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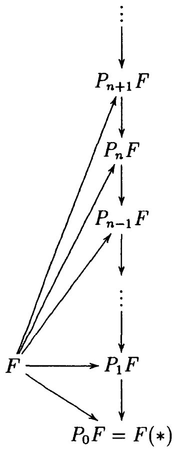
# TAYLOR TOWERS FOR Γ-MODULES

**by Birgit RICHTER**

## 1. Introduction.

Similarly to the Taylor series for smooth functions in differential calculus T. Goodwillie [G] associates a Taylor tower




to endofunctors of topological spaces $F$ which preserve weak homotopy equivalences. This approximation converges to $F$ and the $P_nF$ are $n$-excisive functors, i.e., they can be thought of as degree $n$ approximations of the functor $F$.

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In this paper we will consider endofunctors of simplicial sets, which arise from $\Gamma$-modules, i.e., functors from the category of finite pointed sets $\Gamma$ to the category of $R$-modules, for some ring $R$. Any $\Gamma$-module can be extended to an endofunctor of simplicial sets by extending it degreewise to a functor from pointed simplicial sets to simplicial modules.

Working in the category of $\Gamma$-modules has the advantage that one can use the tools of homological algebra in order to understand the homology of the approximation steps $H_k P_n F : \Gamma \to R$ and the homology of the homotopy fibre $D_n F := \mathrm{hfib}(P_n F \to P_{n-1} F)$ and to describe these homologies in terms of derived functors.

Fitting to the Taylor tower there exists a tower of model category structures, which are localizations of the usual model structure for $\Gamma$-chain complexes at the functors $P_n(-)$. Homologically degree $n$ functors are then the fibrant objects in the $n$-th model category structure.

The homology of the homotopy fibre can be determined in characteristic zero for an important family $(\psi_\ell)_\ell$ of $\Gamma$-modules. This leads to an explicit calculation of higher order Hochschild homology of the truncated polynomial algebra $\mathbb{K}[x]/x^2$.

As an important example we consider the functor $St : \Gamma \to \mathbb{F}_2 - \mathrm{Vect}$ from finite pointed sets to $\mathbb{F}_2$-vector spaces given by $St[n] := \mathbb{F}_2\{\overline{\mathbb{F}_2}[n]\}$. The homotopy of this functor when prolonged to pointed simplicial sets gives the $\mathbb{F}_2$-homology of Eilenberg-MacLane spaces. With the isomorphism $H_* P_1(St)[1] \cong \pi_*^{st}(St)$ we see that $H_* P_1(St)[1]$ is the dual of the Steenrod algebra over $\mathbb{F}_2$.

We prove that this functor splits as $St = \bigoplus \psi_\ell$. We calculate the homology of the homotopy fibres $D_n(\psi_\ell)$ in the approximation of these functors for $\ell$ small and for a certain range of approximation with the help of a spectral sequence which we develop in Section 6.

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## 2. Γ-modules.

Let $\Gamma$ be the category of finite pointed sets and pointed maps. We assume that the objects of $\Gamma$ are the pointed sets $[n] = \{0, 1, \ldots, n\}$ with 0 as a basepoint. If the cardinality of the set is not important we denote an object of $\Gamma$ by $X_+$ where $X$ is a finite set and $+$ is an added basepoint. We call functors $F$ from $\Gamma$ in a category of $R$-modules just $\Gamma$-modules. Here $R$ is a commutative ring with unit. For an arbitrary set $S$ we denote by $R\{S\}$ the free $R$-module generated by $S$.

### 2.1. Examples.

1. The functors $\Gamma^n : \Gamma \to R\text{-mod}$, $n \geqslant 0$, which take $[m] \in \Gamma$ to the free $R$-module generated by the morphisms from $[n]$ to $[m]$,

$$\Gamma^n[m] = R\{\Gamma([n], [m])\}$$

are projective generators of the category of all Γ-modules. The contravariant representable functors $\Gamma_n$ are defined analogously

$$\Gamma_n[m] = R\{\Gamma([m], [n])\}.$$

2. The cokernel of the transformation from $\Gamma^0$ to $\Gamma^1$,

$$L := \text{coker}(\Gamma^0 \to \Gamma^1)$$

sends a finite pointed set $[n]$ to the free $R$-module generated by the elements of $\{1, \ldots, n\}$.

3. The pointwise tensor product of two Γ-modules $F$ and $G$ is defined as

$$(F \otimes G)[n] := F[n] \otimes G[n].$$

Important examples of Γ-modules are the $n$-fold tensor product $L^{\otimes n}$ of $L$, the $n$-th exterior product $\Lambda^n \circ L$ and the $n$-th symmetric product $\text{Sym}^n \circ L$.

### 2.2. Prolongation to simplicial sets.

Every Γ-module $F$ gives rise to a functor from pointed simplicial sets to simplicial $R$-modules: On an arbitrary pointed set $X_+$ the functor $F$ is defined via colimits

$$F(X_+) = \operatorname{colim}_{\substack{S_+ \subset X_+ \\ |S_+| < \infty}} F(S_+),$$

and for a pointed simplicial set $X_\bullet$ the simplicial $R$-module $F(X)_\bullet$ is defined as $F(X)_n := F(X_n)$.

### 2.3. Cross-effects.

The $n$-th cross-effect $cr_n F$ of a $\Gamma$-module $F$ is a functor

$$cr_n F : \underbrace{\Gamma \times \ldots \times \Gamma}_{n} \to R - \text{mod}$$

which is defined as

$$cr_n F(X^1_+, \ldots, X^n_+) := \text{Image}\left(\sum_{k=1}^{n} (-1)^{n-k} \sum_{i_1 < \ldots < i_k} F(\pi_{(X^{i_1}_+ \vee \ldots \vee X^{i_k}_+)})\right).$$

Here $\{i_1, \ldots, i_k\}$ is a subset of $\{1, \ldots, n\}$ and the

$$\pi_{(X^{i_1}_+ \vee \ldots \vee X^{i_k}_+)} : F(X^1_+ \vee \ldots \vee X^n_+) \to F(X^1_+ \vee \ldots \vee X^n_+)$$

are the projection maps on the components $X^{i_1}_+$, ..., $X^{i_k}_+$. An alternative and useful definition is the following: Let $r_i : [n] \to [n-1]$ be the map which is given by

$$r_i(j) = \begin{cases} j, & j < i \\ 0, & j = i \\ j-1, & j > i. \end{cases}$$

Then the above definition of the $n$-th cross-effect evaluated on $([1], \ldots, [1])$ is equivalent to $cr_n F([1], \ldots, [1]) = \bigcap_{i=1}^{n} \ker F(r_i)$.

The functor $cr_n F$ measures the failure of $F$ to be of degree $n-1$. For endofunctors of abelian groups cross-effects are defined in [EM2, §§8,9].

*Examples.* — Linear functors $F$ from $\Gamma$ to abelian groups give rise to Eilenberg-MacLane spectra: The evaluation of such an $F$ on a simplicial model of the $n$-sphere $\mathbb{S}^n$ is a $K(F[1], n)$. Typical degree $n$ functors are $L^{\otimes n}$, $\Lambda^n \circ L$ and $\text{Sym}^n \circ L$.

### 2.4. Dold-Kan correspondence.

Let $\Omega$ denote the category of nonempty finite sets and surjections. We denote the objects of $\Omega$ by $\underline{n} = \{1,\ldots,n\}$. There is a Dold-Kan correspondence (see [P2]) between the category of functors from $\Gamma$ to $R$-mod and the category of functors from $\Omega$ to $R$-mod. For every $\Gamma$-module $F : \Gamma \to R$-mod there is a corresponding functor $cr(F) : \Omega \to R$-mod defined by

$$cr(T)(\underline{n}) = cr_n T([1],\ldots,[1]).$$

A surjection $f : \underline{n} \to \underline{m}$ gives rise to a map of finite pointed sets $f^+$ and such a map sends the intersection of the kernels of the $T(r_i)$ again to an intersection of such kernels and hence $f$ induces a map $cr(f^+) : cr(F)(\underline{n}) \to cr(F)(\underline{m})$. In [P2, 3.1] Pirashvili shows that this transformation

$$cr : R - \mathrm{mod}^{\Gamma} \to R - \mathrm{mod}^{\Omega}$$

is an equivalence of categories.

## 3. Taylor approximation for Γ-modules.

Let $\mathrm{Ch}(R)$ denote the category of non-negative chain complexes of $R$-modules. We will build an approximation tower for $\Gamma$-modules:

$$\cdots \to P_nF \to P_{n-1}F \to \cdots \to P_1F \to P_0F$$

where the functors $P_nF : \Gamma \to \mathrm{Ch}(R)$ are homologically of degree $n$.

