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Pascal HUBERT & Thomas A. SCHMIDT

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INVARIANTS OF TRANSLATION SURFACES

by P. HUBERT and T.A. SCHMIDT

0. Introduction.

A translation surface is a real 2-dimensional manifold with conical singularities equipped with an atlas for which transition functions are translations. The study of Euclidean *billiards*, the straight-line flow within subsets of the Euclidean plane, quickly leads to translation surfaces. Indeed, each holomorphic 1-form on a Riemann surface induces a translation structure on the surface. There is a classic construction [KZ] which passes from an Euclidean polygon to an associated translation surface, determining a complex structure on the surface along with a holomorphic 1-form.

The study of billiards leads rather naturally to the more technical ground of quadratic differentials and Teichmüller theory. The use of this theory has resulted in deep results on the metric theory of the billiard flow. Fundamental results obtained in this manner include those of Kerckhoff-Masur-Smillie [KMS], H. Masur [M] and Eskin-Masur [EM]. The surfaces associated to quadratic differentials are naturally 1/2-translation surfaces, see [GJ2] for a discussion of these. Each such surface admits a translation surface as a double cover. Thus the restriction to the translation surfaces loses virtually none of this rich theory, but it does allow fairly significant simplification of language.

Interest in translation surfaces has been greatly increased by the deep results of W. Veech [Ve1]. Veech introduced the study of the diffeomorphisms of translation surfaces (punctured at the singularities) which are locally affine with respect to the translation structure. The

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differentials of these diffeomorphisms form a group, now called the Veech group. Veech showed that this group has discrete image in $\mathrm{PSL}(2,\mathbb{R})$. He also showed that if the image is a lattice (*i.e.*, if the group has finite covolume), then the billiard flow on the surface is particularly attractive, one has what has become to be known as the *Veech alternative*: In each direction this flow is either periodic, or it is uniquely ergodic.

There are a few examples of translation surfaces with lattice Veech groups, including those found by Veech himself, see [Vo], [EG], [Wa], [KS]. There are also several results indicating that the lattice property is rare amongst Veech groups. Surfaces with lattice Veech groups are of measure zero with respect to a natural measure on the space of these surfaces (of each fixed type) [GJ1] and amongst the translation surfaces given by the [KZ] construction applied to acute nonisosceles triangles, there only three within reasonable computability bounds which have lattice Veech groups [KS]. On the other hand, the translation surfaces having arithmetic lattice Veech groups are dense amongst all translation surfaces (of each fixed type). (We discuss arithmeticity in §1.11.)

The question of which Fuchsian groups can be realized as Veech groups seems completely open, other than a restriction that such a group cannot be cocompact. By extending results of E. Gutkin and C. Judge on arithmetic Veech groups [GJ2] we will show that there are groups which can *never* occur as Veech groups: Any arithmetic Fuchsian group which is not conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{Q})$ cannot be realized as a Veech group. Thus, for example, whereas each of the Hecke triangle groups of odd index were shown by Veech [Ve1] to occur as Veech groups, we have the following result.

THEOREM 1. — *The Hecke triangle groups of index* 4 *and* 6 *are not realizable as Veech groups. Furthermore, for each nonsquare natural number* N, *the* $\mathrm{PSL}(2,\mathbb{R})$ *normalizer of the congruence group* $\Gamma_0(N)$ *is not realizable as a Veech group.*

By way of a simple example given in Section 2, we show that, with minor accessory information (in our case, the genus of the surface), the isomorphism class of the Veech group of a translation surface can uniquely determine the surface. This determination is a priori up to *affine equivalence*, *i.e.*, up to a standard $\mathrm{SL}(2,\mathbb{R})$ action. In a general setting one can hope for no better, as any two affinely equivalent surfaces have isomorphic Veech groups. In our example there are distinguished elements, and hence one can determine that surface up to translation equivalence. We will show by example that, in general, Veech groups are far from sufficient to uniquely determine translation surfaces.

It is natural to study maps between translation surfaces. There are two notions of coverings of translation surfaces in the literature. The first of these is that for which E. Gutkin [G] has suggested the name *balanced* translation covering. Here, not only does the covering map respect the translation structures of the surfaces, but furthermore it is restricted such that the singularities of the two surfaces are "aligned" — the map sends singularities to singularities and the inverse images of singularities are singularities. In the more general translation covering, one simply requires that singularities be sent to singularities. Balanced coverings are better adjusted to questions of Veech groups, as Ya. Vorobets [Vo] and Gutkin and Judge [GJ1], [GJ2] have shown that commensurability classes of Veech groups are preserved by such coverings. This implies in particular that Veech's fundamental lattice property is preserved in this setting. This is not true under general translation coverings, as various examples in the literature show, see say [Vo] or [HS].

It is frequently desirable to mark points on translation surfaces other than true cone singularities — already with the case of genus one it is in some sense necessary to mark at least one point. One then considers Veech groups arising from affine diffeomorphisms on the surfaces with all of the marked points removed. We show that the marking of additional points preserves the lattice property if and only if these points are contained in finite orbits under the original group of affine diffeomorphisms. This allows us to point out the dramatic fact that if a translation surface has a lattice Veech group, then the marking at random of an additional point will, with probability one, result in a nonlattice Veech group.

We are particularly interested in the efficacy of lattice Veech groups in identifying translation surfaces in the setting of balanced coverings of translation surfaces. As we have already stated, the commensurability class of the Veech group is an invariant in this setting. By refining the information in the *parabolic directions* (see Definition 3 below), we define new invariants of these coverings. Our invariants are based upon the connections between singularities of a translation surface, thus are not far from the Veech group itself, nor from such notions as the holonomy field as used by [KS] nor the very related cross-ratio and trace fields as used by [GJ2]. These other invariants are weaker than the Veech group — two surfaces having the same Veech group must have the same holonomy, cross-ratio and trace fields.

In their construction, these other invariants emphasize two-dimensional aspects of the surfaces. Our invariants are closer to techniques used by Vorobets [Vo] in that their main ingredient is the comparison of connections in a single direction. However, we take a union over the parabolic directions. This allows a global nature to the construction, and indeed our invariants refine the invariant which is the Veech group, whenever the Veech group has parabolic elements. As our examples will show, our invariants are in practice quite easy to evaluate. In fact, one of our invariants is simply a set of integral vectors.

We first use the invariants to show there are affinely inequivalent translation surfaces which have isomorphic Veech groups. In order to do this, we apply our invariants to exactly determine certain Veech groups — each of the Veech groups in question were shown to contain lattice groups, but their discoverers [Vo], [KS] did not show equality of the Veech group and their indicated lattice subgroup. Our invariants allow us to very easily show these equalities.

As we have already stated, results of Vorobets and of Gutkin-Judge show that the commensurability class of Veech groups is an invariant of balanced coverings. We apply our invariants to certain examples to conclude that, in contrast to the arithmetic case, in general there is no "final object" amongst translation surfaces having Veech groups of the same commensurability class — that is, we show that there are translation surfaces with commensurable lattice Veech groups which are each minimal with respect to balanced coverings and yet are not affinely equivalent surfaces.

Having seen that balanced coverings taken singly are insufficient to account for all translations surfaces of a commensurability class of Veech groups, we turn to *trees* of such coverings, see the definition of §5. Our invariant allows us to show that there exist translation surfaces which have isomorphic Veech groups but which cannot lie in any common tree of balanced affine coverings.

One then must ask if perhaps at least each tree of balanced coverings is identified by a Veech group which is a maximal Fuchsian group (this is the case for the setting of arithmetic surfaces). Here again, our invariant applied to particular examples shows that not even this is true.

Thus, we have the following theorems.

THEOREM 2. — *The translation surface arising from the Euclidean triangle of angles* ($\pi/2n, \pi/2n, (n-1)\pi/n$) *cannot share a common tree of balanced affine coverings with any surface which has a maximal Fuchsian group as Veech group.*

THEOREM 3. — *The translation surfaces arising from the Euclidean triangles of angles* ($\frac{1}{18}\pi, \frac{1}{18}\pi, \frac{8}{9}\pi$) *and* ($\frac{2}{9}\pi, \frac{1}{3}\pi, \frac{4}{9}\pi$) *have isomorphic Veech groups but cannot share a common tree of balanced affine coverings.*

In our study of balanced coverings, we naturally discovered certain facts about non-balanced coverings. Indeed, the aforementioned exclusion of certain Fuchsian groups as Veech groups arose from studying various coverings. As well, we show that there exist translation surfaces which cover no translation surface whose Veech group is a maximal Fuchsian group. Again, this is in contrast to the arithmetic setting.

In this introduction, we have mentioned the arithmetic Veech groups several times. Here we briefly and very roughly indicate the state of knowledge about them, see also §1.11. A noncocompact Fuchsian group is called *arithmetic* if it admits a finite index subgroup which is $\mathrm{PSL}(2,\mathbb{R})$ conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{Z})$. The fundamental result is that of Gutkin and Judge [GJ2, Theorem 5.5]: A translation surface has an arithmetic Veech group if and only if the surface is a translation covering of a torus with one marked point.

One can think of this result of Gutkin and Judge as an analog of an unpublished but rather famous result of J. Franks (see say, [F]): A pseudo-Anosov diffeomorphism which has quadratic eigenvalue is the ramified covering of a linear Anosov automorphism of a torus; see [BC] for related results. There is an example of P. Arnoux and A. Fathi [AF] which shows the failure of the natural generalization of this to the setting of eigenvalues of degree greater than two. Our results in the non-balanced case can be seen as analogs of their example.

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1. Background.

For convenience, we collect in this section various basic notions, notation and background results. We repeat here some of [HS], §1.

1.1. Translation surfaces from billiards.

A surface is said to be a translation surface if it is equipped with an atlas for which the transition functions are translations in $\mathbb{R}^2$.

A construction apparently due to [KZ] associates to each polygon in $\mathbb{R}^2$ a translation surface. Briefly, one follows the straight line trajectory of a billiard on the polygon, reflecting the polygon when the billiard reaches an edge. If the polygon is *rational angled*, that is with all angles being rational multiples of π, of say with least common denominator N, then $2N$ copies of the polygon suffice to follow any billiard (whose trajectory does not end in a vertex) by straight line path segments — one must identify certain pairs of parallel edges by translation. This [KZ]-construction then gives a finite genus translation surface possibly with singularities amongst the vertices of the copies of the original polygon. The total angle about each singularity is an integral multiple of 2π.

