



# ANNALES

DE

# L'INSTITUT FOURIER

Jonathan ARAZY & Miroslav ENGLIŠ

**Iterates and the boundary behavior of the Berezin transform**

Tome 51, n° 4 (2001), p. 1101-1133.

[http://aif.cedram.org/item?id=AIF\\_2001\\_\\_51\\_4\\_1101\\_0](http://aif.cedram.org/item?id=AIF_2001__51_4_1101_0)

© Association des Annales de l'institut Fourier, 2001, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>

## ITERATES AND THE BOUNDARY BEHAVIOR OF THE BEREZIN TRANSFORM

by J. ARAZY and M. ENGLIŠ

---

### 0. Introduction.

Let  $\mathbb{D}$  be the unit disc in the complex plane  $\mathbb{C}$ ,  $dm$  the Lebesgue measure on  $\mathbb{D}$  normalized so that  $m(\mathbb{D}) = 1$ , and  $A^2(\mathbb{D}, dm)$  the Bergman space of all holomorphic functions in  $L^2(\mathbb{D}, dm)$ . It is well known that  $A^2$  is a closed subspace of  $L^2$  (hence, a Hilbert space in its own right) and that the point evaluations are continuous functionals on  $A^2$ , so that there exists a reproducing kernel  $K(x, y)$  such that

$$(0.1) \quad \begin{cases} K(\cdot, y) \in A^2 \text{ for each fixed } y \in \mathbb{D}, \\ K(y, x) = \overline{K(x, y)}, \text{ and} \\ f(y) = \int_{\mathbb{D}} f(x) K(y, x) dm(x) \quad \forall f \in A^2 \forall y \in \mathbb{D}. \end{cases}$$

Explicitly,  $K(x, y)$  is given by

$$K(x, y) = \frac{1}{(1 - x\bar{y})^2}.$$

---

The first named author was partially supported by the Czech Academy of Sciences during his visits to Prague in October 1998 and January 2000, during which part of this work was done.

The second author's research was supported by GA AV ČR grant A1019005.

**Keywords:** Berezin transform – Geodesic symmetry – Cartan domain – Stochastic operator.

**Math. classification:** 47B38 – 32M15 – 46E22 – 60J05.

The *Berezin transform* is the integral operator on  $\mathbb{D}$  given by

$$(0.2) \quad \begin{aligned} Bf(y) &= \int_{\mathbb{D}} f(x) \frac{|K(x, y)|^2}{K(y, y)} dm(x) \\ &= \int_{\mathbb{D}} f(x) \frac{(1 - |y|^2)^2}{|1 - x\bar{y}|^4} dm(x). \end{aligned}$$

It is clear from (0.1) that this integral converges, for instance, for any bounded measurable function  $f$ . Further, by (0.1)

$$(0.3) \quad Bf = f \quad \text{for } f \text{ a bounded holomorphic function on } \mathbb{D}$$

and also

$$(0.4) \quad B\bar{f} = \overline{Bf}$$

and

$$(0.5) \quad f \geq 0 \implies Bf \geq 0.$$

The operator  $B$  behaves nicely under holomorphic self-maps of  $\mathbb{D}$ . Namely, any biholomorphic map  $\phi : \mathbb{D} \rightarrow \mathbb{D}$  may be written in the form  $\phi(z) = \phi_a(\epsilon z)$ , where  $a = \phi(0) \in \mathbb{D}$ ,  $|\epsilon| = 1$ , and

$$\phi_a(z) := \frac{a - z}{1 - \bar{a}z}.$$

The real Jacobian of the mapping  $\phi_a$  is

$$\frac{dm(\phi_a(z))}{dm(z)} = |\phi'_a(z)|^2 = \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4}.$$

Thus (0.2) can be equivalently rewritten as

$$(0.6) \quad Bf(y) = \int_{\mathbb{D}} f(\phi(x)) dm(x) \quad \text{for any } \phi \in G \text{ with } \phi(0) = y,$$

where  $G$  stands for the group of all biholomorphic self-maps of  $\mathbb{D}$ . In other words,  $Bf$  can be interpreted as a certain *invariant mean value* of the function  $f$  with respect to the measure  $dm$ . Further, let  $K$  stand for the isotropy subgroup of the origin in  $G$  (that is,  $K$  consists of all rotations  $z \mapsto \epsilon z$ ,  $|\epsilon| = 1$ ). Then  $G$  and  $K$  are Lie groups,  $\mathbb{D}$  can be identified with the homogeneous space  $G/K$ , and (0.6) is precisely the definition of the convolution of a function  $f$  with the  $K$ -invariant measure  $m$  in the group-theoretic sense:

$$(0.7) \quad Bf = f * m.$$

From the formulas (0.2) and (0.6) it is possible to establish the following properties of  $B$ :

- (1°) If  $f$  is continuous on the closed disc  $\overline{\mathbb{D}}$ , then so is  $Bf$ , and  $Bf$  and  $f$  have the same boundary values on  $\partial\mathbb{D}$ .
- (2°) If  $F \in C(\partial\mathbb{D})$ , then there exists a unique  $f \in C(\overline{\mathbb{D}})$  such that  $f|_{\partial\mathbb{D}} = F$  and  $Bf = f$ .
- (3°) If  $f \in C(\overline{\mathbb{D}})$ , then as  $k \rightarrow \infty$ ,  $B^k f$  tends pointwise and uniformly on  $\mathbb{D}$  to the (unique, by 2°) function  $g \in C(\overline{\mathbb{D}})$  such that  $g$  has the same boundary values as  $f$  and  $Bg = g$ .

The first assertion can be gleaned from (0.6) using the Dominated Convergence Theorem and the observation that

$$(0.8) \quad \phi_a(z) \rightarrow b \quad \forall z \in \mathbb{D} \quad \text{if } a \rightarrow b \in \partial\mathbb{D}.$$

The second one follows from (0.6), (0.3) and (0.4) (or from (0.7) using the much deeper result from [Fü], cf. Section 3 below) — the function  $f$  is the harmonic function on  $\mathbb{D}$  with boundary values  $F$  (the Poisson extension of  $F$ ). The last assertion was obtained by Zhu [Zh].

In this paper, we generalize 1°–3° in two different directions:

- ( $\alpha$ ) Firstly, to operators  $B$  of the form (0.2) with  $\mathbb{D}$  replaced by an arbitrary bounded domain  $\Omega$  in  $\mathbb{C}^n$ ,  $dm$  by a nonnegative regular Borel measure  $d\mu$  on it, and  $K(x, y)$  by the reproducing kernel of the corresponding Bergman space  $A^2(\Omega, d\mu)$ .
- ( $\beta$ ) Secondly, to operators  $B$  as in (0.6) (= (0.7)) with an arbitrary Cartan domain  $\Omega = G/K$  instead of  $\mathbb{D}$ , and any  $K$ -invariant absolutely continuous probability measure  $\mu$  in place of  $m$ .

Also, thirdly, in the latter case we need to study a generalization of (0.8) to arbitrary Cartan domains, which leads to some results that we believe are of interest in their own right.