Definition 3.1. — *A functor $F : \Gamma \to \mathrm{Ch}(R)$ is called* homologically of degree $\leqslant n$ *if $cr_{n+1}(F)$ is acyclic.*

We will give an explicit construction of this tower for $\Gamma$-modules. The properties of the approximation are straighforward to see with the help of the cotriple approach of Johnson and McCarthy [JMcC2, 2.8]:

- The $n$-th approximation $P_nF$ is homologically of degree $n$.
- The map $P_nF \to P_nP_nF$ is a weak equivalence.
- A $\Gamma$-chain complex is homologically of degree $n$ iff $F \to P_nF$ is a weak equivalence.
- The transformation $P_n : R - \mathrm{mod}^{\Gamma} \to \mathrm{Ch}(R)^{\Gamma}$ is exact.

### 3.1. Construction of the approximation steps.

The construction we give here is similar to the one in [JMcC1] and for the readers who are familiar with the work of B. Johnson and R. McCarthy we cite the corresponding definitions and statements from [JMcC1] in the appropriate places. Before we define the approximation steps $P_nF$ we will give an unnormalized version $P'_nF$. The unnormalized approximation in degree $n$ is a functor $P'_nF$ from $\Gamma$ to chain complexes whose $k$-dimensional component is given by

$$P'_nF(X_+)_k := F\Big(\bigvee_{[n]^k} X_+\Big).$$

We define $(P'_nF(X_+))_0$ as $F(X_+)$. The boundary map

$$\delta : P'_nF(X_+)_k = F\Big(\bigvee_{[n]^k} X_+\Big) \to F\Big(\bigvee_{[n]^{k-1}} X_+\Big) = P'_nF(X_+)_{k-1}$$

is induced by maps on set level

$$\bigvee_{[n]^k} X_+ \to \bigvee_{[n]^{k-1}} X_+.$$

Let $\pi_1^j$ be the projection,

$$\pi_1^j : \bigvee_{[n_1]\times\ldots\times[n_j]\times\ldots\times[n_k]} X_+ \to \bigvee_{[n_1]\times\ldots\times[n_j-1]\times\ldots\times[n_k]} X_+$$

which maps all components $X_+$ with labels $(x_1,\ldots,x_{j-1},n_j,x_{j+1},\ldots,x_k)$ to the basepoint and let $\pi_2^j$ be the map that projects everything labelled by $(x_1,\ldots,x_{j-1},n_j-1,x_{j+1},\ldots,x_k)$ to the basepoint. All other elements are mapped identically. In addition we consider $\pi_>^j$ which overlaps the components with labels $n_j$ and $n_j-1$ in the $j$-th place and is the identity on all other elements (compare [JMcC1, Def. 1.1, 1.2]). Now define the $j$-th part of the boundary $\delta$ as

$$\delta_j := -F(\pi_1^j) - F(\pi_2^j) + F(\pi_>^j).$$

The full boundary map

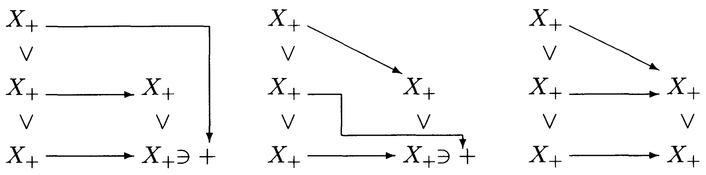
$$\delta : F\Big(\bigvee_{[n]^k} X_+\Big) \longrightarrow F\Big(\bigvee_{[n]^{k-1}} X_+\Big)$$

is the alternating sum of $n$-fold iterations of $\delta_j$,

$$\delta := \sum_{j=1}^{k} (-1)^j \delta_j^n.$$

A proof that this defines in fact a boundary map for the unnormalized approximation $P'_n F$ is an easy modification of the proof in [JMcC1, Prop. 1.4] and can be found in [R, pp. 31,32].

*Example.* — For $n = 2$ the boundary map $\delta : (P'_2 F)_1 \to (P'_2 F)_0$ is induced by the projections $\pi_1^1$ and $\pi_2^1$ and the map $\pi_>^1$ which folds the second and third component




and by the iteration of these maps.

We normalize the chain complexes $P'_n F(X_+)$ and build the quotient concerning "$i$-slabs" and "diagonal" elements. Let us denote the corresponding subsets in $[n]^k$ with the same name. The family of subsets which build $i$-slabs are

$$S_j^i := \{(x_1, \ldots, x_k) \mid x_i \neq j\}, \quad 0 \leqslant j \leqslant n$$

whereas the diagonals are the subsets of the form

$$D_i := \{(x_1, \ldots, x_k) \mid x_i = x_{i+1}\}.$$

DEFINITION 3.2. — *The $n$-th Taylor approximation $P_n F$ of an $\Gamma$-module $F$ in dimension $k$ is the cokernel*

$$(P_n F(X_+))_k := \operatorname{coker}\Big( \bigoplus_{T \in \{\{S_j^i\},\{D_i\}\}} F(\bigvee_{t \in T} X_+) \longrightarrow F(\bigvee_{s \in [n]^k} X_+)\Big).$$

*The maps for this cokernel are induced by the inclusions $T \subset [n]^k$.*

*Remark 3.3.* — It is straightforward to check that the boundary map $\delta$ is well-defined on the quotient (see [JMcC1, Lemma 1.11]).

*Remark 3.4.* — This normalization is necessary for the properties the $P_nF$ should have: The unnormalized approximation $P'_n$ is given by evaluating $F$ on a sum. From the definition of the cross-effect it follows that such a term is isomorphic to a sum of cross-effects. If we consider the normalized term $P_nF$ then in the $\ell$-th chain degree this is isomorphic to

$$(P_nF)_\ell \cong \bigoplus_{S \subset [n]^\ell} cr_{|S|}F$$

where $S$ runs over all non-degenerate subsets, i.e., subsets which are neither slabs nor diagonals, of $[n]^\ell$ (see [JMcC1, Prop. 3.8]). As a subset $S$ with $|S| \leqslant n$ is always degenerate, this ensures among other things that $F \to P_nF$ is a weak equivalence iff $F$ is of homological degree $n$.

### 3.2. The cubical construction.

Let $A$ be an abelian group. We consider the functor $F = \mathbb{Z}\{\overline{A}\{-\}\}$, which maps a finite pointed set $[n]$ to the free abelian group generated by the elements of $A^n$. The Eilenberg-MacLane cubical construction $Q(A)$ which was defined in [EM1, §12, p. 321] is the first Taylor approximation of $\mathbb{Z}\{\overline{A}\{-\}\}$ evaluated on $[1] \in \Gamma$:

$$P_1(\mathbb{Z}\{\overline{A}\{-\}\})[1] = Q_*(A).$$

Hence $Q_*(A)$ is a chain complex of abelian groups which is a quotient of $\mathbb{Z}\{A^{2^m}\}$ in degree $m$.

In degree zero the nontrivial generators are of the form $(a)$ with $0 \neq a \in A$; in degree one the generators are $(a, b)$ with $a, b \neq 0$. In degrees zero, one and two the boundary map looks as follows:

$$\begin{aligned} \delta(a) &= 0, \\ \delta(a,b) &= (a+b) - (a) - (b), \\ \delta\binom{a,b}{c,d} &= -(a,b) - (c,d) + (a+c, b+d) \\ &\quad +(a,c) + (b,d) - (a+b, c+d). \end{aligned}$$

We abbreviate the $n$-th Taylor approximation of $F = \mathbb{Z}\{\overline{A}\{-\}\}$ evaluated at $[1]$ with

$$Q^n_*(A) := P_n(\mathbb{Z}\{\overline{A}\{-\}\})[1]$$

and call this chain complex the $n$-th cubical construction on $A$.

### 3.3. The connections inside the tower.

As in the additive case there are natural transformations $F \to P_nF$ and $P_nF \to P_{n-1}F$ for every $n$ (compare [JMcC1, Lemma 1.15]). From now on we assume that our functors $F$ are reduced i.e., $F[0] = 0$. This is no actual restriction because the part $F[0]$ always splits as a direct summand and we can consider the reduced part $F'$ defined by $F(X_+) \cong F'(X_+) \oplus F[0]$. The Taylor approximation of $F$ is then essentially the one of $F'$ because $P_nF \cong F[0] \oplus P_nF'$.

PROPOSITION 3.5. — *The maps $F([\ell]) \mapsto (P_nF([\ell]))_0$ induce natural transformations*

$$p_n : F \to P_nF. \qquad \square$$

The different layers of the Taylor tower can be connected by the following maps:

PROPOSITION 3.6. — *There are natural transformations*

$$q_n : P_nF \to P_{n-1}F$$

*such that the triangles*

$$\begin{array}{ccc} & & P_nF \\ & \nearrow & \downarrow \\ F & \to & P_{n-1}F \end{array}$$

*commute.*

*Proof.* — In degree zero the map $q_n$ is just the identity because the modules $(P_nF(X_+))_0$ and $(P_{n-1}F(X_+))_0$ are the same. For degree $k$ greater than 0 we define the map

$$(q_n)_k : (P_nF)_k \to (P_{n-1}F)_k$$

as the following composition:

$$(q_n)_k = \delta_1 \circ \cdots \circ \delta_k.$$

It is easy to see, that these maps are in fact chain maps. A proof can be found in [R]. It works similar to the proof of [JMcC1, Lemma 1.7]. $\square$

### 3.4. Model category structures.

The aim of this part is to give model category structures on $\Gamma$-chain complexes, which fit to the Taylor tower, i.e., the $n$-th approximation step should be fibrant in the right setting.