As this construction takes place on $\mathbb{R}^2$, the natural 1-form dz induces a 1-form on the translation surface. There is a unique complex structure for which this 1-form is holomorphic.

1.2. Translation surfaces from 1-forms.

One can as well begin with a holomorphic 1-form on a Riemann surface. Integration gives local coordinates off of the zeros of the 1-form. The charts so defined have translations for transition functions; one obtains a translation surface. At a zero of the 1-form which is of multiplicity $m-1$, there is a singularity of angle $2m\pi$.

Multiplying a given 1-form by a nonzero complex constant has negligible effect. Multiplication by a real constant simply scales the area of the translation surface; multiplication by a complex constant of norm one merely rotates the charts, in effect inducing a different choice of the standard, say vertical, direction.

Note also that there is no canonical direction in the [KZ] process — dividing out by at least the natural action of the rotation group $\mathrm{SO}(2,\mathbb{R})$ is completely natural here. Again, real constant scaling of a fixed polygon clearly lead to scaled versions of the same translation surface.

Thus, we will actually work in projective spaces of holomorphic 1-forms, identifying all nonzero complex multiples of a 1-form.

1.3. Near Teichmüller theory.

As mentioned in the introduction, our topic is closely allied with Teichmüller theory. Each holomorphic quadratic differential on a Riemann surface induces local coordinates; if the quadratic form is the square of a holomorphic 1-form, then these coordinates are simply given in the aforementioned manner: integration of the 1-form. Any real surface admits an action of $\mathrm{SL}(2,\mathbb{R})$ on the set of its atlases – given an atlas, post composition of its local coordinate functions with $A \in \mathrm{SL}(2,\mathbb{R})$ defines a new atlas. Note that this action preserves the set of translation atlases.

Any given holomorphic 1-form determines a translation atlas; the $\mathrm{SL}(2,\mathbb{R})$-orbit of this translation atlas is comprised of translation atlases arising from 1-forms. (This last – and much more – may be precisely proven by combining results of [KMS] and of I. Kra [Kr].) In fact, [KMS] show that the set of the squares of these 1-forms gives a Teichmüller disk of quadratic differentials. Now, a Teichmüller disk admits a hyperbolic metric; with respect to this metric, $\mathrm{PSL}(2,\mathbb{R})$ acts faithfully as the oriented isometry group.

1.4. Fuchsian groups — discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

We have just seen a hint that $\mathrm{PSL}(2,\mathbb{R})$ plays a fundamental role in the theory of our topic. Let us review some of the basics of the theory of Fuchsian groups, the set of discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$. Recall that $\mathrm{PSL}(2,\mathbb{R})$ acts as oriented isometries on the hyperbolic plane in its Poincaré half-plane model by way of fractional linear transformations.

Of fundamental importance for us will be the class of *lattices.* A Fuchsian group is a lattice if it is of finite covolume (that is, the quotient of the hyperbolic plane by the group has finite area).

Amongst the lattices are (Schwarz) *triangle groups.* We fix a triangle of angles π/p, π/q and π/r in the (extended) hyperbolic plane and consider the reflections through the sides of the triangle. Each reflection is orientation reversing; the group generated by the words of even length in these reflections forms a Fuchsian group, see say, [B]. We call (p,q,r) the signature of the triangle group. In fact, any two triangle groups of the same signature are $\mathrm{PSL}(2,\mathbb{R})$-conjugate. This being the case, we will sometimes speak loosely and say the triangle group $\Delta(p,q,r)$ to mean some group of this signature. (Indeed, in the theory of Veech groups it is conjugation classes of Fuchsian groups which naturally arise.) Another important fact is that triangle groups are maximal amongst Fuchsian groups: A triangle group can only be contained in triangle (Fuchsian) groups, see say [B].

A particular class of triangle groups arises quite frequently. These are the Hecke groups, of signature $(2, q, \infty)$. The Hecke group of index $q = 3$ is nothing other than the modular group $\mathrm{PSL}(2, \mathbb{Z})$. Each Hecke group is a maximal Fuchsian group.

Recall from the introduction that a noncocompact Fuchsian group is called *arithmetic* if it admits a finite index subgroup which is $\mathrm{PSL}(2, \mathbb{R})$ conjugate to a finite index subgroup of $\mathrm{PSL}(2, \mathbb{Z})$. It is a result of A. Leutbecher [L] that the Hecke groups are arithmetic for exactly the indices $q = 3, 4, 6$.

Fuchsian groups are said to be *commensurate* or *strictly commensurable* if they share a common subgroup of finite index in each. They are said to be *commensurable* if a finite index subgroup of one conjugates within $\mathrm{PSL}(2, \mathbb{R})$ to give a finite index subgroup in the other.

Warning — We follow the definitions of [GJ2] here. It is also common to use the term commensurable to denote what they call commensurate!

A deep result of G. Margulis [M] see also [MR], implies that within the strict commensurability class of a nonarithmetic Fuchsian group there is a single maximal group. (Note that this does not hold in the arithmetic setting: Each of the Hecke groups of index $q = 3, 4, 6$ is maximal.) Since we will often identify groups only up to $\mathrm{PSL}(2, \mathbb{R})$-conjugacy, let us note that the Margulis result clearly extends to hold for any Fuchsian group conjugate to a nonarithmetic group.

We also will need to consider some aspects of the internal structure of Fuchsian groups. Given an element A of $\mathrm{PSL}(2, \mathbb{R})$, we will simply represent A by one of its corresponding elements in $\mathrm{SL}(2, \mathbb{R})$. Naturally, the trace is then only defined up to absolute value. Recall that $\mathrm{PSL}(2, \mathbb{R})$ acts upon the hyperbolic plane in its Poincaré half-plane model by way of fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d} \cdot$$

One solves for fixed points to find that an element A fixes exactly one point on the boundary $\mathbb{R} \cup \{\infty\}$ if and only if A has absolute value of trace equal to 2. Such elements are called *parabolic.* We say that a direction vector θ in $\mathbb{R}^2$ is *fixed* by a parabolic element if one of the corresponding elements of $\mathrm{SL}(2,\mathbb{R})$ fixes θ. Each parabolic element is $\mathrm{PSL}(2,\mathbb{R})$ conjugate to a translation. Parabolicity is defined by trace, thus the conjugacy class within any given Fuchsian group of a parabolic element is comprised of parabolic elements. Furthermore, if an element is parabolic, then so is any power of it. A *primitive* element of a group is one which is not the positive power of any other element of the group. A *maximal parabolic conjugacy class* of a Fuchsian group is a conjugacy class of primitive parabolic elements. Of importance here is that each lattice group has a finite number of maximal parabolic conjugacy classes. Furthermore, a result of C. Siegel, see say [K], gives that any lattice is finitely generated.

1.5. Affine functions and Veech groups.

Let us fix a holomorphic 1-form ω on a Riemann surface M and let $Z(\omega)$ denote the set of the zeros of ω. Let $M' := M \setminus Z(\omega)$. A diffeomorphism $f : M' \to M'$ which extends to a homeomorphism from M to itself is called *affine* with respect to the translation structure on M induced by ω if the derivative of f is constant in the charts of ω and is given by some fixed element $A \in \mathrm{SL}(2,\mathbb{R})$. Note that this definition requires that the extension of f and that of its inverse send $Z(\omega)$ to itself (permutation of this set is allowed).

Away from zeros of ω, locally $f(z) = Az + c_i$, where the c_i depend only on the chart of z. The set of all such functions is called the affine group of ω, $\mathrm{Aff}(\omega)$. The Veech group, $\Gamma(\omega)$, is the subgroup of $\mathrm{SL}(2,\mathbb{R})$ representing the derivatives of the affine functions. In fact, Veech [V1] shows that the object of main interest is this group taken up to projective equivalence; that is, we need only consider the image of Γ in $\mathrm{PSL}(2,\mathbb{R})$. In what follows, we will indeed simply write $\Gamma(\omega)$ for this corresponding subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

This projective group $\Gamma(\omega)$ acts on the Teichmüller disk given by the quadratic differentials which are the squares of the 1-forms associated (by way of the various affinely related translation structures) to ω^2. Veech showed that $\Gamma(\omega)$ acts discontinuously on this hyperbolic disk; that is, $\Gamma(\omega)$ is a Fuchsian group.

The quotient of the disk by $\Gamma(\omega)$ is a Riemann surface (its hyperbolic structure identifies a complex structure) inside the Riemann moduli space of M. It is a remark of Veech [Ve1], see also [Vo], Proposition 3.3, that this surface cannot be compact; in other words, $\Gamma(\omega)$ must be noncocompact. See [H1], [H2] for some remarks on the possible relationship of this new Riemann surface to M.

The fundamental *Veech alternative* states that if the Veech group of a translation surface is a lattice, then in each direction the billiard flow is either periodic, or it is uniquely ergodic.

A fundamental open problem is to characterize which Fuchsian groups are Veech groups.

1.6. Veech groups and automorphisms.

Given a rational angled polygon, the $2N$ (as in §1.1) copies which comprise its associated translation surface are given by an action of the dihedral group of order $2N$. That is, one takes an original copy of the polygon and a copy which is a reflection about an edge. One creates N copies of this doubled polygon by applying the powers of the rotation of angle $2\pi/N$. These are all glued together to give the translation surface. Thus there is an oriented self-map on the surface whose differential is a rotation of order N. Now, if N is odd then there is an element of order N in the Veech group of the translation surface; if $N = 2k$ is even, then there is an element of order k in the (projective) Veech group.

In fact, self-maps of the type above induce automorphisms of the underlying Riemann surface. Indeed, these are diffeomorphisms which are locally rotations (up to translation); they are isometries. They clearly preserve the conformal structure, and hence the complex structure of the underlying Riemann surface.

1.7. Results on Veech groups from rational triangles.

We establish some notation to be used in the remainder of this paper.

Notation. — Let $\mathcal{T}(p,q,r)$ be the rational Euclidean triangle whose angles are $p\pi/n$, $q\pi/n$, $r\pi/n$, where $n = p+q+r$ and $1 = \gcd(p,q,r)$.

Let $X(p,q,r)$ and $\omega(p,q,r)$ be the Riemann surface and its holomorphic 1-form associated to the billiard flow on the Euclidean triangle $\mathcal{T}(p,q,r)$. Furthermore, let $\Gamma(p,q,r)$ be the Veech group of $\omega(p,q,r)$. Let $\Delta(p,q,r)$ be the Fuchsian triangle group for the angles $\pi/p, \pi/q, \pi/r$ (see say, [B]).

