational boundary components.

A boundary component is called rational if the corresponding isotropic space F is defined over $\mathbb{Q}$. The union of $\mathcal{H}_l$ with all rational boundary components is denoted by $\mathcal{H}_l^*$. The rational orthogonal group

$$\mathrm{O}'(L \otimes_{\mathbb{Z}} \mathbb{Q}) = \mathrm{O}(L \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \mathrm{O}'(2, l)$$

acts on $\mathcal{H}_l^*$.

Denote by $\mathrm{O}(L)$ the integral orthogonal group of L and put $\mathrm{O}'(L) = \mathrm{O}(L) \cap \mathrm{O}'(2, l)$. Let $\Gamma = \Gamma(L)$ be the discriminant kernel of $\mathrm{O}'(L)$. This is the subgroup of finite index consisting of all elements, which act as the identity on the discriminant group L'/L. Observe that $\Gamma(L)$ is functorial in the following sense: If $\tilde{L} \subset L$ is a sublattice, then $\Gamma(\tilde{L}) \subset \Gamma(L)$.

By the theory of Baily-Borel, the quotient

$$X_\Gamma = \mathcal{H}_l^*/\Gamma$$

carries the structure of a (compact) projective variety, which contains $\mathcal{H}_l/\Gamma$ as a Zariski open subvariety. The topology of X_Γ is the quotient topology of a certain topology on $\mathcal{H}_l^*$. We will describe part of it a little later. If $s \in \mathcal{H}_l^*$ is any point and Γ_s its stabilizer in Γ, the canonical map

$$\mathcal{H}_l^*/\Gamma_s \longrightarrow \mathcal{H}_l^*/\Gamma$$

defines an open embedding of a small neighbourhood of the image of s.

We recall another fact. If s, t are points in the same boundary component and if $g \in \mathrm{O}'(2,l)$ is an element with $g(s) = t$, then g normalizes the boundary component. In particular Γ_s is contained this normalizer. Therefore the local structure of a cusp depends on the normalizer of the boundary component containing the cusp.

The normalizer of a one-dimensional boundary component.

In the following we abbreviate $V = L \otimes_\mathbb{Z} \mathbb{R}$ and $G = \mathrm{O}'(2,l)$. Let $F \subset V$ be a two dimensional isotropic subspace. We write

$$P = \{g \in G; \quad g(F) = F\}$$

for the normalizer of F and

$$P_0 = \{g \in G; \quad g \text{ acts on } F \text{ as identity}\}$$

for its centralizer. As in (2.1) we choose four isotropic elements e_1, e_2, e_3, e_4 such that $F = \mathbb{R}e_1 + \mathbb{R}e_3$ and such that $H_1 = \mathbb{R}e_1 + \mathbb{R}e_2$ and $H_2 = \mathbb{R}e_3 + \mathbb{R}e_4$ are two orthogonal hyperbolic planes. We obtain an orthogonal decomposition

$$V = H_1 \oplus H_2 \oplus W,$$

where W is a negative definite subspace (of dimension $l-2$). If $x_1, \dots, x_4 \in \mathbb{R}$ and $\mathfrak{x} \in W$, we briefly write $(x_1, x_2, x_3, x_4, \mathfrak{x})$ instead of $x_1e_1 + \ldots + x_4e_4 + \mathfrak{x}$. All elements of P_0 have to fix the components x_2, x_4. We give four examples by describing the image of $X = (x_1, x_2, x_3, x_4, \mathfrak{x}) \in V$. The first three are given by *Eichler transformations.* (Recall that if $u \in V$

is an isotropic vector and $v \in V$ is orthogonal on u, then the Eichler transformation

$$a \mapsto E(u,v)(a) = a + (a,u)v - (a,v)u - q(v)(a,u)u \qquad (a \in V)$$

defines an element of G.)

1. $X \mapsto E(e_3, te_1)(X) = (x_1 + tx_4, x_2, x_3 - tx_2, x_4, \mathfrak{x}) \quad (t \in \mathbb{R})$,

2. $X \mapsto E(e_3, b)(X) = (x_1, x_2, x_3 - (b, \mathfrak{x}) - q(b)x_4, x_4, \mathfrak{x} + x_4 b) \quad (b \in W)$,

3. $X \mapsto E(e_1, a)(X) = (x_1 - (a, \mathfrak{x}) - q(a)x_2, x_2, x_3, x_4, \mathfrak{x} + x_2 a) \quad (a \in W)$.

4. The orthogonal group $\mathrm{O}(W)$ is embedded into G (acting trivially on the two hyperbolic planes).

It is useful to introduce adapted coordinates of $\mathcal{H}_l$. If $[Z] = [z_1, z_2, z_3, z_4, \mathfrak{z}]$ is an element of $\mathcal{H}_l$, then z_4 is different from zero and we can normalize $z_4 = 1$. Since the norm of Z is zero, the coordinate z_3 is determined by the other ones. We simply identify

$$(z_1, z_2, \mathfrak{z}) \longleftrightarrow [z_1, z_2, *, 1, \mathfrak{z}].$$

(This is a tube domain realization of $\mathcal{H}_l$ in $\mathbb{C}^l$.) An easy calculation shows that the image of $(z_1, z_2, \mathfrak{z})$ under the transformations 1–4 (in the same order) is given by

1. $(z_1 + t, z_2, \mathfrak{z})$,
2. $(z_1, z_2, \mathfrak{z} + b)$,
3. $(z_1 - (a, \mathfrak{z}) - q(a)z_2, z_2, \mathfrak{z} + z_2 a)$,
4. $(z_1, z_2, k(\mathfrak{z}))$, $\quad (k \in \mathrm{O}(W))$.

The transformations of type 1 and 2 commute, they generate a group, which is isomorphic to the additive group of the vector space $\mathbb{R} \times W$. We denote the orthogonal transformation, which corresponds to the element $(t, b) \in \mathbb{R} \times W$ by $T_{t,b}$. The transformations of type 3 form a group isomorphic to the additive group of W. The orthogonal transformation of type 3, which corresponds to $a \in W$, is denoted by R_a. A simple calculation yields the commutation rule

$$R_a T_{t,b} R_a^{-1} = T_{t-(a,b),b}.$$

This means that the set of all products $R_a T_{t,b}$ is a group, which we denote by U. Obviously U is a semi-direct product, and we have an exact sequence

$$0 \longrightarrow \mathbb{R} \times W \longrightarrow U \longrightarrow W \longrightarrow 0.$$

If we use the abbreviation

$$[a, t, b] = R_a \cdot T_{t,b},$$

the group law is given by

$$[a, t, b] \cdot [a', t', b'] = [a + a', t + t' + (a', b), b + b'].$$

Under the action of $[a, t, b]$ on $\mathcal{H}_l$ the point $(z_1, z_2, \mathfrak{z})$ is mapped to

$$(z_1 + t - (a, \mathfrak{z}) - (a, b) - q(a)z_2,\ z_2,\ \mathfrak{z} + z_2 a + b).$$

We also mention that the subgroup of all elements $[0, t, 0]$ is a normal subgroup of U with trivial action of U. The natural projection $[a, t, b] \mapsto (a, b)$ gives rise to the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow U \longrightarrow W \times W \longrightarrow 0.$$

Here $\mathbb{R}$ and $W \times W$ are understood as additive groups. We will see that this sequence does not split.