For $\Gamma$-chain complexes there is a standard model category structure, because the category of $\Gamma$-chain complexes is the same as the category of chain complexes of $\Gamma$-modules and this category has the usual structure such that

- weak equivalences are isomorphisms in homology,
- cofibrations are degreewise splitting monomorphisms with projective cokernel and
- fibrations are morphisms which are surjective in positive degrees.

So typically the cokernel of a cofibration is a direct sum of standard projective generators $\Gamma^k$.

*Remark 3.7.* — It is straightforward to see that this model category structure is proper.

### 3.5. Model category structures fitting to the Taylor tower.

We want to define a fitting model category structure for each approximation step in the Taylor tower. "Fitting" means that the weak equivalences should be the $H_*P_n$-isomorphisms. To this end we have to localize the standard model category on $\Gamma$-chain complexes. O. Renaudin gained in [Re] similar structures via constructing localizing subcategories.

The model category of $\Gamma$-chain complexes is proper. We change the model structure and call a map $f$ a $P_n(-)$-equivalence if $P_n(f)$ is a weak equivalence in the standard structure. A map $f$ is a $P_n$-fibration if it has the right lifting property concerning all ordinary cofibrations which are $P_n(-)$-equivalences. We denote this structure by $\mathcal{T}_n$. The functors $P_n(-)$ have three especially good properties:

1. The functors $P_n(-)$ preserve weak equivalences.
2. The maps $(p_n)_{P_n(F)} : P_nF \to P_nP_nF$ are weak equivalences.
3. The class of the $P_n(-)$-equivalences is closed concerning pullbacks along $P_n(-)$-fibrations and pushouts along cofibrations.

The third property is a consequence of the exactness of the functors $P_n(-)$; hence $H_*P_n(-)$ is a homology theory. In this situation $\mathcal{T}_n$ is again a proper model category. Bousfield and Friedlander [BF] proved this for simplicial model categories; Goerss and Jardine give a more general proof in [GoJ].

THEOREM 3.8 [GoJ, X, Theorem 4.1] *Given a closed and proper model category $\mathcal{C}$ and given an endofunctor $Q : \mathcal{C} \to \mathcal{C}$ with a natural transformation $\eta : \mathrm{Id} \to Q$ and with the properties 1,2 and 3 above, there is a proper and closed model category structure on $\mathcal{C}$, so that the weak equivalences are the $Q(-)$-equivalences, the cofibrations stay as they were in the former structure and the fibration are determined by the right lifting property.*

The fibrant objects in the model category for $Q = P_n$ are exactly the functors $F$, for which the transformation $(p_n)_F : F \to P_nF$ is a weak equivalence of Γ-chain complexes (see [GoJ], X, Corollary 4.12), hence the functors of homological degree $n$ are fibrant. In particular the approximation steps $P_nF$ are fibrant in $\mathcal{T}_n$.

## 4. Homology of the approximation steps.

### 4.1. Higher cubical constructions in the set context.

For the interpretation of the homology of the usual Eilenberg-MacLane cubical construction $Q_*$ it was helpful to introduce an auxiliary complex $SQ_*$. T. Pirashvili related in [P1] this complex to the homology of $Q_*(A)$ - which is the homology of the first approximation $P_1$ applied to the functor $\mathbb{Z}\{\overline{A}\{-\}\}$ - and to the stable homology of Eilenberg-MacLane spaces. The generalized version of $SQ_*$ which we introduce in this section will help us to gain a description of the homology of the higher cubical constructions.

Let $X_+ \in \Gamma$. Define $\widetilde{SQ}_k^{(n)}(X_+)$ to be the free module generated by $[n]^k$-tuples of pairwise disjoint subsets of $X$, i.e., $\widetilde{SQ}_0^{(n)}(X_+)$ is generated by $(S), S \subset X$ and $\widetilde{SQ}_1^{(n)}(X_+)$ has elements like $\chi = (S_0, \dots, S_n)$ with $S_i \cap S_j = \varnothing$, $(i \neq j)$, $S_i \subset X$ as a basis. We denote elements of $\widetilde{SQ}_k^{(n)}(X_+)$ as functions $\chi$ from the set $[n]^k$ to the power set of $X$ whose images of the points $(\varepsilon_1, \dots, \varepsilon_k) \in [n]^k$ are pairwise disjoint.

The functor $\widetilde{SQ}_*^{(n)}$ is contravariant: For a morphism $f : X_+ \to Y_+$ the induced map from $\widetilde{SQ}_*^{(n)}(Y_+)$ to $\widetilde{SQ}_*^{(n)}(X_+)$ takes the preimage of the subsets. The case $n = 1$ has already been defined in [P1].

The boundary map $\delta : \widetilde{SQ}_k^{(n)}(X_+) \to \widetilde{SQ}_{k-1}^{(n)}(X_+)$ is defined analogously to the one for $P_nF$ and we use similar projection maps: The map $\pi_1^i$ removes all subsets in $\widetilde{SQ}_k^{(n)}(X_+)$ with label $n$ in the $i$-th coordinate whereas $\pi_2^i$ removes the ones labelled by $n-1$ in the $i$-th place. The map $\pi_>^i$ takes the union of the subsets labelled by $n$ and $n-1$ in the coordinate $\varepsilon_i$. Again let $\delta_i$ denote $\pi_>^i - \pi_1^i - \pi_2^i$. The boundary $\delta$ is the alternating sum of the $n$-fold iterations of the $\delta_i$:

$$\delta = \sum_{i=1}^{k} (-1)^i \delta_i^n.$$

For $k = 1$ and $n = 2$, $\delta$ maps a generator $(S, T, U)$ to

$$\delta(S,T,U) = (S \cup T \cup U) - (S \cup T) - (S \cup U) - (T \cup U) + (S) + (T) + (U).$$

The final complex $SQ_k^{(n)}(X_+)$ is the quotient of $\widetilde{SQ}_k^{(n)}(X_+)$ by all elements, which have the empty set as a value in one hyperplane, i.e., generators $\chi$ with $\chi(\varepsilon_1, \ldots, \varepsilon_{i-1}, j, \varepsilon_{i+1}, \ldots, \varepsilon_k) = \varnothing$ for a $j \in [n]$ and by diagonal elements, i.e., $\chi$ with $\chi(\varepsilon_1, \ldots, \varepsilon_k) = \varnothing$ if $(\varepsilon_1, \ldots, \varepsilon_k) \in [n]^k$, and $\varepsilon_i \neq \varepsilon_{i+1}$.

### 4.2. The homological degree of $SQ_*^{(n)}$.

The higher cubical constructions in the set context $SQ_*^{(n)}$ are crucial for the calculation of the homology of the higher cubical constructions for functors. Recall from 3.1 that a functor is homologically of degree $\leqslant n$ if its $(n+1)$-st cross-effect is acyclic.

Proposition 4.1. — *The functor $SQ_*^{(n)} : \Gamma^{op} \to \mathrm{Ch}(R)$ is homologically of degree $n$.*

*Proof.* — There is an explicit chain homotopy which proves the claim. For a generator

$$\chi(\epsilon_1, \ldots, \epsilon_k) \in SQ_k^{(n)}(X_+^0 \vee \ldots \vee X_+^n)$$

we define a map $\mathcal{H}$ to $SQ^{(n)}_{k+1}(X^0_+ \vee \ldots \vee X^n_+)$ via

$$\mathcal{H}(\chi)(i, \epsilon_1, \ldots, \epsilon_k) := \chi(\epsilon_1, \ldots, \epsilon_k) \cap X^i \quad \text{for } 0 \leqslant i \leqslant n.$$

This gives a homotopy between the zero map and the map whose image is the $(n+1)$-st cross-effect. □

### 4.3. The homology of the higher $Q$-constructions.

DEFINITION 4.2. — *The $n$-th $Q$-construction of a functor $F : \Gamma \to R$-mod is defined as*

$$Q^{(n)}_*(F) := SQ^{(n)}_* \otimes_\Gamma F \in \mathrm{Ch}(R).$$

Here $(-) \otimes_\Gamma (-)$ denotes the tensor product of Γ-modules: For a contravariant Γ-module $T$ and a covariant Γ-module $F$ this tensor product is given as the coend of the bifunctor $T \otimes F : \Gamma^{op} \times \Gamma \to R$-mod, i.e., it is the coequalizer of $\bigoplus_{[n] \xrightarrow{f} [m])} T([n]) \otimes F([m]) \underset{\mathrm{id} \otimes f_*}{\overset{f^* \otimes \mathrm{id}}{\rightrightarrows}} \bigoplus_{[n]} T([n]) \otimes F([n])$ where $f : [n] \to [m]$ is a pointed map. It is easy to see that

$$P_n F[1] \cong Q^{(n)}_*(F)$$

because $\widetilde{SQ}^{(n)}_\ell \otimes_\Gamma F \cong \Gamma_{[n]^\ell} \otimes_\Gamma F \cong F([n]^\ell)$ and the submodules generated by the relations map isomorphically.

