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**$L^p$ – $L^q$  ESTIMATES FOR FUNCTIONS  
OF THE LAPLACE–BELTRAMI OPERATOR  
ON NONCOMPACT SYMMETRIC SPACES. III**

by M. COWLING, S. GIULINI & S. MEDA

*Dedicated to the memory of Nanda*

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This paper is the third of a series on semigroups of operators related to the Laplace–Beltrami operator on a symmetric space of the noncompact type. The heat and the Poisson semigroups, together with some variants and applications, were studied in [CGM1] and [CGM2]. Here we study the Poisson semigroup in complex time.

Let  $G$  and  $K$  be a connected noncompact semisimple Lie group with finite centre and a maximal compact subgroup thereof, and consider the symmetric space  $G/K$ , which we also denote by  $X$ . There is a canonical invariant Riemannian metric on  $X$ ; denote by  $-\mathcal{L}_0$  the associated Laplace–Beltrami operator. By general nonsense,  $\mathcal{L}_0$  is positive and essentially self-adjoint on  $C_c^\infty(X)$ ; let  $\mathcal{L}$  be the unique self-adjoint extension of  $\mathcal{L}_0$  and  $\{\mathcal{E}_\zeta\}$  the spectral resolution of the identity for which

$$\mathcal{L}f = \int_b^\infty \zeta \, d\mathcal{E}_\zeta f \quad \forall f \in \text{Dom}(\mathcal{L}),$$

where  $b = \langle \rho, \rho \rangle$ ,  $\rho$  being the usual half-sum of the positive roots.

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For  $\theta$  in  $[0, 1]$ , the (complex-time)  $\theta$ -Poisson semigroup  $(\mathcal{P}_{\tau,\theta})_{\text{Re } \tau \geq 0}$  is defined by

$$\mathcal{P}_{\tau,\theta} f = \int_b^\infty \exp(-\tau(\zeta - \theta b)^{1/2}) d\mathcal{E}_\zeta f \quad \forall f \in L^2(X).$$

If  $1 \leq p, q \leq \infty$  and an operator  $\mathcal{T}$  satisfies a norm inequality of the form

$$\|\mathcal{T}f\|_q \leq C\|f\|_p \quad \forall f \in L^2(X) \cap L^p(X),$$

then  $\mathcal{T}$  is said to be  $L^p$ - $L^q$ -bounded. If  $1 \leq p, q \leq \infty$ , we denote by  $\|\mathcal{T}\|_{p,q}$  the norm of the linear operator  $\mathcal{T}$  from  $L^p(X)$  to  $L^q(X)$ . We write  $\|\mathcal{T}\|_p$  instead of  $\|\mathcal{T}\|_{p,p}$ . As usual,  $p'$  will denote the index  $p/(p - 1)$  conjugate to  $p$ .

The operators  $\mathcal{P}_{\tau,\theta}$ , for  $\tau$  in  $\mathbb{R}^+$ , form the Poisson semigroup considered by J.-Ph. Anker [A1], by Anker and L. Ji [AJ], and in [CGM2]. In [AJ], optimal upper and lower bounds for the kernel of  $\mathcal{P}_{\tau,\theta}$  were found, while in [CGM2], the behaviour of  $\|\mathcal{P}_{\tau,\theta}\|_{p,q}$  as  $\tau$  tends to 0 and to  $\infty$  for all  $p$  and  $q$  in  $[1, \infty]$  for which  $\mathcal{P}_{\tau,\theta}$  is  $L^p$ - $L^q$ -bounded was described. In this paper we consider the operators  $\mathcal{P}_{\tau,\theta}$  in the case where  $\theta = 1$  and  $\tau$  is complex and  $\text{Re } \tau > 0$ , and study the behaviour of  $\|\mathcal{P}_{\tau,\theta}\|_{p,q}$ . This involves finding explicit formulae for the kernel, so in some sense our work also develops that of [AJ]. We remark that the shifted Laplace–Beltrami operator  $-\mathcal{L} + b$ , corresponding to the case where  $\theta = 1$ , occurs naturally in geometry, as it is conformally invariant; see, e.g., S. Helgason [H2]. An analysis of the operators  $\mathcal{P}_{\tau,\theta}$  in the case where  $0 \leq \theta < 1$  will be carried out in a forthcoming paper.

For brevity, in the rest of this paper we write  $\mathcal{P}_\tau$  instead of  $\mathcal{P}_{\tau,1}$ .

Our main theorem describes the behaviour of  $\|\mathcal{P}_\tau\|_{p,q}$  for various possible values of  $p$  and  $q$  and for  $\tau$  in various subsets of the right half of the complex plane. This description is nearly complete, but when  $p < 2 < q$  and  $|\tau|$  is large but  $\tau$  is nearly imaginary, our methods do not yield good estimates. To formulate the theorem, we need three definitions:  $\mathbf{H} = \{\tau \in \mathbb{C} : \text{Re } \tau > 0\}$ , while if  $T$  is in  $\mathbb{R}^+$ , then  $\mathbf{D}_T$  and  $\mathbf{H}_T$  are defined by

$$\mathbf{D}_T = \{\tau \in \mathbf{H} : |\tau| \leq T\} \quad \text{and} \quad \mathbf{H}_T = \left\{ \tau \in \mathbf{H} : (\text{Re } \tau)^2 \geq \left( \frac{\text{Im } \tau}{t} \right)^2 + 1 \right\}.$$

The first of the regions is a half disc, while the second is the area to the right of a hyperbola.

It is obvious if  $\tau \in \overline{\mathbf{H}}$ , then  $\mathcal{P}_\tau$  is bounded on  $L^2(X)$ , and

$$\|\mathcal{P}_\tau\|_2 = 1.$$

THEOREM 1. — Suppose that  $1 \leq p, q \leq \infty$  and  $T \in \mathbb{R}^+$ . Then the following hold:

- (i) if  $\tau \in \overline{\mathbf{H}} \setminus \{0\}$  and  $\mathcal{P}_\tau$  is  $L^p$ - $L^q$ -bounded, then  $p \leq 2 \leq q$
- (ii) if  $\tau \in \overline{\mathbf{H}}$ , and  $p < 2 = q$  or  $p = 2 < q$ , then

$$\|\mathcal{P}_\tau\|_{p;q} = \|\mathcal{P}_{\operatorname{Re} \tau}\|_{p;q} \sim \begin{cases} (\operatorname{Re} \tau)^{-n(1/p-1/q)} & \text{if } \operatorname{Re} \tau \in (0, 1] \\ (\operatorname{Re} \tau)^{-\nu/2} & \text{if } \operatorname{Re} \tau \in [1, \infty) \end{cases}$$

- (iii) if  $\tau \in \mathbf{H}$  and  $1 \leq p < 2 < q \leq \infty$ , then

$$\|\mathcal{P}_\tau\|_{p;q} \leq C \begin{cases} (\operatorname{Re} \tau)^{-n(1/p-1/q)} & \text{if } \operatorname{Re} \tau \in (0, 1] \\ (\operatorname{Re} \tau)^{-\nu/2} & \text{if } \operatorname{Re} \tau \in [1, \infty) \end{cases}$$

and further

$$\|\mathcal{P}_\tau\|_{p;q} \sim (\operatorname{Re} \tau)^{-\nu} \quad \forall \tau \in \mathbf{H}_T$$

- (iv) if  $1 \leq p \leq 2 \leq q \leq \infty$ , then

$$\|\mathcal{P}_\tau\|_{p;q} \sim \frac{|\tau|^{(n-1)(|1/q-1/p|-|1/p-1/q|)/2}}{(\operatorname{Re} \tau)^{(n-1)|1/q-1/p|+(n+1)|1/p-1/q|/2}} \quad \forall \tau \in \mathbf{D}_T$$

- (v) if  $\tau \in i\mathbb{R} \setminus \{0\}$  and  $\mathcal{P}_\tau$  is  $L^p$ - $L^q$ -bounded, then  $p = 2 = q$ .

We use two distinct approaches to describe the behaviour of  $\|\mathcal{P}_\tau\|_{p;q}$ . For the case where  $\tau$  is in  $\mathbf{H}_T$ , we use spectral methods, similar to those of our previous papers [CGM1] and [CGM2]. The proofs of Theorem 1 (i)–(iii) are given in Section 2.

When  $\tau$  is in  $\mathbf{D}_T$ , we write the kernel of  $\mathcal{P}_\tau$  in terms of the wave propagation operator  $\cos(s(\mathcal{L} - b)^{1/2})$ . This technique was developed by J. Cheeger, M. Gromov and M. Taylor [CGT] and by Taylor [T], in their studies of functional calculus for the Laplace operator on general Riemannian manifolds with bounded geometry. Subsequently, J.-Ph. Anker [A2] adapted this method to estimate the kernels of certain functions of  $\mathcal{L}$  on noncompact symmetric spaces. However, the estimates of these authors follow from the finite propagation speed of the wave operator, while we need more refined information to estimate the local part of the kernel. Essentially, we write the kernel of the wave propagation operator as a sum of certain explicit distributions and a remainder term which is relatively smooth and well behaved. The main results we obtain using wave equation methods are Lemma 3.3 about the “local part” of the operator  $\mathcal{P}_\tau$  when  $\tau$  is in  $\mathbf{D}_T$ , and Propositions 3.6 and 3.7 establishing Theorem 1 (iv) and (v). We obtain upper estimates for  $\|\mathcal{P}_\tau\|_{p;q}$  for all  $\tau$ , but we do not know how

good these are when the imaginary part of  $\tau$  is arbitrarily large, because we are not able to expand the kernel of the wave propagation operator with uniform estimates of the remainder over arbitrarily large sets. We believe that this is a very interesting problem.