We proceed to describe the contents of the paper in more detail.

In Section 1, we establish some general results concerning 1°–3° for any stochastic operator  $B$  which fixes holomorphic functions, i.e. for an operator satisfying (0.3) and (0.5) which maps the space of bounded continuous functions on  $\Omega$  into itself; the operators  $B$  in both ( $\alpha$ ) and ( $\beta$ ) above are of this type. The main result (Theorem 1.4) is that 1° and 3° are true except that the boundary values are preserved by  $B$  only on  $\partial_p\Omega$ , the subset of *peak points of holomorphic functions* on  $\partial\Omega$ , and the convergence

of  $B^k f$  is uniform only on compact subsets of  $\Omega \cup \partial_p \Omega$ . The corresponding analog of  $2^\circ$  is also true, with  $\partial\Omega$  replaced by  $\partial_p \Omega$  (Corollary 1.5), but without the uniqueness.

In Section 2 we apply the machinery of Section 1 to the part ( $\alpha$ ) above, i.e. to the Berezin transform  $B$  on a domain  $\Omega \subset \mathbb{C}^n$  with a measure  $\mu$  on  $\Omega$  satisfying some mild conditions. If  $n = 1$  and  $\Omega$  has  $C^1$  boundary, or if  $\Omega$  is strictly pseudoconvex with  $C^3$  boundary, then we obtain the full analogs of  $1^\circ$ – $3^\circ$  (Theorem 2.3). As a special case this contains Zhu's result for the disc mentioned above, and also the main result of [AL] which concerns  $2^\circ$  for  $\Omega \subset \mathbb{C}$  and  $\mu$  the Lebesgue measure.

In the remaining Sections 3–5 we deal with ( $\beta$ ), i.e. we consider  $B$  the convolution operator (0.7) on Cartan domains. In that case the bounded functions on  $\Omega$  satisfying  $Bf = f$  are precisely the bounded harmonic functions (in the sense of Godement), and the analog of  $2^\circ$  — with  $\partial\Omega$  replaced by the Shilov boundary  $\partial_e \Omega$ , since we are now in several complex variables — is known to hold from the work of Fürstenberg [Fü]. (Strictly speaking, [Fü] establishes  $2^\circ$  only for  $L^\infty(\partial_e \Omega)$  and  $L^\infty(\Omega)$  instead of our  $C(\partial_e \Omega)$  and  $C(\overline{\Omega})$ , respectively; we settle this in Section 5 below.) Applying the machinery from Section 1, we give the analog of  $3^\circ$ , with  $f$  a bounded continuous function on  $\Omega \cup \partial_e \Omega$  and the convergence locally uniform on  $\Omega \cup \partial_e \Omega$ , as Theorem 3.2. The analog of  $1^\circ$  turns out to be more delicate, and we first have to analyze the behavior of the geodesic symmetries  $\phi_a$  on  $\Omega$  as  $a$  approaches a point of  $\partial\Omega$ . This question has been studied by Kaup and Sauter [KS]; we develop their results for our needs in Section 4. In particular, we show that if  $v \in \partial\Omega$  is a tripotent,  $\beta \in D_0(v)$ , and  $a \in \Omega$  approaches the point  $v + \beta \in \partial\Omega$ , then the geodesic symmetry  $\phi_a(z)$  tends, pointwise and locally uniformly in  $z \in \Omega$ , to  $v + \phi_\beta(\rho^v(z))$ , where  $\rho^v$  is a holomorphic retraction of  $\Omega$  onto the boundary face  $v + D_0(v)$ . (See Section 4 for the notation.) A number of other useful results concerning the symmetries  $\phi_a$  and the transvections  $g_a(z) = \phi_a(-z)$  are also established, which we believe are interesting in their own right (for some readers perhaps even more than the main results concerning the operators  $B$ ). Finally, in Section 5 we use these results to settle completely the analog of  $1^\circ$  (Theorem 5.2): it turns out that  $B$  still maps  $C(\overline{\Omega})$  into itself, however, it preserves boundary values only on the Shilov boundary  $\partial_e \Omega$ , while on the other boundary faces it induces certain “boundary Berezin transforms” — that is, the restriction of  $Bf$  to the boundary face  $v + D_0(v)$  (where  $v$  is a tripotent) is uniquely determined by the restriction of  $f$  to this face, and the mapping  $f|_{v+D_0(v)} \mapsto Bf|_{v+D_0(v)}$  is again an operator of the form

(0.7) but with the convolution taken in the bounded symmetric domain  $D_0(v)$  and the measure  $\mu$  replaced by an appropriate measure  $\mu_v$  on  $D_0(v)$  (uniquely determined by  $\mu$  and  $v$ ). As another application of the results of Section 4 we establish the fact (which seems not to be treated in [Fü]), mentioned above, that if  $F \in C(\partial_e \Omega)$  then the harmonic extension of  $F$  into  $\Omega$  is actually continuous on the closure  $\bar{\Omega}$  (Theorem 5.3). Everything is finally combined together to conclude (Theorem 5.4) that for  $f \in C(\bar{\Omega})$  and  $B$  the operator (0.7), the iterates  $B^k f$  converge in fact uniformly on  $\bar{\Omega}$ , thus settling completely the analog of 3°.

The last Section 6 contains some concluding remarks (and an open problem). In particular, we observe that for bounded functions continuous on  $\Omega$  but not on  $\bar{\Omega}$ , not only the iterates  $B^k f$  need not converge (pointwise) in general, but nor need even their Cesàro means, a thing the authors at one time suspected might be true.

Throughout the text, the word “measure” will mean a nonnegative regular Borel measure, and similarly all functions are always assumed to be Borel measurable.

## 1. Stochastic operators.

Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$ ,  $L^\infty = L^\infty(\Omega)$  the space of bounded (Borel-) measurable functions on  $\Omega$ , and  $BC(\Omega) = C(\Omega) \cap L^\infty(\Omega)$  the subspace of bounded continuous functions on  $\Omega$ . We will say that an operator  $B : L^\infty \rightarrow BC(\Omega)$  is *stochastic* if

$$(1.1) \quad Bf \geq 0 \quad \text{whenever} \quad f \geq 0, \quad \text{and}$$

$$(1.2) \quad B\mathbf{1} = \mathbf{1};$$

and that it *fixes holomorphic functions* if

$$(1.3) \quad Bf = f \quad \text{for all bounded holomorphic } f.$$

Throughout the rest of this paper, we will assume that  $B : L^\infty \rightarrow BC(\Omega)$  is a stochastic operator which fixes holomorphic functions, and will be interested in the limiting behavior of the iterates  $B^k$ ,  $k \rightarrow \infty$ .

The following characterization of stochastic operators is well-known.

**PROPOSITION 1.1.** — *An operator  $B : L^\infty \rightarrow BC$  is stochastic if and only if for each  $y \in \Omega$  there exists a (nonnegative regular Borel) measure*

$\mu_y$  on  $\Omega$  such that

$$(1.4) \quad \mu_y \text{ is a probability measure, } \forall y \in \Omega$$

and

$$(1.5) \quad Bf(y) = \int_{\Omega} f(x) d\mu_y(x).$$

Clearly, the measures  $\mu_y$  are uniquely determined by  $B$ .