One immediately checks that a transformation of type 4 is contained in U only if it is the identity. Moreover, it is easily seen that $\mathrm{O}(W)$ acts by conjugation on U. Hence we may form the semi-direct product $U \ltimes \mathrm{O}(W)$. It is a subgroup of the centralizer P_0.

Lemma 2.2.— *The centralizer P_0 of the two-dimensional isotropic subspace $F \subset V$ is generated by the transformations of type 1-4. Furthermore, one has the exact sequences*

$$\begin{gathered} 1 \longrightarrow U \longrightarrow P_0 \longrightarrow \mathrm{O}(W) \longrightarrow 1, \\ 0 \longrightarrow \mathbb{R} \times W \longrightarrow U \longrightarrow W \longrightarrow 0, \\ 0 \longrightarrow \mathbb{R} \longrightarrow U \longrightarrow W \times W \longrightarrow 0. \end{gathered}$$

The first two of these sequences split (as semi-direct products).

The full normalizer P is easy to describe. We have a natural homomorphism $P \to \mathrm{GL}(F) \cong \mathrm{GL}(2, \mathbb{R})$. By definition P is a subgroup of the spinor kernel. Thus the image is only $\mathrm{GL}_+(F)$. We obtain the exact sequence

$$1 \longrightarrow P_0 \longrightarrow P \longrightarrow \mathrm{GL}_+(F) \longrightarrow 1.$$

The group $\mathrm{GL}_+(F)$ acts on the boundary component and this action can be identified with the standard action of $\mathrm{GL}_+(2, \mathbb{R})$ on the upper half plane.

3. Local cohomology.

Let us fix some notation for the rest of this paper. We suppose that the two dimensional isotropic subspace F is defined over $\mathbb{Q}$. Then F corresponds to a rational one-dimensional boundary component B_F of $\mathcal{H}_l^*$.

Since $L\otimes_{\mathbb{Z}}\mathbb{Q}$ splits two hyperbolic planes, we may further assume that the vectors e_1, e_3 (see (2.1)) are primitive elements of the lattice L and that e_2, e_4 are contained in $L\otimes_{\mathbb{Z}}\mathbb{Q}$. Let N_j $(j = 1,3)$ be the uniquely determined positive integers such that $(e_j, L) = N_j\mathbb{Z}$. Then e_j/N_j are contained in L'. We denote by $\mathcal{L}$ the subgroup

$$\mathcal{L} = \{\beta \in L'/L;\quad (\beta, e_j) \equiv 0 \pmod{N_j} \text{ for } j = 1, 3\} \tag{3.1}$$

of the discriminant group L'/L.

Let D be the lattice $L \cap F^{\perp} \cap \tilde{F}^{\perp}$. Then D is negative definite of dimension $l-2$ and $W = D \otimes_{\mathbb{Z}} \mathbb{R}$. If $\lambda \in V$, then we write λ_D for the orthogonal projection of λ to W. If λ is even contained in L' then λ_D belongs to D'. (But note that D' is in general not a sublattice of L'.)

Recall that $\Gamma = \Gamma(L)$ is the discriminant kernel of the integral orthogonal group $\mathrm{O}'(L)$. It can be easily verified that for $t \in \mathbb{Z}$ and $a, b \in D$ the transformations $T_{t,b}$ and R_a are contained in Γ.

Notation 3.1.— We write Γ_∞ for the subgroup of Γ, which is generated by the transformations $T_{t,b}$ and R_a with $t \in \mathbb{Z}$ and $a, b \in D$. The subgroup of Γ_∞ generated by the $T_{t,b}$ is denoted by $\mathfrak{t}$. The subgroup generated by the R_a is denoted by Λ.

The group Γ_∞ is a subgroup of finite index in the stabilizer Γ_s of any generic boundary point $s \in B_F$. Moreover, Γ_∞ is the semi-direct product of the normal subgroup $\mathfrak{t}$ and Λ. We have a natural exact sequence

$$0 \longrightarrow \mathfrak{t} \longrightarrow \Gamma_\infty \longrightarrow \Lambda \longrightarrow 0,$$

which splits, i.e., there is a section

$$\Lambda \longrightarrow \Gamma_\infty, \quad a \mapsto [a, 0, 0].$$

The groups $\mathfrak{t} = \mathbb{Z} \times D$ and $\Lambda = D$ are obviously Abelian. We now consider the disjoint union $\mathcal{H}_l \cup \mathcal{H}$. We identify a point $\tau \in \mathcal{H}$ with the boundary point $(1, 0, -\tau, 0) \in B_F$. This means that $\mathcal{H}_l \cup \mathcal{H}$ is considered as a part of $\mathcal{H}_l^*$. We define a topology on $\mathcal{H}_l \cup \mathcal{H}$ (which is induced from the topology on

$\mathcal{H}_l^*$, which leads to the Baily-Borel compactification, and which we don't introduce here).

We consider a point $\tau \in \mathcal{H}$. Let $\varepsilon > 0$ and denote by $V_\varepsilon(\tau)$ the ε-ball

$$V_\varepsilon(\tau) = \{z_2 \in \mathcal{H}; \quad |z_2 - \tau| < \varepsilon\}.$$

We define

$$U_\varepsilon(\tau) = \left\{(z_1, z_2, \mathfrak{z}); \quad z_2 \in V_\varepsilon(\tau),\ y_1 y_2 + q(\mathfrak{y}) > \varepsilon^{-1}\right\}.$$

We will often simply write V_ε resp. U_ε instead of $V_\varepsilon(\tau)$ resp. $U_\varepsilon(\tau)$, if the point τ is clear from the context. The group Γ_∞ acts on the set $U_\varepsilon(\tau) \cup V_\varepsilon(\tau)$.

DEFINITION 3.2.— *A set $U \subset \mathcal{H}_l \cup \mathcal{H}$ is called open, if its intersection with $\mathcal{H}_l$ is open in the usual sense, and if for every $\tau \in \mathcal{H} \cap U$ there exists an $\varepsilon > 0$ such that*

$$U_\varepsilon(\tau) \cup V_\varepsilon(\tau) \subset U.$$

Baily's criterion for the extension of a complex space (see [Ca], [Fr1] chapter II for a simplified version) applies to show that

$$(\mathcal{H}_l \cup \mathcal{H})/\Gamma_\infty = \mathcal{H}_l/\Gamma_\infty \cup \mathcal{H}$$

is a normal complex space. It is holomorphically convex in the following sense:

An arbitrary point $a \in \mathcal{H}_l/\Gamma_\infty \cup \mathcal{H}$ is an isolated point of the set of common zeros of finitely many analytic functions on $\mathcal{H}_l/\Gamma_\infty \cup \mathcal{H}$.

This can be proved by standard constructions using Poincaré series (compare [Fr2] chapter 2 §4).

PROPOSITION 3.3.— *The sets*

$$U_\varepsilon/\Gamma_\infty \ \cup \ V_\varepsilon \qquad (\varepsilon > 0)$$

define a fundamental system of Stein neighbourhoods of $\tau \in \mathcal{H}_l/\Gamma_\infty \cup \mathcal{H}$. The space $\mathcal{H}_l/\Gamma_\infty \cup \mathcal{H}$ is also a Stein space.