Applications of our results to the study of the regularity properties of the wave equation for fixed time will appear in a forthcoming paper.

## 1. Notation and background material.

We use the standard notation of the theory of Lie groups and symmetric spaces, as in the book of Helgason [H1]. Our notation here is also consistent with our papers [CGM1] and [CGM2], to which we refer several times. In particular,  $G$  denotes a noncompact semisimple Lie group with finite centre,  $NAK$  an Iwasawa decomposition of  $G$ , and  $X$  the Riemannian symmetric space  $G/K$ . We identify functions on  $X$  with  $K$ -right-invariant functions on  $G$  in the usual way. Thus, if  $f$  is a  $K$ -right-invariant function on  $G$ , we say that  $f$  is supported in the ball  $\overline{B}(o, 1)$ , the closed ball in  $X$  centred at the origin  $o$  in  $X$  of radius 1, when the corresponding function on  $X$  has this property. Similarly, for  $x$  in  $G$ , we define  $|x|$  to be the geodesic distance between  $xK$  and  $o$  in  $X$ . We denote by  $n$  the dimension of  $X$ , by  $\ell$  its rank, and by  $\nu$  the ‘‘pseudo-dimension’’  $2|\Sigma_0^+| + \ell$ , where  $|\Sigma_0^+|$  is the cardinality of the set of the positive indivisible roots of  $(\mathfrak{g}, \mathfrak{a})$ .

For any  $x$  in  $G$ , we denote by  $A(x)$  the element of  $\mathfrak{a}$  such that  $x$  is in  $N \exp A(x)K$ . For any linear form  $\lambda : \mathfrak{a} \rightarrow \mathbb{C}$ , the elementary spherical function  $\phi_\lambda$  is defined by the rule

$$\phi_\lambda(x) = \int_K \exp[(i\lambda + \rho)A(kx)] dk \quad \forall x \in G.$$

The spherical transform  $\tilde{f}$  of an  $L^1(G)$ -function  $f$  is defined by the formula

$$\tilde{f}(\lambda) = \int_G f(x) \phi_{-\lambda}(x) dx \quad \forall \lambda \in \mathfrak{a}^*.$$

Harish-Chandra’s inversion formula and Plancherel formula state that

$$f(x) = \int_{\mathfrak{a}^*} \tilde{f}(\lambda) \phi_\lambda(x) d\mu(\lambda) \quad \forall x \in G$$

for ‘‘nice’’  $K$ -bi-invariant functions  $f$  on  $G$ , and

$$\|f\|_2 = \left[ \int_{\mathfrak{a}^*} |\tilde{f}(\lambda)|^2 d\mu(\lambda) \right]^{1/2} \quad \forall f \in L^2(K \backslash G / X),$$

where  $d\mu(\lambda) = c_G |\langle \lambda \rangle|^{-2} d\lambda$ , and  $\mathbf{c}$  denotes the Harish-Chandra  $\mathbf{c}$ -function. For the details, see, for instance, [H1], IV.7.

We often deal with the inversion formula and the Plancherel formula with radial integrands. Using polar co-ordinates in  $\mathfrak{a}^*$ , it is easy to see that there exists a real analytic function  $\kappa$  on  $\mathbb{R}^+$  such that if  $g : [0, \infty) \rightarrow \mathbb{C}$  and  $f(\lambda) = g(|\lambda|)$  for all  $\lambda$  in  $\mathfrak{a}^*$ , then

$$\int_{\mathfrak{a}^*} f(\lambda) d\mu(\lambda) = \int_0^\infty g(r) \kappa(r) dr,$$

provided the integrals converge. In [CGM1], §2, it is shown that

$$\kappa(r) \sim r^{\nu-1}(r+1)^{n-\nu} \quad \forall r \in \mathbb{R}^+.$$

Let  $\mathbf{W}_1$  be the interior of the convex hull in  $\mathfrak{a}^*$  of the images of  $\rho$  under the action of the Weyl group of  $(\mathfrak{g}, \mathfrak{a})$ . For  $\delta$  in  $(0, 1)$ , we denote by  $\mathbf{W}_\delta$  the dilate of  $\mathbf{W}_1$  by  $\delta$  and by  $\mathbf{T}_\delta$  the tube over the polygon  $\mathbf{W}_\delta$ , i.e.,  $\mathbf{T}_\delta = \mathfrak{a}^* + i\delta\mathbf{W}_1$ ;  $\overline{\mathbf{W}}_\delta$  and  $\overline{\mathbf{T}}_\delta$  denote the closures of these sets in  $\mathfrak{a}^*$  and  $\mathfrak{a}_\mathbb{C}^*$  respectively.

For  $\tau$  in  $\overline{\mathbf{H}}$ , we denote by  $p_\tau$  the  $K$ -bi-invariant function (or distribution) on  $G$  such that  $\tilde{p}_\tau = P_\tau$ , where

$$P_\tau(\lambda) = \exp\left(-\tau \langle \lambda, \lambda \rangle^{1/2}\right) \quad \forall \lambda \in \mathfrak{a}^*.$$

Then

$$\mathcal{P}_\tau f = f * p_\tau \quad \forall \tau \in \overline{\mathbf{H}} \quad \forall f \in L^2(X).$$

Note that  $\tilde{p}_\tau$  does not continue analytically to the tube  $\mathbf{T}_\epsilon$ , for any positive  $\epsilon$ . Define (formally) the operator  $\mathcal{L}_1$  by the formula

$$\mathcal{L}_1 = \mathcal{L} - b.$$

By spectral theory,  $\mathcal{P}_\tau f = \exp(-\tau \mathcal{L}_1^{1/2})f$  for all  $f$  in  $L^2(X)$ .

Frequently, we deal with operators which are defined by convolution with  $K$ -invariant kernels on  $G$ ; when we do, we usually indicate the kernel and spherical Fourier transform corresponding to the operator  $\mathcal{R}$  by  $r$  and  $R$  respectively.

Positive constants are denoted by  $C$ ; these may differ from one line to another, and may depend on any quantifiers written, implicitly or explicitly, before the relevant formula, and on any written explicitly after. The expression

$$A(t) \sim B(t) \quad \forall t \in \mathbf{D},$$

where  $\mathbf{D}$  is some subset of the domains of  $A$  and of  $B$ , means that there exist constants  $C$  and  $C'$  such that

$$C|A(t)| \leq |B(t)| \leq C'|A(t)| \quad \forall t \in \mathbf{D}.$$

The expression  $A(t) \asymp B(t)$  as  $t \rightarrow t_0$  means that  $A(t)/B(t) \rightarrow 1$  as  $t \rightarrow t_0$ . We denote the integer part function by  $\llbracket \cdot \rrbracket$ .

### 2. Norm estimates for $\mathcal{P}_\tau$ .

In this section we prove Theorem 1 (i)–(iii). For the readers' convenience, we recall the statements.

**THEOREM 1.** — *Suppose that  $1 \leq p, q \leq \infty$  and  $T \in \mathbb{R}^+$ . Then the following hold:*

(i) *if  $\tau \in \overline{\mathbf{H}} \setminus \{0\}$  and  $\mathcal{P}_\tau$  is  $L^p$ – $L^q$ -bounded, then  $p \leq 2 \leq q$*

(ii) *if  $\tau \in \overline{\mathbf{H}}$  and  $p < 2 = q$  or  $p = 2 < q$ , then*

$$\|\mathcal{P}_\tau\|_{p;q} = \|\mathcal{P}_{\operatorname{Re} \tau}\|_{p;q} \sim \begin{cases} (\operatorname{Re} \tau)^{-n(1/p-1/q)} & \text{if } \operatorname{Re} \tau \in (0, 1] \\ (\operatorname{Re} \tau)^{-\nu/2} & \text{if } \operatorname{Re} \tau \in [1, \infty) \end{cases}$$

(iii) *if  $\tau \in \mathbf{H}$  and  $1 \leq p < 2 < q \leq \infty$ , then*

$$\|\mathcal{P}_\tau\|_{p;q} \leq C \begin{cases} (\operatorname{Re} \tau)^{-n(1/p-1/q)} & \text{if } \operatorname{Re} \tau \in (0, 1] \\ (\operatorname{Re} \tau)^{-\nu} & \text{if } \operatorname{Re} \tau \in [1, \infty), \end{cases}$$

and further

$$\|\mathcal{P}_\tau\|_{p;q} \sim (\operatorname{Re} \tau)^{-\nu} \quad \forall \tau \in \mathbf{H}_T.$$

*Remark.* — To prove (ii) and (iii) we use estimates for  $\|\mathcal{P}_t\|_{p;q}$  for real  $t$ , namely,

$$\|\mathcal{P}_t\|_{p;q} \sim t^{-n(1/p-1/q)} \quad \forall t \in (0, 1),$$

and

$$\|\mathcal{P}_t\|_{p;q} \sim \begin{cases} t^{-\nu} & \text{if } p < 2 < q \\ t^{-\nu/2} & \text{if either } p = 2 < q \text{ or } p < 2 = q \end{cases} \quad \forall t \in (1, \infty).$$

These are corrected versions of estimates in [CGM2]. In [CGM2], we assumed implicitly that  $\theta < 1$  in the proofs of the results (if not in the statements). The correct statement of [CGM2], Lemma 3, for the case where  $\theta = 1$  is that

$$\|p_{t,1}^\sigma\|_2 \sim t^{-\nu/2} \quad \forall t \in [1, \infty)$$

and

$$\|p_{t,1}^\sigma\|_\infty \sim t^{-\nu} \quad \forall t \in [1, \infty).$$

The proof of these estimates is easier than the proof of the corresponding estimates for  $p_{t,\theta}^\sigma$  when  $0 \leq \theta < 1$  given in [CGM2], Lemma 3. When these corrected estimates are used, the estimates for  $\|\mathcal{P}_t\|_{p;q}$  given above follow by the argument used to prove [CGM2], Theorem 1.