*Proof.* — Obviously (1.4) and (1.5) imply (1.1) and (1.2). Conversely, by the Riesz representation theorem, (1.1) implies that for each  $y \in \Omega$  the linear functional  $f \mapsto Bf(y)$  determines a nonnegative regular Borel measure  $\mu_y$  on  $\Omega$ . By (1.2), the total mass of this measure is 1.  $\square$

It is immediate from (1.4) and (1.5) that

$$(1.6) \quad \|Bf\|_{\infty} \leq \|f\|_{\infty},$$

$$(1.7) \quad B\bar{f} = \overline{Bf}$$

and

$$(1.8) \quad |f| \leq g \implies |Bf| \leq Bg$$

(in particular,

$$(1.9) \quad f \leq g \implies Bf \leq Bg).$$

From (1.8) and (1.3) we also have

$$(1.10) \quad |f| \leq B|f| \quad \text{for all bounded holomorphic } f.$$

From (1.1) and (1.2) it further follows that if  $B : L^{\infty} \rightarrow BC$  is stochastic, then so are its iterates  $B^k$ ,  $k = 1, 2, \dots$ . By the last proposition, there exist measures  $\mu_{k,y}$  ( $y \in \Omega, k = 1, 2, \dots$ ) so that

$$(1.11) \quad \mu_{k,y} \text{ are probability measures on } \Omega,$$

and

$$(1.12) \quad B^k f(y) = \int_{\Omega} f(x) d\mu_{k,y}(x).$$

Let  $\partial\Omega$  and  $\partial_e\Omega$  denote the topological and the Shilov boundary of  $\Omega$ , respectively, and let  $\partial_p\Omega$  stand for the set of all *peak-points* of the algebra

$$A(\overline{\Omega}) := \{f \in C(\overline{\Omega}) : f \text{ is holomorphic on } \Omega\}.$$

That is,  $\partial_p\Omega$  consists of all points  $p \in \partial\Omega$  for which there exists  $f \in A(\overline{\Omega})$  such that  $f(p) = 1$  and  $|f| < 1$  on  $\overline{\Omega} \setminus \{p\}$ . It is known that  $\partial_p\Omega \subset \partial_e\Omega \subset \partial\Omega$ , and  $\partial_p\Omega$  is dense in  $\partial_e\Omega$  (see [Ga], §II.11). The following lemma and proposition are taken from [AL], where they are proved for a very special choice of  $\Omega$  and  $\mu_y$ ; the proof extends to the general case with only trivial modifications.

LEMMA 1.2. — Assume that  $B$  is a stochastic operator from  $L^\infty$  into  $BC$  which fixes holomorphic functions. If  $p \in \partial_p\Omega$  and  $U$  is any neighborhood of  $p$ , then as  $y \rightarrow p$ ,

$$\int_{\Omega \setminus U} d\mu_{k,y}(x) \rightarrow 0 \quad \text{uniformly in } k.$$

Proof. — Let  $f$  be a peaking function for  $p$  and  $\epsilon > 0$ . Replacing  $f$  by  $f^m$  with  $m$  large enough, we can assume that  $|f| < \epsilon/2$  on  $\Omega \setminus U$ . Choose then  $\delta > 0$  such that  $|f(y)| > 1 - \frac{\epsilon}{2}$  in the ball  $|y - p| < \delta$ . Since  $f$  is holomorphic and  $|f| \leq 1$  on  $\Omega$ , by (1.3) and (1.12) we have for  $|y - p| < \delta$  and any  $k$ ,

$$\begin{aligned} \int_U d\mu_{k,y}(x) &\geq \left| \int_U f(x) d\mu_{k,y}(x) \right| \\ &= \left| B^k f(y) - \int_{\Omega \setminus U} f(x) d\mu_{k,y}(x) \right| \\ &\geq |f(y)| - \int_{\Omega \setminus U} |f(x)| d\mu_{k,y}(x) \quad (\text{as } B^k f = f) \\ &\geq |f(y)| - \frac{\epsilon}{2} \int_{\Omega} d\mu_{k,y}(x) \\ &= |f(y)| - \frac{\epsilon}{2} \\ &\geq \left(1 - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} = 1 - \epsilon, \end{aligned}$$

which together with (1.11) implies

$$\int_{\Omega \setminus U} d\mu_{k,y}(x) \leq \epsilon \quad \text{for } |y - p| < \delta \text{ and any } k. \quad \square$$

PROPOSITION 1.3. — Assume that  $B$  is a stochastic operator from  $L^\infty$  into  $BC$  which fixes holomorphic functions. If  $p \in \partial_p\Omega$ ,  $f \in L^\infty$  and  $f(z) \rightarrow a$  as  $z \rightarrow p$ , then also  $B^k f(z) \rightarrow a$  as  $z \rightarrow p$  ( $k = 1, 2, \dots$ ), uniformly in  $k$ .



*Proof.* — Let  $U$  be a neighborhood of  $p$  where  $|f - a|$  is small. Then by (1.11) and (1.12)

$$|B^k f(y) - a| = |B^k(f - a)(y)| \leq \int_U |f(x) - a| d\mu_{k,y}(x) + \int_{\Omega \setminus U} |f(x) - a| d\mu_{k,y}(x).$$

The first integral is small (uniformly in  $k$ ) because  $|f - a|$  is small on  $U$  and the total mass of  $\mu_{k,y}$  is 1. The second integral is small (uniformly in  $k$ ) if  $y$  is close to  $p$  because of Lemma 1.2 and the boundedness of  $f - a$ . Hence the assertion follows.  $\square$

We now prove a general result concerning the existence of the limit of  $B^k f$ ,  $k \rightarrow \infty$ .

**THEOREM 1.4.** — *Assume that  $B$  is a stochastic operator from  $L^\infty$  into  $BC$  which fixes holomorphic functions. Then for each  $f \in C(\bar{\Omega})$ , the sequence  $B^k f$  converges pointwise and uniformly on compact subsets of  $\Omega \cup \partial_p \Omega$  to a function  $g \in BC(\Omega \cup \partial_p \Omega)$  satisfying  $g|_{\partial_p \Omega} = f|_{\partial_p \Omega}$  and  $Bg = g$ .*

*Proof.* — Assume first that  $f = |\phi|$ , with  $\phi \in A(\bar{\Omega})$ . Then by (1.10) and (1.6)  $f \leq Bf \leq \|f\|_\infty$ . Using (1.9) we obtain by iteration

$$f \leq Bf \leq B^2 f \leq \dots \leq \|f\|_\infty.$$

By the Bolzano-Weierstrass theorem,  $B^k f \nearrow g$  for some (bounded, lower semicontinuous) function  $g$  on  $\Omega$ . By the Monotone Convergence Theorem,  $Bg = \lim B^{k+1} f = g$  (hence, in particular,  $g \in BC(\Omega)$ ). By Proposition 1.3,  $g$  extends continuously to  $\Omega \cup \partial_p \Omega$ , coincides with  $f$  on  $\partial_p \Omega$ , and  $B^k f \nearrow g$  on  $\Omega \cup \partial_p \Omega$ . By Dini's theorem, the convergence is therefore uniform on compact subsets of  $\Omega \cup \partial_p \Omega$ . Thus all the assertions of Theorem 1.4 hold for  $f = |\phi|$ .