The proof follows from the fact that the function

$$|z_2 - \tau|^2 + (y_1 y_2 + q(\mathfrak{y}))^{-1}$$

is plurisubharmonic on $\mathcal{H}_l/\Gamma_\infty$ and can be extended to a continuous function on $\mathcal{H}_l/\Gamma_\infty \cup \mathcal{H}$ [1].

The singular locus of $\mathcal{H}_l/\Gamma_\infty \cup \mathcal{H}$ is exactly the boundary component $\mathcal{H}$. If we remove the singularities from our Stein neighbourhoods $U_\varepsilon/\Gamma_\infty \cup V_\varepsilon$, we simply get $U_\varepsilon/\Gamma_\infty$. We want to determine the group of analytic line bundles

$$\operatorname{Pic}(U_\varepsilon/\Gamma_\infty) = H^1(U_\varepsilon/\Gamma_\infty, \mathcal{O}^*).$$

Every analytic line bundle on U_ε is trivial because this is a contractible Stein space. Hence

$$\operatorname{Pic}(U_\varepsilon/\Gamma_\infty) = H^1(\Gamma_\infty, \mathcal{O}(U_\varepsilon)^*).$$

We use the abbreviation

$$R(\varepsilon) = \mathcal{O}(U_\varepsilon).$$

The exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow R(\varepsilon) \longrightarrow R(\varepsilon)^* \longrightarrow 0$$

induces the exact sequence of cohomology groups

$$\begin{aligned} H^1(\Gamma_\infty, \mathbb{Z}) \longrightarrow H^1(\Gamma_\infty, R(\varepsilon)) &\longrightarrow H^1(\Gamma_\infty, R(\varepsilon)^*) \\ &\longrightarrow H^2(\Gamma_\infty, \mathbb{Z}) \longrightarrow H^2(\Gamma_\infty, R(\varepsilon)). \end{aligned} \tag{3.2}$$

The group Γ_∞ contains $\mathfrak{t}$ as a normal subgroup, and the factor group is Λ. Since $U_\varepsilon/\mathfrak{t}$ is a Stein space we have

$$H^p(\Gamma_\infty, R(\varepsilon)) = H^p(\Lambda, \mathcal{P}(\varepsilon)),$$

where $\mathcal{P}(\varepsilon)$ denotes the space of holomorphic functions on U_ε, which are periodic with respect to $\mathfrak{t}$. The elements of this space admit Fourier expansions and the "constant" coefficients are functions which only depend on z_2. If we denote by $\mathcal{P}_0(\varepsilon)$ all elements whose constant Fourier coefficient vanishes we get a splitting

$$H^p(\Lambda, \mathcal{P}(\varepsilon)) = H^p(\Lambda, \mathcal{O}(V_\varepsilon)) \oplus H^p(\Lambda, \mathcal{P}_0(\varepsilon)).$$

[1] Grauert and Remmert [GrRe] introduced the notion of a plurisubharmonic function $p : X \to \mathbb{R}$ on an arbitrary complex space X. If X is smooth and p is a $\mathcal{C}^\infty$-function such that the matrix $\partial^2 p/\partial z_i \partial \bar{z}_j$ is positive definite in any point for some local coordinates z_i then p is plurisubharmonic. If p is a continuous function on a complex space and plurisubharmonic outside some thin analytic set, then it is plurisubharmonic everywhere. If X is a holomorphically convex complex space and p a plurisubharmonic function on X such that all $X_\varepsilon = \{x;\ p(x) < \varepsilon\}$ are relatively compact then X_ε and X are Stein spaces.

A special case of a more general result due to Ballweg [Ba] states:

PROPOSITION 3.4 (Ballweg). — *Assume that $p < l-2$. For sufficiently small ε the group $H^p(\Lambda, \mathcal{P}_0(\varepsilon))$ vanishes.*

For the rest of this section we assume that $l \geqslant 4$ and that $\varepsilon > 0$ is sufficiently small in the sense of Proposition 3.4.

By Ballweg's result we know that

$$H^1(\Gamma_\infty, R(\varepsilon)) = H^1(\Lambda, \mathcal{O}(V_\varepsilon)) = \operatorname{Hom}(\Lambda, \mathcal{O}(V_\varepsilon)). \tag{3.3}$$

Using (3.3) and the isomorphism $\operatorname{Hom}(\Gamma_\infty, \mathbb{Z}) \cong H^1(\Gamma_\infty, \mathbb{Z})$, we may rewrite the exact sequence (3.2) in the following way:

$$\begin{aligned} \operatorname{Hom}(\Gamma_\infty, \mathbb{Z}) \longrightarrow \operatorname{Hom}(\Lambda, \mathcal{O}(V_\varepsilon)) &\longrightarrow H^1(\Gamma_\infty, R(\varepsilon)^*) \\ &\longrightarrow H^2(\Gamma_\infty, \mathbb{Z}) \longrightarrow H^2(\Gamma_\infty, R(\varepsilon)). \end{aligned} \tag{3.4}$$

The natural projection $\Gamma_\infty \to \Lambda$ induces a homomorphism $\operatorname{Hom}(\Lambda, \mathbb{Z}) \to \operatorname{Hom}(\Gamma_\infty, \mathbb{Z})$. It can be used to obtain from (3.4) the exact sequence

$$\begin{aligned} \operatorname{Hom}(\Lambda, \mathcal{O}(V_\varepsilon))/\operatorname{Hom}(\Lambda, \mathbb{Z}) &\longrightarrow H^1(\Gamma_\infty, R(\varepsilon)^*) \\ &\longrightarrow H^2(\Gamma_\infty, \mathbb{Z}) \longrightarrow H^2(\Gamma_\infty, R(\varepsilon)). \end{aligned} \tag{3.5}$$

We also have the exact sequence

$$0 \longrightarrow \operatorname{Hom}(\Lambda, \mathbb{Z}) \longrightarrow \operatorname{Hom}(\Lambda, \mathcal{O}(V_\varepsilon)) \longrightarrow \operatorname{Hom}(\Lambda, \mathcal{O}(V_\varepsilon)^*) \longrightarrow 0, \tag{3.6}$$

because Λ is a free group. If we combine (3.5) and (3.6) we finally find that

$$\operatorname{Hom}(\Lambda, \mathcal{O}(V_\varepsilon)^*) \longrightarrow \operatorname{Pic}(U_\varepsilon/\Gamma_\infty) \longrightarrow H^2(\Gamma_\infty, \mathbb{Z}) \longrightarrow H^2(\Gamma_\infty, R(\varepsilon)) \tag{3.7}$$

is exact.

We now derive some information about $H^2(\Gamma_\infty, \mathbb{Z})$. Every bilinear form

$$B : D \times D \longrightarrow \mathbb{Z}$$

defines a 2-cocycle[2] of Γ_∞ acting trivially on $\mathbb{Z}$. We denote this cocycle by B, too. It is given by

$$B([a, t, b], [a', t', b']) = B(a, b').$$

[2] Throughout we use the inhomogeneous standard complex of group cohomology as in [Sh] chapter 8.