W. Veech [V1] showed that

THEOREM A (Veech). — *For each $n \geq 5$,*

$$\Gamma(1,1,n-2) = \begin{cases} \Delta(2,n,\infty) & \textit{odd } n, \\ \Delta(m,\infty,\infty) & \textit{if } n = 2m. \end{cases}$$

Earle and Gardiner [EG] show, in our notation, that

$$\Gamma(2,2,1) = \Delta(5,\infty,\infty).$$

Indeed, by inspection of their examples, they actually show the following theorem.

THEOREM B (Earle-Gardiner). — *Let the integer* $k \geq 2$. *Then*

$$\Gamma(2k-1, 2k-1, 2) = \Delta(2k,\infty,\infty),$$

$$\Gamma(k,k,1) = \Delta(2k+1,\infty,\infty).$$

There are various results indicating that Veech groups are rarely lattices. Perhaps the most striking is the following.

THEOREM C (Kenyon-Smillie). — *Let* $\mathcal{T} = \mathcal{T}(p,q,r)$ *be an acute nonisosceles triangle with* $p+q+r \leq 10{,}000$. *Then* $\Gamma(p,q,r)$ *is a lattice group if and only if* $\mathcal{T}$ *is one of the following*:

$$(a) \quad \mathcal{T}(3,4,5), \quad (b) \quad \mathcal{T}(3,5,7), \quad \textit{or} \quad (c) \quad \mathcal{T}(2,3,4).$$

The first of these two cases were shown to give lattice groups by Vorobets.

LEMMA D (Vorobets). — *One has that*

$$\Gamma(3,4,5) \supset \Delta(6,\infty,\infty) \quad \textit{and} \quad \Gamma(3,5,7) \supset \Delta(15,\infty,\infty).$$

Kenyon and Smillie proved that the third of these gives a lattice group.

LEMMA E (Kenyon-Smillie). — *One has that* $\Gamma(2,3,4) \supset \Delta(9,\infty,\infty)$.

1.8. Marking extra points.

It is convenient to consider translation structures with some removable singularities marked. We introduce notation for this purpose.

Notation. — Let $\mathcal{P}(\omega;\{p_1,\ldots,p_n\})$ denote the translation structure on a surface M given by the 1-form ω and having marked points p_1 through p_n in addition to the zeros of ω. Given $\mathcal{P}$ of this sort, let M'' be M (having the structure of ω) with both $Z(\omega)$ and the set of the p_i removed. The affine group, $\mathrm{Aff}(\mathcal{P})$, for such a marked translation structure is the group of the affine diffeomorphisms which restrict so as to take M'' to itself.

The Veech group, $\Gamma(\mathcal{P})$, is then the (projective image of the) derivatives of these affine diffeomorphisms.

For a fixed surface M, and marked structures $\mathcal{P}$ and $\mathcal{Q}$, we write $\mathcal{P} \subset \mathcal{Q}$ if the marked structures have the same underlying 1-form, and the marked points of $\mathcal{P}$ are amongst those of $\mathcal{Q}$.

LEMMA F (see [HS]). — *Let $\mathcal{P}$ and $\mathcal{Q}$, $\mathcal{P} \subset \mathcal{Q}$, be as above. Then both $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ are subgroups of $\Gamma(\omega)$. Furthermore, there is a finite index subgroup of $\Gamma(\mathcal{Q})$ which is contained in $\Gamma(\mathcal{P})$. If $\Gamma(\mathcal{Q})$ is a lattice, then so are $\Gamma(\mathcal{P})$ and $\Gamma(\omega)$.*

1.9. Translation and affine coverings, equivalence.

We say that a map $f : M \to N$ gives a *translation covering* of $(N, \mathcal{Q})$ by $(M, \mathcal{P})$ if the restriction $f : M'' \to N''$ is such that $\psi \circ f \circ \phi^{-1}$ are translations where ψ and ϕ are the (various appropriate choices of the) local coordinates for the atlases of $\mathcal{P}$ and $\mathcal{Q}$ respectively. Note that a translation covering is in particular a holomorphic (ramified) covering of the corresponding Riemann surfaces.

Similarly, we say that a map f gives an *affine covering* of $(N, \mathcal{Q})$ by $(M, \mathcal{P})$ if the restriction $f : M'' \to N''$ is such that the aforementioned compositions are of the form $Az + c_{i,j}$ where A is a fixed matrix in $\mathrm{SL}(2, \mathbb{R})$, but the translation vectors $c_{i,j}$ may vary with the choice of charts. Note that an affine covering is in particular a quasi-conformal (ramified) covering of the corresponding Riemann surfaces.

Let B be any matrix in $\mathrm{SL}(2, \mathbb{R})$. We define $(M, B \circ \mathcal{P})$ by replacing the coordinate functions of the translation structure of $(M, \mathcal{P})$ by their post-composition with B. Let f give an affine covering of $(N, \mathcal{Q})$ by $(M, \mathcal{P})$. If A is the matrix of the derivative of f, then we define f^A to be the covering of $(N, \mathcal{Q})$ by $(M, A \circ \mathcal{P})$. Similarly, we define f_A to be the covering of $(N, A^{-1} \circ \mathcal{Q})$ by $(M, \mathcal{P})$. The following can be found in [Vo].

LEMMA G (Vorobets). — *Let f give an affine covering of $(N,\mathcal{Q})$ by $(M,\mathcal{P})$. Let A be the matrix of the derivative of f. Then both f^A and f_A are translation coverings.*

If there is a degree one translation covering of one translation surface by another, then the covering map admits an inverse, and we say that the translation surfaces are *translation equivalent.* Similarly a degree one affine covering gives an *affine equivalence* of translation surfaces.

1.10. Commensurability results.

Given a general translation or affine covering of $(N, \mathcal{Q})$ by $(M, \mathcal{P})$, it seems unclear as to exactly how $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ are related. There is, however, some vague knowledge of their relationship.

THEOREM H (Vorobets; Gutkin-Judge). — *If there is a translation covering of* $(N,\mathcal{Q})$ *by* $(M,\mathcal{P})$, *then* $\Gamma(\mathcal{P})$ *and* $\Gamma(\mathcal{Q})$ *are commensurate.*

COROLLARY I (Gutkin-Judge). — *If there is an affine covering of* $(N,\mathcal{Q})$ *by* $(M,\mathcal{P})$, *then* $\Gamma(\mathcal{P})$ *and* $\Gamma(\mathcal{Q})$ *are commensurable.*

In particular settings, these results can be strengthened. The following observation of [AH] will be of use.

LEMMA J (Arnoux–Hubert). — *The translation surface* $\mathcal{M}_{2n}$ *formed by identifying opposite sides of a regular planar* $2n$*-gon is balanced translation double covered by* $\mathcal{T}(1,1,2n-2)$. *The Veech groups of these surfaces are isomorphic.*

1.11. Arithmetic groups.

The fundamental result for arithmetic Veech groups is that of Gutkin and Judge [GJ2], Theorem 5.5.

THEOREM K (Gutkin–Judge). — *A translation surface has an arithmetic Veech group if and only if the surface is a translation covering of a torus with one marked point.*

Note that the translation coverings of Theorem K are not necessarily balanced.

We will show that there are arithmetic Fuchsian groups which can never occur as the Veech group of any translation surface. For this the following notions will be helpful. Fix a $N \in \mathbb{N}$, then

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) \mid N \mid c \right\}.$$

Let $\Gamma_B(N)$ be the $\mathrm{PSL}(2,\mathbb{R})$ normalizer of $\Gamma_0(N)$. Each conjugacy class of arithmetic groups has a representative inside some $\Gamma_B(N)$, as shown by H. Helling [H], see also [C].

THEOREM L (Helling). — *Given a noncocompact arithmetic Fuchsian group, there exists* $N \in \mathbb{N}$ *such that* $\Gamma_B(N)$ *contains a* $\mathrm{PSL}(2,\mathbb{R})$*-conjugate of the given group.*

Given a natural number N, let

$$\tau = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}.$$

It is easy to see that τ is in $\Gamma_B(N)$. But then one also notices that say

$$\begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & -\sqrt{N} \end{pmatrix}$$

is in $\Gamma_B(N)$. Thus if N is nonsquare, $\Gamma_B(N)$ is not a subgroup of $\mathrm{PSL}(2,\mathbb{Q})$.

2. Identifying translation surfaces by Veech groups.

We start with a rather special example to motivate our discussion of the efficacy of the Veech group in identifying translation surfaces.

For various reasons most of the examples in the literature of translation surfaces are the [KZ] surfaces of triangles. The following type of construction seems well-known to the experts in the field, but has yet to appear in the literature.

Definition 1. — The translation surface formed by taking a symmetric rectangular cross of minor side length 1 and of internal length $\lambda \geq 1$ (see Figure 1) and identifying opposite sides is called the *cross of translation* λ.

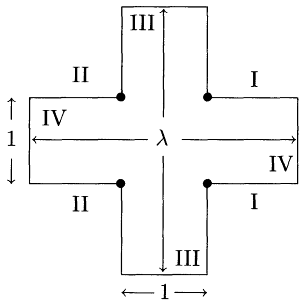

Figure 1. The cross of translation λ

LEMMA 1. — *For each positive real $\lambda \geq 1$, the cross of translation λ is a genus two translation surface with a unique singular point. The surface has an element of order two, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, in its (projective) Veech group; the corresponding affine diffeomorphism fixes the surface.*

Proof. — One easily finds that the internal vertices (again, see Figure 1) are identified to give a single point of angle 6π. There are no other singular points, hence the genus is indeed two. The visible rotational symmetry of the cross gives the element of order two in $\mathrm{PSL}(2,\mathbb{R})$. □

LEMMA 2. — *The cross of translation $\lambda_5 = \frac{1}{2}(1+\sqrt{5}\,)$ has as its Veech group a triangle group $\Delta(2,5,\infty)$.*

Proof. — Whenever $\lambda(\lambda-1)$ is rational, one easily succeeds with the Veech construction of parabolic elements, see [Ve1], Prop. 2.4, for the vertical direction of the cross. Since $\lambda_5(\lambda_5-1) = 1$, one finds the parabolic element $S = \left(\begin{smallmatrix} 1 & 0 \\ \lambda_5 & 1 \end{smallmatrix}\right)$.