Set now

$$\mathcal{E} = \{f \in C(\bar{\Omega}) : \exists g \in BC(\Omega \cup \partial_p \Omega) \text{ such that } g|_{\partial_p \Omega} = f|_{\partial_p \Omega}, Bg = g \text{ and } B^k f \rightarrow g \text{ as } k \rightarrow \infty, \text{ uniformly on compact subsets of } \Omega \cup \partial_p \Omega\}.$$

Evidently  $\mathcal{E}$  is a linear subspace of  $C(\bar{\Omega})$ , and a routine check using (1.6) reveals that it is closed. In view of the previous paragraph, functions of the form  $f = |\phi|$  with  $\phi \in A(\bar{\Omega})$  belong to  $\mathcal{E}$ . Thus  $\mathcal{E} \supset \mathcal{G}$ , where  $\mathcal{G}$  is the vector

space of all functions on  $\overline{\Omega}$  of the form

$$f = \sum_{\text{finite}} c_j |\phi_j|, \quad c_j \in \mathbb{C}, \phi_j \in A(\overline{\Omega}).$$

The set  $\mathcal{G}$  is closed under pointwise multiplication and complex conjugation, i.e. is a  $*$ -subalgebra of  $C(\overline{\Omega})$ . It also separates points: if  $x \neq y$ , then  $x, y$  must differ in some coordinate, say,  $x_1 \neq y_1$ ; but then  $f(z) = |z_1 - x_1|$  satisfies  $f(x) = 0, f(y) \neq 0$  and  $f \in \mathcal{G}$ . By the Stone-Weierstrass theorem,  $\mathcal{G}$  is dense in  $C(\overline{\Omega})$ . As  $\mathcal{G} \subset \mathcal{E}$  and  $\mathcal{E}$  is closed, we conclude that  $\mathcal{E} = C(\overline{\Omega})$ , and the desired assertion follows.  $\square$

As  $\partial_e \Omega$  is the closure of  $\partial_p \Omega$ , the following corollary is immediate from Theorem 1.4 and Tietze's extension theorem.

**COROLLARY 1.5.** — *Assume that  $B$  is a stochastic operator from  $L^\infty$  into  $BC$  which fixes holomorphic functions. Then for any  $\phi \in C(\partial_e \Omega)$ , there exists  $g \in BC(\Omega \cup \partial_p \Omega)$  such that  $Bg = g$  and the restrictions of  $g$  and  $\phi$  to  $\partial_p \Omega$  coincide.*

The last theorem gives a fairly good description of the limiting behavior of  $B^k f$  on  $\Omega \cup \partial_p \Omega$ , for any  $f \in C(\overline{\Omega})$ . In general, controlling this behavior even on  $\partial_e \Omega \setminus \partial_p \Omega$  seems much more difficult. The following theorem may be useful in some situations in this regard.

**THEOREM 1.6.** — *Assume that  $B$  is a stochastic operator from  $L^\infty$  into  $BC$  which fixes holomorphic functions. Introduce the following assumptions:*

(A) *There exists a function  $h \in C(\overline{\Omega})$  such that  $h = 0$  on  $\partial_e \Omega, h > 0$  on  $\overline{\Omega} \setminus \partial_e \Omega$ , and  $Bh \leq h$  on  $\Omega$ .*

(B) *There exists a function  $h$  as in (A) such that in addition  $B^k h \rightarrow 0$  uniformly on  $\Omega$ .*

(C) *For any  $\phi \in C(\partial_e \Omega)$  there exists a unique  $g \in BC(\Omega \cup \partial_e \Omega)$  such that  $Bg = g$  and  $g|_{\partial_e \Omega} = \phi$ . (We call  $g$  the  $B$ -Poisson extension of  $\phi$ .)*

*If (A) and (C) hold, then for any  $f \in BC(\Omega \cup \partial_e \Omega)$ , the limit  $\lim_{k \rightarrow \infty} B^k f = g$  exists pointwise and locally uniformly on  $\Omega \cup \partial_e \Omega$ , and  $g$  is the  $B$ -Poisson extension of  $f|_{\partial_e \Omega}$ .*

*If in addition (B) holds, then  $B^k f \rightarrow g$  even uniformly on  $\Omega$ .*

We shall see presently that (C) already implies that the function  $h$  in (A) satisfies  $B^k h \searrow 0$  pointwise on  $\Omega \cup \partial_e \Omega$ .

*Proof.* — Let  $g$  be the  $B$ -Poisson extension of  $f$ . Replacing  $f$  by  $f - g$  we may assume that  $g = 0$ , i.e. that  $f$  vanishes on the Shilov boundary  $\partial_e\Omega$ . We then want to show that  $B^k f \rightarrow 0$  pointwise or uniformly. By (1.6) and an obvious approximation argument, we may even assume that  $f$  vanishes in some neighborhood of  $\partial_e\Omega$ . Then there exists a constant  $c > 0$  such that  $|f| \leq ch$ , where  $h$  is the function from the assumption (A). By (1.8),  $|B^k f| \leq c \cdot B^k h$  for all  $k$ . If (B) holds, it thus follows that  $B^k f \rightarrow 0$  uniformly on  $\Omega$ , which settles the second part of the theorem. For the first part, it suffices to show that  $B^k h \rightarrow 0$  pointwise. As  $h \geq 0$  and  $Bh \leq h$ , we have by (1.9) again  $B^{k+1}h \leq B^k h$  and  $B^k h \geq 0 \forall k$ , so

$$h \geq Bh \geq B^2h \geq \dots \geq 0 \quad \text{on } \Omega.$$

Therefore, by the Bolzano-Weierstrass theorem,  $B^k h \searrow g$  for some function  $g$  on  $\Omega$ ,  $g \geq 0$ . By the Monotone Convergence Theorem,  $Bg = \lim B^{k+1}h = g$ ; this also implies that  $g \in BC(\Omega)$ . As  $h \geq g \geq 0$  and  $h|_{\partial_e\Omega} = 0$ , the function  $g$  extends by continuity to  $\Omega \cup \partial_e\Omega$  and vanishes on  $\partial_e\Omega$ . By the uniqueness part of assumption (C),  $g = 0$ , which completes the proof.  $\square$

*Remark.* — The proof shows that (A) and (C) in fact imply that  $B$  maps  $BC(\Omega \cup \partial_e\Omega)$  into itself and preserves the boundary values on  $\partial_e\Omega$ , i.e.,  $Bf(x) = f(x) \forall x \in \partial_e\Omega$ .