*Proof.* — We first prove (i). The condition that  $p \leq q$  is a generalization of [Hö1], Theorem 1.1. Denote by  $h_1$  the fundamental solution of the heat equation at time 1. If  $q < 2$  and  $\mathcal{P}_\tau$  is  $L^p-L^q$ -bounded, then  $(\mathcal{P}_\tau h_1)^\sim$  must be analytic in  $\mathbf{T}_{2/q-1}$ . This implies that  $q \geq 2$ . The condition that  $p \leq 2$  follows by duality.

To prove (ii), we first consider the case where  $p < 2 = q$ . By spectral theory,  $\mathcal{P}_{i\text{Im}\tau}$  is an isometry on  $L^2(X)$ , so  $\|\mathcal{P}_\tau f\|_2 = \|\mathcal{P}_{\text{Re}\tau} f\|_2$  for all  $f$  in  $L^p(X)$ . Thus,

$$\|\mathcal{P}_\tau\|_{p;2} = \|\mathcal{P}_{\text{Re}\tau}\|_{p;2} \sim \begin{cases} (\text{Re}\tau)^{-n(1/p-1/q)} & \text{if } \text{Re}\tau \in (0, 1] \\ (\text{Re}\tau)^{-\nu/2} & \text{if } \text{Re}\tau \in [1, \infty) \end{cases}$$

as required. We argue similarly in the case where  $p = 2 < q$ . We now prove the upper estimate for part (iii). Clearly,

$$\mathcal{P}_\tau = \mathcal{P}_{\text{Re}\tau/2} \mathcal{P}_{i\text{Im}\tau} \mathcal{P}_{\text{Re}\tau/2}.$$

Then, if  $\text{Re}\tau \geq 0$ ,

$$\begin{aligned} \|\mathcal{P}_\tau\|_{p;q} &\leq \|\mathcal{P}_{\text{Re}\tau/2}\|_{p;2} \|\mathcal{P}_{i\text{Im}\tau}\|_2 \|\mathcal{P}_{\text{Re}\tau/2}\|_{2;q} \\ &= \|\mathcal{P}_{\text{Re}\tau/2}\|_{p;2} \|\mathcal{P}_{\text{Re}\tau/2}\|_{2;q} \\ &\sim \|\mathcal{P}_{\text{Re}\tau}\|_{p;q}. \end{aligned}$$

Finally, we prove the lower estimate for (iii). As a preliminary, note that if  $m$  is in  $\mathbb{R}^+$  and  $|k(r)| \leq C r^m$  for all  $r$  in  $\mathbb{R}^+$ , then

$$\begin{aligned} \left| \int_0^\infty e^{-\tau r} k(r) \, dr \right| &\leq C \int_0^\infty e^{-\text{Re}\tau r} r^m \, dr \\ &= C \Gamma(m+1) (\text{Re}\tau)^{-m-1} \quad \forall \tau \in \mathbf{H}. \end{aligned}$$

Now observe that

$$\|\mathcal{H}_1\|_{1;p} \|\mathcal{P}_\tau\|_{p;q} \|\mathcal{H}_1\|_{q;\infty} \geq \|h_2 * p_\tau\|_\infty \geq |h_2 * p_\tau(e)|.$$

By passing to polar co-ordinates, we see that

$$|h_2 * p_\tau(e)| = \left| \int_{\mathfrak{a}^*} \tilde{h}_2(\lambda) \tilde{p}_\tau(\lambda) \, d\mu(\lambda) \right| = e^b \left| \int_0^\infty e^{-\tau r - 2r^2} \kappa(r) \, dr \right|.$$



For an appropriate constant  $c$ ,  $\kappa(r) \asymp cr^{\nu-1}$  as  $r \rightarrow 0$ , and so

$$e^{-2r^2} \kappa(r) = cr^{\nu-1} + [e^{-2r^2} \kappa(r) - cr^{\nu-1}],$$

where  $|e^{-2r^2} \kappa(r) - cr^{\nu-1}| \leq Cr^\nu$  for all  $r$  in  $\mathbb{R}^+$ . Thus

$$\left| \int_0^\infty e^{-\tau r - 2r^2} \kappa(r) dr - c \int_0^\infty e^{-\tau r} r^{\nu-1} dr \right| \leq C\Gamma(\nu + 1) (\operatorname{Re} \tau)^{-\nu-1},$$

and so

$$\int_0^\infty e^{-\tau r - 2r^2} \kappa(r) dr \asymp c\Gamma(\nu) \tau^{-\nu}$$

as  $\tau$  tends to  $\infty$  in  $\mathbf{H}_T$ . Combining these estimates, it follows that  $\|\mathcal{P}_\tau\|_{p,q} \geq C(\operatorname{Re} \tau)^{-\nu}$  for all  $\tau$  in  $\mathbf{H}_T$  such that  $|\tau|$  is large enough; it follows that this estimate holds for all  $\tau$  in  $\mathbf{H}_T$  by a compactness argument.  $\square$

### 3. Analysis of $\mathcal{P}_\tau$ when $\tau$ is small.

In this section, we study  $\mathcal{P}_\tau$  when  $\tau$  is small. Our approach is like that of [CGT], but we use a more sophisticated device than finite propagation speed. We prove the last parts of Theorem 1 in Propositions 3.6 and 3.7. We begin with a little real analysis.

For  $\tau$  in  $\mathbf{H}$ , we define the function  $p_\tau^{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{C}$  by the formula

$$p_\tau^{\mathbb{R}}(v) = \frac{\tau}{\pi(\tau^2 + v^2)}.$$

Clearly

$$\widehat{p_\tau^{\mathbb{R}}}(v) = \exp(-\tau |v|).$$

By spherical Fourier analysis

$$\tilde{p}_\tau(\lambda) = \int_{\mathbb{R}} p_\tau^{\mathbb{R}}(s) \cos(s|\lambda|) ds = 2 \int_0^\infty p_\tau^{\mathbb{R}}(s) \cos(s|\lambda|) ds \quad \forall \lambda \in \mathfrak{a}^*.$$

Thus, by spectral theory,

$$\mathcal{P}_\tau = 2 \int_0^\infty p_\tau^{\mathbb{R}}(s) \cos(s\sqrt{\mathcal{L}_1}) ds.$$

We will expand  $\cos(s|\lambda|)$  using the Hadamard parametrix, and then integrate term by term. In order to control the errors, we have to restrict the range of integration, which involves decomposing  $p_\tau^{\mathbb{R}}$  into a local part, where it may be large, and a part “at infinity”, where it is uniformly small. We will need some results about  $p_\tau^{\mathbb{R}}$ , which we present in Lemma 3.1, and some

results about the distributions which appear in the Hadamard parametrix, which we present in Lemma 3.2. Then we return to the symmetric space  $X$ .

LEMMA 3.1. — Suppose that  $\gamma$  is in  $\mathbb{R}^+$  and  $n$  is in  $\mathbb{N}$ , such that  $\gamma = 1$  and  $n \geq 2$  or  $\gamma > 1$  and  $n > 2\gamma$ , and, for  $T$  in  $\mathbb{R}^+$  and  $\tau$  in  $\mathbf{D}_T$ , define  $F(n, \gamma, \tau, T)$  by the formula

$$F(n, \gamma, \tau, T) = \begin{cases} \log \frac{T}{\operatorname{Re} \tau} + 1 & \text{if } \gamma = 1 \text{ and } n = 2 \\ \log \frac{|\tau|}{\operatorname{Re} \tau} + \left(\frac{T}{|\tau|}\right)^{n-2} & \text{if } \gamma = 1 \text{ and } n > 2 \\ \left(\frac{|\tau|}{\operatorname{Re} \tau}\right)^{\gamma-1} & \text{if } \gamma > 1 \text{ and } n < 2\gamma. \end{cases}$$

Then

$$C_1 F(n, \gamma, \tau, T) \leq |\tau|^{2\gamma-n} \int_0^{3T} |\tau^2 + r^2|^{-\gamma} r^{n-1} dr \leq C_2 F(n, \gamma, \tau, T),$$

uniformly for  $T$  in  $\mathbb{R}^+$  and  $\tau$  in  $\mathbf{D}_T$ , with bounds  $C_1$  and  $C_2$  which depend on  $n$  and  $\gamma$ .