But, S and T generate the Hecke group of index $q = 5$, see say [B]. Since each Hecke group is a maximal Fuchsian group, we have indeed determined the Veech group. Finally, this Hecke group is indeed a triangle group of the indicated signature. □

The following is well-known, see say [S], or for a textbook discussion [BL], p. 347.

LEMMA 3. — *Up to isomorphism there is exactly one nonsingular compact Riemann surface of genus two which admits an automorphism of order 5. This is the Riemann surface of equation $y^2 = 1 - x^5$.*

Recall that Veech [Ve1] showed (in particular) that the Veech group of the translation surface $X(1,1,3)$ is (up to $\mathrm{PSL}(2,\mathbb{R})$-conjugation) the Hecke group of index $q = 5$. In that group there is a unique conjugacy class of elements of order two. Thus, there is up to translation equivalence a unique translation surface which is fixed by an affine diffeomorphism whose derivative is (projectively) an element of order two. For ease of statement, let us say that this surface is fixed by the element of order two in the Veech group of $X(1,1,3)$.

LEMMA 4. — *The cross of translation $\lambda = \frac{1}{2}(1+\sqrt{5}\,)$ is translation equivalent to the surface fixed by the element of order two in the Veech group of $X(1,1,3)$.*

Proof. — There is an element of order five in the Hecke group of index $q = 5$. The subgroup generated by this element is unique up to conjugation in the group. Now, the element of order five in the Veech group of the cross of translation $\lambda = \frac{1}{2}(1 + \sqrt{5}\,)$ fixes a translation surface.

By Lemma 3, this new fixed translation surface has as its underlying Riemann surface that of equation $y^2 = 1 - x^5$. Automorphisms of Riemann surfaces act linearly on the vector space of holomorphic 1-forms of the surface. In this case, a generator for the subgroup of automorphisms of order 5 sends (x, y) to $(\zeta_5 x, y)$, where ζ_5 is a primitive fifth root of unity. Clearly, the forms $\mathrm{d}x/y$ and $x\,\mathrm{d}x/y$ are eigenvectors of *distinct* eigenvalues for the action of these automorphisms. Hence, up to negligible constants, the only 1-forms fixed by an automorphism of order five are these two. (Recall the discussion of §1.2 of such negligible constants.) Therefore, our apparently new fixed translation surface must in fact be translation equivalent to the translation surface defined by one of these 1-forms.

Now, [EG] have shown that the Veech group of $x\,\mathrm{d}x/y$ is isomorphic to a triangle group $\Delta(5, \infty, \infty)$. Since the Hecke group of index five is a $\Delta(2, 5, \infty)$, our translation surface cannot be translation equivalent to its surface. Veech [Ve1] showed that the Veech group of $X(1, 1, 3)$ is a $\Delta(2, 5, \infty)$, and furthermore, that this translation surface is exactly that given by the holomorphic 1-form $\mathrm{d}x/y$.

We have identified the Teichmüller disk of the cross of translation $\lambda = \frac{1}{2}(1 + \sqrt{5}\,)$. Up to translation equivalence there is exactly one translation structure in this disk which is fixed by the element of order 2. □

Remark 1. — There are various ways to simplify the above proof. In particular, one could simply give the passage from the cross to $X(1, 1, 3)$ as an explicit affine map. We prefer the present proof, as it hints at some of the algebraic aspects of Veech groups.

Remark 2. — The Riemann surface of equation $y^2 = 1 - x^n$ admits an automorphism of order n which has distinct eigenvalues. This automorphism is the exact analogy of that mentioned in the proof of the preceding lemma. Furthermore, for n odd Veech [Ve2] gives a geometric property which indicates that these Riemann surfaces are quite special amongst all those of the same genus. Could it be that any genus $g = \lfloor \frac{1}{2}(n - 1) \rfloor$ translation surface which has Veech group isomorphic to that of $X(1, 1, n)$ — recall that the underlying Riemann surface of $X(1, 1, n)$ is indeed of equation $y^2 = 1 - x^n$ — must actually be affinely equivalent to $X(1, 1, n)$?

3. Refining the Veech group – affine invariants of parabolic directions.

The commensurability class of Veech groups is preserved under balanced affine coverings, see [Vo], [GJ1], [GJ2] and for some related discussion [HS]. We ask for additional invariants in this setting.

We introduce the ingredients for our invariants, see [Vo] for related notions. Until stated otherwise, we fix a translation surface (M, ω) and a marking $\mathcal{P}(\omega; \{p_1, \ldots, p_n\})$.

Definition 2. — A simple geodesic connecting two marked points (passing through no others) is called a *connection.* (Recall that the zeros of ω are always amongst the marked points. Simple geodesics connecting these are traditionally known as *saddle connections.*)

A direction vector θ in $\mathbb{R}^2$ is a *connection direction* if there is a connection in this direction on (M, ω). (Note that each connection thus defines two connection directions, θ and $-\theta$.)

The atlas of (M, ω) allows one to associate to each connection a $\vec{v} \in \mathbb{R}^2$ (one uses the so-called *holonomy* of (M, ω), see [KS] for discussion of this notion). We call $\vec{v} \in \mathbb{R}^2$ a *connection vector* if there is a connection whose vector is $\vec{v}$. Let $\mathcal{V}(\mathcal{P}, \theta)$ be the ordered *n-tuple* of all connection vectors in the direction θ, where the ordering is by length. Let $\overline{\mathcal{V}}(\mathcal{P}, \theta)$ be the *set* formed by these vectors.

For each $\vec{v}$ in $\overline{\mathcal{V}}(\mathcal{P}, \theta)$, let $n(\vec{v})$ be the number of occurrences of $\vec{v}$ in $\mathcal{V}(\mathcal{P}, \theta)$. Let $N(\mathcal{P}, \theta)$ be the sum of these $n(\vec{v})$. Let $\vec{v}_0$ be a shortest vector in $\overline{\mathcal{V}}(\mathcal{P}, \theta)$, and for each $\vec{v}$ in $\overline{\mathcal{V}}(\mathcal{P}, \theta)$, let

$$\tilde{\ell}(\vec{v}) := \frac{\|\vec{v}\|}{\|\vec{v}_0\|}$$

be the scaled length of $\vec{v}$. We define $R(\mathcal{P}, \theta)$, the *scaling vector*, to be the ordered $N(\mathcal{P}, \theta)$-tuple of the real numbers $\tilde{\ell}(\vec{v})$. Here for each $\vec{v}$ in $\overline{\mathcal{V}}(\mathcal{P}, \theta)$, $\tilde{\ell}(\vec{v})$ occurs $n(\vec{v})$ times, and the ordering is by size of positive real numbers. Finally, we define $\mathcal{N}(\mathcal{P}, \theta)$, the *counting vector*, to be the ordered n-tuple of natural numbers which, for each $\vec{v}$ in $\overline{\mathcal{V}}(\mathcal{P}, \theta)$, reports the multiplicity $n(\vec{v})$ with ordering again by the size of the $\tilde{\ell}(\vec{v})$.

Note that since $\mathcal{P}$ has only finitely many marked points, both $R(\mathcal{P}, \theta)$ and $\mathcal{N}(\mathcal{P}, \theta)$ are indeed finite.

Remark 3. — To summarize the above definition, the scaling vector $R(\mathcal{P},\theta)$ is the N-tuple of reals by which one must scale a shortest connection vector in the direction θ to obtain all of the connection vectors in this direction. The counting vector $\mathcal{N}(\mathcal{P},\theta)$ gives the number of connection vectors in this direction which are of each possible length.

As shown by [Vo] and [GJ1], the two Veech groups involved in any given balanced translation covering are commensurate — there is a common subgroup of finite index in each. The following restatement of a result of [Vo] is easily proved with the above ideas and definitions.

LEMMA 5 (Vorobets). — *Let M, $\mathcal{P}$ and θ as above. If there exists $\vec{v} \in \overline{\mathcal{V}}(\mathcal{P},\theta)$ such that $n(\vec{v}) = 1$, then any balanced translation covering $f\colon(N,\mathcal{Q}) \to (M,\mathcal{P})$ is such that $\Gamma(\mathcal{Q}) \subset \Gamma(\mathcal{P})$.*

Proof. — This is [Vo], Proposition 5.3 restated in the present vocabulary. □

Notation. — Recall that for all $A \in \mathrm{SL}(2,\mathbb{R})$, whereas $\mathrm{Aff}(\mathcal{P})$ is exactly equal to $\mathrm{Aff}(A \circ \mathcal{P})$, one has that the corresponding Veech groups are conjugate. Let us denote general *conjugation* by A, AGA^{-1}, with $A \cdot G$, then $\Gamma(A \circ \mathcal{P}) = A \cdot \Gamma(\mathcal{P})$.

Notation. — There is a natural linear action of $\mathrm{SL}(2,\mathbb{R})$ on ordered n-tuples of 2-vectors given by extending the action on the individual vectors. Let us denote this in our context by $A \star \mathcal{V}(\mathcal{P},\theta)$.

The following lemma shows that $\mathrm{SL}(2,\mathbb{R})$ acts equivariantly on the vectors of connections, $\mathcal{V}(\mathcal{P},\theta)$.