In fact, it is easily checked that this is a 2-cocycle. Hence we get a map

$$\mathrm{Bil}(D) \longrightarrow H^2(\Gamma_\infty, \mathbb{Z})$$

from the group $\mathrm{Bil}(D)$ of bilinear forms on D to $H^2(\Gamma_\infty, \mathbb{Z})$. Only the image is important for our applications. It is a basic fact that this map is not injective:

PROPOSITION 3.5. — *The kernel of the map* $\mathrm{Bil}(D) \to H^2(\Gamma_\infty, \mathbb{Z})$ *equals the cyclic subgroup generated by the bilinear form* $(\cdot,\cdot)$. *The image of this map is contained in the kernel of* $H^2(\Gamma_\infty, \mathbb{Z}) \longrightarrow H^2(\Gamma_\infty, R(\varepsilon))$.

Proof. — The first statement follows from the so-called "Five Term Exact Sequence" of group cohomology. If G is a group which acts on an Abelian group A and if H is a normal subgroup then there is a certain homomorphism

$$\mathrm{tg} : H^1(H, A)^{G/H} \to H^2(G/H, A^H),$$

called "transgression", which has the property that the sequence

$$H^1(H, A)^{G/H} \longrightarrow H^2(G/H, A^H) \longrightarrow H^2(G, A)$$

is exact. We apply this to the situation $G = \Gamma_\infty$, $A = \mathbb{Z}$ (trivial operation), $H = \mathbb{Z}$, and the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_\infty \longrightarrow D \times D \longrightarrow 0.$$

In this case $H^1(H, \mathbb{Z}) \cong \mathbb{Z}$. A straightforward computation shows that $\mathrm{tg}(1) \in H^2(D \times D, \mathbb{Z})$ is the bilinear form which maps the pair $([a, 0, b], [a', 0, b'])$ to $-(a', b)$. But this bilinear form differs from $(\cdot,\cdot)$ only by a coboundary. Since $\mathrm{Bil}(D) \to H^2(\Gamma_\infty, \mathbb{Z})$ factors through $H^2(D \times D, \mathbb{Z})$ we obtain the assertion.

It remains to show that the image of a bilinear form B vanishes in $H^2(\Gamma_\infty, R(\varepsilon))$. A cochain that trivializes B is given by

$$f([a, t, b], Z) = B(a, \mathfrak{z}) - \frac{z_2}{2} B(a, a)$$

for $[a, t, b] \in \Gamma_\infty$ and $Z = (z_1, z_2, \mathfrak{z}) \in R(\varepsilon)$. Here B is extended $\mathbb{C}$-bilinearly. □

One can define the trace $\mathrm{tr}(B)$ of an Element $B \in \mathrm{Bil}(D)$ as the trace of a Gram matrix of B with respect to an orthonormal basis of the quadratic space $(D \otimes_{\mathbb{Z}} \mathbb{R}, -q)$. The trace of $(\cdot,\cdot)$ equals $2-l$. Proposition 3.5 implies

Remark 3.6. — An element $B \in \mathrm{Bil}(D)$ defines a torsion element in $H^2(\Gamma_\infty, \mathbb{Z})$ if and only if

$$B(h,h) + \frac{\mathrm{tr}(B)}{l-2}(h,h) = 0$$

for any $h \in D$.

The groups $\mathrm{Pic}(U_\varepsilon/\Gamma_\infty)$ and $H^2(\Gamma_\infty, \mathbb{Z})$ are usually not torsion free. However, in the present paper we ignore these torsion problems.

4. Local Heegner divisors.

Recall the notation introduced at the beginning of Section 3. Let X be a normal irreducible complex space. By a divisor D on X we mean a formal linear combination $D = \sum n_Y Y$ $(n_Y \in \mathbb{Z})$ of irreducible closed analytic subsets Y of codimension 1 such that the support $\bigcup_{n_Y \neq 0} Y$ is a closed analytic subset of everywhere pure codimension 1. We write $\mathrm{Div}(X)$ for the divisor group of X.

For any vector $\lambda \in L'$ of negative norm the orthogonal complement of λ in $\mathcal{H}_l$ defines a divisor $\lambda^\perp$ on $\mathcal{H}_l$. Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$. Then

$$H(\beta, m) = \sum_{\substack{\lambda \in L' \\ q(\lambda)=m \\ \lambda + L = \beta}} \lambda^\perp \tag{4.1}$$

is a Γ-invariant divisor on $\mathcal{H}_l$. It is the inverse image under the canonical projection of an algebraic divisor on the quotient X_Γ (which will also be denoted by $H(\beta, m)$). The multiplicities of all irreducible components equal 2, if $2\beta = 0$, and 1, if $2\beta \neq 0$ in L'/L. Following Borcherds we call this divisor the *Heegner divisor* of discriminant (β, m). Note that $H(\beta, m) = H(-\beta, m)$.

In the present paper we are interested in the contribution of $H(\beta, m)$ to the local Picard groups at generic boundary points of X_Γ. Let s be a generic point in B_F and denote its image in X_Γ by s, too. Since Γ_∞ has finite index in the stabilizer Γ_s of s, the local Picard group $\mathrm{Pic}(X_\Gamma, s)$

as defined in the introduction can be described by means of the groups $\mathrm{Pic}(U_\varepsilon(s)/\Gamma_\infty)$ up to torsion. The group Γ_s/Γ_∞ acts on $\varinjlim\,(\mathrm{Pic}(U_\varepsilon/\Gamma_\infty))$ and the invariant part satisfies

$$\varinjlim\left(\mathrm{Pic}(U_\varepsilon/\Gamma_\infty)\otimes_{\mathbb{Z}}\mathbb{Q}\right)^{\Gamma_s/\Gamma_\infty} = \mathrm{Pic}(X_\Gamma, s)\otimes_{\mathbb{Z}}\mathbb{Q}. \tag{4.2}$$

The natural inclusion and projection maps induce the commutative diagram

$$\begin{array}{ccc}
\mathrm{Div}(\mathcal{H}_l/\Gamma) & \longrightarrow & \mathrm{Div}(U_\varepsilon/\Gamma_\infty) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Div}(\mathcal{H}_l) & \longrightarrow & \mathrm{Div}(U_\varepsilon)
\end{array}$$

of divisor groups. The image of $H(\beta,m)$ in $\mathrm{Div}(U_\varepsilon/\Gamma_\infty)$ is denoted by $H_F(\beta,m)$ and will be called a *local Heegner divisor.* Its image in $\mathrm{Div}(U_\varepsilon)$ is a Γ_∞-invariant divisor, which will also be denoted by $H_F(\beta,m)$. (Note that our definitions of local divisor groups and local Heegner divisors also depend on the choice of the complementary isotropic subspace $\tilde{F}$.)

The proof of the next lemma will be left to the reader.

LEMMA 4.1. — *Let $\lambda\in L'$ be a vector of negative norm. Then $\lambda^\perp$ has non-zero intersection with U_ε for every $\varepsilon>0$, if and only if λ is orthogonal to F.*

Thus, if ε is sufficiently small, the local Heegner divisor $H_F(\beta,m)\in\mathrm{Div}(U_\varepsilon)$ is given by

$$H_F(\beta,m) = \sum_{\substack{\lambda\in L'\cap F^\perp\\ q(\lambda)=m\\ \lambda+L=\beta}} \lambda^\perp. \tag{4.3}$$

In particular if $H_F(\beta,m)\neq 0$ in $\mathrm{Div}(U_\varepsilon)$, then β belongs to the subgroup $\mathcal{L}$ of L'/L. Observe that for any $\beta\in\mathcal{L}$ there exists a representative $\dot\beta\in L'\cap F^\perp$. For the rest of this paper we fix such a representative $\dot\beta$ for every $\beta\in\mathcal{L}$. The assignment $\beta\mapsto\dot\beta_D$ induces a surjective homomorphism

$$\pi:\mathcal{L}\longrightarrow D'/D.$$

Throughout we will assume that ε is small enough such that (4.3) holds.