It is not true in general that  $B$  preserves the boundary values on all of  $\partial\Omega$ . For  $B$  the Berezin transform with respect to the Lebesgue measure (cf. the next section), the simplest example is the function  $f(z) = |z_1|$  on the bidisc  $\mathbb{D} \times \mathbb{D}$ .  $\square$

## 2. Berezin transform on planar domains and strictly pseudoconvex domains.

Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$ ,  $d\mu$  a measure on  $\Omega$ , and consider the Bergman space  $A^2(\Omega, d\mu)$  of all analytic functions on  $\Omega$  square-integrable with respect to  $d\mu$ . We shall assume throughout that the point evaluations are continuous on this space, so that there exists a reproducing kernel  $K(x, y)$ , and that  $K(x, x) > 0 \forall x \in \Omega$ . (Both assumptions are fulfilled, for instance, if the measure  $d\mu$  is finite and has a continuous positive density with respect to the Lebesgue measure on  $\Omega$ ; on the other hand, as we shall see below they imply that  $\text{supp}\mu \supset \partial_e\Omega$ .) One can then define the

integral operator on  $\Omega$ ,

$$(2.1) \quad Bf(y) = \int_{\Omega} f(x) \frac{|K(x, y)|^2}{K(y, y)} d\mu(x)$$

called the *Berezin transform* with respect to  $\mu$ . Observe that the operator (2.1) is of the form (1.5) with  $d\mu_y(x) = |K(x, y)|^2 K(y, y)^{-1} d\mu(x)$ . The next proposition shows that this operator satisfies the hypothesis from the previous section.

PROPOSITION 2.1. — *The operator (2.1) maps  $L^\infty$  into  $BC(\Omega)$ , is stochastic and fixes holomorphic functions.*

*Proof.* — The last two assertions are immediate consequences of the reproducing property of the kernel  $K(x, y)$ . For the first, consider more generally the function of two variables

$$\tilde{B}f(y, z) := \int_{\Omega} f(x) K(y, x) K(x, z) d\mu(x).$$

The integral converges for any  $y, z \in \Omega$  because  $f \in L^\infty$  and  $K(\cdot, y), K(\cdot, z)$  are in  $L^2(\Omega, d\mu)$ . For  $z \in \Omega$  let  $U \subset \Omega$  be a small disc centered at  $z$ . For any closed contour  $\gamma \subset U$  we have

$$\begin{aligned} \oint_{\gamma} \int_{\Omega} |f(x) K(y, x) K(x, z)| d\mu(x) dy &\leq \|f\|_{\infty} \cdot \|K(\cdot, z)\| \cdot \sup_{y \in \gamma} \|K(\cdot, y)\| \cdot \left( \oint_{\gamma} dy \right) \\ &= \|f\|_{\infty} \cdot \text{length}(\gamma) \cdot K(z, z)^{1/2} \cdot \sup_{y \in \gamma} K(y, y)^{1/2} \\ &< \infty, \end{aligned}$$

since  $K(y, y)$  is a continuous function of  $y$  and  $\gamma$  is a compact set. Thus we have by Fubini

$$\oint_{\gamma} \tilde{B}f(y, z) dy = \int_{\Omega} f(x) K(x, z) \oint_{\gamma} K(y, x) dy d\mu(x) = 0.$$

By Morera's theorem, this implies that  $\tilde{B}f$  is holomorphic in  $y$ . As  $\tilde{B}f(z, y) = \overline{\tilde{B}f(y, z)}$ , it follows that  $\tilde{B}f(y, y)/K(y, y) = Bf(y)$  is not only continuous, but even real-analytic on  $\Omega$ . □

We can now present the first application of the results from the previous section.

LEMMA 2.2. — *Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$  and  $\mu$  a measure on  $\Omega$  such that the point evaluations are continuous on the Bergman space  $A^2(\Omega, d\mu)$  and its reproducing kernel satisfies  $K(x, x) > 0 \forall x \in \Omega$ . Then  $\text{supp}\mu_y = \text{supp}\mu \forall y \in \Omega$ , and  $\text{supp}\mu \supset \partial_e\Omega$ .*

*Proof.* — For any  $y \in \Omega$ , the hypothesis  $K(y, y) > 0$  guarantees that the holomorphic function  $K(\cdot, y)$  cannot vanish on an open set, so  $d\mu_y(x) = K(y, y)^{-1}|K(x, y)|^2 d\mu(x)$  has indeed the same support as  $d\mu$ . For the second assertion, let  $p \in \partial_p\Omega$  and assume that there exists a neighborhood  $U$  of  $p$  disjoint from  $\text{supp}\mu$ . From  $\text{supp}\mu = \text{supp}\mu_y$  and Lemma 1.2 we then have

$$1 = \int_{\Omega} d\mu_y = \int_{\Omega \setminus U} d\mu_y \rightarrow 0 \quad \text{as } y \rightarrow p,$$

a contradiction. Hence  $p \in \text{supp}\mu$ , so  $\partial_p\Omega \subset \text{supp}\mu$ . As  $\partial_e\Omega$  is the closure of  $\partial_p\Omega$ , the second assertion follows.  $\square$

THEOREM 2.3. — *Let  $\Omega$  be either a bounded domain in the complex plane  $\mathbb{C}$  with  $C^1$  boundary, or a strictly pseudoconvex domain in  $\mathbb{C}^n$  with  $C^3$  boundary, and  $\mu$  a measure on  $\Omega$  such that the point evaluations are continuous on the Bergman space  $A^2(\Omega, d\mu)$  and its reproducing kernel satisfies  $K(x, x) > 0 \forall x \in \Omega$ . Then*

- (a)  *$B$  maps  $C(\overline{\Omega})$  into itself and preserves the boundary values.*
- (b) *For any  $f \in C(\overline{\Omega})$ , the sequence  $B^k f$  converges uniformly on  $\overline{\Omega}$  to a function  $g \in C(\overline{\Omega})$  satisfying  $Bg = g$  and  $g|_{\partial\Omega} = f|_{\partial\Omega}$ .*
- (c) *In particular, for any  $\phi \in C(\partial\Omega)$  there exists a unique  $g \in C(\overline{\Omega})$  satisfying  $Bg = g$  and  $g|_{\partial\Omega} = \phi$  (the  $B$ -Poisson extension of  $\phi$ ).*

*Proof.* — It is known that for domains  $\Omega$  of this kind  $\partial_p\Omega = \partial\Omega$ . (For the planar domains, this is easy: if  $p \in \partial\Omega$  then  $f(z) = n\epsilon / ((p + n\epsilon) - z)$  peaks at  $p$ , where  $n$  is the outward unit normal to  $\partial\Omega$  at  $p$  and  $\epsilon > 0$  is sufficiently small. For the strictly pseudoconvex domains, see e.g. Theorem 5.2.15 in [Kr].) Consequently, also  $\partial_e\Omega = \partial\Omega$  and  $\Omega \cup \partial_p\Omega = \overline{\Omega}$ . Hence (a) follows from Proposition 1.3, (b) from Theorem 1.4, and the existence part of (c) from Corollary 1.5. The uniqueness part of (c) follows by a standard maximum principle argument from the fact that  $Bf(y)$  is an average of  $f$  against the probability measure  $\mu_y$  and  $\text{supp}\mu_y \supset \partial\Omega$ .  $\square$

For  $\Omega$  the unit disc in the complex plane and  $\mu$  the Lebesgue measure on it, the functions satisfying  $Bg = g$  are precisely the harmonic functions

(see the next section). In this context the assertion (b) of the last theorem was proved, by entirely different means, by Zhu [Zh].