LEMMA 6. — *Let $(M,\mathcal{P})$ be a translation surface and θ some direction on this surface. Let $A \in SL(2,\mathbb{R})$. Then $A\theta$ is a connection direction on $(M,A\circ\mathcal{P})$. Furthermore, $\mathcal{V}(A\circ\mathcal{P},A\theta) = A\star\mathcal{V}(\mathcal{P},\theta)$. As well, the equalities $\mathcal{N}(A\circ\mathcal{P},A\theta) = \mathcal{N}(\mathcal{P},\theta)$ and $R(A\circ\mathcal{P},A\theta) = R(\mathcal{P},\theta)$ hold.*

Proof. — That $\psi = A\theta$ is a connection direction and the equality of $\mathcal{V}(A\circ\mathcal{P},A\theta) = A\star\mathcal{V}(\mathcal{P},\theta)$ follow directly from the *linearity* of the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$. The other equalities follow from this first one. □

LEMMA 7. — *Let M and $\mathcal{P}$ be as above. Let $\mathfrak{P}$ be a parabolic conjugacy class of the Veech group $\Gamma(\mathcal{P})$. Suppose that θ and ψ are* *connection directions for $\mathcal{P}$ fixed by corresponding elements of $\mathfrak{P}$. Then there exists $A \in \Gamma(\mathcal{P})$ such that $\mathcal{V}(\mathcal{P},\psi) = A \star \mathcal{V}(\mathcal{P},\theta)$. Furthermore, the equalities $\mathcal{N}(\mathcal{P},\psi) = \mathcal{N}(\mathcal{P},\theta)$ and $R(\mathcal{P},\psi) = R(\mathcal{P},\theta)$ hold.*

Proof. — Conjugate elements in $\mathfrak{P}$ fix θ and ψ respectively. Thus, there is an $A \in \Gamma(\mathcal{P})$ such that $\psi = A\theta$. Hence, ψ is a connection direction for $A \circ \mathcal{P}$. By the previous lemma $\mathcal{V}(A \circ \mathcal{P}, \psi) = A \star \mathcal{V}(\mathcal{P}, \theta)$. Thus our conclusion holds once we have shown that $\mathcal{V}(A \circ \mathcal{P}, \psi) = \mathcal{V}(\mathcal{P}, \psi)$.

There is at least one $f \in \mathrm{Aff}(\mathcal{P})$ whose derivative gives A^{-1}. From Lemma G, any such f gives a degree one balanced translation covering of $(M, A \circ \mathcal{P})$ by $(M, \mathcal{P})$. Hence, the set of connection vectors is the same for both surfaces. Furthermore, the n-tuples of connection vectors in any fixed direction is the same for both surfaces. In particular, they are the same for the direction ψ and therefore $\mathcal{V}(A \circ \mathcal{P}, \psi) = \mathcal{V}(\mathcal{P}, \psi)$. □ □

Definition 3. — Suppose that M and $\mathcal{P}$ are as above and $\mathfrak{P}$ is a parabolic conjugacy class of the Veech group $\Gamma(\mathcal{P})$. A connection direction θ is called a *parabolic direction* for $\mathfrak{P}$ if θ is fixed by some element of this conjugacy class. Choose any such θ. Using the previous lemma, we have the definitions $\mathcal{N}(\mathfrak{P}, \mathcal{P}) := \mathcal{N}(\mathcal{P}, \theta)$ and $R(\mathfrak{P}, \mathcal{P}) := R(\mathcal{P}, \theta)$.

PROPOSITION 1. — *Let $\mathfrak{P}$ be a parabolic conjugacy class of the Veech group of a translation surface $(M,\mathcal{P})$. Then for any $A \in SL(2,\mathbb{R})$, $\mathcal{N}(\mathfrak{P},\mathcal{P}) = \mathcal{N}(A \cdot \mathfrak{P}, A \circ \mathcal{P})$ and $R(\mathfrak{P},\mathcal{P}) = R(A \cdot \mathfrak{P}, A \circ \mathcal{P})$.*

Proof. — By the previous lemma, we can choose connection directions θ for $\mathcal{P}$ and $\psi = A\theta$ for $A \circ \mathcal{P}$ which are fixed by elements of $\mathfrak{P}$ and $A \cdot \mathfrak{P}$, respectively and such that one has

$$\mathcal{N}(\mathfrak{P}, \mathcal{P}) = \mathcal{N}(\mathcal{P}, \theta) \quad \text{and} \quad \mathcal{N}(A \cdot \mathfrak{P}, A \circ \mathcal{P}) = \mathcal{N}(A \circ \mathcal{P}, \psi);$$

and also

$$R(\mathfrak{P}, \mathcal{P}) = R(\mathcal{P}, \theta) \quad \text{and} \quad R(A \cdot \mathfrak{P}, A \circ \mathcal{P}) = R(A \circ \mathcal{P}, \psi).$$

Therefore, Lemma 7 applies. □

Definition 4. — Suppose that M and $\mathcal{P}$ are as above. Let $\{\mathfrak{P}_i\}$ be the set of maximal parabolic conjugacy classes of the Veech group $\Gamma(\mathcal{P})$. Define $R(\mathcal{P})$ to be the set of the $R(\mathfrak{P}_i, \mathcal{P})$ and $\mathcal{N}(\mathcal{P})$ to be the set of the counting vectors $\mathcal{N}(\mathfrak{P}_i, \mathcal{P})$. We call $R(\mathcal{P})$ the *set of scaling vectors* and $\mathcal{N}(\mathcal{P})$ the *set of counting vectors.*

Now, let $\{\theta_i\}$ be corresponding parabolic directions and let $c(\mathcal{P})$ be the greatest common divisor of the multiplicities of vectors in these directions. That is, let

$$c(\mathcal{P}) = \gcd\big(\{n(\vec{v}) \mid \vec{v} \in \overline{\mathcal{V}}(\theta), \theta \text{ parabolic direction for } \mathcal{P}\}\big).$$

For each i and each $\vec{v} \in \overline{\mathcal{V}}(\theta_i)$, let

$$\widetilde{n}(\vec{v}) = \frac{n(\vec{v})}{c(\mathcal{P})}.$$

For each i, we define $\widetilde{R}(\mathfrak{P}_i)$, the *weighted scaling vector*, to be the ordered $N(\mathfrak{P})/c(\mathcal{P})$-tuple of the $\widetilde{\ell}(\vec{v})$, ordering by size of positive real numbers. We define $\widetilde{\mathcal{N}}(\mathfrak{P}_i)$, the *weighted counting vector* similarly. Note that $\widetilde{R}(\mathfrak{P}_i)$ differs from $R(\mathfrak{P}_i)$ only in that a common repetition of values has been suppressed and similarly for $\widetilde{\mathcal{N}}(\mathfrak{P}_i)$ and $\mathcal{N}(\mathfrak{P}_i)$.

Finally, let $\widetilde{R}(\mathcal{P})$ be the *set* of the $\widetilde{R}(\mathfrak{P}_i)$. Define $\widetilde{\mathcal{N}}(\mathcal{P})$ similarly. We call $\widetilde{R}(\mathcal{P})$ the *set of weighted scaling vectors* and $\widetilde{\mathcal{N}}(\mathcal{P})$ the *set of weighted counting vectors*.

From the preceding lemmas, it is clear that each of $\widetilde{R}(\mathcal{P})$, $R(\mathcal{P})$, $\widetilde{\mathcal{N}}(\mathcal{P})$ and $\mathcal{N}(\mathcal{P})$ is well-defined.

Remark 4. — To summarize the above definition, given a marking $\mathcal{P}$ on M, $R(\mathcal{P})$ is the set of the scaling vectors for the parabolic directions of $\mathcal{P}$ and $\widetilde{R}(\mathcal{P})$ is the set of these except that common repetition, counted by $c(\mathcal{P})$, is eliminated. In particular, $R(\mathcal{P})$ is a finer invariant of the "disk" of $(M, \mathcal{P})$. However, we will show that $\widetilde{R}(\mathcal{P})$ is an invariant of balanced coverings. Similarly for $\widetilde{\mathcal{N}}(\mathcal{P})$ and $\mathcal{N}(\mathcal{P})$. Note also that our invariants are independent of multiplication of 1-forms (giving the underlying structure of the surface) by nonzero complex constants.

THEOREM 4. — *If $f:(M,\mathcal{P}) \to (N,\mathcal{Q})$ is a balanced affine covering, then both $\widetilde{R}(\mathcal{P}) = \widetilde{R}(\mathcal{Q})$ and $\widetilde{\mathcal{N}}(\mathcal{P}) = \widetilde{\mathcal{N}}(\mathcal{Q})$.*

Proof. — By Proposition 1 and Lemma G, we may conjugate either $\mathcal{P}$ or $\mathcal{Q}$ so as to assume that f is a balanced translation covering.

Now, if $f : (M, \mathcal{P}) \to (N, \mathcal{Q})$ is a balanced translation covering, then in particular f takes the marked points of $\mathcal{P}$ to those of $\mathcal{Q}$, and the pre-images of the marked points of $\mathcal{Q}$ are contained within those of $\mathcal{P}$. Thus, f gives an unramified covering, of say degree d, of N'' by M''. Here double primes are used to denote the same translation surfaces, but with all marked points deleted. Let $\vec{v}$ be a connection vector for $\mathcal{Q}$. The preimage under f of each connection for $\mathcal{Q}$ of vector $\vec{v}$ is exactly d connections for $\mathcal{P}$. Since f is a translation covering, each of these preimages is also of vector $\vec{v}$.

The translation structures $\mathcal{P}$ and $\mathcal{Q}$ clearly share the same set of connection directions. They also have the same set of parabolic directions. Now, again since f is a balanced translation covering, the Veech groups $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ are commensurate, see [Vo] or [GJ1]. Thus given $\mathfrak{p}$ an element in some parabolic conjugacy class $\mathfrak{P}$ of $\Gamma(\mathcal{P})$, there exists $n = n(\mathfrak{p}) \in \mathbb{N}$ such that $\mathfrak{p}^n \in \Gamma(\mathcal{Q})$; similarly for $\mathfrak{q}$ parabolic in $\Gamma(\mathcal{Q})$. Since $\mathfrak{p}^n$ fixes a direction θ if and only if $\mathfrak{p}$ does, one indeed deduces that the sets of parabolic directions for $\mathcal{P}$ and $\mathcal{Q}$ are one and the same. (The proofs of the commensurateness of the Veech groups also are based upon use of the d-sheeted unramified cover of the punctured surfaces.)

From the preceding paragraph, the set of parabolic directions for $\mathcal{P}$ is exactly the set of parabolic directions for $\mathcal{Q}$. Furthermore, in each direction, each connection vector $\vec{v}$ occurs exactly d times as often as a connection vector for $\mathcal{P}$ as it does for $\mathcal{Q}$. However, $\widetilde{R}(\mathcal{Q})$, is defined such that each $\tilde{\ell}(\vec{v})$ occurs only $\tilde{n}(\vec{v})$ times; thus, the possibly complicating multiple d has been factored out. Finally, each parabolic direction is a parabolic direction for some maximal parabolic class. Since the invariant $\widetilde{R}$ is defined as a union over these maximal parabolic classes, we find that the sets $\widetilde{R}(\mathcal{Q})$ and $\widetilde{R}(\mathcal{P})$ are indeed equal.