Let $\lambda\in L'\cap F^\perp$. Then the group Γ_∞ acts on the set $\lambda+\mathbb{Z}e_1+\mathbb{Z}e_3\subset L'\cap F^\perp$ with finitely many orbits. Thus the divisor

$$H_\infty(\lambda) = \sum_{\nu_1,\nu_3\in\mathbb{Z}} (\lambda+\nu_1 e_1+\nu_3 e_3)^\perp$$

is invariant under Γ_∞. It defines an element of $\operatorname{Div}(U_\varepsilon/\Gamma_\infty)$. For $\beta \in \mathcal{L}$ the local Heegner divisor $H_F(\beta, m)$ can be written as a finite sum

$$H_F(\beta, m) = \sum_{\substack{\lambda \in D \\ q(\lambda+\dot\beta)=m}} H_\infty(\lambda + \dot\beta). \tag{4.4}$$

Recall that $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$ can be described by *automorphy factors* in the following way. An automorphy factor of Γ_∞ on U_ε is a 1-cocycle of Γ_∞ with values in $R(\varepsilon)^*$, i.e., a holomorphic function

$$J : \Gamma_\infty \times U_\varepsilon \longrightarrow R(\varepsilon)^* \tag{4.5}$$

with the property $J(gg', Z) = J(g, g'Z)J(g', Z)$ for $g, g' \in \Gamma_\infty$. An automorphy factor of the form $J(g, Z) = h(gZ)/h(Z)$ with $h \in R(\varepsilon)^*$ is called trivial. The group $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty) = H^1(\Gamma_\infty, R(\varepsilon)^*)$ is the factor group of the group of all automorphy factors modulo the subgroup of trivial automorphy factors. Special automorphy factors are given by invertible holomorphic functions, which do not depend on z_1 and $\mathfrak{z}$. These are simply homomorphisms $\Gamma_\infty \to \mathcal{O}(V(\varepsilon))^*$.

If H is a divisor in $\operatorname{Div}(U_\varepsilon/\Gamma_\infty)$ then its image in $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$ can be determined as follows: Let f be a holomorphic function on U_ε whose divisor equals the inverse image of H in $\operatorname{Div}(U_\varepsilon)$. Then

$$J(g, Z) = f(gZ)/f(Z)$$

is an automorphy factor of Γ_∞ on U_ε. Its class in $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$ is the image of H.

We shall now determine the position of $H_\infty(\lambda)$ in the Picard group $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$. It will turn out that up to torsion it is completely determined by the Chern class in $H^2(\Gamma_\infty, \mathbb{Z})$ of $H_\infty(\lambda)$.

Definition 4.2. — *Let $\lambda \in L' \cap F^\perp$ be a vector of negative norm. Then we define the local Borcherds product attached to $H_\infty(\lambda)$ by*

$$\Psi_\lambda(Z) = \prod_{n \in \mathbb{Z}} \left[1 - e\left(\sigma_n(nz_2 + (\lambda, Z))\right)\right]$$

for $Z = (z_1, z_2, \mathfrak{z}) \in \mathcal{H}_l$. Here

$$\sigma_n = \begin{cases} +1, & \text{if } n \geqslant 0, \\ -1, & \text{if } n < 0, \end{cases}$$

and $e(z) := e^{2\pi i z}$ for $z \in \mathbb{C}$.

The sign σ_n ensures that the infinite product converges normally on the whole generalized upper half plane $\mathcal{H}_l$. It is invariant under the subgroup $\mathfrak{t}$ of Γ_∞. However, it is not invariant under the subgroup Λ.

It is easily checked that the divisor of Ψ_λ is exactly $H_\infty(\lambda)$. There is a strong analogy between functions of the above type and Borcherds' automorphic products [Bo1] or more precisely the generalized Borcherds products attached to Heegner divisors which were introduced in [Br1], [Br2].

The properties of Ψ_λ imply that the image of $H_\infty(\lambda)$ in $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$ is given by the automorphy factor

$$J_\lambda([a,t,b],Z) = \Psi_\lambda(R_a Z)/\Psi_\lambda(Z) \qquad ([a,t,b] \in \Gamma_\infty). \tag{4.6}$$

Let us compute $J_\lambda([a,t,b],Z)$ more explicitly. Using the formula for the action of R_a we find

$$\begin{aligned} J_\lambda([a,t,b],Z) &= \frac{\prod\limits_{n\in\mathbb{Z}} [1 - e(\sigma_n((n + (\lambda_D, a))z_2 + (\lambda, Z)))]}{\prod\limits_{n\in\mathbb{Z}} [1 - e(\sigma_n(nz_2 + (\lambda, Z)))]} \\ &= \prod_{n\in\mathbb{Z}} \frac{1 - e\left(\sigma_{n-(\lambda_D,a)}(nz_2 + (\lambda, Z))\right)}{1 - e(\sigma_n(nz_2 + (\lambda, Z)))}. \end{aligned}$$

In the latter product only those n give a contribution different from 1, which satisfy

i) $0 \leqslant n < (\lambda_D, a)$, or

ii) $0 > n \geqslant (\lambda_D, a)$.

Hence the product is actually finite.

The first case can only occur if $(\lambda_D, a) > 0$. We may use the elementary identity

$$\frac{1 - e(z)}{1 - e(-z)} = -e(z)$$

to obtain

$$\begin{aligned} J_\lambda([a,t,b],Z) &= \prod_{0\leqslant n<(\lambda_D,a)} [-e(-nz_2 - (\lambda, Z))] \\ &= e\left(\frac{1}{2}(\lambda_D, a) - \frac{z_2}{2}(\lambda_D, a)((\lambda_D, a) - 1) - (\lambda_D, a)(\lambda, Z)\right). \end{aligned} \tag{4.7}$$

One immediately checks that the same formula holds in the second case, too.

We briefly recall the construction of the Chern class of a class $[J]$ of automorphy factors in $H^1(\Gamma_\infty, R(\varepsilon)^*)$. Let J be an automorphy factor as in (4.5) representing $[J]$. For any $g \in \Gamma_\infty$ let $A(g,Z)$ be a holomorphic function on U_ε such that

$$J(g,Z) = e(A(g,Z)).$$

Then

$$c(g,g') = A(gg',Z) - A(g,g'Z) - A(g',Z)$$

is an integral constant for all $g, g' \in \Gamma_\infty$. The function $(g,g') \mapsto c(g,g')$ defines a 2-cocycle of Γ_∞ acting trivially on $\mathbb{Z}$. Obviously each $A(g,Z)$ is only determined up to an integral additive constant. It is easily seen that a different choice of the functions $A(g,Z)$ only changes $c(g,g')$ by a coboundary. Moreover, any 2-cocycle corresponding to a trivial automorphy factor is a coboundary. Thus, if we map $[J]$ to the image of $c(\cdot,\cdot)$ in $H^2(\Gamma_\infty,\mathbb{Z})$, we get a well defined homomorphism. This is an explicit construction of the Chern class map δ, the connecting homomorphism in the exact sequence (3.2).