The homology of the $n$-th $Q$-construction $Q^{(n)}_*(F)$ has an interpretation as a Tor-functor, because $SQ^{(n)}_*$ is a projective resolution of a functor, which we define now:

Let $\mathcal{B}_n : \Gamma \to R$-mod be the functor which maps a finite pointed set $X_+$ to the free module generated by all nonempty subsets of $X$ with cardinality less or equal to $n$. On morphisms $f : X_+ \to Y_+$ in Γ a generator $S \subset X$ in $\mathcal{B}_n(X_+)$ is taken to its image $f(S)$ if this image does not contain zero; if $0 \in f(S)$ then the induced map is zero. Now define $t^n : \Gamma^{op} \to R$-mod as the dual functor of $\mathcal{B}_n$, i.e., $t^n(S_+) = \mathrm{Hom}_R(\mathcal{B}_n(S_+), R)$.

LEMMA 4.3. — *The functor $t^n$ is of degree $n$.*

*Proof.* — We will prove this claim by an inductive argument. For $n = 1$ it is obvious that $t = t^1$ is additive. Consider the canonical exact sequence

$$0 \to \mathcal{B}_{n-1} \to \mathcal{B}_n \to \mathcal{B}_n / \mathcal{B}_{n-1} \to 0.$$

The dual of this sequence is again exact. Hence the functors $t^n$ fit in the following exact sequence of functors from $\Gamma^{op}$ to $R$-mod:

$$0 \to \theta^n \longrightarrow t^n \longrightarrow t^{n-1} \to 0$$

where $\theta^n$ is the dual of the quotient $\mathcal{B}_n/\mathcal{B}_{n-1}$, i.e., $\theta^n(S_+)$ is dual to the free module generated by subsets of $S$ of cardinality exactly $n$. Clearly the functor $\theta^n$ is of degree $n$. With the exactness of the cross-effect it follows that $cr_{n+1}t^n = 0$. □

LEMMA 4.4. — *The $\Gamma$-chain complex $SQ_*^{(n)}$ is a projective resolution of a degree $n$ functor which is isomorphic to $t^n$.*

*Proof.* — The projectivity of the $SQ_*^{(n)}$ is clear because they are quotients of the projective generators $\Gamma_S$ with $S = [n]^i$ where $\Gamma_S(X_+) = R\{\Gamma(X_+, S)\}$ and there is a section from $SQ_*^{(n)}$ to $\widetilde{SQ}_*^{(n)}$ (compare the argument in [JP], 2.3). We have to prove that the homology of $SQ_*^{(n)}$ vanishes in positive dimensions and that its zeroth homology is a functor isomorphic to $t^n$.

It follows from the definition of $SQ_*^{(n)}$ that

$$SQ_i^{(n)}[\ell] = 0 \quad \forall i > 0 \quad \text{and} \quad \ell \leqslant n.$$

In general $SQ_*^{(n)}$ has only homology in degree 0 because its homology is polynomial of degree $n$ and for $[1], \ldots, [n]$ there is no homology in higher degrees.

We now prove that

$$H_0 SQ_*^{(n)} \cong t^n.$$

With the Dold-Kan correspondence of 2.4 we have to show that this zeroth homology and $t^n$ have the same cross effects. On an object $X_+ \in \Gamma$ the module $H_0(SQ_*^{(n)})(X_+)$ is generated by subsets $S \subset X$ with cardinality $|S| \leqslant n$. If $f : X_+ \longrightarrow Y_+$ is a morphism in $\Gamma$, then the induced map $f^* : H_0(SQ_*^{(n)})(Y_+) \longrightarrow H_0(SQ_*^{(n)})(X_+)$ takes the equivalence class of the preimages: If the cardinality of $f^{-1}(S)$ is less or equal to $n$, then $f^*(S) = f^{-1}(S)$; otherwise $f^{-1}(S) \in H_0(SQ_*^{(n)})(X_+)$ is equivalent to some alternating sum of subsets with cardinality $\leqslant n$. The degree of $H_0 SQ_*^{(n)}$ is $n$. Its $i$-th cross effect $cr_i H_0(SQ_*^{(n)})(X_+^0, \ldots, X_+^i)$ is given by the cokernel of

$$\bigoplus_{j=0}^{i} H_0(SQ_*^{(n)})(X_+^0 \vee \ldots \vee \check{X}_+^j \vee \ldots \vee X_+^i) \xrightarrow{\bigoplus r_j^*} H_0(SQ_*^{(n)})(X_+^0 \vee \ldots \vee X_+^i).$$

It is easy to see that the cross effects $cr_i(H_0(SQ_*^{(n)}))([1], \ldots, [1])$ are one-dimensional for $i \leqslant n$. They vanish for $i > n$ because $H_0(SQ_*^{(n)})$ is of degree $n$. The identification of $\{1, \ldots, i\}$ in $cr_i(H_0(SQ_*^{(n)}))$ with $\{1, \ldots, i\}^*$ in $cr_i(t^n)$ induces a natural isomorphism. □

As we have this projective resolution the homology of the higher $Q$-constructions is given by

THEOREM 4.5.

$$H_* Q_*^{(n)}(F) = H_*(SQ_*^{(n)} \otimes_\Gamma F) = \mathrm{Tor}_*^\Gamma(t^n, F) \qquad \square$$

For a chain complex $C_*$ we denote by $sh_1 C_*$ the shifted chain complex, i.e., $(sh_1 C_*)_k = C_{k-1}$.

DEFINITION 4.6. — *The $n$-th homotopy fibre of the Taylor tower is defined as* $D_n F_* = \mathrm{cone}_{*+1}(q_n : P_n F \to P_{n-1} F)$.

PROPOSITION 4.7. — *The homology of the $n$-th homotopy fibre $D_n F_*$ is given as a derived functor of a Γ-module of degree $n$, namely*

$$H_*(D_n F[1]) = \mathrm{Tor}_*^\Gamma(\theta^n, F).$$

*Proof.* — We will prove that the functor $F \mapsto H_*(D_n F[1])$ fulfills the universal properties of the Tor-functor, i.e., we have to show

1) $H_*(D_n(-)[1])$ maps short exact sequences of functors to long exact sequences.

2) The functors $H_*(D_n(-)[1])$ and $\mathrm{Tor}_*^\Gamma(\theta^n, -)$ coincide on projective objects.

1) Let $0 \to F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ be a short exact sequence of functors. We have to show that $H_*(D_n(-)[1])$ maps this sequence to a long exact sequence

$$\begin{array}{ccccccc} & & & & \cdots & \longrightarrow & H_{\ell+1}(D_n(F'')[1]) \\ \longrightarrow & H_\ell(D_n(F')[1]) & \longrightarrow & H_\ell(D_n(F)[1]) & & \longrightarrow & H_\ell(D_n(F'')[1]) \\ \longrightarrow & H_{\ell-1}(D_n(F')[1]) & \longrightarrow & \cdots & & & \end{array}$$

The functor $F \mapsto P_nF[1]$ is exact, hence for every short exact sequence as above we have the following array of commutative diagrams:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{n-1}F'[1] & \longrightarrow & P_{n-1}F[1] & \longrightarrow & P_{n-1}F''[1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & sh_1D_nF'[1] & \longrightarrow & sh_1D_nF[1] & \longrightarrow & sh_1D_nF''[1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & sh_1P_nF'[1] & \longrightarrow & sh_1P_nF[1] & \longrightarrow & sh_1P_nF''[1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

As the composition $D_nF'[1] \to D_nF''[1]$ vanishes, we can apply the $3 \times 3$-lemma and obtain that $0 \to D_nF'[1] \to D_nF[1] \to D_nF''[1] \to 0$ is a short exact sequence of chain complexes and the claim is proved.

2) To make the two functors coincide we have to show that they have the same value on projectives. The short exact sequence

$$0 \to P_{n-1}F[1] \to sh_1D_nF[1] \to sh_1P_nF[1] \to 0$$

leads to the usual long exact sequence in homology

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H_{k+1}P_{n-1}F[1] & \longrightarrow & H_kD_nF[1] & \longrightarrow & H_kP_nF[1] \longrightarrow \\
 & \longrightarrow & H_kP_{n-1}F[1] & \longrightarrow & \cdots & &
\end{array}$$

Thus it is clear that $H_kD_nF[1]$ vanishes for $k > 0$ if $F$ is projective. For $k = 0$ we obtain that $H_0D_nF[1] \cong (t^n \otimes_\Gamma F)/(t^{n-1} \otimes_\Gamma F)$ and this can be identified with $\theta^n \otimes_\Gamma F$. □

## 5. Calculation of $\mathrm{Tor}_*(\theta^m, \psi_n)$ in characteristic zero.

In this section the ground field $\mathbb{K}$ is of characteristic zero. We will give a complete calculation of the homology of the homotopy fibres $D_n$ of the functors $\psi_m = \mathcal{B}_m/\mathcal{B}_{m-1}$. These $\Gamma$-vector spaces possess a Koszul-like resolution (see [P3]). For odd degrees ($m = 2k+1$) one has

$$0 \to \Lambda^k \circ L \otimes \mathrm{Sym}^1 \circ L \to \cdots \to \mathrm{Sym}^{2k+1} \circ L \to \psi_{2k+1} \to 0$$

whereas for even degrees ($m = 2k$) the resolution looks as follows:

$$0 \to \Lambda^k \circ L \to \Lambda^{k-1} \circ L \otimes \mathrm{Sym}^2 \circ L \to \cdots \to \mathrm{Sym}^{2k} \circ L \to \psi_{2k} \to 0.$$