Of course, the same is true for $\widetilde{\mathcal{N}}(\mathcal{P})$ and $\widetilde{\mathcal{N}}(\mathcal{Q})$. □

Remark 5. — The maximal parabolic conjugacy classes of commensurate Fuchsian groups are in general in many-to-many correspondence. Indeed, a parabolic class of some group Γ may well split into several classes in the finite index *subgroup* which is common with a commensurate $\widetilde{\Gamma}$. These parabolic classes may then unite amongst themselves (or even with other parabolic classes) to form one or more parabolic classes in $\widetilde{\Gamma}$.

In terms of Veech groups, given a balanced translation covering $f : (M, \mathcal{P}) \to (N, \mathcal{Q})$, $\Gamma(\mathcal{Q})$ and $\Gamma(\mathcal{P})$ are commensurate. We have shown that the two sets of parabolic directions are the same. The respective maximal parabolic conjugacy classes of the groups partition the parabolic directions by way of the equivalence relation of being a parabolic direction for a particular maximal parabolic conjugacy class. These two partitions may be quite different; however, the invariants $\widetilde{R}(\mathcal{P})$ and $\widetilde{R}(\mathcal{Q})$ are indeed equal.

Definition 5. — We call a translation surface $(M,\mathcal{P})$ *minimal with respect to balanced affine coverings* if the existence of a balanced affine covering $f : (M,\mathcal{P}) \to (N,\mathcal{Q})$ implies that f is of degree one and hence that $(M,\mathcal{P})$ and $(N,\mathcal{Q})$ are affinely equivalent.

LEMMA 8. — *If $(M,\mathcal{P})$ is a translation surface such that $c(\mathcal{P}) = 1$, then $(M,\mathcal{P})$ is minimal with respect to balanced affine coverings.*

Proof. — If $f : (M,\mathcal{P}) \to (N,\mathcal{Q})$ is a balanced affine covering, then f is a topological covering for the surfaces with the marked points removed. Thus, if f is of degree d, then d divides $c(\mathcal{P})$. Hence, if $c(\mathcal{P}) = 1$, then f must indeed be of degree 1. □

4. Applications to the arithmetic setting.

Recall that a Fuchsian group with parabolic elements is called *arithmetic* if it is commensurable with $\mathrm{PSL}(2,\mathbb{Z})$. Gutkin and Judge [GJ2], Theorem 5.5, have shown that a translation surface $(M,\mathcal{P})$ has an arithmetic Veech group if and only if there is a covering of translation surfaces of some torus with one marked point by the $(M,\mathcal{P})$. Their result does not require that this covering be balanced, but simply that the marked points of $\mathcal{P}$ be sent to the marked point on the torus. The inverse image of the marked point of the torus may strictly contain the marked points of $\mathcal{P}$.

Definition 6. — We call a translation surface $(M,\mathcal{P})$ *balanced arithmetic* if it admits a balanced covering to a torus with one marked point.

With this definition in hand, Lemma 5 with Theorem K implies the following.

COROLLARY 1. — *The Veech group of any balanced arithmetic translation surface is conjugate to a subgroup of* $\mathrm{PSL}(2,\mathbb{Z})$.

Proof. — By the Gutkin-Judge result (Theorem K) and Lemma 5 (or simply Vorobets' own [Vo], Proposition 5.3), the Veech group of any balanced arithmetic translation surface is contained in that of its corresponding torus with one marked point. But, each such torus has its Veech group conjugate to $\mathrm{PSL}(2,\mathbb{Z})$. □

Which arithmetic groups might arise as Veech groups in the non-balanced case? An easy calculation shows that the arithmetic Hecke groups G_4 and G_6 cannot. This leads to the following characterization. A fairly direct consequence of the result of Gutkin and Judge, this gives the first obstruction to noncocompact Fuchsian groups appearing as Veech groups.

THEOREM $1'$. — *The Veech group of any arithmetic translation surface is* $\mathrm{PSL}(2,\mathbb{R})$ *conjugate to a subgroup of* $\mathrm{PSL}(2,\mathbb{Q})$. *In particular, the Hecke groups G_4 and G_6 never occur as Veech groups. Furthermore, if N is a nonsquare natural number, then $\Gamma_B(N)$ (as defined in* §1.11) *can never occur as a Veech group.*

Proof. — By an affine change and thus a conjugation of the Veech group, we may assume that we have an arithmetic translation surface $(M,\mathcal{P})$ which covers the square torus. Consider the marking $\mathcal{Q}$ which contains $\mathcal{P}$ and such that $(M,\mathcal{Q})$ is a balanced arithmetic translation surface. By Corollary 1, $\Gamma(\mathcal{Q})$ is a finite index subgroup of $\mathrm{PSL}(2,\mathbb{Z})$. Furthermore, $\Gamma(\mathcal{Q})$ admits a finite index subgroup which is also a subgroup of $\Gamma(\mathcal{P})$. Thus, of course, $\Gamma(\mathcal{P})$ is an arithmetic group.

Now, the lattice $\Lambda_0(\mathcal{P})$ in $\mathbb{R}^2$ generated by the connection vectors of $\mathcal{P}$ is clearly a finite index sublattice of the corresponding $\Lambda_0(\mathcal{Q})$, see [KS] for the notion of Λ_0. Due to the translation covering, both $\Lambda_0(\mathcal{P})$ and $\Lambda_0(\mathcal{Q})$ are of finite index in $\mathbb{Z}^2$. Since the Veech group $\Gamma(\mathcal{P})$ sends $\Lambda_0(\mathcal{P})$ to itself, $\Gamma(\mathcal{P}) \subset \mathrm{PSL}(2,\mathbb{Q})$.

The elements

$$\begin{pmatrix} \sqrt{2} & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sqrt{3} & -1 \\ 1 & 0 \end{pmatrix}$$

belong to G_4 and G_6, respectively. Since conjugation preserves traces, neither of these groups can be conjugate to subgroups of $\mathrm{PSL}(2,\mathbb{Q})$. However, G_4 and G_6 are arithmetic [L], see [K] for a textbook discussion. The Gutkin-Judge result implies that these groups can only appear as the Veech group of arithmetic translation surfaces. We conclude that they do not in fact ever occur as Veech groups.

Recall that when N is a nonsquare natural number $\Gamma_B(N)$ is not contained in $\mathrm{PSL}(2,\mathbb{Q})$. Thus, the theorem follows. □

Remark 6. — From Helling's result, Theorem L, we have virtually located all of the arithmetic groups which can fail to be Veech groups due to their traces being nonrational. To obtain this explicitly, one need determine exactly which subgroups of the $\Gamma_B(N)$ fail to have rational traces.

Remark 7. — After completing this work, we noticed that an application of Theorem 28 of [KS] easily shows that if A is a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$ such that the field $\mathbb{Q}(\operatorname{tr} A^2)$ does not contain $\operatorname{tr} A$ (where tr indicates the absolute value of the standard trace), then A cannot be an element of a Veech group.

Each Hecke group of even index has such an element: let $A = \left(\begin{smallmatrix} 3\lambda & -1 \\ 1 & 0 \end{smallmatrix}\right)$, where $\lambda = 2\cos\pi/q$. One checks that A is a hyperbolic element of the Hecke group of index q. The field extension degree $[\mathbb{Q}(\operatorname{tr} A) : \mathbb{Q}(\operatorname{tr} A^2)] = 2$ if and only if the index q is even. We conclude that no Hecke group of even index can be realized as a Veech group. Of course, Veech [Ve1] showed that every Hecke group of odd index is realized as a Veech group.

5. Trees of balanced coverings.

A tree of balanced coverings gives a way to pass from one translation surface to another, without requiring that there be a map between these two surfaces.

Definition 7. — A *tree of balanced affine coverings* is a lattice of morphisms connecting objects in the category of finite genus translation surfaces with morphisms being balanced affine coverings. That is, it is a collection of translation surfaces $(M_i, \mathcal{P}_i)$ and of balanced affine coverings $f_{\alpha,\beta} : (M_\alpha, \mathcal{P}_\alpha) \to (M_\beta, \mathcal{P}_\beta)$ such that for each pair of consecutive integers $(i, i+1)$ in the index set, one has exactly one of $f_{i,i+1}$ or $f_{i+1,i}$.

COROLLARY 2. — *Both the set of weighted scaling vectors, $\widetilde{R}(\mathcal{P})$, and the set of weighted counting vectors, $\widetilde{N}(\mathcal{P})$, are invariants of lattices of balanced affine coverings.*

The following lemma follows directly from the case of a single covering, [Vo], [GJ1].

LEMMA 9. — *Both the property of having a Veech group which is a lattice Fuchsian group and that of having a Veech group with parabolic elements is preserved within trees of balanced affine coverings.*

COROLLARY 3. — *Suppose that the marked translation surface $(M,\mathcal{P})$ lies within a tree of balanced affine coverings which includes a marked translation surface whose Veech group is a Fuchsian group with exactly one maximal parabolic conjugacy class. Then each of the sets $R(\mathcal{P})$, $\widetilde{R}(\mathcal{P})$, $\mathcal{N}(\mathcal{P})$, and $\widetilde{\mathcal{N}}(\mathcal{P})$ is a singleton set.*

Proof. — If $(N,\mathcal{Q})$ is such that $\Gamma(\mathcal{Q})$ has a single maximal parabolic conjugacy class, then $\widetilde{R}(\mathcal{Q})$ is a singleton set. If $(N,\mathcal{Q})$ shares a common tree with $(M,\mathcal{P})$, then by Theorem 4, $\widetilde{R}(\mathcal{P})$ is also a singleton set. But then even the $R(\mathfrak{P})$ must be the same for all maximal parabolic conjugacy classes $\mathfrak{P}$ of the Veech group $\Gamma(\mathcal{P})$. $\square$

Definition 8. — Given $n \geq 2$, let $\mathcal{M}_{2n}$ be the translation surface formed by identifying opposite sides of a regular planar $2n$-gon.

Theorem B, first given in [EG], shows in particular that the Veech group of each $\mathcal{M}_{2n}$ has two parabolic conjugacy classes. The following result on $\mathcal{N}$-invariants is then obtained by a simple induction with respect to quantities from elementary plane geometry, see Figure 2 and Figure 3.

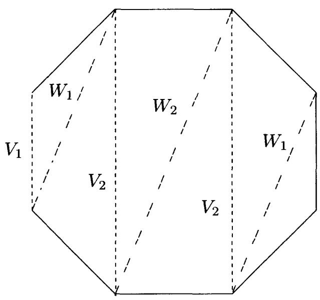

Figure 2. Computing $\mathcal{N}(\omega)$ for surfaces of regular $4k$-gons; here $k=2$.