Proof. — We write  $\tau = |\tau| e^{i\phi}$ . The change of variables  $r = |\tau| s$  shows that

$$\int_0^{3T} |\tau^2 + r^2|^{-\gamma} r^{n-1} dr = |\tau|^{n-2\gamma} \int_0^{3T/|\tau|} |s^2 + e^{2i\phi}|^{-\gamma} s^{n-1} ds.$$

Observe that

$$\begin{aligned} s^2 + 1 \geq |s^2 + e^{2i\phi}| &= (s^4 + 2s^2 \cos(2\phi) + 1)^{1/2} \\ &= ((s^2 - 1)^2 + 4s^2 \cos^2 \phi)^{1/2} \geq |s^2 - 1|. \end{aligned}$$

Therefore, if  $0 \leq s \leq 1/2$ , then  $3/4 \leq |s^2 + e^{2i\phi}| \leq 5/4$ , so that

$$\int_0^{1/2} |s^2 + e^{2i\phi}|^{-\gamma} s^{n-1} ds \sim 1.$$

Similarly, if  $s \geq 3/2$ , then  $5s^2/9 \leq |s^2 + e^{2i\phi}| \leq 13s^2/9$ , so that

$$\begin{aligned} \int_{3/2}^{3T/|\tau|} |s^2 + e^{2i\phi}|^{-\gamma} s^{n-1} ds &\sim \int_{3/2}^{3T/|\tau|} s^{n-2\gamma-1} ds \\ &\sim \begin{cases} 1 & \text{if } n < 2\gamma \\ 1 + \log \frac{T}{|\tau|} & \text{if } n = 2\gamma \\ \left(\frac{T}{|\tau|}\right)^{n-2\gamma} & \text{if } n > 2\gamma. \end{cases} \end{aligned}$$

Finally,

$$\begin{aligned} \int_{1/2}^{3/2} |s^2 + e^{2i\phi}|^{-\gamma} s^{n-1} ds &= \int_{1/2}^{3/2} |(s^2 - 1)^2 + 4s^2 \cos^2 \phi|^{-\gamma/2} s^{n-1} ds \\ &= 2^{-\gamma} \int_{1/2}^{3/2} \left| \left( \frac{s^2 - 1}{2s} \right)^2 + \cos^2 \phi \right|^{-\gamma/2} s^{n-\gamma-1} ds \\ &\sim \int_{1/2}^{3/2} \left| \left( \frac{s^2 - 1}{2s} \right)^2 + \cos^2 \phi \right|^{-\gamma/2} \frac{s^2 + 1}{2s^2} ds \\ &= \int_{-3/4}^{5/12} |t^2 + \cos^2 \phi|^{-\gamma/2} dt \\ &= \cos^{1-\gamma} \phi \int_{-3/4 \cos \phi}^{5/12 \cos \phi} |u^2 + 1|^{-\gamma/2} du. \end{aligned}$$

It is easy to check that

$$\int_{-3/4 \cos \phi}^{5/12 \cos \phi} |u^2 + 1|^{-\gamma/2} du \sim \begin{cases} \cos^{\gamma-1} \phi & \text{if } \gamma < 1 \\ 1 + \log \frac{1}{\cos \phi} & \text{if } \gamma = 1 \\ 1 & \text{if } \gamma > 1, \end{cases}$$

and the above estimates combine to give the required result. □

We define the analytic family of locally integrable functions  $\{\chi_+^z : \text{Re } z > 0\}$  on  $\mathbb{R}$  by the rule

$$\chi_+^z(s) = \begin{cases} \Gamma(z)^{-1} s^{z-1} & \text{if } s > 0 \\ 0 & \text{if } s \leq 0. \end{cases}$$

This family admits an analytic continuation as distributions to the whole complex plane.

Given a bounded continuous function  $f : \mathbb{R}^+ \rightarrow \mathbb{C}$  such that  $|f(v)| = O(v^{-\gamma})$ , where  $\gamma$  is in  $(1/2, \infty)$ , we may define the function  $\check{\chi}_+^{1/2} \star f : \mathbb{R}^+ \rightarrow \mathbb{C}$  by the formula

$$\check{\chi}_+^{1/2} \star f(u) = \frac{1}{\Gamma(1/2)} \int_u^\infty (v - u)^{-1/2} f(v) dv.$$

Later, we will also use  $\star$  to denote convolution on  $\mathbb{R}$ ; it will be clear from the context which use of the symbol  $\star$  is intended.

LEMMA 3.2. — *Suppose that the function  $f$  and its derivative  $f'$  are both continuous, and that  $|f(v)| = O(v^{-\gamma})$  and  $|f'(v)| = O(v^{-\gamma})$ , where  $\gamma$  is in  $(1/2, \infty)$ . Then*

$$(\check{\chi}_+^{1/2} \star f)' = \check{\chi}_+^{1/2} \star f'.$$

If  $\tau$  is in  $\mathbf{D}_T$  and  $f = p_\tau^\mathbb{R}$ , then

$$(\tilde{\chi}_+^{1/2} \star f)(u) = \frac{\tau}{\sqrt{\pi(\tau^2 + u)}} \quad \forall u \in \mathbb{R}^+.$$

*Proof.* — The first result is a standard convolution manipulation:

$$\begin{aligned} (\tilde{\chi}_+^{1/2} \star f)'(u) &= \lim_{h \rightarrow 0} \frac{1}{h} ((\tilde{\chi}_+^{1/2} \star f)(u + h) - (\tilde{\chi}_+^{1/2} \star f)(u)) \\ &= \lim_{h \rightarrow 0} \frac{1}{\Gamma(1/2)} \int_u^\infty (v - u)^{-1/2} \frac{f(v + h) - f(v)}{h} dv \\ &= (\tilde{\chi}_+^{1/2} \star f')(u). \end{aligned}$$

Now suppose that  $\tau$  is in  $\mathbf{D}_T$  and  $f = p_\tau^\mathbb{R}$ . The change of variables  $w = (\tau^2 + u)^{-1}(v - u)$  shows that

$$\begin{aligned} (\tilde{\chi}_+^{1/2} \star f)(u) &= \frac{1}{\pi^{3/2}} \int_u^\infty (v - u)^{-1/2} \frac{\tau}{\tau^2 + v} dv \\ &= \frac{1}{\pi^{3/2}} \frac{\tau}{\sqrt{\tau^2 + u}} \int_0^\infty \frac{1}{w^{1/2}(1 + w)} dw. \end{aligned}$$

The last integral is equal to  $\pi$ , and the desired formula is proved. □

Now we work in the symmetric space  $X$ . We denote by  $\mathfrak{b}_r$  the ball in  $\mathfrak{p}$  with centre 0 and radius  $r$  and by  $S_\mathfrak{p}$  the unit sphere in  $\mathfrak{p}$ , equipped with surface measure  $\sigma$ ; the exponential map  $\exp : \mathfrak{p} \rightarrow X$  gives geodesic co-ordinates in  $X$  centred at  $o$ , and  $\exp \mathfrak{b}_r$  is the ball of radius  $r$  centred at  $o$  in  $X$ . Let  $J$  denote the Jacobian of the exponential map from  $\mathfrak{p}$  equipped with Lebesgue measure to  $X$  equipped with Riemannian measure.

Suppose that  $U$  is in  $C^\infty(\mathfrak{p})$  and  $s$  is in  $\mathbb{R}^+$ . Given  $z$  in  $\mathbb{C}$ , we define  $R_s(U, z)$  to be the distribution on  $X$  which acts on smooth compactly supported functions  $\psi$  by the formula

$$\begin{aligned} \langle R_s(U, z), \psi \rangle &= \int_{\mathfrak{p}} \chi_+^z(s^2 - |Y|^2) \psi \circ \exp(Y) U(Y) J(Y) dY \\ &= \int_0^\infty \chi_+^z(s^2 - r^2) r^{n-1} \int_{S_\mathfrak{p}} \psi \circ \exp(rY) U(rY) J(rY) d\sigma(Y) dr. \end{aligned}$$

This formula is defined as a convergent integral when  $\operatorname{Re} z > 0$ , and by analytic continuation (using the second expression) for other values of  $z$ . When  $\operatorname{Re} z > 0$ , then  $R_s(U, z)$  is given by an integrable function on  $X$ , and

$$R_s(U, z)(\exp Y) = U(Y) \chi_+^z(s^2 - |Y|^2) \quad \forall Y \in \mathfrak{p}.$$

For  $s$  in  $\mathbb{R}^+$ , let  $\Phi_s$  denote the  $K$ -invariant distribution on  $X$ , whose spherical Fourier transform (when viewed as a  $K$ -bi-invariant distribution on  $G$ ) is given by

$$\tilde{\Phi}_s(\lambda) = \cos(s|\lambda|) \quad \forall \lambda \in \mathfrak{a}^*.$$

The Hadamard parametrix construction, applied to the hyperbolic operator  $\partial_t^2 + \mathcal{L}_1$ , gives the following formula for  $\Phi_s$ :

$$\Phi_s = \sum_{j=0}^{\infty} s R_s \left( U_j, \frac{2j - n + 1}{2} \right),$$

where  $U_j$  is a smooth  $K$ -invariant function on  $X$  for all nonnegative integers  $j$  (see [B], Proposition 27 or, for more details about the method, [Hö2], 17.4–17.5). Further, the function  $U_0$  is positive and bounded above and away from zero on all compact sets.