The maps $\Lambda^{i+1} \circ L \otimes \mathrm{Sym}^j \circ L \to \Lambda^i \circ L \otimes \mathrm{Sym}^{j+2} \circ L$ in the sequences are

$$(x_0 \wedge \ldots \wedge x_i) \otimes (y_1 \cdots y_j) \mapsto \sum_{h=0}^{i} (x_0 \wedge \ldots \wedge \check{x}_h \wedge \ldots \wedge x_i) \otimes (x_h^2 y_1 \cdots y_j)$$

and

$$\begin{array}{rcl} \mathrm{Sym}^m \circ L & \to & \psi_m \\ y_1 \cdots y_m & \mapsto & \{y_1, \ldots, y_m\} \quad \mathrm{mod} \quad \mathcal{B}_{m-1}. \end{array}$$

Proposition 5.1. — *Let $i + j$ be $m$. The tensor products which are needed for the calculation of* $\mathrm{Tor}_*^\Gamma(\theta^n, \psi_m)$ *are*

$$\theta^n \otimes_\Gamma (\Lambda^i \circ L \otimes \mathrm{Sym}^j \circ L) = \begin{cases} \mathbb{K} & \text{for} \quad m = n, i = 0, 1 \\ 0 & \text{else.} \end{cases}$$

*Proof.* — If $m < n$ then the degree of $F_{i,j} := \Lambda^i \circ L \otimes \mathrm{Sym}^j \circ L$ with $i + j = m$ is $m$ and hence smaller than $n$. This gives us

$$\theta^n \otimes_\Gamma F_{i,j} \cong cr(\theta^n) \otimes_\Omega cr(F_{i,j}) \cong (\mathbb{K} \otimes_\Omega cr(F_{i,j})(\underline{n}))/\sim \quad = 0.$$

Now let $m$ be greater than $n$. There is a sequence of surjections

$$t^{\otimes n} \twoheadrightarrow \mathrm{Sym}^n \circ t \twoheadrightarrow \theta^n.$$

As we assumed that the characteristic of $\mathbb{K}$ is zero $F_{i,j}$ is a direct summand of $L^{\otimes m}$ and we obtain a surjection

$$t^{\otimes n} \otimes_\Gamma L^{\otimes m} \twoheadrightarrow t^{\otimes n} \otimes_\Gamma F_{i,j} \twoheadrightarrow \theta^n \otimes_\Gamma F_{i,j}.$$

For a contravariant functor $T$ the cross-effect is $cr_m(T) = \mathrm{coker}(\bigoplus_{i=1}^m T[m-1] \to T[m])$. Taking the exact sequence

$$\bigoplus_{i=1}^{m} \Gamma^{m-1} \to \Gamma^m \to L^{\otimes m} \to 0$$

and tensoring it with $T$ gives

$$\begin{array}{ccccccc} T \otimes_\Gamma (\bigoplus_{i=1}^m \Gamma^{m-1}) & \longrightarrow & T \otimes_\Gamma \Gamma^m & \longrightarrow & T \otimes_\Gamma L^{\otimes m} & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & & & \\ \bigoplus_{i=1}^m T[m-1] & \longrightarrow & T[m] & \longrightarrow & T \otimes_\Gamma L^{\otimes m} & \longrightarrow & 0. \end{array}$$

Thus $T\otimes_\Gamma L^{\otimes m} \cong cr(T)(\underline{m})$. Therefore $t^{\otimes n}\otimes_\Gamma L^{\otimes m} \cong cr(t^{\otimes n})(\underline{m}) = 0$ because $t^{\otimes n}$ is of degree $n$. Hence the tensor products $\theta^n \otimes_\Gamma F_{i,j}$ vanish if $i + j = m \neq n$. The case which is left to consider is $m = n$. In this case

$$\theta^m\otimes_\Gamma F_{i,j} \cong (\mathbb{K}\otimes cr(F_{i,j})(\underline{m}))/(f^*(x)\otimes y \sim x\otimes f_*(y)) \cong \mathbb{K}\otimes_{\Sigma_m} cr(F_{i,j})(\underline{m}).$$

Hence this cross-effect gives the coinvariants of the $\Sigma_m$-module $cr(\Lambda^i \circ L \otimes \mathrm{Sym}^j \circ L)(\underline{m})$. As $F_{i,j}$ is a quotient of $L^{\otimes m}$ and as taking cross effects is exact, we get

$$(*) \qquad cr(L^{\otimes m}) \twoheadrightarrow cr(F_{i,j}).$$

Lemma 5.2. — *As a $\Sigma_m$-module $cr_m(L^{\otimes m})$ is isomorphic to $\mathbb{K}[\Sigma_m]$.*

*Proof.* — By definition $L^{\otimes m}([n])$ is the free vector space generated by all $m$-tuples of elements in $\{1,\dots,n\}$. The kernel of the map $L^{\otimes m}(r_i)$ : $L^{\otimes m}([n]) \to L^{\otimes m}([n-1])$ consists of all $m$-tuples which contain $i$, because $L^{\otimes m}(r_i)((x_1,\dots,x_m)) = (r_i(x_1),\dots,r_i(x_m))$. Hence the intersection of all of these kernels are the $m$-tuples which contain each $i \in \{1,\dots,m\}$ exactly once. □

Thus the sequence $(*)$ reduces to $\mathbb{K}[\Sigma_m] \twoheadrightarrow cr_m(F_{i,j})$. Taking coinvariants shows that the coinvariants of $cr_m(F_{i,j})$ are isomorphic to $\mathbb{K}$ or to 0, because $\mathbb{K} \twoheadrightarrow \mathbb{K} \otimes_{\Sigma_m} cr_m(F_{i,j})$. This statement can be made more precise:

Lemma 5.3.

$$\mathbb{K} \otimes_{\Sigma_m} cr_m(F_{i,j}) \cong \begin{cases} \mathbb{K} & i = 0,1 \\ 0 & \textit{else.} \end{cases}$$

*Proof.* — Let $\pi_1 : L^{\otimes i} \longrightarrow \Lambda^i \circ L$ and $\pi_2 : L^{\otimes j} \longrightarrow \mathrm{Sym}^j \circ L$ be the canonical maps. The natural transformation $L^{\otimes m} \xrightarrow{\pi_1\otimes\pi_2} F_{i,j}$ gives a map on the corresponding cross effects by restriction. But for $i > 1$ the we get $\pi_1 \otimes \pi_2(x_1,\dots,x_m) = x_1 \wedge \dots \wedge x_i \otimes x_{i+1} \cdots x_m$. Taking coinvariants this term is equivalent to its negative and hence it vanishes. □

This finishes the proof of our proposition and leads to the calculation of the Koszul resolutions: Applying our results there are only two nontrivial sequences, namely $0 \to \theta^n \otimes_\Gamma (\Lambda^1 \otimes \mathrm{Sym}^{(n+1)-2}) \circ L \to \theta^n \otimes_\Gamma \mathrm{Sym}^{n+1} \circ L \to \theta^n \otimes_\Gamma \psi_{n+1} \to 0$ which gives $0 \to \mathbb{K} \to 0 \to \theta^n \otimes_\Gamma \psi_{n+1} \to 0$ and

$0 \to \mathbb{K} \to \theta^n \otimes_\Gamma \psi_n \to 0$ with the $\mathbb{K}$ arising from $\theta^n \otimes_\Gamma \mathrm{Sym}^n \circ L$. The corresponding Tor-groups are

$$\mathrm{Tor}_\ell^\Gamma(\theta^n, \psi_m) = \begin{cases} \mathbb{K} & \ell = 0, \quad m = n \\ \mathbb{K} & \ell = 1, \quad m = n+1 \\ 0 & \text{else.} \end{cases}$$

### 5.1. Application to higher order Hochschild homology.

In this section the ground ring $\mathbb{K}$ is a field of characteristic zero. For a commutative $\mathbb{K}$-algebra $A$ and an $A$-module $M$ let $\mathcal{L}(A, M)$ be the $\Gamma$-module which sends a finite pointed set $[n]$ to $M \otimes A^{\otimes n}$. A morphism $f : [n] \to [m]$ in $\Gamma$ maps an element $a_0 \otimes \cdots \otimes a_n$ in $M \otimes A^{\otimes n}$ to $b_0 \otimes \cdots \otimes b_m$ with $b_i = \prod_{f(j)=i} a_j$. Hochschild homology of order $d$ is defined via the evaluation on the simplicial $d$-sphere

$$H_*^{[d]}(A, M) := \pi_*(\mathcal{L}(A, M)(\mathbb{S}^d)).$$

As a concrete example we will calculate $H_*^{[d]}(\mathbb{K}[x]/x^2, \mathbb{K})$ for $d$ even. For $d$ odd the calculation of $H_*^{[d]}(\mathbb{K}[x]/x^2, \mathbb{K}[x]/x^2)$ is done in [P3], 5.4. As it is shown in 1.8 of [P3] there is a splitting

$$\mathcal{L}(\mathbb{K}[x]/x^2, \mathbb{K}) \cong \bigoplus_k \psi_k$$

which leads to a decomposition

$$H_n^{[d]}(\mathbb{K}[x]/x^2, \mathbb{K}) \cong \bigoplus_{i+dj=n} \bigoplus_k \mathrm{Tor}_i(\theta^j, \psi_k)$$

for $d$ even. Hence the calculation reduces to the determination of the Tor-groups which we have just calculated. Thus we proved the following