As a special case of Theorem 2.3(c) we obtain the following result of Axler and Lech [AL].

**COROLLARY 2.4.** — *Let  $\Omega$  be a bounded  $(n + 1)$ -connected domain in  $\mathbb{C}$  with  $C^1$  boundary and  $d\mu$  the Lebesgue measure on  $\Omega$ . Denote by  $\gamma_0, \dots, \gamma_n$  the connected components of  $\partial\Omega$ , with  $\gamma_0$  the boundary of the unbounded component of  $\mathbb{C} \setminus \Omega$ . Then there exist real-valued functions  $u_1, \dots, u_n \in C(\overline{\Omega})$  such that*

1.  $u_i|_{\gamma_j} = \delta_{ij}$  ( $i = 1, \dots, n, j = 0, \dots, n$ );
2.  $Bu_i = u_i$  ( $i = 1, \dots, n$ ); and
3. any real-valued function  $f$  continuous on  $\overline{\Omega}$  and fixed by  $B$  can be uniquely represented as

$$f = \sum_{j=1}^n c_j u_j + \operatorname{Re} u$$

where  $c_1, \dots, c_n \in \mathbb{R}$  and  $u \in A(\overline{\Omega})$ .

*Proof.* — According to part (c) of the last theorem, there exist unique functions  $u_1, \dots, u_n \in C(\overline{\Omega})$  satisfying 1 and 2. The uniqueness together with (1.6) also implies that  $u_i$  are real-valued. If  $f \in C(\overline{\Omega})$  is real-valued and fixed by  $B$ , the function  $g = f - \sum_1^n a_j u_j$  will have the same property, for any  $a_1, \dots, a_n \in \mathbb{R}$ . Owing to 1 we can choose  $a_j$  in such a way that

$$\oint_{\gamma_j} g(z) dz = \oint_{\gamma_0} g(z) dz \quad \forall j = 1, \dots, n.$$

It is then known that there exists  $u \in A(\overline{\Omega})$  such that  $\operatorname{Re} u = g$  on  $\partial\Omega$ . By (1.7) and (1.3), the function  $\operatorname{Re} u$  is also fixed by  $B$ . Thus  $g$  and  $\operatorname{Re} u$  are two  $B$ -fixed functions having the same boundary values; by the uniqueness part of (c) again, this implies  $g \equiv \operatorname{Re} u$ . This completes the proof.  $\square$

### 3. Convolution operators on Cartan domains.

Let  $\Omega = G/K$  be a Cartan domain in  $\mathbb{C}^n$  in its Harish-Chandra realization (i.e.  $\Omega$  is circular and convex) and  $\mu$  a  $K$ -invariant probability measure on  $\Omega$  absolutely continuous with respect to the Lebesgue measure.

For  $a \in \Omega$  we denote by  $\phi_a$  the geodesic symmetry interchanging  $a$  and the origin. The convolution operator

$$(3.1) \quad B_\mu f := f * \mu$$

can be written in terms of  $\phi_a$  as

$$(3.2) \quad B_\mu f(a) = \int_\Omega f(\phi_a(z)) d\mu(z).$$

As any  $g \in G$  mapping 0 into  $a$  is of the form  $g = \phi_a \circ k$  with  $k \in K$ , the  $K$ -invariance of  $\mu$  implies that one even has

$$(3.3) \quad B_\mu f(a) = \int_\Omega f(g(z)) d\mu(z)$$

for any element  $g$  of  $G$  with  $g(0) = a$ .

The operator  $B_\mu$  is clearly of the form (1.5) with  $d\mu_y(x) = d\mu(\phi_y(x))$ . The fact that  $\mu$  is a probability measure implies that  $B_\mu \mathbf{1} = \mathbf{1}$  and  $B_\mu f \geq 0$  when  $f \geq 0$ . Further, if  $f$  is holomorphic, then  $B_\mu f(a) = f(a)$  by the  $K$ -invariance of  $\mu$  and the mean value property of holomorphic functions. (In fact, the same argument gives  $B_\mu f = f$  for all bounded harmonic (in the sense discussed below) functions as well.) The following proposition therefore implies that  $B_\mu$  is an operator of the type considered in Section 1.

PROPOSITION 3.1. — *The operator (3.1) maps  $L^\infty$  into  $BC(\Omega)$ .*

*Proof.* — As  $\|B_\mu f\|_\infty \leq \|f\|_\infty$ , it is enough to show that  $B_\mu f \in BC(\Omega)$  for  $f$  the characteristic function of a Borel set  $E$  (linear combinations of such functions are dense in  $L^\infty$ ). For such an  $f$ ,

$$B_\mu f(a) = \int_{\phi_a(E)} d\mu.$$

In view of the absolute continuity of  $\mu$ , it is in turn sufficient to show that as  $a \rightarrow b$ , the Lebesgue measures of the sets  $\phi_a(E) \setminus \phi_b(E)$  and  $\phi_b(E) \setminus \phi_a(E)$  tend to zero. We give the proof for  $\phi_a(E) \setminus \phi_b(E)$ . For each  $a \in \Omega$ , it is known that  $\phi_a : \Omega \rightarrow \Omega$  is a rational function whose coefficients depend smoothly on  $a$ ; in particular,  $\phi_a$  extends biholomorphically to a neighborhood of  $\bar{\Omega}$ , and as  $a \rightarrow b \in \Omega$ ,  $\phi_a \rightarrow \phi_b$  uniformly on  $\Omega$ . Hence for any  $\epsilon > 0$ , if  $a$  is close enough to  $b$  the set  $\phi_a(E)$  will lie in the  $\epsilon$ -neighborhood (with respect to the Euclidean metric) of  $\phi_b(E)$ . Denoting  $F(z) = \text{dist}_{\text{Euclidean}}(z, \phi_b(E))$ , it therefore suffices to show that

$$\int_{\{z: 0 < F(z) < \epsilon\}} dm(z) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

( $dm$  stands for the Lebesgue measure). However, as  $F$  is a continuous function (even Lipschitz, with the Lipschitz constant equal 1, by the triangle inequality), this is certainly true.  $\square$

By the results of Fürstenberg [Fü], any bounded function  $f$  satisfying  $B_\mu f = f$  must be harmonic (in the sense of Godement [Gd]), has a.e. radial limits on the Shilov boundary  $\partial_e \Omega$ , and can be recovered from these boundary values by the Poisson integral:  $f = P[f|_{\partial_e \Omega}]$ , with

$$(3.4) \quad PF(z) := \int_{\partial_e \Omega} F(g(\zeta)) d\sigma(\zeta) \quad \text{for } F : \partial_e \Omega \rightarrow \mathbb{C},$$

where  $g$  is any element of  $G$  such that  $g(0) = z$ , and  $d\sigma$  is the unique  $K$ -invariant probability measure on  $\partial_e \Omega$ . Moreover, for a bounded symmetric domain the peak-set  $\partial_p \Omega$  coincides with the Shilov boundary  $\partial_e \Omega$ , and they consist precisely of the points in  $\bar{\Omega}$  of maximal Euclidean distance from the origin. (Except the case of  $\Omega$  the unit ball in  $\mathbb{C}^n$ , the latter is a proper subset of  $\partial \Omega$ .) Combining these facts with Theorem 1.4 we thus arrive at the following theorem.