LEMMA 10. — *The translation surface $\mathcal{M}_{2n}$, with its natural marked structure, has set of counting vectors*

$$\mathcal{N}(\mathcal{M}_{2n}) = \begin{cases} \{(1, \underbrace{2,\dots,2}_{k-1 \text{ times}}), (\underbrace{2,\dots,2}_{k-1\text{ times}},1)\} & \textit{if } n = 2k; \\ \{(1, \underbrace{2,\dots,2}_{k-1 \text{ times}},1), (\underbrace{2,\dots,2}_{k\text{ times}})\} & \textit{if } n = 2k+2. \end{cases}$$

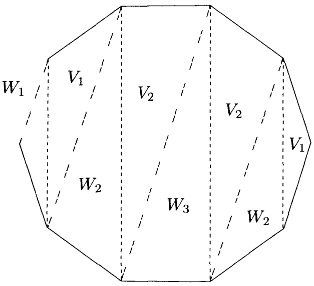

Figure 3. Computing $\mathcal{N}(\omega)$ for regular $(4k+2)$-gons; here $k = 2$.

THEOREM 2′. — *The translation surface $X(1,1,n-2)$ with $n = 2m$ cannot share a common tree of balanced affine coverings with any surface which has as its Veech group a maximal Fuchsian group.*

Proof. — Theorem A gives Veech's result: If $n = 2m$ is even, then the Veech group of $X(1,1,n-2)$ is a $\Delta(m,\infty,\infty)$. This is an index 2 subgroup of a $\Delta(2,2m,\infty)$, it is thus not a maximal Fuchsian group.

Now, $X(1,1,2m-2)$ is a translation double cover of $\mathcal{M}_{2m}$, as Lemma J states. But, Lemma 10 shows that the set of weighted scaling vectors for the $\mathcal{M}_{2m}$ are not singletons. Hence, the same is true for the maximal parabolic weighted relation vectors for these $X(1,1,n-2)$. Corollary 3 now applies to finish our proof. □

Recall that there are three exceptional acute nonisosceles triangle translation surfaces within reasonable computability bounds [KS]. Vorobets [Vo] determined that the Veech groups of two of these were contained in certain maximal triangle Fuchsian groups, Kenyon and Smillie determined that the third of these has Veech group also contained in some maximal triangle Fuchsian group. In fact, in all three cases it was shown that the Veech group in question is either of index two or is the maximal triangle Fuchsian group. It turns out that the index two subgroups in question all have more than one (in fact two) maximal parabolic conjugacy class, but each corresponding maximal triangle Fuchsian group only has exactly one such class. Thus, we are able to apply Corollary 3 and show that in all of these cases, the Veech group is actually of index two in its (unique due to nonarithmeticity) maximal Fuchsian group.

LEMMA 11. — *The translation surfaces $X(2,3,4)$, $X(3,4,5)$ and $X(3,5,7)$ have Veech groups $\Delta(9,\infty,\infty)$, $\Delta(6,\infty,\infty)$ and $\Delta(15,\infty,\infty)$ respectively. Each of these is a non-maximal Fuchsian group.*

Proof. — The proof for each of these three cases is virtually the same. Let us treat $X(3,5,7)$ in detail.

Vorobets showed that the Veech group $\Gamma(3,5,7)$ contains the triangle group $\Delta(15,\infty,\infty)$. This group is an index 2 subgroup of a maximal triangle group, a $\Delta(2,30,\infty)$. Since the nonarithmetic Hecke group of index $q=30$ is of signature $(2,30,\infty)$, any $\Delta(15,\infty,\infty)$ can be strictly contained in at most one Fuchsian group, thus in a $\Delta(2,30,\infty)$.

Since a $\Delta(2,30,\infty)$ group contains exactly one maximal parabolic conjugacy class, it suffices to show that the Veech group of $X(3,5,7)$ contains two distinct maximal parabolic conjugacy classes (or as [Ve1] writes, two noncommuting idempotents). For then this Veech group must in fact be a $\Delta(15,\infty,\infty)$.

Now, Vorobets has already determined the connection vectors (and more) in a certain direction. For the convenience of the reader, we have copied [Vo], Figure 5, as our Figure 4, adding in labels for vertical connection vectors. The determination of the cylinders in the vertical direction by Vorobets allows one to conclude that the corresponding $\mathcal{N}(\mathcal{P},\theta)$ is $(1,1,2,3)$.

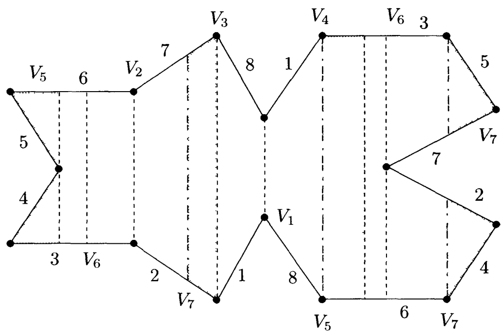

Figure 4. $X(3,5,7)$, *from* [Vo]; $\mathcal{N}(\mathcal{P},\theta)=(1,1,2,3)$.

We now consider the horizontal direction, see Figure 5. It is sufficient to notice that there are two shortest connection vectors in this direction to conclude that the counting vectors for these two directions are not equal. This implies the desired result for the case of $X(3,5,7)$.

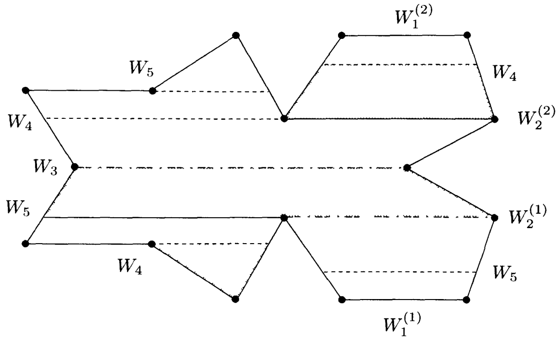

Figure 5. $X(3,5,7)$, the horizontal connections. $\mathcal{N}(\mathcal{P},\theta') = (2,2,1,2)$.

For the case of $X(2,3,4)$, we use [KS], Figure 7, and the related discussion of [KS] to find that in the vertical direction the counting vector is $(1,2,3)$. Study of the horizontal direction gives that both shortest vector and second shortest vectors are singletons. Therefore, we have found a distinct counting vector.

For the remaining case, that of $X(3,4,5)$, we use [Vo], Figure 4 and find that in the vertical direction there, the counting vector is $(2,3)$. In the horizontal direction we will certainly find the same (there is a rotation of $\frac{1}{2}\pi$ in the group already determined). However, we consider the direction angle $\frac{1}{4}\pi$ from the horizontal and find $(1,4)$. □

The following gives an example of two translation surfaces of isomorphic Veech groups but which cannot be placed in a common lattice of balanced coverings.

THEOREM 3′. — *The translation surfaces $X(2,3,4)$ and $X(1,1,16)$ have isomorphic Veech groups but cannot share a common tree of balanced coverings. The same is true for $X(3,4,5)$ and $X(1,1,10)$ as well as for $X(3,5,7)$ and $X(1,1,28)$.*

Proof. — The isomorphisms of the groups follows from the preceding lemma and Theorem A.

It now suffices to note that the $\widetilde{\mathcal{N}}$-invariants are unequal for each of the indicated pairs. Again by Lemma J, each of the $X(1,1,2n)$ is a balanced covering of the corresponding $\mathcal{M}_{2m(n)}$. The $\widetilde{\mathcal{N}}$-invariants for the $X(1,1,2n)$ are thus deduced from Lemma 10. On the other hand, the $\mathcal{N}$- and hence $\widetilde{\mathcal{N}}$-invariants for the exceptional acute triangles' surfaces are determined in the proof of Lemma 11. These invariants are easily compared and seen to be distinct. Therefore, an application of Corollary 2 proves our result. □

6. Nonminimality in general translation coverings.

If a translation surface has an arithmetic Veech group, then the Gutkin-Judge result states that this surface admits a translation covering of a marked torus. We now show that there are two translation surfaces both of which are minimal with respect to translation coverings, but which have isomorphic Veech groups. We use the following two lemmas.

LEMMA 12. — *If $(M,\mathcal{P})$ is a translation surface with exactly one marked point and whose Veech group is nonarithmetic, then $(M,\mathcal{P})$ cannot be the translation cover of a translation torus $(\mathbb{T},\mathcal{Q})$ with any number of marked points.*

Proof. — Suppose that a general translation surface $(M,\mathcal{P})$ translation covers a torus $(\mathbb{T},\mathcal{Q})$. Since all marked points on the torus are removable singularities, this $(M,\mathcal{P})$ also translation covers the torus $(\mathbb{T},\mathcal{Q}')$ for every marking $\mathcal{Q}'\subset\mathcal{Q}$ of $\mathbb{T}$ which contains all of the images of the marked points of $(M,\mathcal{P})$. In the case that $(M,\mathcal{P})$ is a translation surface with exactly one marked point, this implies that $(M,\mathcal{P})$ can in fact translation cover a translation torus with a single marked point. But, by Theorem K, the Veech group of such an $(M,\mathcal{P})$ would then be arithmetic. We have reached a contradiction: There can be no torus translation covered by our $(M,\mathcal{P})$. □

LEMMA 13. — *Let $(M,\mathcal{P})$ be a translation surface of genus 3. If the angles of the singularities of (M,ω) are not equal in pairs, then $(M,\mathcal{P})$ can be a translation cover of no translation surface of genus 2.*

Proof. — By the Riemann-Hurwitz formula, a Riemann surface of genus 3 can only cover a Riemann surface of genus 2 by a degree 2 unramified map. Now, the pull-back of singularities under such a map doubles the number of the singularities but preserves the angles. (For an earlier use of these arguments, see [AF].) □

THEOREM 5. — *The translation surfaces $X(3,4,5)$ and $\mathcal{M}_2$ are each of genus 3 and of a single singularity, have isomorphic Veech groups, are not affinely equivalent and each is minimal with respect to affine coverings.*

Proof. — Theorem B and Lemma 11 show that the Veech groups are indeed isomorphic. The respective $\mathcal{N}$-invariants are given in Lemma 10 and in the proof of Lemma 11; they are easily seen to be distinct. Theorem 4 hence shows that these translation surfaces are not affine equivalent.