THEOREM 5.4. — *For $d$ even the Hochschild homology of order $d$ of $\mathbb{K}[x]/x^2$ with coefficients in $\mathbb{K}$ is*

$$H_n^{[d]}(\mathbb{K}[x]/x^2; \mathbb{K}) \cong \bigoplus_{i+dj=n} \bigoplus_k \mathrm{Tor}_i^\Gamma(\theta^j, \psi_k) \cong \begin{cases} \mathbb{K} & \text{if } d \mid n \\ \mathbb{K} & \text{if } d \mid n-1 \\ 0 & \text{else.} \end{cases}$$

## 6. A spectral sequence for $H_*D_nF[1]$.

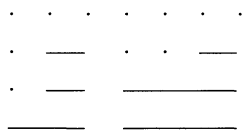
We saw that the homology of the homotopy fibre $D_nF_*[1] = \text{cone}_{*+1}(P_nF \longrightarrow P_{n-1}F)[1]$ is determined by $\text{Tor}_*^{\Gamma}(\theta^n, F)$. In [P2] Pirashvili developed a hyperhomology spectral sequence with abutment $\text{Tor}_*^{\Gamma}(t, F)$. A generalization of this approach leads to a spectral sequence which converges to $\text{Tor}_*^{\Gamma}(\theta^n, F)$. In the following part let $k$ be a field.

### 6.1. Iterated partitions.

For the definition of the spectral sequence we need the notion of iterated partitions. The term *p-partition* is defined inductively. A 1-partition (or simply partition) of some natural number $m$ is a sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\ell \geqslant 1$ and $1 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_\ell$ where the $\lambda_i$ add up to $m = \lambda_1 + \cdots + \lambda_\ell$. Here $\ell$ is called the length of the partition $\lambda$ and is denoted by $\ell(\lambda)$. For $n \geqslant 2$ an $n$-partition is a partition $\lambda = (\lambda_1, \ldots, \lambda_j)$ together with $(n-1)$-partitions $\lambda^i$ of $\lambda_i$ for $1 \leqslant i \leqslant j$. The length of an $n$-partition is an $n$-tuple of natural numbers

$$\ell(\lambda) = (\ell(\lambda^1) + \cdots + \ell(\lambda^j), j).$$

The set of $p$-partitions of $n_0$ with length $(n_1, \ldots n_{p-1}, n)$ is denoted by $\Pi(n_0, \ldots, n_{p-1}, n)$. For instance a $\lambda$ in $\Pi(7, 5, 3, 2)$ can be the following:



((1,2),(1,1,2))
((1,2), 4)
(3,4).

The partition $(3, 4)$ of 7 is refined taking the partition of 3 into $(1, 2)$ and the partition of 4 into $(1, 1, 2)$.

We associate to any $n$-partition a group of permutations: A 1-partition $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ yields the group $\Sigma_\lambda = \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_\ell}$ and to an $n$-partition $\mu = (\mu^1, \ldots, \mu^j)$ we associate iteratively the group $\Sigma_\mu = \Sigma_{\mu^1} \times \cdots \times \Sigma_{\mu^j}$. In the example above we obtain $\Sigma_\lambda = \Sigma_1 \times \Sigma_2 \times \Sigma_1 \times \Sigma_1 \times \Sigma_2 \cong \Sigma_2 \times \Sigma_2$.

### 6.2. The spectral sequence.

In the following we will outline the construction of the hyperhomology spectral sequence. All functors $F$ will have values in $k$-vector spaces.

THEOREM 6.1. — *There is a spectral sequence* $E^1_{p,q} \Longrightarrow \mathrm{Tor}^{\Gamma}_{p+q}(\theta^m, F)$ *of the following form:*

$$E^1_{pq} = \bigoplus_{\substack{n_0 > \dots > n_{p-1} > n \\ \lambda \in \Pi(n_0, \dots, n_{p-1}, n)}} H_q(\Sigma_\lambda, cr(F)(\underline{n}_0)).$$

*Remark 6.2.* — If we work over $k = \mathbb{Q}$ then this spectral sequence just consists of the zeroth row.

### 6.3. A chain complex of right Ω-modules.

Using the Dold-Kan correspondence between Γ-modules and Ω-modules, we can work in the category of Ω-modules to calculate $\mathrm{Tor}^{\Gamma}_*(\theta^n, F)$. The following result relates these derived functors with Tor-groups for Ω-modules.

PROPOSITION 6.3. — *For* Γ*-modules there is an isomorphism*

$$\mathrm{Tor}^{\Gamma}_*(T, F) \cong \mathrm{Tor}^{\Omega}_*(cr(T), cr(F)).$$

We will only indicate how the proof works. One uses that the equivalence $cr$ maps the family of reduced projective generators $(L^{\otimes m})_m$ to the family of representable functors in the category of Ω-modules. The Yoneda lemma implies that the assumption holds for these generators. A full proof is given in [R, Prop. 2.5.1].

An EI-category is a small category with all endomorphisms being automorphisms; Ω is an EI-category with the permutation groups $\Sigma_n$ as endomorphisms of $\underline{n}$. In such a situation there is a standard resolution $\mathcal{K}_*(U)$ of every $U : \Omega^{op} \to k-\mathrm{Vect}$ (as in [P2, 4.3] or [L]). With $\Omega(\underline{n}, \underline{m})$ we denote the morphisms in Ω from $\underline{n}$ to $\underline{m}$. This set has a canonical right $\Sigma_n$ and a left $\Sigma_m$ action. In degree $\ell$ the chain complex of right Ω-modules $\mathcal{K}_*(U)$ looks as follows:

$$\mathcal{K}_\ell(U)(-) := \bigoplus U(\underline{n}_\ell) \otimes_{\Sigma_{n_\ell}} k\{\Omega(\underline{n}_{\ell-1}, n_\ell)\} \otimes_{\Sigma_{n_{\ell-1}}} \dots \otimes_{\Sigma_{n_1}} k\{\Omega(\underline{n}_0, \underline{n}_1)\} \otimes_{\Sigma_{n_0}} k\{\Omega(-, \underline{n}_0)\}$$

where the sum runs over all $n_0 > \dots > n_\ell$.

The boundary map $d$ on an element $(a; f_\ell, \dots, f_0)$ with $a \in U(\underline{n}_\ell)$ is defined as

$$d(a; f_\ell, \dots, f_0) = \sum_{i=0}^{\ell-1} (-1)^i (a; f_\ell, \dots, f_{i+1} f_i, \dots, f_0) \\ + (-1)^\ell (f_\ell^*(a); f_{\ell-1}, \dots, f_0).$$

A proof that this complex is a resolution of $U$ can be found in [P2, Lemma 4.4], hence for every $\Omega$-module $F$ we obtain a hyperhomology spectral sequence

$$E^1_{pq} = \mathrm{Tor}^\Omega_q(\mathcal{K}_p(U), F) \Rightarrow \mathrm{Tor}^\Omega_{p+q}(U, F).$$

### 6.4. Simplification of the spectral sequence for $U = \theta^n$.

For $\theta^n : \Gamma^{op} \to k$-Vect and $F : \Gamma \to k$-Vect, the above spectral sequence gives

$$E^1_{pq} = \mathrm{Tor}^\Omega_q(\mathcal{K}_p(cr(\theta^n)), cr(F)) \Rightarrow \mathrm{Tor}^\Omega_{p+q}(cr(\theta^n), cr(F)) \cong \mathrm{Tor}^\Gamma_{p+q}(\theta^n, F).$$

But we know that $cr(\theta^n)(\underline{i}) = 0$ if $i \neq n$. Hence the above chain complex reduces to

$$\mathcal{K}_\ell(U) = \bigoplus k \otimes_{\Sigma_n} k\left\{\Omega(\underline{n}_{\ell-1}, \underline{n})\right\} \otimes_{\Sigma_{n_{\ell-1}}} \\ \dots \otimes_{\Sigma_{n_1}} k\left\{\Omega(\underline{n}_0, \underline{n}_1)\right\} \otimes_{\Sigma_{n_0}} k\left\{\Omega(-, \underline{n}_0)\right\}.$$