Write

$$\Phi_s = \sum_{j=0}^{n+1} s R_s \left( U_j, \frac{2j - n + 1}{2} \right) + \varrho_{n+1,s}.$$

If  $S$  is in  $\mathbb{R}^+$ , then, by [Hö2], 17.5.4 and the considerations thereabouts, there exists a constant  $C_S$  such that

$$\|\varrho_{n+1,s}\|_{\infty} \leq C_S s^{n+2} \quad \forall s \in [0, S].$$

If  $\lfloor n/2 \rfloor + 1 \leq j \leq n + 1$ , then  $R_s(U_j, \frac{2j-n+1}{2})$  is in  $L^{\infty}(X)$  and

$$\begin{aligned} \left\| R_s \left( U_j, \frac{2j - n + 1}{2} \right) \right\|_{\infty} &= \left\| U_j \chi_{+\frac{2j-n+1}{2}}(s^2 - |\cdot|^2) \right\|_{\infty} \\ &\leq \sup\{|U_j(Y)| : |Y| \leq s\} \frac{s^{2j-n-1}}{\Gamma(\frac{2j-n+1}{2})}. \end{aligned}$$

Thus we may write

$$\Phi_s = \sum_{j=0}^{\lfloor n/2 \rfloor} s R_s \left( U_j, \frac{2j - n + 1}{2} \right) + \varrho_{\lfloor n/2 \rfloor, s},$$

where

$$(1) \quad \|\varrho_{\lfloor n/2 \rfloor, s}\|_{\infty} \leq C_S \quad \forall s \in [0, S].$$

We decompose  $p_{\tau}$ . Let  $\omega : \mathbb{R} \rightarrow [0, 1]$  be a smooth even function, supported in  $[-3T, 3T]$ , which is equal to 1 in  $[-2T, 2T]$ ; in order to establish estimates which are uniform in  $T$ , we also assume that the derivatives of  $\omega$  satisfy  $\|\omega^{(j)}\|_{\infty} \leq C_N T^{-j}$  when  $j = 0, 1, \dots, N$ . Let  $a_{\tau}$  and

$b_\tau$  denote the  $K$ -bi-invariant functions whose spherical Fourier transforms are defined by

$$A_\tau(\lambda) = \int_{\mathbb{R}} \omega(s) p_\tau^{\mathbb{R}}(s) \cos(s|\lambda|) ds \quad \forall \lambda \in \mathfrak{a}^*$$

and

$$B_\tau(\lambda) = \int_{\mathbb{R}} (1 - \omega(s)) p_\tau^{\mathbb{R}}(s) \cos(s|\lambda|) ds \quad \forall \lambda \in \mathfrak{a}^*.$$

Notice that  $p_\tau = a_\tau + b_\tau$  and that  $a_\tau$  is supported in  $\overline{B}(o, 3T)$ , by finite propagation speed. The analysis of  $a_\tau$  is quite difficult and is carried out in Lemma 3.3. This is done by using the asymptotic expansion of the wave propagator near the point  $o$ . The idea is simple but the details are rather involved. The analysis of the mapping properties of the operator  $\mathcal{B}_\tau$  hinges on easy estimates of  $b_\tau$  and  $\tilde{b}_\tau$ ; this is carried out in Proposition 3.4. The analysis of the operator  $\mathcal{P}_\tau$  is carried out in Proposition 3.5 and Proposition 3.6.

We now prove our main result concerning  $a_\tau$ .

LEMMA 3.3. — *Suppose that  $T$  is in  $\mathbb{R}^+$ . The function  $a_\tau$  is smooth,  $K$ -bi-invariant and supported in  $\overline{B}(o, 3T)$ , for all  $\tau$  in  $\mathbf{D}_T$ . Further,*

(2)

$$a_\tau(\exp Y) = \tau \sum_{j=0}^{\llbracket n/2 \rrbracket} \frac{\Gamma(n/2 - j + \epsilon)}{\pi} U_j(Y) (\tau^2 + |Y|^2)^{j-(n+1)/2} + E(Y, \tau)$$

$$\forall Y \in \mathfrak{b}_{3T},$$

where  $\epsilon = 1/2$  if  $n$  is even and 1 if  $n$  is odd, and the error term  $E$  satisfies the inequality

$$(3) \quad \|E(\cdot, \tau)\|_\infty \leq C_T \left( |\tau|^{n-2} \log \frac{|\tau|}{\operatorname{Re} \tau} + 1 \right),$$

uniformly for  $\tau$  in  $\mathbf{D}_T$ . Finally, for all  $q$  in  $[1, \infty]$ ,

$$\|a_\tau\|_q \sim |\tau|^{(n-1)(1/q-1/2)} (\operatorname{Re} \tau)^{1/q-(n+1)/2} \quad \forall \tau \in \mathbf{D}_T.$$

*Proof.* — First we show that  $a_\tau$  is smooth. The spherical Fourier multiplier  $A_\tau$  corresponding to  $a_\tau$  is given by the formula  $A_\tau(\lambda) = \alpha_\tau(|\lambda|)$ , where

$$\alpha_\tau(t) = \int_{\mathbb{R}} \omega(s) p_\tau^{\mathbb{R}}(s) \cos(st) ds;$$

then  $\alpha_\tau$  is the Fourier transform of a  $C_c^\infty(\mathbb{R})$ -function, hence lies in the Schwartz space  $\mathcal{S}(\mathbb{R})$ , and is even. It follows that  $A_\tau$  lies in the Schwartz

space  $\mathcal{S}(\mathfrak{a})$ , whence  $a_\tau$  lies in Harish-Chandra’s Schwartz space  $\mathcal{S}_2(G)$ , so is smooth.

Observe that

$$a_\tau(\exp Y) = 2 \sum_{j=0}^{\lfloor n/2 \rfloor} U_j(Y) \int_0^\infty \omega(s) p_\tau^{\mathbb{R}}(s) \chi_+^{(2j-n+1)/2}(s^2 - |Y|^2) s \, ds + 2 \int_0^\infty \omega(s) p_\tau^{\mathbb{R}}(s) \varrho_{\lfloor n/2 \rfloor, s}(\exp Y) \, ds \quad \forall Y \in \mathfrak{p},$$

where  $\chi_+^{(2j-n+1)/2}$  is interpreted distributionally by analytic continuation. Define  $E_1(Y, \tau)$  by the formula

$$E_1(Y, \tau) = 2 \int_0^\infty \omega(s) p_\tau^{\mathbb{R}}(s) \varrho_{n, s}(\exp Y) \, ds.$$

Lemma 3.1 and formula (1) imply that

$$\|E_1(\cdot, \tau)\|_\infty \leq C_T |\tau|^{n-2} \left[ \log \frac{|\tau|}{\operatorname{Re} \tau} + \left( \frac{T}{|\tau|} \right)^{n-2} \right] \quad \forall \tau \in \mathbf{D}_T$$

if  $n > 2$ ; if  $n = 2$  then the expression in square parentheses on the right hand side of the inequality must be replaced by  $\log(T/\operatorname{Re} \tau) + 1$ .

We now consider

$$2 \int_0^\infty \omega(s) p_\tau^{\mathbb{R}}(s) \chi_+^{(2j-n+1)/2}(s^2 - |Y|^2) s \, ds.$$

Define the functions  $q_\tau^{\mathbb{R}} : \mathbb{R}^+ \rightarrow \mathbb{C}$  and  $r_\tau^R : \mathbb{R}^+ \rightarrow \mathbb{C}$  by the formulae

$$q_\tau^{\mathbb{R}}(v) = \omega(\sqrt{v}) p_\tau^{\mathbb{R}}(\sqrt{v}) \quad \text{and} \quad r_\tau^R(v) = (1 - \omega(\sqrt{v})) p_\tau^{\mathbb{R}}(\sqrt{v}).$$

Suppose that  $h$  is in  $\mathbb{N}$ . To simplify the formulae, we often write  $u$  in place of  $|Y|^2$  in the rest of the proof. Differentiation with respect to a real variable is denoted by  $D$ , or by  $D_v$  to indicate that the variable is  $v$ . If  $\operatorname{Re} z > 0$ , then

$$(4) \quad \begin{aligned} 2 \int_0^\infty \omega(s) p_\tau^{\mathbb{R}}(s) \chi_+^z(s^2 - u) s \, ds &= \int_0^\infty q_\tau^{\mathbb{R}}(v) \chi_+^z(v - u) \, dv \\ &= (-1)^h D_u^h \int_0^\infty q_\tau^{\mathbb{R}}(v) \chi_+^{z+h}(v - u) \, dv, \end{aligned}$$

by definition of  $\chi_+^z$ .

Suppose that  $n$  is even. By analytic continuation, the definition of  $q_\tau^{\mathbb{R}}$  and  $r_\tau^R$ , and Lemma 3.2, it follows that

$$2 \int_0^\infty \omega(s) p_\tau^{\mathbb{R}}(s) \chi_+^{1/2-h}(s^2 - u) s \, ds$$

$$\begin{aligned}
 &= (-1)^h D_u^h \int_0^\infty q_\tau^{\mathbb{R}}(v) \chi_+^{1/2}(v-u) \, dv \\
 &= (-1)^h D_u^h \int_0^\infty (p_\tau^{\mathbb{R}}(\sqrt{v}) - r_\tau^{\mathbb{R}}(v)) \chi_+^{1/2}(v-u) \, dv \\
 &= (-1)^h D_u^h \frac{\tau}{\sqrt{\pi(\tau^2+u)}} - (-1)^h D^h (\tilde{\chi}_+^{1/2} \star r_\tau^{\mathbb{R}})(u) \\
 &= \tau \frac{\Gamma(h+1/2)}{\pi} (\tau^2+u)^{-h-1/2} - (-1)^h (\tilde{\chi}_+^{1/2} \star D^h r_\tau^{\mathbb{R}})(u).
 \end{aligned}$$

We take  $h$  equal to  $(n-2j)/2$  in the preceding formula, multiply by  $U_j$ , and sum over  $j$ , to conclude that, for even  $n$ ,

$$\begin{aligned}
 &2 \sum_{j=0}^{n/2} U_j(Y) \int_0^\infty \omega(s) p_\tau^{\mathbb{R}}(s) \chi_+^{(2j-n+1)/2} (s^2 - |Y|^2) s \, ds \\
 &= \tau \left( \sum_{j=0}^{n/2} U_j(Y) \frac{\Gamma((n-2j+1)/2)}{\pi} (\tau^2 + |Y|^2)^{j-(n+1)/2} \right) + E_2(Y, \tau),
 \end{aligned}$$

where

$$E_2(Y, \tau) = - \sum_{j=0}^{n/2} U_j(Y) (-1)^{(n-2j)/2} (\tilde{\chi}_+^{1/2} \star D^{(n-2j)/2} r_\tau^{\mathbb{R}})(|Y|^2).$$

Define  $E(Y, \tau)$  to be  $E_1(Y, \tau) + E_2(Y, \tau)$ . Then, in order to prove formulae (2) and (3) for the case where  $n$  is even, it remains only to estimate  $E_2(\cdot, \tau)$ .