Proposition 4.3. — *The Chern class $\delta(H_\infty(\lambda)) \in H^2(\Gamma_\infty,\mathbb{Z})$ of the local Heegner divisor $H_\infty(\lambda) \in \operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$ is given by the 2-cocycle*

$$c([a,t,b],[a',t',b']) = (\lambda_D, a)(\lambda_D, b')$$

of Γ_∞. In particular, $\delta(H_\infty(\lambda))$ belongs to the image of $\operatorname{Bil}(D) \to H^2(\Gamma_\infty,\mathbb{Z})$.

Proof. — The divisor $H_\infty(\lambda)$ is represented by the automorphy factor (4.7) as an element of $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$. If we carry out the construction outlined above, we get the assertion. □

Proposition 4.4. — *The Chern class $\delta(H)$ of a finite linear combination*

$$H = \sum_{\substack{\lambda \in L' \cap F^\perp \\ q(\lambda)<0}} c_\lambda H_\infty(\lambda)$$

($c_\lambda \in \mathbb{Z}$) of local Heegner divisors is a torsion element of $H^2(\Gamma_\infty,\mathbb{Z})$, if and only if

$$\sum_{\substack{\lambda \in L' \cap F^\perp \\ q(\lambda)<0}} c_\lambda \left[(\lambda_D, h)^2 - \frac{(\lambda_D,\lambda_D)}{l-2}(h,h) \right] = 0$$

for any $h \in D$.

Proof. — The trace of the bilinear form $B(a,b') = (\lambda_D, a)(\lambda_D, b')$ is $-(\lambda_D, \lambda_D)$. Hence the assertion follows from Remark 3.6 and Proposition 4.3. □

The bilinear form $(\cdot,\cdot)$ is non-degenerated, whereas $(\cdot,\lambda_D)(\lambda_D,\cdot)$ is obviously degenerated. Thus every individual local Heegner divisor $H_\infty(\lambda)$ is non-zero in $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)\otimes_{\mathbb{Z}}\mathbb{Q}$.

The Chern class of $H_\infty(\lambda)$ only depends on the projection $\lambda_D \in D'$. In fact, using the above results, it is easily seen that $H_\infty(\lambda) = H_\infty(\lambda')$ in $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)\otimes_{\mathbb{Z}}\mathbb{Q}$, if $\lambda_D = \lambda'_D$.

Our main interest lies in the divisors $H_F(\beta,m)$. In the following theorem we describe their position in $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$ up to torsion. Let $\gamma \in D'/D$ and $m \in \mathbb{Z}+q(\gamma)$ with $m<0$. We write $P_{\gamma,m}$ for the bilinear form

$$(4.8)\qquad P_{\gamma,m}(a,b) = \sum_{\substack{\lambda\in D'\\ \lambda+D=\gamma\\ q(\lambda)=m}} \left[(a,\lambda)(\lambda,b) - \frac{(\lambda,\lambda)}{l-2}(a,b)\right] \qquad (a,b\in D).$$

THEOREM 4.5. — *A finite linear combination*

$$H = \frac{1}{2}\sum_{\beta\in\mathcal{L}}\sum_{\substack{m\in\mathbb{Z}+q(\beta)\\ m<0}} c(\beta,m)H_F(\beta,m)$$

(with integral coefficients $c(\beta,m)$ satisfying $c(\beta,m) = c(-\beta,m)$) is a torsion element of $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$*, if and only if*

$$(4.10)\qquad \sum_{\beta\in\mathcal{L}}\sum_{\substack{m\in\mathbb{Z}+q(\beta)\\ m<0}} c(\beta,m)P_{\pi(\beta),m}(a,a) = 0$$

for all $a \in D$.

Proof. — If the linear combination H is a torsion element of $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$, then (4.10) follows by Proposition 4.4.

Conversely, assume that (4.10) holds for all $a\in D$. Then we also have

$$(4.11)\qquad \sum_{\beta\in\mathcal{L}}\sum_{\substack{m\in\mathbb{Z}+q(\beta)\\ m<0}} c(\beta,m)P_{\pi(\beta),m}(a,b) = 0$$

for all $a,b \in D \otimes_{\mathbb{Z}} \mathbb{C}$. According to (4.7), an automorphy factor J representing H in $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$ is given by

$$\begin{aligned} J(g,Z) &= \prod_{\substack{\beta\in\mathcal{L}\\ m\in\mathbb{Z}+q(\beta)\\ m<0}} \prod_{\substack{\lambda\in D\\ q(\lambda+\dot\beta)=m}} J_{\lambda+\dot\beta}(g,Z)^{\frac{c(\beta,m)}{2}} \\ &= \prod_{\substack{\beta\in\mathcal{L}\\ m\in\mathbb{Z}+q(\beta)\\ m<0}} \prod_{\substack{\lambda\in D\\ q(\lambda+\dot\beta)=m}} \\ &\qquad e\Big(-\frac{z_2}{2}(\lambda+\dot\beta_D,a)(\lambda+\dot\beta_D,a)-(\lambda+\dot\beta_D,a)(\lambda+\dot\beta,Z)\Big)^{\frac{c(\beta,m)}{2}} \end{aligned}$$

for $g=[a,t,b]\in\Gamma_\infty$. The linear terms in $\lambda+\dot\beta_D$ cancel out, because $c(\beta,m)=c(-\beta,m)$. In the latter equation we write $(\lambda+\dot\beta,Z)=(\lambda+\dot\beta_D,\mathfrak{z})+(\dot\beta-\dot\beta_D,Z)$. Here the second scalar product is just a rational linear combination of 1 and z_2. We now use (4.11) to rewrite $J(g,Z)$ as follows:

$$\begin{aligned} J(g,Z) &= e((\mu_1,a)l+(\mu_2,a)z_2) \\ &\quad\times \prod_{\substack{\beta\in\mathcal{L}\\ m\in\mathbb{Z}+q(\beta)\\ m<0}} \prod_{\substack{\lambda\in D'\\ \lambda+D=\pi(\beta)\\ q(\lambda)=m}} e\Big(-\frac{(\lambda,\lambda)}{l-2}\,(q(a)z_2+(a,\mathfrak{z}))\Big)^{\frac{c(\beta,m)}{2}}, \end{aligned}$$

where μ_1,μ_2 are suitable vectors in $D\otimes_{\mathbb{Z}}\mathbb{Q}$. The invertible functions $u_1(Z)=e((\mu_2,\mathfrak{z}))$ and $u_2(Z)=e(z_1)$ give rise to the trivial automorphy factors

$$\begin{aligned} j_1(g,Z) &= u_1(gZ)/u_1(Z) = e\left((\mu_2,a)z_2+(\mu_2,b)\right), \\ j_2(g,Z) &= u_2(gZ)/u_2(Z) = e\left(t-(a,b)-q(a)z_2-(a,\mathfrak{z})\right). \end{aligned}$$

If we multiply J by appropriate rational powers of j_1 and j_2, we find that J is equivalent to

$$J'(g,Z) = e((a,\mu_1)-(b,\mu_2)+r((a,b)-t))$$

with a suitable constant $r\in\mathbb{Q}$. Thus J is a torsion element of $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$. □

Note that by (4.2) a linear combination of local Heegner divisors as in (4.9) is a torsion element in $\operatorname{Pic}(U_\varepsilon/\Gamma_\infty)$ for some ε, if and only if it is a torsion element of $\operatorname{Pic}(X_\Gamma,s)$.