**THEOREM 3.2.** — *For any  $f \in BC(\Omega \cup \partial_e \Omega)$ , the iterates  $B_\mu^k f$  converge locally uniformly on  $\Omega \cup \partial_e \Omega$  to the Poisson extension (defined by (3.4)) of  $f|_{\partial_e \Omega}$ .*

*Remark.* — Let  $p$  be the genus and  $h(x, y)$  the Jordan triple determinant of the Cartan domain  $\Omega$  (see the next section for details). For  $\nu > p - 1$ , the measures  $d\mu_\nu(z) = c_\nu h(z, z)^{\nu - p} dm(z)$  on  $\Omega$ , where  $dm$  stands for the Lebesgue measure and  $c_\nu > 0$ , are finite,  $K$ -invariant and absolutely continuous with respect to the Lebesgue measure. Fixing  $c_\nu$  so that  $d\mu_\nu$  has total mass 1, it turns out that the corresponding weighted Bergman spaces  $A^2(\Omega, d\mu_\nu)$  have continuous point evaluations, their reproducing kernels are equal to  $h(x, y)^{-\nu}$ , and one can define the associated Berezin transform  $B_\nu$  in the same way as in the preceding section. These operators were, in fact, our original objects of interest when writing this paper. It turns out that they are of the form (3.1) with  $\mu = \mu_\nu$ ; thus, in particular, the last theorem applies to them too.  $\square$

We conclude this section by giving another proof of Theorem 3.2, based on Theorem 1.6. Recall that under the action of the compact Lie group  $K$ , the space  $\mathcal{P}$  of holomorphic polynomials on  $\mathbb{C}^n$  has the Peter-Weyl decomposition

$$\mathcal{P} = \bigoplus_m \mathcal{P}_m$$



into  $K$ -irreducible subspaces; here  $\mathbf{m}$  ranges over all *signatures*, i.e.  $r$ -tuples of integers  $m_1 \geq m_2 \geq \dots \geq m_r \geq 0$ , where  $r$  is the rank of  $\Omega$ . Equipped with the *Fock inner product*

$$\langle p, q \rangle_F := p \left( \frac{\partial}{\partial z} \right) \overline{q(\bar{z})}|_{z=0},$$

each  $\mathcal{P}_{\mathbf{m}}$  becomes a (finite-dimensional) reproducing kernel Hilbert space, and we denote by  $K_{\mathbf{m}}(x, y)$  the corresponding reproducing kernel. Further, it is known that there exist vectors  $e_1, \dots, e_r \in \mathbb{C}^n$  (a *Jordan frame*) such that every  $z \in \mathbb{C}^n$  can be written in the form

$$(3.5) \quad z = k \sum_{j=1}^r t_j e_j$$

with  $k \in K$  and  $t_1, \dots, t_r \geq 0$ , and then

$$K_{\mathbf{1}^r}(z, z) = c \prod_{j=1}^r t_j^2$$

where  $c$  is a nonzero constant depending only on  $\Omega$  and  $\mathbf{1}^r$  stands for the signature  $\mathbf{m} = (1, \dots, 1)$  consisting of all 1's. Also, the element (3.5) belongs to  $\overline{\Omega}$  if and only if  $0 \leq t_j \leq 1 \forall j$ , and belongs to  $\partial_e \Omega$  if and only if  $t_j = 1 \forall j$ , i.e. if and only if  $z = ke$  where  $e = e_1 + \dots + e_r$ . See [FK].

Now consider the function  $h$  on  $\overline{\Omega}$  given by

$$(3.6) \quad h(z) = K_{\mathbf{1}^r}(e, e) - K_{\mathbf{1}^r}(z, z).$$

For  $z$  given by (3.5),  $h(z) = c(1 - \prod_1^r t_j^2)$ , and in view of the last remarks it follows that  $h(z) \geq 0$  on  $\overline{\Omega}$ , with equality occurring if and only if  $z \in \partial_e \Omega$ . On the other hand, if  $\{\psi_j\}_j$  is an orthonormal basis for the polynomial space  $\mathcal{P}_{\mathbf{1}^r}$  of the Peter-Weyl decomposition, then by the familiar formula for a reproducing kernel

$$K_{\mathbf{1}^r}(z, z) = \sum_j |\psi_j(z)|^2,$$

and as by (1.10)  $B_\mu |\psi_j|^2 \geq |\psi_j|^2 \forall j$  and by (1.2)  $B_\mu \mathbf{1} = \mathbf{1}$ , it follows that  $B_\mu h \leq h$ . Thus the function  $h$  satisfies the assumption (A) of Theorem 1.6. As the assumption (C) there is fulfilled in view of the results of Fürstenberg mentioned above, Theorem 3.2 follows. □

In the rest of this paper, we stick to the case of  $\Omega$  a Cartan domain and will describe the behavior of the convolution operator (3.1) and its

iterates on  $\partial\Omega \setminus \partial_e\Omega$ , where we lack control so far. In view of (3.2), this is clearly tantamount to the investigation of the geodesic symmetries  $\phi_a$  as  $a$  tends to the boundary. This is what we undertake in the next section.

#### 4. Boundary limits of holomorphic automorphisms.

As in the previous section, let  $\Omega$  be a Cartan domain in  $\mathbb{C}^n$  of type  $(r, a, b)$ . Thus  $\Omega = G/K$ , with  $G = \text{Aut}_0(\Omega)$  the identity connected component of the group of holomorphic automorphisms, and  $K \subset G$  the subgroup stabilizing the origin. Let  $G = KAN$  be the Iwasawa decomposition relative to the Cartan involution  $\theta(g) = s_0gs_0$ , where  $s_0(z) = -z$  is the symmetry at the origin. Let  $s_z = gs_0g^{-1}$  be the symmetry at  $z$ , where  $g \in G$  is any element for which  $g(0) = z$ .