We claim that  $\tilde{\chi}_+^{1/2} \star r_\tau^{\mathbb{R}}$  is smooth and that  $\tilde{\chi}_+^{1/2} \star D^h r_\tau^{\mathbb{R}}$  is bounded in  $\mathbb{R}^+$  for all  $h$  in  $\mathbb{N}$ . To see this, note first that, by Leibniz's rule,  $D^h r_\tau^{\mathbb{R}}$  is a sum of terms involving derivatives of  $p_\tau^{\mathbb{R}}(\sqrt{\cdot})$  and  $1 - \omega(\sqrt{\cdot})$ , and recall that  $1 - \omega(s)$  vanishes unless  $|s| > 2T$  and  $\|1 - \omega\|_\infty = 1$ , while  $D^j \omega(s)$  vanishes unless  $2T < |s| < 3T$  and  $\|D^j \omega\|_\infty \leq C_N T^{-j}$  if  $j \leq N$ . Note also that, if  $|v| \geq 4T^2$ , then

$$\begin{aligned}
 |D_v^j p_\tau^{\mathbb{R}}(\sqrt{v})| &= \left| D_v^j \frac{\tau}{\pi(\tau^2+v)} \right| \\
 &= \left| \frac{\tau j!}{\pi(\tau^2+v)^{j+1}} \right| \\
 &\leq \frac{5^{j+1} T j!}{3^{j+1} \pi (T^2+v)^{(j+1)}}.
 \end{aligned}$$



It therefore follows that, if  $h \leq N$ , then

$$\begin{aligned} |D^h r_\tau^R(v)| &\leq C_N \sum_{j=0}^h |D^j(1 - \omega(\sqrt{v}))| \left| D^{h-j} p_\tau^R(\sqrt{v}) \right| \\ &\leq C_N \sum_{j=0}^h T^{1-2j} \chi_{[4T^2, \infty)}(v) (T^2 + v)^{-(h-j+1)} \\ &\leq C_N T^{1-2h} \chi_{[4T^2, \infty)}(v) (T^2 + v)^{-1}. \end{aligned}$$

We deduce that

$$\begin{aligned} \left| \check{\chi}_+^{1/2} \star D^h r_\tau^R(u) \right| &\leq C_N T^{1-2h} \int_u^\infty (v - u)^{-1/2} \chi_{[4T^2, \infty)}(v) (T^2 + v)^{-1} dv \\ &\leq C_N T^{1-2h} \int_0^\infty v^{-1/2} \chi_{[4T^2, \infty)}(v) (T^2 + v)^{-1} dv \\ &\leq C_N T^{1-2h} \int_{4T^2}^\infty v^{-3/2} dv = C_N T^{-2h}. \end{aligned}$$

This proves our claim, from which it follows immediately that

$$\|E_2(\cdot, \tau)\|_\infty \leq C_T,$$

as required to complete our proof of formulae (2) and (3) in the case where  $n$  is even.

If  $n$  is odd, the argument to prove (2) and (3) is similar but easier, and we just outline it. By analytic continuation of formula (4), if  $h$  is in  $\mathbb{N}$ , then

$$\begin{aligned} &2 \int_0^\infty \omega(s) p_\tau^R(s) \chi_+^{-h}(s^2 - u) s ds \\ &= (-1)^h D^h \int_0^\infty q_\tau^R(v) \chi_+^0(v - u) dv \\ &= (-1)^h D^h q_\tau^R(u) \\ &= (-1)^h D^h p_\tau^R(\sqrt{u}) - (-1)^h D^h r_\tau^R(u) \\ &= (-1)^h D_u^h \frac{\tau}{\pi(\tau^2 + u)} - (-1)^h D^h r_\tau^R(u) \\ &= \tau \frac{\Gamma(h + 1)}{\pi} (\tau^2 + u)^{-h-1} - (-1)^h D^h r_\tau^R(u). \end{aligned}$$

We take  $h$  equal to  $(n - 2j - 1)/2$  in the preceding formula, multiply by  $U_j$ , and sum over  $j$ , to conclude that, for odd  $n$ ,

$$\begin{aligned} &2 \sum_{j=0}^{\lfloor n/2 \rfloor} U_j(Y) \int_0^\infty \omega(s) p_\tau^R(s) \chi_+^{(2j-n+1)/2}(s^2 - |Y|^2) s ds \\ &= \tau \left( \sum_{j=0}^{\lfloor n/2 \rfloor} U_j(Y) \frac{\Gamma(n - 2j + 1)/2}{\pi} (\tau^2 + |Y|^2)^{j-(n+1)/2} \right) + E_2(Y, \tau), \end{aligned}$$

where

$$E_2(Y, \tau) = - \sum_{j=0}^{n/2} U_j(Y) (-1)^{(n-2j+1)/2} D^{(n-2j)/2} r_\tau^R (|Y|^2).$$

Define  $E(Y, \tau)$  to be  $E_1(Y, \tau) + E_2(Y, \tau)$ . Then, in order to prove formulae (2) and (3) for the case where  $n$  is odd, it remains only to estimate  $E_2(\cdot, \tau)$ . This is a simpler version of the estimate for  $E_2(\cdot, \tau)$  in the case where  $n$  is even.

We now prove the rest of the lemma. It is easy to check that the term in which  $j = 0$  dominates all the other terms in the expression for  $a_\tau$ . Further,

$$\inf_{|Y| \leq 3T} |\tau^2 + |Y|^2| \sim |\tau| \operatorname{Re} \tau \quad \forall \tau \in \mathbf{D}_T.$$

Therefore

$$\|a_\tau\|_\infty \sim |\tau|^{-(n-1)/2} (\operatorname{Re} \tau)^{-(n+1)/2} \quad \forall \tau \in \mathbf{D}_T.$$

If  $1 \leq q < \infty$ , we integrate in polar co-ordinates and use (2) to obtain

$$\begin{aligned} \|a_\tau\|_q &= \left[ \int_G |a_\tau(x)|^q dx \right]^{1/q} \\ &\sim |\tau| \left[ \int_0^{3T} |p_\tau^{\mathbb{R}}(r)|^{(n+1)q/2} r^{n-1} dr \right]^{1/q} \\ &\sim |\tau|^{(n-1)(1/q-1/2)} (\operatorname{Re} \tau)^{1/q-(n+1)/2} \quad \forall \tau \in \mathbf{D}_T, \end{aligned}$$

as required. □

To complete the analysis of  $\mathcal{P}_\tau$  for  $\tau$  small, we have to analyse  $\mathcal{B}_\tau$ . We do this by using spectral methods.

**PROPOSITION 3.4.** — *If  $\tau$  is in  $\mathbf{H}$  and  $\mathcal{B}_\tau$  is  $L^p-L^q$ -bounded, then  $1 \leq p \leq 2 \leq q \leq \infty$ . Further, if  $T \in \mathbb{R}^+$  and  $1 \leq p \leq 2 \leq q \leq \infty$ , then there exists a constant  $C_T$  such that*

$$\|\mathcal{B}_\tau\|_{p,q} \leq C_T \quad \forall \tau \in \mathbf{D}_T.$$

*Proof.* — Since  $\mathcal{P}_\tau = \mathcal{A}_\tau + \mathcal{B}_\tau$ , and  $\mathcal{A}_\tau$  is  $L^p-L^q$ -bounded whenever  $1 \leq p \leq q \leq \infty$  while  $\mathcal{P}_\tau$  is not  $L^p-L^q$ -bounded unless  $1 \leq p \leq 2 \leq q \leq \infty$ , it follows that if  $\mathcal{B}_\tau$  is  $L^p-L^q$ -bounded, then  $1 \leq p \leq 2 \leq q \leq \infty$ .

The spherical Fourier multiplier  $B_\tau$  corresponding to  $\mathcal{B}_\tau$  is given by the formula  $B_\tau(\lambda) = \beta_\tau(|\lambda|)$ , where

$$\beta_\tau(t) = \int_{\mathbb{R}} (1 - \omega(s)) p_\tau^{\mathbb{R}}(s) \cos(st) ds.$$

We claim that  $|\beta_\tau(t)| = O((t^2 + 1)^{-N})$  for all  $N$  in  $\mathbb{N}$ , uniformly for  $\tau$  in  $\mathbf{D}_T$ .

Assuming our claim, it is immediate to check that  $B_\tau$  lies in  $L^1(\mathfrak{a}; \mu)$ , the Lebesgue space relative to the Plancherel measure, and in  $L^2(\mathfrak{a}; \mu)$  and  $L^\infty(\mathfrak{a}; \mu)$ , uniformly for  $\tau$  in  $\mathbf{D}_T$ . The inversion formula and the Plancherel formula for the spherical Fourier transformation then imply that  $b_\tau$  lies in  $L^2(G)$  and in  $L^\infty(G)$ , and that convolution with  $b_\tau$  is bounded on  $L^2(G)$ , uniformly for  $\tau$  in  $\mathbf{D}_T$ . From this, by [CGM1], Theorem 2.2, it follows that  $\mathcal{B}_\tau$  is  $L^p$ - $L^q$ -bounded for all  $p$  and  $q$  such that  $1 \leq p \leq 2 \leq q \leq \infty$ , and that all the corresponding operator norms are uniformly bounded for  $\tau$  in  $\mathbf{D}_T$ .