Here the sum is taken over all $n_0 > \dots > n_{\ell-1} > n$. According to Lemma 4.6 in [P2] we obtain

$$\mathrm{Tor}^\Omega_*(M \otimes_{\Sigma_n} k\left\{\Omega(-, \underline{n})\right\}, T) \cong \mathrm{Tor}^{\Sigma_n}_*(M, T(\underline{n}))$$

for all $\Sigma_n$-modules $M$ and all $\Omega$-modules $T$. We gain an even stronger simplification by applying Lemma 4.8 in [P2]: The $\Sigma_n$-module

$$k \otimes_{\Sigma_n} k\left\{\Omega(\underline{n}_{p-1}, \underline{n})\right\} \otimes_{\Sigma_{n_{p-1}}} \dots \otimes_{\Sigma_{n_1}} k\left\{\Omega(\underline{n}_0, \underline{n}_1)\right\}$$

is isomorphic to

$$\bigoplus_{\lambda \in \Pi(n_0, \dots, n)} k\left[\Sigma_{n_0}/\Sigma_\lambda\right].$$

After all these reductions our spectral sequence looks as follows:

$$E^1_{pq} = \mathrm{Tor}^\Omega_q(\mathcal{K}_p(cr(\theta^n), cr(F)) = \bigoplus_{\substack{n_0 > \dots > n_{p-1} > n \\ \lambda \in \Pi(n_0, \dots, n_{p-1}, n)}} \mathrm{Tor}^{\Sigma_{n_0}}_q(k[\Sigma_{n_0}/\Sigma_\lambda]; (cr(F))(\underline{n}_0)).$$

Using the Shapiro lemma in group homology we obtain

$$\begin{aligned}\mathrm{Tor}_q^{\Sigma_{n_0}}(k[\Sigma_{n_0}/\Sigma_\lambda]; cr(F)(\underline{n}_0)) &\cong \mathrm{Tor}_q^{\Sigma_{n_0}}(k; k[\Sigma_{n_0}/\Sigma_\lambda] \otimes cr(F)(\underline{n}_0)) \\ &\cong H_q(\Sigma_{n_0}, \Sigma_{n_0}/\Sigma_\lambda \otimes cr(F)(\underline{n}_0)) \\ &\cong H_q(\Sigma_\lambda, cr(F)(\underline{n}_0)).\end{aligned}$$

Therefore the final version of our spectral sequence for the homology of the $n$-th homotopy fibre $\mathrm{Tor}_*^\Gamma(\theta^n, F)$ is

$$E^1_{pq} = \bigoplus_{\substack{n_0 > \cdots > n_{p-1} > n \\ \lambda \in \Pi(n_0, \ldots, n_{p-1}, n)}} H_q(\Sigma_\lambda, cr(F)(\underline{n}_0)).$$

*Remark 6.4.* — Note that the $E^1_{0,q}$-term for $\mathrm{Tor}(\theta^n, \psi_m)$ consists of $\mathrm{Tor}_q(k \otimes_{\Sigma_n} \Omega_n, cr(\psi_m)) \cong \mathrm{Tor}_q^{\Sigma_n}(k, cr(\psi_m)(\underline{n})) \cong \mathrm{Tor}_q^{\Sigma_n}(k, k)$ iff $m = n$. Otherwise this term is trivial.

## 7. The dual of the Steenrod algebra.

### 7.1. Decomposition as a Γ-vector space.

As an example for an explicit calculation we will consider the functor

$$\begin{aligned} St: \quad \Gamma &\longrightarrow \mathbb{F}_2 - \mathrm{Vect} \\ [n] &\mapsto \mathbb{F}_2\{\overline{\mathbb{F}}_2\{[n]\}\} \end{aligned}$$

from finite pointed sets to $\mathbb{F}_2$-vector spaces, which takes a finite pointed set to the reduced vector space on the elements and takes then the free $\mathbb{F}_2$-vector space of this. Prolonging this functor to simplicial pointed sets and evaluating it on an arbitrary simplicial model for a sphere leads to the homology of Eilenberg-MacLane spaces. As $\pi_*^{st}(St) \cong \mathrm{Tor}_*^\Gamma(t, St)$ (see [P3, Prop 2.2]), we gain that the homology of the first approximation evaluated at [1] is nothing but the dual of the Steenrod algebra.

Recall that we denote by $\mathcal{B}_\ell : \Gamma \to \mathbb{F}_2$-Vect the functor that maps a finite pointed set $S_+$ to the free $\mathbb{F}_2$-vector space which is generated by all nonempty subsets of $S$ with cardinality less or equal to $\ell$ and that we abbreviate the functor $\mathcal{B}_\ell/\mathcal{B}_{\ell-1}$ by $\psi_\ell$.

For our calculations the whole functor $St$ would be too complicated to deal with, but we can show that this functor splits into homogenous pieces, namely $St$ is a sum of the functors $\psi_\ell$ which are easier to handle.

THEOREM 7.1. — *As a $\Gamma$-module $St$ splits into*

$$St \cong \bigoplus_{\ell \geqslant 0} \psi_\ell,$$

*where $\psi_0$ is the constant functor with value $\mathbb{F}_2$ on every object.*

We will prove that $cr(St)$ is a split functor in the category of $\Omega$-modules. Let $St_n$ abbreviate $cr_n St([1], \ldots, [1])$.

PROPOSITION 7.2. — *Every map $f : \underline{n} \to \underline{n-1}$ in $\Omega$ induces the trivial map $f_* : St_n \to St_{n-1}$.*

LEMMA 7.3. — *All vector spaces $St_n$ are of dimension one*

$$\dim_{\mathbb{F}_2} St_n = 1.$$

*Proof.* — We will prove $cr_n St([1], \ldots, [1]) \cong \mathbb{F}_2$ and $cr_n St([1], \ldots, [1], [k]) \cong \mathbb{F}_2^{2^k - 1}$ inductively. In degree one we have that

$$St[0] \oplus cr_1 St[1] = St[1] = \mathbb{F}_2\left[\overline{\mathbb{F}_2}[1]\right],$$

hence $cr_1 St[1] \cong \mathbb{F}_2$ and

$$St[k] = St[0] \oplus cr_1 St[k] \cong \mathbb{F}_2^{2^k},$$

so $cr_1 St[k] \cong \mathbb{F}_2^{2^k - 1}$. Assume the two claims are true for all $i < n$. From the definition of the cross-effect of $St$ and the assumption we gain

$$\begin{aligned} St_n &\cong cr_2(cr_{n-1} St([1], \ldots, [1], \_))([1], [1]) \\ &\cong \ker(cr_{n-1} St([1], \ldots, [1], [2]) \to St_{n-1} \times St_{n-1}) \\ &\cong \ker(\mathbb{F}_2^3 \to \mathbb{F}_2 \times \mathbb{F}_2) \\ &\cong \mathbb{F}_2. \end{aligned}$$

The last isomorphism comes from the fact that the map from $\mathbb{F}_2^3$ to $\mathbb{F}_2 \times \mathbb{F}_2$ is surjective. The second claim is also straightforward to see:

$$\begin{aligned} cr_n St([1], \ldots, [k]) &\cong \ker(cr_{n-1} St([1], \ldots, [1], [k+1]) \\ &\qquad \to St_{n-1} \times cr_{n-1} St([1], \ldots, [1], [k])) \\ &\cong \ker(\mathbb{F}_2^{2^{k+1}-1} \to \mathbb{F}_2 \times \mathbb{F}_2^{2^k - 1}) \\ &\cong \mathbb{F}_2^{2^k - 1}, \end{aligned}$$

and hence the proof is completed. □

*Proof of the proposition.* — As $St_n$ is one dimensional we have only to prove that the basis element is mapped trivially to $St_{n-1}$. Let $e_1, \dots, e_n$ denote the canonical basis of $\overline{\mathbb{F}}_2[n]$. Then we claim that the element

$$E_n := (e_1 + \cdots + e_n) + \sum_{i=1}^{n} (e_1 + \cdots + \check{e}_i \cdots + e_n) + \cdots + (e_1) + \cdots + (e_n)$$

is a basis of $St_n$. Here the inner sums are taken in $\mathbb{F}_2\{\underline{n}\}$ whereas the sums outside of the parenthesis are taken in $\mathbb{F}_2\{\overline{\mathbb{F}}_2\{-\}\}$. We have to prove that $E_n$ is in the intersections of the kernels of the $St(r_i)$. Since $E_n$ is symmetric with respect to the action of the symmetric group, it is enough to show that $St(r_n)(E_n) = 0$. Let $(E_{n-1} + e_n)$ denote the sum over all summands in $E_n$ which contain $e_n$ and some term from $E_{n-1}$. It is obvious that we can decompose $E_n$ in

$$E_n = (E_{n-1} + e_n) + E_{n-1} + (e_n).$$

But then it is easy to see that $St(r_n)(E_n) = E_{n-1} + E_{n-1} = 0$.

Now let $f : \underline{n} \to \underline{n-1}$ be a map in $\Omega$. Without loss of generality we can assume that $f(n) = f(n-1) = n-1$. But then $f_*(E_n) = 2E_{n-1} = 0$.