The translation surface $X(3,4,5)$ is one of the exceptional examples of [Vo]. Vorobets already pointed out that it has one singularity. An easy angle calculation shows that the surface is of genus 3. Thus Lemma 13 shows that it can cover no genus two translation surface. Its Veech group is nonarithmetic and hence Lemma 12 shows that it can cover no genus one translation surface.

Lemma J states that $\mathcal{M}_{12}$ is balanced translation double covered by $X(1,1,10)$ and has the same Veech group. Thus in particular it has a nonarithmetic Veech group. Furthermore, an easy calculation shows that $\mathcal{M}_{12}$ is a genus 3 surface with one singularity. Again, Lemmas 12 and 13 show that it is minimal with respect to affine coverings. □

7. Preserving the lattice property.

In attempting to force an arbitrary covering of translation surfaces to be balanced, one may well need to add points to the covered structure. Suppose one is in the most interesting case, where the Veech groups are lattices. What are the possible sets of additional points which one can mark on a surface while preserving the property of having a lattice Veech group?

THEOREM 6. — *If $(M,\mathcal{P})$ is a translation surface of genus at least 2 such that $\Gamma(\mathcal{P})$ is a lattice and $\mathcal{Q}$ is a marking of M containing $\mathcal{P}$, then $\Gamma(\mathcal{Q})$ is a lattice if and only if no marked point of $\mathcal{Q}$ lies in an infinite $Aff(\mathcal{P})$-orbit.*

Proof. — By definition, the marked points of $\mathcal{Q}$ form a finite set. If none of these lies in an infinite $\mathrm{Aff}(\mathcal{P})$-orbit, then the union S of all of their $\mathrm{Aff}(\mathcal{P})$-orbits is still a finite set and $\mathrm{Aff}(\mathcal{P})$ clearly acts on this finite set S. The subgroup of $\mathrm{Aff}(\mathcal{P})$ which acts trivially on S then has finite index in $\mathrm{Aff}(\mathcal{P})$. But this subgroup is also a subgroup of $\mathrm{Aff}(\mathcal{Q})$. As the derivatives of this finite index subgroup of $\mathrm{Aff}(\mathcal{P})$ form a finite index subgroup of the lattice $\Gamma(\mathcal{P})$, these derivatives form a lattice group. Thus we have found a subgroup of $\Gamma(\mathcal{Q})$ which is a lattice. Therefore, $\Gamma(\mathcal{Q})$ itself must be a lattice.

Now, suppose that $\Gamma(\mathcal{Q})$ is a lattice. Since the marked points of $\mathcal{P}$ form a finite subset of those of $\mathcal{Q}$, the subgroup $\mathrm{Stab}_{\mathrm{Aff}(\mathcal{Q})}\,\mathcal{P}$ of $\mathrm{Aff}(\mathcal{Q})$

which stabilizes the marked points of $\mathcal{P}$ has finite index in $\mathrm{Aff}(\mathcal{Q})$. Hence the corresponding derivatives form a finite index subgroup of $\Gamma(\mathcal{Q})$. Since $\Gamma(\mathcal{Q})$ is a lattice, so is this subgroup. But, $\mathrm{Stab}_{\mathrm{Aff}(\mathcal{Q})}\,\mathcal{P}$ is also a subgroup of $\mathrm{Aff}(\mathcal{P})$. Therefore the lattice group of its derivatives is a subgroup of $\Gamma(\mathcal{P})$; it must be of finite index, as any noncocompact lattice group must be of finite index in any Fuchsian group containing it.

We would like to use the finiteness of the index of the derivatives of $\mathrm{Stab}_{\mathrm{Aff}(\mathcal{Q})}\,\mathcal{P}$ in $\Gamma(\mathcal{P})$ to show that $\mathrm{Stab}_{\mathrm{Aff}(\mathcal{Q})}\,\mathcal{P}$ itself is of finite index in $\mathrm{Aff}(\mathcal{P})$. In order to do this we need to contemplate the group homomorphism from $\mathrm{Aff}(\mathcal{P})$ to $\mathrm{PSL}(2,\mathbb{R})$ which is given by taking derivatives (and then projectivizing). The kernel of this homomorphism is a subgroup of the automorphism group of the underlying Riemann surface of M – if an affine diffeomorphism has trivial derivative, it certainly preserves angles, thus is a conformal map. But, since the genus of M is at least 2, M has a finite automorphism group. Hence, the kernel of the map from $\mathrm{Aff}(\mathcal{Q})$ to $\mathrm{PSL}(2,\mathbb{R})$ is finite. (For an earlier use of this argument, see [Ve1].) Therefore, the index of $\mathrm{Stab}_{\mathrm{Aff}(\mathcal{Q})}\,\mathcal{P}$ in $\mathrm{Aff}(\mathcal{P})$ must also be finite.

Now choose in $\mathrm{Aff}(\mathcal{P})$ finitely many $\mathrm{Stab}_{Aff(\mathcal{Q})}\,\mathcal{P}$ coset representatives. The $\mathrm{Aff}(\mathcal{P})$-image of the set of marked points of $\mathcal{Q}$ is simply the union of its images under the coset representatives. But the set of coset representatives is finite. Therefore, this image is a finite set. But, it contains the $\mathrm{Aff}(\mathcal{P})$-orbit of each marked point and thus each of these orbits is finite. □

Remark 8. — Note that in the above the restriction on the genus was only used to ensure that the number of automorphisms of the surface be finite. This is true under the hypothesis of the surface having sufficently many marked points in the genus $g = 0$ and 1 cases.

As a corollary to the above, we have the dramatic fact that adding almost any point to a marking which has a lattice Veech group will cause the loss of the lattice property.

COROLLARY 4. — *Given $(M,\mathcal{P})$ a translation surface with lattice Veech group $\Gamma(\mathcal{P})$, scale the natural measure given by the area form of M so as to obtain a probability measure on M. Choose a point q at random with respect to this measure and let $\mathcal{Q}$ be the marking containing $\mathcal{P}$ and with the added marked point q. Then with probability one with respect to q, $\Gamma(\mathcal{Q})$ is not a lattice.*

Proof. — Recall that any Veech group is noncocompact, thus a lattice Veech group must have parabolic elements. We fix a maximal parabolic element of $\Gamma(\mathcal{P})$. By the fundamental Veech criterion [Ve1], there is a decomposition of M into cylinders in the fixed direction of the parabolic element such that the appropriate powers of the linear Dehn twists in the cylinders patch together to give an element of $\mathrm{Aff}(\mathcal{P})$.

By a transformation of finite Jacobian, we can bring any cylinder to the form of the square. Hence, consider the linear Dehn twist on a single square cylinder of side length one: $T(x, y) = (x, x + y \bmod 1)$. The twist fixes the vertical sides and has finite orbits along the line segments of equation $y = mx + b$ with m, b rational numbers. The union of these countably many line segments is of course of measure zero. We thus find that the points of finite orbits for any of our linear Dehn twists form a set of zero area. Taking the sum over finitely many cylinders still gives area zero.

Therefore, the set of points of M which have finite $\mathrm{Aff}(\mathcal{P})$-orbits is clearly of measure zero. □

The following underlines this difficulty of marking points and preserving the lattice property.

LEMMA 14. — *Let $(M,\mathcal{P})$ be a translation surface such that $\Gamma(\mathcal{P})$ is a lattice and let $\mathcal{Q}$ be a marking of M containing $\mathcal{P}$. If $\Gamma(\mathcal{Q})$ is a lattice then the set of connection directions of $(M,\mathcal{P})$ equals that of $(M,\mathcal{Q})$.*

Proof. — Let θ be a connection direction for $\mathcal{Q}$. Thus there is a parabolic element, say T, of $\Gamma(\mathcal{Q})$ fixing θ. By Lemma H, $\Gamma(\mathcal{Q})$ has a finite index subgroup contained in $\Gamma(\mathcal{P})$. But, this means in particular that some power of T must be in $\Gamma(\mathcal{P})$. Powers of parabolic elements of course fix the same directions as the original parabolic elements. Therefore, θ is a connection direction for $\mathcal{P}$.

Now suppose that ψ is a connection direction for $\mathcal{P}$. Since $\Gamma(\mathcal{Q})$ is a lattice, the finite index subgroup of it which is also a subgroup of $\Gamma(\mathcal{P})$ must also be a lattice. This implies that this subgroup must also be of finite index in $\Gamma(\mathcal{P})$. Let $U \in \Gamma(\mathcal{P})$ fix the direction ψ. There is a finite power of U in $\Gamma(\mathcal{Q})$. Since this power also fixes the direction ψ, we conclude that ψ is a connection direction for $\mathcal{Q}$. □

9. Final comments and further questions.

The main open questions about Veech groups are: 1) Which Fuchsian groups appear as Veech groups? and: 2) How do Veech groups change under affine coverings?

We have shown that (conjugacy classes of) Veech groups do not uniquely determine trees of balanced affine coverings. Could they possibly uniquely determine trees of general affine coverings? We doubt this; we view the example given in Theorem 5 as reason for our doubts. Does the Veech group with the additional information of our invariants uniquely determine such trees?

Is the Veech group of any covering surface actually a subgroup of the Veech group of a surface which is minimal for that tree? We have no counterexamples to this, and of course the answer is affirmative for generic balanced covers, as one will generically have some connection vector which is a singleton (see Lemma 5).

We would like to see further results on exactly which points can be marked on a translation surface of lattice Veech group and still preserve this property. In particular, it would be interesting to have algebraic characterizations of these points as well as an algorithm for determining them.

The relationship between uniformization of a surface by a Fuchsian group and related Veech groups remains mysterious. Remarks of [H1], [H2] clearly indicate that there are deep arithmetic connections.

Similarly, it would be interesting to see if one could characterize the points of the (usual, say) boundary of Riemann moduli space which can arise as cusps of quotients of Teichmüller disks by (lattice) Veech groups.

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Pascal HUBERT,
Institut de Mathématiques de Luminy
163 avenue de Luminy, case 907
13288 Marseille Cedex 09 (France).
hubert@iml.univ–mrs.fr
&
Thomas A. SCHMIDT,
Oregon State University
Department of Mathematics
Kidder Hall 368
Corvallis OR 97331–4605 (USA).
toms@math.orst.edu