It suffices therefore to prove our claim. By classical Fourier analysis, it suffices to show that  $(D^2 + 1)^N (1 - \omega) p_\tau^{\mathbb{R}}$  is in  $L^1(\mathbb{R})$ , uniformly for  $\tau$  in  $\mathbf{D}_T$ . This is very similar to what we did in estimating  $D^j r_\tau^{\mathbb{R}}$  in the proof of Lemma 3.3, and we omit the details.  $\square$

Shortly, we will consider  $\mathcal{P}_\tau p_\sigma$ . This will lead to terms  $\mathcal{P}_\tau a_\sigma$  and  $\mathcal{P}_\tau b_\sigma$ . It will be important to know about the operator  $\mathcal{P}_\tau b_\sigma$ , and our next lemma is about this.

PROPOSITION 3.5. — *Suppose that  $T$  is in  $\mathbb{R}^+$ . There exists a constant  $C_T$  such that*

$$\|\mathcal{P}_\tau \mathcal{B}_\sigma\|_{p,q} \leq C_T$$

for all  $p$  and  $q$  such that  $1 \leq p \leq 2 \leq q \leq \infty$  and all  $\sigma$  and  $\tau$  in  $\mathbf{D}_T$ .

*Proof.* — The spherical Fourier multiplier  $\exp(-\tau |\cdot|) \beta_\sigma(|\cdot|)$  corresponds to the operator  $\mathcal{P}_\tau \mathcal{B}_\sigma$ . As with the estimate of  $\|\mathcal{B}_\tau\|_{p,q}$  in the previous proposition, it will suffice to show that

$$|\exp(-\tau |t|) \beta_\sigma(|t|)| = O((t^2 + 1)^{-N})$$

for all  $N$  in  $\mathbb{N}$ , uniformly for  $\sigma$  and  $\tau$  in  $\mathbf{D}_T$ . This in turn is an immediate corollary of the estimates  $|\exp(-\tau |t|)| \leq 1$  and  $|\beta_\sigma(t)| = O((t^2 + 1)^{-N})$  for all  $t$  in  $\mathbb{R}$ ; the first of these is trivial and the second is already proved.  $\square$

PROPOSITION 3.6. — *Suppose that  $T$  is in  $\mathbb{R}^+$ . The following hold:*

(i) *if  $1 \leq p \leq q \leq \infty$ , then*

$$\|\mathcal{A}_\tau\|_{p,q} \sim \frac{|\tau|^{(n-1)(|1/q-1/p'|-|1/p-1/q|)/2}}{(\operatorname{Re} \tau)^{(n-1)|1/q-1/p'|+(n+1)|1/p-1/q|/2}} \quad \forall \tau \in \mathbf{D}_T$$

(ii) if  $1 \leq p \leq 2 \leq q \leq \infty$ , then

$$\|\mathcal{P}_\tau\|_{p;q} \sim \frac{|\tau|^{(n-1)(|1/q-1/p'|-|1/p-1/q|)/2}}{(\operatorname{Re} \tau)^{((n-1)|1/q-1/p'|+(n+1)|1/p-1/q|)/2}} \quad \forall \tau \in \mathbf{D}_T.$$

*Proof.* — Observe first that  $\mathcal{A}_\tau = \mathcal{P}_\tau - \mathcal{B}_\tau$ , and that both  $\|\mathcal{P}_\tau\|_2$  and  $\|\mathcal{B}_\tau\|_2$  are uniformly bounded, so that  $\|\mathcal{A}_\tau\|_2$  is uniformly bounded as  $\tau$  varies over  $\mathbf{D}_T$ . Further,  $\|\mathcal{A}_\tau\|_{1;q} = \|a_\tau\|_q$ , so by Lemma 3.3

$$\|\mathcal{A}_\tau\|_{1;q} \sim |\tau|^{(n-1)(1/q-1/2)} (\operatorname{Re} \tau)^{1/q-(n+1)/2} \quad \forall \tau \in \mathbf{D}_T.$$

From this, by interpolation and duality, we deduce that

$$\|\mathcal{A}_\tau\|_{p;q} \leq C \frac{|\tau|^{(n-1)(|1/q-1/p'|-|1/p-1/q|)/2}}{(\operatorname{Re} \tau)^{((n-1)|1/q-1/p'|+(n+1)|1/p-1/q|)/2}} \quad \forall \tau \in \mathbf{D}_T.$$

Since  $\mathcal{P}_\tau = \mathcal{A}_\tau + \mathcal{B}_\tau$ , it follows from this and Proposition 3.4 that, if  $1 \leq p \leq 2 \leq q \leq \infty$ , then

$$\|\mathcal{P}_\tau\|_{p;q} \leq C \frac{|\tau|^{(n-1)(|1/q-1/p'|-|1/p-1/q|)/2}}{(\operatorname{Re} \tau)^{((n-1)|1/q-1/p'|+(n+1)|1/p-1/q|)/2}} \quad \forall \tau \in \mathbf{D}_T.$$

To prove the converse inequalities, we consider first  $\mathcal{P}_\tau a_\sigma$ , where  $\sigma$  and  $\tau$  both lie in  $\mathbf{D}_T$ . By the semigroup property,

$$\mathcal{P}_\tau a_\sigma = \mathcal{P}_\tau p_\sigma - \mathcal{P}_\tau b_\sigma = p_{\tau+\sigma} - \mathcal{P}_\tau b_\sigma,$$

whence

$$\begin{aligned} \|\mathcal{P}_\tau\|_{p;q} &\geq \frac{\|p_{\tau+\sigma} - \mathcal{P}_\tau b_\sigma\|_q}{\|a_\sigma\|_p} \\ &= \frac{\|\mathcal{P}_{\tau+\sigma} - \mathcal{P}_\tau \mathcal{B}_\sigma\|_{1;q}}{\|a_\sigma\|_p} \\ (5) \quad &\geq \frac{\|\mathcal{A}_{\tau+\sigma} + \mathcal{B}_{\tau+\sigma} - \mathcal{P}_\tau \mathcal{B}_\sigma\|_{1;q}}{\|a_\sigma\|_p} \\ &\geq \frac{\|a_{\tau+\sigma}\|_q - C_0}{\|a_\sigma\|_p}, \end{aligned}$$

by Proposition 3.4 and Proposition 3.5. Since  $\|\mathcal{P}_\tau\|_{p;q} = \|\mathcal{P}_\tau\|_{q';p'}$ , there is no loss of generality in assuming that  $1/p + 1/q \geq 1$ . In this case, we take  $\sigma$  to be  $\operatorname{Re} \tau$ . Then  $|\tau| \leq |\sigma + \tau| \leq 2|\tau|$ , and  $\operatorname{Re}(\sigma + \tau) = 2 \operatorname{Re} \tau$ . We estimate  $\|a_{\tau+\sigma}\|_q$  using Lemma 3.3 (but with  $2T$  in place of  $T$ ), and deduce that

$$\|\mathcal{P}_\tau\|_{p;q} \geq \frac{C_1 |\tau|^{(n-1)(1/q-1/2)} (\operatorname{Re} \tau)^{1/q-(n+1)/2} - C_0}{C_2 (\operatorname{Re} \tau)^{(n-1)(1/p-1/2)+1/p-(n+1)/2}}.$$

Now if  $\tau$  is in  $\mathbf{D}_T$  and  $\operatorname{Re} \tau < [2C_0 / (C_1 T^{(n-1)(1/q-1/2)})]^{2q/(2-nq-q)}$ , it follows that

$$C_1 |\tau|^{(n-1)(1/q-1/2)} (\operatorname{Re} \tau)^{1/q-(n+1)/2} \geq C_1 T^{(n-1)(1/q-1/2)} (\operatorname{Re} \tau)^{1/q-(n+1)/2} \geq 2C_0,$$

whence

$$\begin{aligned} \|\mathcal{P}_\tau\|_{p,q} &\geq \frac{C_1 |\tau|^{(n-1)(1/q-1/2)} (\operatorname{Re} \tau)^{1/q-(n+1)/2}}{2C_2 (\operatorname{Re} \tau)^{n/p-n}} \\ &= C \frac{|\tau|^{(n-1)(|1/q-1/p'|-|1/p-1/q|)/2}}{(\operatorname{Re} \tau)^{((n-1)|1/q-1/p'|+(n+1)|1/p-1/q|)/2}}. \end{aligned}$$

This proves the converse inequality for  $\mathcal{P}_\tau$ .

Since  $\mathcal{A}_\tau = \mathcal{P}_\tau - \mathcal{B}_\tau$ , and  $\|\mathcal{B}_\tau\|_{p,q}$  is bounded, it follows that  $\|\mathcal{A}_\tau\|_{p,q}$  satisfies the same inequality, at least when  $1 \leq p \leq 2 \leq q \leq \infty$ . To prove the converse inequality for  $\|\mathcal{A}_\tau\|_{p,q}$  for general  $p$  and  $q$  such that  $1 \leq p \leq q \leq \infty$ , it suffices to observe that, by interpolation,

$$\|\mathcal{A}_\tau\|_{p^*,q^*}^2 \leq \|\mathcal{A}_\tau\|_{p,q} \|\mathcal{A}_\tau\|_{1,\infty},$$

where  $1/p^* = (1+1/p)/2$  and  $1/q^* = 1/2q$ . Since  $1 \leq p^* \leq 2 \leq q^* \leq \infty$ , we have a lower bound for  $\|\mathcal{A}_\tau\|_{p^*,q^*}$ , which, combined with the upper bound for  $\|\mathcal{A}_\tau\|_{1,\infty}$ , yields the required lower bound for  $\|\mathcal{A}_\tau\|_{p,q}$ .  $\square$

The last result of this section proves Theorem 1 (v). Before we state the result, observe that the definitions of the kernels  $a_\tau$  and  $b_\tau$  and the corresponding multipliers  $A_\tau$  and  $B_\tau$  extend to the case where  $\tau$  is in  $i\mathbb{R}$  and  $|\tau| \leq T$ . In the  $a_\tau$  case,  $\omega p_{\tau+\epsilon}^{\mathbb{R}}$  converges distributionally as  $\epsilon$  in  $\mathbb{R}^+$  tends to 0, and  $A_{\tau+\epsilon}$  converges locally uniformly as  $\epsilon$  tends to 0+, while in the  $b_\tau$  case,  $(1-\omega) p_{\tau+\epsilon}^{\mathbb{R}}$  converges uniformly and in  $L^1(\mathbb{R})$  as  $\epsilon$  tends to 0+, and  $B_{\tau+\epsilon}$  converges uniformly. It therefore makes sense to consider the operators  $\mathcal{A}_\tau$  and  $\mathcal{B}_\tau$  in this case.