5. Modular forms.

In this section we investigate the relation between the obstructions occurring in Theorem 4.5 and the coefficients of certain vector valued theta series of weight $1+l/2$. We find interesting connections to Borcherds' global "theory of obstructions" for the construction of automorphic products [Bo2].

We write $\mathrm{Mp}_2(\mathbb{R})$ for the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$. The elements of $\mathrm{Mp}_2(\mathbb{R})$ are pairs $(M, \phi(\tau))$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, and ϕ denotes a holomorphic function on $\mathcal{H}$ with $\phi(\tau)^2 = c\tau + d$. The product of $(M_1, \phi_1(\tau)), (M_2, \phi_2(\tau)) \in \mathrm{Mp}_2(\mathbb{R})$ is given by

$$\big(M_1, \phi_1(\tau)\big)\big(M_2, \phi_2(\tau)\big) = \big(M_1 M_2, \phi_1(M_2\tau)\phi_2(\tau)\big),$$

where $M\tau = \frac{a\tau+b}{c\tau+d}$ denotes the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$.

Let $\mathrm{Mp}_2(\mathbb{Z})$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. It is well known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right),$$
$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right).$$

One has the relations $S^2 = (ST)^3 = Z$, where $Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right)$ is the standard generator of the center of $\mathrm{Mp}_2(\mathbb{Z})$.

Let N be an even lattice of signature (b^+, b^-) equipped with a bilinear form $(\cdot,\cdot)$. We write q for the corresponding quadratic form $q(x) = \frac{1}{2}(x,x)$ and denote the dual lattice by N'. (In our latter applications N will be L or D.)

Recall that there is a unitary representation ρ_N of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[N'/N]$. If we denote the standard basis of $\mathbb{C}[N'/N]$ by $(\mathfrak{e}_\gamma)_{\gamma \in N'/N}$ then ρ_N can be defined by the action of the generators $S, T \in \mathrm{Mp}_2(\mathbb{Z})$ as follows (see also [Bo1], where the dual of ρ_N is used):

$$\rho_N(T)\mathfrak{e}_\gamma = e(-q(\gamma))\mathfrak{e}_\gamma \tag{5.1}$$

$$\rho_N(S)\mathfrak{e}_\gamma = \frac{\sqrt{i}^{\,b^+-b^-}}{\sqrt{|N'/N|}} \sum_{\delta \in N'/N} e((\gamma,\delta))\mathfrak{e}_\delta. \tag{5.2}$$

This representation is essentially the Weil representation attached to the quadratic module $(N'/N, q)$ (cf. [No]). It factors through a finite quotient of $\mathrm{Mp}_2(\mathbb{Z})$. Note that $\rho_N(Z)\mathfrak{e}_\gamma = i^{b^- - b^+}\mathfrak{e}_{-\gamma}$.

Let $k \in \frac{1}{2}\mathbb{Z}$ and $f : \mathcal{H} \to \mathbb{C}[N'/N]$ be a holomorphic function. Then f is called *modular form* of weight k with respect to ρ_N and $\mathrm{Mp}_2(\mathbb{Z})$ if

i) $f(M\tau) = \phi(\tau)^{2k}\rho_N(M,\phi)f(\tau)$ for all $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$,

ii) f is holomorphic at ∞.

Here the second condition has the following meaning: Condition (i) implies that f has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma \in N'/N} \sum_{n \in \mathbb{Z} - q(\gamma)} a(\gamma, n)\mathfrak{e}_\gamma(n\tau),$$

where we have abbreviated $\mathfrak{e}_\gamma(\tau) := e(\tau)\mathfrak{e}_\gamma$. As usual, f is called holomorphic at ∞, if all coefficients $a(\gamma, n)$ with $n < 0$ vanish. Moreover, if all $a(\gamma, n)$ with $n \leqslant 0$ vanish, then f is called *cusp form*. The $\mathbb{C}$-vector space of modular forms of weight k with respect to ρ_N and $\mathrm{Mp}_2(\mathbb{Z})$ is denoted by $M_{k,N}$, the subspace of cusp forms by $S_{k,N}$.

For the rest of this paper let $k = 1 + l/2$. Recall that $D = L \cap F^\perp \cap \tilde{F}^\perp$ is a negative definitive lattice of rank $l - 2$. Special modular forms in the space $S_{k,D}$ can be constructed by means of theta series with harmonic polynomials. The homogeneous polynomial

$$Q(u, v) = (u, v)^2 - \frac{(u, u)(v, v)}{l - 2} \qquad (u, v \in W)$$

is harmonic in u and v. For any fixed $v \in W$ we have the $\mathbb{C}[D'/D]$-valued theta series

$$\Theta_D(\tau, v) = \sum_{\lambda \in D'} Q(\lambda, v)\mathfrak{e}_\lambda(-q(\lambda)\tau). \tag{5.3}$$

By the usual Poisson summation argument it can be shown that $\Theta_D(\tau, v)$ is a cusp form in $S_{k,D}$ (see for instance [Bo1] Theorem 4.1). We denote by $S_{k,D}^{\Theta}$ the subspace of $S_{k,D}$, which is generated by the $\Theta_D(\tau, v)$ when v varies through W.

The point is that the polynomials $P_{\gamma,m}$ defined in (4.8) are precisely the Fourier coefficients of $\Theta_D(\tau, v)$:

$$\Theta_D(\tau, v) = \sum_{\gamma \in D'/D} \sum_{\substack{m \in \mathbb{Z} - q(\gamma) \\ m < 0}} P_{\gamma,m}(v, v)\mathfrak{e}_\gamma(-m\tau).$$

Therefore Theorem 4.5 can also be stated as follows.

PROPOSITION 5.1. — *A finite linear combination*

$$H = \frac{1}{2} \sum_{\beta \in \mathcal{L}} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) H_F(\beta, m) \tag{5.4}$$

(with integral coefficients $c(\beta, m)$ satisfying $c(\beta, m) = c(-\beta, m)$) is a torsion element of $\mathrm{Pic}(U_\varepsilon/\Gamma_\infty)$, *if and only if*

$$\sum_{\beta \in \mathcal{L}} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) a(\pi(\beta), -m) = 0 \tag{5.5}$$

for any cusp form $f \in S_{k,D}^{\Theta}$ with Fourier coefficients $a(\gamma, n)$ ($\gamma \in D'/D$ and $n \in \mathbb{Z} - q(\gamma)$).

Borcherds constructed a lifting from certain vector valued modular forms of weight $1 - l/2$ for $\mathrm{Mp}_2(\mathbb{Z})$ to meromorphic modular forms for the group $\Gamma(L)$ (cf. [Bo1] Theorem 13.3). Since these lifts have certain interesting infinite product expansions they are called automorphic products or Borcherds products. Their divisors are linear combinations of Heegner divisors. We now compare Proposition 5.1 with the following condition for the existence of Borcherds products for the group $\Gamma(L)$ with prescribed divisor (cf. [Bo2] Theorem 3.1).