As $cr(St)$ and $\bigoplus cr(\psi_i)$ have the same values on objects and as they have only trivial transformations, they coincide. Using the Dold-Kan correspondence between Γ-modules and Ω-modules we obtain a splitting for $St$. □

As a concrete example we will calculate the homology of the homotopy fibre of the quadratic approximation of the functors $\psi_\ell$, for $\ell = 2, 3, 4$ over some field $k$.

**7.1.1.** $\mathrm{Tor}_*^\Gamma(\theta^2, \psi_\ell)$, $\quad \ell = 2, 3, 4$

a) From remark 6.4 we see that $\mathrm{Tor}_q^\Gamma(\theta^2, \psi_2) \cong H_q(\Sigma_2; k)$.

b) $\mathrm{Tor}_*^\Gamma(\theta^2, \psi_3)$.

The hyperhomology spectral sequence for $\mathrm{Tor}_*^\Gamma(\theta^2, \psi_3)$ is

$$E^1_{pq} = \bigoplus_{\substack{n_1 > \quad > n_p > 2 \\ \lambda \in \Pi(n_1, \quad , n_p, 2)}} H_q(\Sigma_\lambda; (cr(\psi_3))(\underline{n}_1)).$$

But $cr(\psi_3)(\underline{n}) = 0$ unless $n = 3$. Having Remark 6.4 in mind we see that the spectral sequence consists only of a nontrivial $E^1_{1q}$ part, namely

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \\ 0 & \bigoplus_{\lambda\in\Pi(3,2)} H_1(\Sigma_\lambda;k) & 0 & \dots \\ 0 & \bigoplus_{\lambda\in\Pi(3,2)} H_0(\Sigma_\lambda;k) & 0 & \dots \end{array}$$

There is only one partition $\lambda \in \Pi(3,2)$ namely $\lambda = (1,2)$ and for this partition we obtain $\Sigma_\lambda = \Sigma_1 \times \Sigma_2 \cong \Sigma_2$. All differentials are trivial in this case, hence for all $n$

$$\mathrm{Tor}^\Gamma_n(\theta^2, \psi_3) = E^1_{1,n-1} = H_{n-1}(\Sigma_2;k).$$

c) $\mathrm{Tor}^\Gamma_*(\theta^2, \psi_4)$.

From now on all coefficients are taken in $k$ without mentioning them anymore. In a similar manner as in the above case we obtain that the $E^1$-term for $\mathrm{Tor}^\Gamma_*(\theta^2, \psi_4)$ reduces to

$$E^1_{pq} = \bigoplus_{\substack{n_1 > \quad > n_p > 2 \\ \lambda\in\Pi(n_1=4, \quad ,n_p,2)}} H_q(\Sigma_\lambda).$$

The resulting $E^1$ is

$$\begin{array}{c|c|c|c|c} \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & H_1(\Sigma_3)\oplus H_1(\Sigma_2\times\Sigma_2) & 3H_1(\Sigma_2) & 0 & \dots \\ 0 & H_0(\Sigma_3)\oplus H_0(\Sigma_2\times\Sigma_2) & 3H_0(\Sigma_2) & 0 & \dots \end{array}$$

The differentials arise from the differentials in the chain complex $\mathcal{K}_*(cr(\theta^2))$. Therefore one copy of $H_*(\Sigma_2)$ which corresponds to the 2-partition $(1,3)$, $(1,1,2)$ is mapped horizontally to the summand $H_*(\Sigma_3)$. The boundary of the other summand which corresponds to the 2-partition $(2,2)$, $(1,1,2)$ is mapped to $H_*(\Sigma_2\times\Sigma_2)$. The $E^2$-term that comes out is

$$\begin{array}{c|c|c|c} \vdots & \vdots & \vdots & \vdots \\ 0 & H_1(\Sigma_3)/H_1(\Sigma_2)\oplus H_1(\Sigma_2\times\Sigma_2)/H_1(\Sigma_2) & H_1(\Sigma_2) & 0 \\ 0 & H_0(\Sigma_3)/H_0(\Sigma_2)\oplus H_0(\Sigma_2\times\Sigma_2)/H_0(\Sigma_2) & H_0(\Sigma_2) & 0 \end{array}$$

Thus the next differential $d^2$ ends up in a zero column and hence $E^2 = E^\infty$ and

$$\begin{aligned} \mathrm{Tor}^\Gamma_n(\theta^2, \psi_4) \quad \cong \quad & H_{n-2}(\Sigma_2;k)\oplus H_{n-1}(\Sigma_2\times\Sigma_2;k)/H_{n-1}(\Sigma_2;k) \\ \oplus \quad & H_{n-1}(\Sigma_3;k)/H_{n-1}(\Sigma_2;k). \end{aligned}$$

*Remark 7.4.* — In a similar manner one can use the spectral sequence to compute $\mathrm{Tor}_*^\Gamma(\theta^n, \psi_m)$ for $m = n, n+1, n+2$.

### 7.2. Homology of posets of partitions.

We saw in Section 5 how to compute $\mathrm{Tor}(\theta^n, \psi_m)$ at least over the rational numbers. These Tor-groups have an interpretation as the homology of a poset of certain unordered partitions, namely those partitions arising as layers of $p$-partitions. Let $\Xi$ be a $p$-partition of $n$, i.e., $\Xi$ is a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ with $1 \leqslant \lambda_1 \leqslant \dots \leqslant \lambda_k$ and $\sum \lambda_i = n$ together with $(p-1)$-partitions of the $\lambda_i$: $(\lambda_i^1, \dots, \lambda_i^{n_i})$. The collection $((\lambda_1^1, \dots, \lambda_1^{n_1}), \dots, (\lambda_k^1, \dots, \lambda_k^{n_k}))$ is no actual partition in general because it might happen that $\lambda_j^{n_j} \geqslant \lambda_{j+1}^1$.

DEFINITION 7.5. — *An $\ell$-tuple $a = (a_1, \dots, a_\ell)$ of natural numbers is an* unordered partition *of $n$ if $1 \leqslant a_i$ and $\sum a_i = n$. Then $\ell$ is called the* length *of $a$.*

In a $p$-partition, $(\lambda_1, \dots, \lambda_k)$ is called the first layer of the $p$-partition $\Xi$, the second layer is $((\lambda_1^1, \dots, \lambda_1^{n_1}), \dots, (\lambda_k^1, \dots, \lambda_k^{n_k}))$ and so on. Hence a $p$-partition $\Xi$ consists of $p$ layers of unordered partitions.

DEFINITION 7.6. — *An unordered partition $\lambda$ of $n$ is called grown if $\lambda$ is a layer of some $p$-partition $\Xi$.*

*Example.* — The unordered partition $(1,2,1,1,2)$ is grown because it is a third layer of the 3-partition $(3,4), (1,2,4)$ and $(1,2,1,1,2)$. A typical example for an unordered partition which is not grown is something of the form $(n, 1)$ with $n > 1$.

DEFINITION 7.7. — *Let $\mu$ and $\nu$ be two grown partitions. Then $\nu$ is a* refinement *of $\mu$ if there is a $p$-partition $\Xi$ and $\nu$ is a higher layer of $\Xi$ than $\mu$.*

Let $\Pi(m, n)$ be the set of all grown partitions of $m$ which arise from an actual partition of length $n$. We say that $\lambda < \mu$ for $\lambda, \mu \in \Pi(m, n)$, if $\lambda$ is a refinement of $\mu$. The homology of the poset $\Pi(m, n)$ can be described as follows:

THEOREM 7.8.

$$H_*(\Pi(m,n);\underline{\mathbb{Q}}) \cong \begin{cases} \mathbb{Q} & *=0, \quad m=n \\ \mathbb{Q} & *=1, \quad m=n+1 \\ 0 & \textit{else.} \end{cases}$$

Here $\underline{\mathbb{Q}}$ is the constant functor, which assigns the field $\mathbb{Q}$ to every unordered partition.

*Proof.* — The spectral sequences which converges to $\mathrm{Tor}_*^\Gamma(\theta^n, \psi_m)$ degenerates at the $E^2$-level over the rational numbers. The $E^1$-term reduces to

$$E^1_{p,0} = \bigoplus_{\substack{n_1=m>n_2> \,..>n_p>n \\ \Xi\in\Pi(m,n_2,\,.,n_p,n)}} \mathbb{Q}$$

because the zeroth group homology of any symmetric group $H_0(\Sigma_\lambda, \mathbb{Q}))$ is $\mathbb{Q}$. The first differential comes from the differential in $K_*(cr(\theta^n))$, hence it just forgets some layers. But this is exactly what the differential does in the complex which computes the homology of the poset $\Pi(m,n)$. A $p$-partition $\Xi \in \Pi(m, n_2, \ldots, n_p, n)$ is nothing but a $p$-string of composable morphisms in $\Pi(m,n)$. □

*Remark 7.9.* — From the proof of the theorem it it clear that for an arbitrary field $k$ it is still true that the $E^2$-term for $\mathrm{Tor}_*^\Gamma(\theta^n, \psi_m)$ consists of the homology of the partition poset with coefficients in the functor which takes a partition to the homology of its corresponding symmetric group

$$E^2_{p,q} \cong H_p(\Pi(m,n); H_q(\Sigma_{(-)}, k)).$$

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