**PROPOSITION 3.7.** — *Suppose that  $T$  is in  $\mathbb{R}^+$  and that  $\tau$  is in  $i\mathbb{R} \setminus \{0\}$  and  $|\tau| \leq T$ . Then  $\mathcal{B}_\tau$  is  $L^p$ - $L^q$ -bounded if and only if  $1 \leq p \leq 2 \leq q \leq \infty$ , and  $\mathcal{A}_\tau$  is  $L^p$ - $L^q$ -bounded if and only if  $p = 2 = q$ . Finally,  $\mathcal{P}_\tau$  is  $L^p$ - $L^q$ -bounded if and only if  $p = 2 = q$ .*

*Proof.* — Since  $A_\tau + B_\tau = \exp(-\tau|\cdot|)$ , and  $A_\tau$  extends to an entire function, being the spherical Fourier transform of a compactly supported distribution,  $B_\tau$  does not extend holomorphically to any tube  $\mathbf{T}_\epsilon$  for any  $\epsilon$  in  $\mathbb{R}^+$ , so, as argued before, if  $\mathcal{B}_\tau$  is  $L^p$ - $L^q$ -bounded then  $1 \leq p \leq 2 \leq q \leq \infty$ .

Conversely, for such  $p$  and  $q$ ,  $\mathcal{B}_\tau$  is  $L^p-L^q$ -bounded because  $B_\tau(\lambda)$  is  $O((|\lambda|^2 + 1)^{-N})$  in  $\mathfrak{a}^*$ , for all  $N$  in  $\mathbb{N}$ .

Next,  $\exp(-\tau|\cdot|)$  is in  $L^\infty(\mathfrak{a}^*)$ , whence  $\mathcal{P}_\tau$  is  $L^2$ -bounded. To show that  $\mathcal{P}_\tau$  is not  $L^p-L^q$ -bounded for other  $p$  and  $q$ , we may suppose that  $1 \leq p \leq 2 \leq q \leq \infty$ , since otherwise  $\mathcal{P}_\tau$  is unbounded by Theorem 1 (i). Suppose therefore that  $1 \leq p \leq 2 \leq q \leq \infty$  and that  $p < q$ . Observe that formula (5) extends to this case, so that, if  $\sigma$  is in  $\mathbf{D}_T$ ,

$$\begin{aligned} \|\mathcal{P}_\tau\|_{p;q} &\geq \frac{\|p_{\tau+\sigma} - \mathcal{P}_\tau b_\sigma\|_q}{\|a_\sigma\|_p} \\ &= \frac{\|\mathcal{P}_{\tau+\sigma} - \mathcal{P}_\tau \mathcal{B}_\sigma\|_{1;q}}{\|a_\sigma\|_p} \\ &\geq \frac{\|\mathcal{A}_{\tau+\sigma} + \mathcal{B}_{\tau+\sigma} - \mathcal{P}_\tau \mathcal{B}_\sigma\|_{1;q}}{\|a_\sigma\|_p} \\ &\geq \frac{\|a_{\tau+\sigma}\|_q - C_0}{\|a_\sigma\|_p}. \end{aligned}$$

We hold  $\tau$  fixed and let  $\sigma$  be a positive real number such that  $\sigma < |\tau|$ . Then

$$\begin{aligned} \|\mathcal{P}_\tau\|_{p;q} &\geq \frac{C_1 |\tau + \sigma|^{(n-1)(1/q-1/2)} \sigma^{1/q-(n+1)/2} - C_0}{C_2 \sigma^{(n-1)(1/p-1/2)+1/p-(n+1)/2}} \\ &\geq \frac{C_1 (2|\tau|)^{(n-1)(1/q-1/2)} \sigma^{1/q-(n+1)/2} - C_0}{C_2 \sigma^{(n-1)(1/p-1/2)+1/p-(n+1)/2}}, \end{aligned}$$

which behaves like  $\sigma^{-n(1/p-1/q)}$  as  $\sigma$  tends to  $0+$ . This shows that  $\mathcal{P}_\tau$  is not  $L^p-L^q$ -bounded unless  $p = 2 = q$ .

Finally, to show that  $\mathcal{A}_\tau$  is not  $L^p-L^q$ -bounded if  $1 \leq p \leq 2 \leq q \leq \infty$  unless  $p = 2 = q$ , we may suppose that  $1 \leq p \leq q < 2$ . Indeed,  $\mathcal{A}_\tau$  cannot be  $L^p-L^q$ -bounded if  $1 \leq p \leq 2 \leq q \leq \infty$ , for in this case  $\mathcal{P}_\tau$  would be  $L^p-L^q$ -bounded, which is false. Further, if  $\mathcal{A}_\tau$  were  $L^p-L^q$ -bounded for some  $(p, q)$  such that  $2 < p \leq q \leq \infty$ , then  $\mathcal{A}_\tau$  would also be  $L^p-L^q$ -bounded for the dual indices  $(q', p')$ , since  $\|\mathcal{P}_\tau\|_{p;q} = \|\mathcal{P}_\tau\|_{q';p'}$ .

Suppose then that  $\mathcal{A}_\tau$  is  $L^p-L^q$ -bounded and that  $1 \leq p \leq q < 2$ . We take  $\sigma$  in  $\mathbf{D}_T$  and observe that, by the semigroup property and the fact that  $a_\sigma$  and  $a_\tau$  are supported in  $B(o, 3T)$ ,

$$\begin{aligned} \mathcal{A}_\tau a_\sigma &= \mathcal{P}_\tau p_\sigma - \mathcal{P}_\tau b_\sigma - \mathcal{B}_\tau a_\sigma \\ &= a_{\tau+\sigma} + b_{\tau+\sigma} - a_\tau * b_\sigma - b_\tau * b_\sigma - b_\tau * a_\sigma \\ &= a_{\tau+\sigma} + \chi_{B(o,6T)} (b_{\tau+\sigma} - a_\tau * b_\sigma - b_\tau * b_\sigma - b_\tau * a_\sigma). \end{aligned}$$

Now  $\|b_{\tau+\sigma}\|_\infty$  is uniformly bounded for  $\sigma$  in  $\mathbf{D}_T$ , so that the  $L^q$ -norm of  $\chi_{B(o,6T)} b_{\tau+\sigma}$  is uniformly bounded. Similarly,  $\mathcal{A}_\tau$  is  $L^2$ -bounded and  $\|b_\sigma\|_2$  is uniformly bounded, so  $\|a_\tau * b_\sigma\|_2$  is uniformly bounded, whence  $\|\chi_{B(o,6T)}(a_\tau * b_\sigma)\|_q$  is uniformly bounded. Further,  $b_\tau$  is in  $L^2$  and  $\|b_\sigma\|_2$  is uniformly bounded, so  $\|b_\tau * b_\sigma\|_\infty$  is uniformly bounded, whence  $\|\chi_{B(o,6T)}(b_\tau * b_\sigma)\|_q$  is uniformly bounded. Finally,  $b_\tau$  is in  $L^{p'}$ , so  $\|b_\tau * a_\sigma\|_\infty$  is bounded by  $C \|a_\sigma\|_p$ , whence  $\|\chi_{B(o,6T)}(b_\tau * a_\sigma)\|_q$  is bounded by a multiple of  $\|a_\sigma\|_p$ . Now we see that

$$\begin{aligned} \|\mathcal{A}_\tau\|_{p;q} &\geq \frac{\|a_{\tau+\sigma} + \chi_{B(o,6T)}(b_{\tau+\sigma} - a_\tau * b_\sigma - b_\tau * b_\sigma - b_\tau * a_\sigma)\|_q}{\|a_\sigma\|_p} \\ &\geq \frac{\|a_{\tau+\sigma}\|_q - C_0 - C_1 \|a_\sigma\|_p}{\|a_\sigma\|_p} \\ &= \frac{\|a_{\tau+\sigma}\|_q - C_0}{\|a_\sigma\|_p} - C_1. \end{aligned}$$

We hold  $\tau$  fixed and let  $\sigma$  be a positive real number such that  $\sigma < |\tau|$ . Then

$$\begin{aligned} \|\mathcal{A}_\tau\|_{p;q} &\geq \frac{C_2 |\tau + \sigma|^{(n-1)(1/q-1/2)} \sigma^{1/q-(n+1)/2} - C_0}{C_3 \sigma^{(n-1)(1/p-1/2)+1/p-(n+1)/2}} - C_1 \\ &\geq \frac{C_2 |\tau|^{(n-1)(1/q-1/2)} \sigma^{1/q-(n+1)/2} - C_0}{C_3 \sigma^{(n-1)(1/p-1/2)+1/p-(n+1)/2}} - C_1 \\ &\sim \sigma^{-n/p+(n-1)/2+1/q} \end{aligned}$$

as  $\sigma$  tends to  $0+$ . This shows that  $\mathcal{A}_\tau$  is not  $L^p$ - $L^q$ -bounded unless  $p = 2 = q$ . □

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