THEOREM 5.2 (Borcherds). — *A finite linear combination of Heegner divisors*

$$\frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) H(\beta, m) \tag{5.6}$$

(with $c(\beta, m) \in \mathbb{Z}$ and $c(\beta, m) = c(-\beta, m)$) is the divisor of a Borcherds product for the group $\Gamma(L)$ (as in [Bo1] Theorem 13.3), if and only if for any cusp form $f \in S_{k,L}$ with Fourier coefficients $a(\gamma, n)$ the equality

$$\sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) a(\beta, -m) = 0 \tag{5.7}$$

holds.

By means of the $\uparrow$-operator, defined in [Br1] Lemma 15.2, it is possible to embed $S_{k,D}$ into $S_{k,L}$. According to Theorem 5.2 the space $S_{k,L}$ carries some information on the subgroup of $\mathrm{Pic}(X_\Gamma)$ generated by the Heegner divisors $H(\beta, m)$. A subspace of $S_{k,L}$, the image of $S_{k,D}^{\Theta}$, encodes the subgroup of the local Picard group $\mathrm{Pic}(X_\Gamma, s) \otimes_{\mathbb{Z}} \mathbb{Q}$, which is generated by the pullbacks of the $H(\beta, m)$'s.

DEFINITION 5.3.— *A divisor H on X_Γ is called trivial at generic boundary points, if for every one-dimensional irreducible component B of the boundary of X_Γ there exists a generic point $s \in B$ such that H is a torsion element of* $\mathrm{Pic}(X_\Gamma, s)$.

If F is a meromorphic modular form for the group $\Gamma(L)$, then the divisor (F) attached to F is trivial at generic boundary points. This is an immediate consequence of the transformation behaviour of modular forms.

Unimodular lattices.

For the rest of this paper we assume that L is unimodular. Then D is a negative definite even unimodular lattice of rank $l-2$. Thus $l \equiv 2 \pmod 8$. Any negative definite even unimodular lattice can be realized as a sublattice of L. This follows from the fact that there exists just one isomorphism class of even unimodular lattices of signature $(2, l)$ (cf. [Wat] chapter 7, §3). The space $S_{k,L} = S_{k,D}$ is the usual space of elliptic cusp forms of weight $k = 1 + l/2$ for $\mathrm{SL}_2(\mathbb{Z})$.

THEOREM 5.4.— *Let*

$$H = \frac{1}{2} \sum_{\substack{m \in \mathbb{Z} \\ m<0}} c(0,m) H(0,m)$$

be a finite linear combination of Heegner divisors $H(0,m)$ (with coefficients $c(0,m) \in \mathbb{Z}$). Then the following statements are equivalent:

i) *H is the divisor of a Borcherds product for the group $\Gamma(L)$ as in [Bo1] Theorem 13.3.*

ii) *H is the divisor of a meromorphic automorphic form for $\Gamma(L)$.*

iii) *H is trivial at generic boundary points.*

Proof.— We only have to prove that (iii) implies (i). Assume that H is trivial at generic boundary points. Then by Proposition 5.1 we have

$$\sum_{\substack{m \in \mathbb{Z} \\ m<0}} c(0,m) a(0,-m) = 0 \tag{5.8}$$

for any cusp form $f \in S_{k,D}^{\Theta}$ with Fourier coefficients $a(0,n)$ and any negative definite even unimodular lattice D of rank $l-2$. In view of Theorem

5.2 it suffices to show that (5.8) holds for any cusp form $f \in S_{k,L}$ with Fourier coefficients $a(0,n)$. Hence it suffices to prove that the set of all theta series $\Theta_D(\tau, v)$ (where D is any negative definite even unimodular lattice and $v \in D \otimes_{\mathbb{Z}} \mathbb{R}$) generates the space $S_{k,L}$. In fact, this is a consequence of a Theorem due to Waldspurger [Wal] (see also [EZ] Theorem 7.4). □

As a corollary we find that any meromorphic modular form for the group $\Gamma(L)$, whose divisor is a linear combination of Heegner divisors, has to be a Borcherds product. This result was already obtained in greater generality (under the weaker condition that L splits two hyperbolic planes over $\mathbb{Z}$) in [Br1]. (See also [Br2] for the $\mathrm{O}(2,2)$-case of Hilbert modular surfaces.) However, in [Br1] a completely different argument is used, which does not say anything about the local Picard groups of X_Γ.

Moreover, as a consequence of Theorem 5.4 and Theorem 5.2 we may infer that the rank of the subgroup of $\mathrm{Pic}(X_\Gamma)$ generated by the Heegner divisors equals $1 + \dim(S_{k,L})$.

BIBLIOGRAPHY

[Ba] L. BALLWEG, Die lokalen Cohomologiegruppen der Baily-Borel-Kompaktifizierung in generischen Randpunkten, Dissertation, University of Heidelberg (1992).

[Bo1] R. E. BORCHERDS, Automorphic forms with singularities on Grassmannians, Invent. Math., 132 (1998), 491–562.

[Bo2] R. E. BORCHERDS, The Gross-Kohnen-Zagier theorem in higher dimensions, Duke Math. J., 97 (1999), 219–233.

[Br1] J. H. BRUINIER, Borcherds products on $\mathrm{O}(2,l)$ and Chern classes of Heegner divisors, preprint (January 2000), http://www.mathi.uni-heidelberg.de/~bruinier/.

[Br2] J. H. BRUINIER, Borcherds products and Chern classes of Hirzebruch-Zagier divisors, Invent. Math., 138 (1999), 51–83.

[Ca] H. CARTAN, Fonctions automorphes, Séminaire No. 10, Paris (1957/58).

[EZ] M. EICHLER and D. ZAGIER, The Theory of Jacobi Forms, Progress in Math. 55 (1985), Birkhäuser.

[Fr1] E. FREITAG, Siegelsche Modulfunktionen, Springer-Verlag, 1983.

[Fr2] E. FREITAG, Hilbert Modular Forms, Springer-Verlag, 1990.

[GrRe] H. GRAUERT and R. REMMERT, Plurisubharmonische Funktionen in komlexen Räumen, Math. Zeitschr., 65 (1956), 175–194.

[No] A. NOBS, Die irreduziblen Darstellungen der Gruppen $\mathrm{SL}_2(\mathbb{Z}_p)$, insbesondere $\mathrm{SL}_2(\mathbb{Z}_2)$. I. Teil, Comment Math. Helvetici, 51 (1976), 465–489.

[Sh] G. SHIMURA, Introduction to the Arithmetic Theory of Automorphic Functions, Princeton University Press, 1971.

[Wal] J.-L. WALDSPURGER, Engendrement par des séries thêta de certains espaces de formes modulaires, Invent. Math., 50 (1979), 135–168.

[Wat] G. L. WATSON, Integral quadratic forms, Cambridge University Press, 1960.

Manuscrit reçu le 16 mai 2000,
accepté le 31 mai 2000.

Jan Hendrik BRUINIER, Eberhard FREITAG,
Universität Heidelberg
Mathematisches Institut
Im Neuenheimer Feld 288
D-69120 Heidelberg (Germany).
bruinier@mathi.uni-heidelberg.de
freitag@mathi.uni-heidelberg.de