Let  $z, w \in \Omega$  and let  $L(z, w)$  be the geodesic line from  $z$  to  $w$  in  $\Omega$  (with respect to the Bergman metric, which is, up to a constant factor, the unique  $G$ -invariant Riemannian metric on  $\Omega$ ). Since the elements of  $G$  are isometries we have

$$g(L(z, w)) = L(g(z), g(w)) \quad \forall g \in G \quad \forall z, w \in \Omega.$$

Let  $d(z, w)$  be the distance induced by the Bergman metric. We denote by  $m(z, w)$  the *geodesic midpoint* between  $z$  and  $w$ . Thus

$$m(z, w) \in L(z, w) \quad \text{and} \quad d(z, m(z, w)) = d(w, m(z, w)) = \frac{1}{2}d(z, w).$$

For convenience we denote also

$$m(z) = m(z, 0) = \text{the mid-point between } z \text{ and } 0.$$

Then

$$\phi_z = s_{m(z)},$$

the symmetry at the geodesic midpoint between 0 and  $z$ . Notice that  $\phi_z(L(0, z)) = L(z, 0)$  and  $\phi_z(m(z)) = m(z)$ . We also let

$$g_z = \phi_z s_0, \quad z \in \Omega.$$

Then  $g_z$  (called *transvection*) is another element of  $G$  for which  $g_z(0) = z$ .

We will use the language of Jordan theory, see [Lo] or [Ar] for details and notation. In particular, we let  $Z(\simeq \mathbb{C}^n)$  stand for the  $JB^*$ -triple

whose unit ball is  $\Omega$ ,  $\{xyz\}$  for the triple product of  $Z$ ,  $D(x, y)$  for the multiplication operators  $D(x, y)z = \{xyz\}$ ,  $Q(x)$  for the quadratic operator  $Q(x)z = \{xzx\}$ , and  $B(x, y)$  for the Bergman operator

$$B(x, y)z = z - 2D(x, y)z + Q(x)Q(y)z.$$

In terms of these operators, the transvections and the geodesic symmetries are given by

$$(4.1) \quad \begin{aligned} g_a(z) &= a + B(a, a)^{1/2}(I + D(z, a))^{-1}z, \\ \phi_a(z) &= a - B(a, a)^{1/2}(I - D(z, a))^{-1}z \quad (a \in \Omega, z \in \bar{\Omega}). \end{aligned}$$

An element  $v \in Z$  is a *tripotent* if  $\{vvv\} = v$ . Two tripotents  $u, v$  are called *orthogonal* if  $D(u, v) = 0$  (this is equivalent to  $D(v, u) = 0$ ). Every  $a \in Z$  can be written in the form

$$(4.2) \quad a = \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_l e_l$$

where  $e_1, \dots, e_l$  are pairwise orthogonal tripotents and  $\lambda_1, \dots, \lambda_l \geq 0$ . Further,  $a$  belongs to  $\Omega, \bar{\Omega}$ , or  $\partial_e \Omega$  if and only if  $\lambda_j < 1, \lambda_j \leq 1$ , and  $\lambda_j = 1$ , respectively, for all  $j$ . (In particular, every tripotent belongs to  $\partial \Omega$ .) Associated to a tripotent  $v \in \bar{\Omega}$  is the Peirce decomposition

$$Z = Z_1(v) \oplus Z_{1/2}(v) \oplus Z_0(v),$$

with  $Z_\nu(v) = \text{Ker}(D(v, v) - \nu)$ ,  $\nu = 0, \frac{1}{2}, 1$ . The subspace  $Z_1(v)$  is a  $JB^*$ -algebra under the product  $(x, y) \mapsto \{xvy\}$ , with unit  $v$  and involution  $z^* = \{vzv\}$ .

Two elements (not necessarily tripotents)  $a, b \in Z$  are said to be *orthogonal* if  $D(a, b) = 0$ . This is equivalent to the existence of a tripotent  $v$  such that  $a \in Z_1(v), b \in Z_0(v)$ . Equivalently,  $a$  and  $a'$  are orthogonal if and only if they can both be represented in the form (4.2), with the same  $e_j$ , and with the corresponding numbers  $\lambda_j, \lambda'_j$  satisfying  $\lambda_j \lambda'_j = 0$  for all  $j$ .

To a system  $e_1, \dots, e_l$  of pairwise orthogonal tripotents, there is similarly associated a *joint Peirce decomposition*

$$Z = \bigoplus_{0 \leq i \leq j \leq l} E_{ij}$$

of  $Z$  into orthogonal subspaces

$$E_{ij} = \left\{ x \in \mathbb{C}^n; D(e_k, e_k)x = \frac{\delta_{ik} + \delta_{kj}}{2} x \quad \forall k = 0, \dots, l \right\}.$$

In terms of the canonical projections  $P_{ij}$  onto  $E_{ij}$ , the operator  $B(a, b)$  for  $a = \sum_1^l \alpha_j e_j$  and  $b = \sum_1^l \beta_j e_j$  is given by

$$(4.3) \quad B(a, b) = \sum_{0 \leq i \leq j \leq l} (1 - \alpha_i \bar{\beta}_i)(1 - \alpha_j \bar{\beta}_j) P_{ij}$$

(where  $\alpha_0 := 0, \beta_0 := 0$ ). Notice that if  $\|a\| < 1, \|b\| \leq 1$  then  $B(a, b)$  is invertible. Similarly,  $\|a\| \leq 1$  and  $\|b\| < 1$  implies that  $B(a, b)$  is invertible (we use  $\|a\| = \max |\alpha_j|, \|b\| = \max |\beta_j|$ ).

For  $f(t)$  an odd complex-valued function of a real variable  $t \in (-\rho, \rho)$  and  $a$  of the form (4.2) with  $\lambda_j < \rho$ , one defines the *odd functional calculus* by setting

$$f(a) = f(\lambda_1)e_1 + \dots + f(\lambda_l)e_l.$$

This gives a well-defined (i.e. independent of the choice of the representation of  $a$  in the form (4.2)) element of  $Z$ . If  $f$  is real-analytic with power series expansion  $f(t) = \sum_0^\infty f_{2k+1} t^{2k+1}$  converging for  $|t| < \rho$ , then  $f(a)$  can equivalently be defined as the sum of the convergent series  $\sum_{k=0}^\infty f_{2k+1} a^{(2k+1)}$ , where  $a^{(2k+1)} = Q(a)^k a$ .

For a tripotent  $v$ , denote

$$D_\nu(v) = \Omega \cap Z_\nu(v) \quad \left( \nu = 0, \frac{1}{2}, 1 \right)$$

and let

$$\Omega(v) = v + D_0(v)$$

be the boundary face of  $\bar{\Omega}$  whose center is  $v$ . Every  $b \in \partial\Omega$  belongs to some  $\Omega(v)$  for a unique tripotent  $v$ , and, conversely,  $\Omega(v) \subset \partial\Omega$  for any nonzero tripotent  $v$ .  $D_0(v)$  is a Cartan domain of type  $(r - \text{rank}(v), a, b)$ . It is known that  $G$  permutes the boundary faces of  $\Omega$ . Thus for a tripotent  $v$  and  $g \in G$  there exists a (unique) tripotent  $\tilde{v}$  such that  $g(\Omega(v)) = \Omega(\tilde{v})$ ; moreover,  $\text{rank}(\tilde{v}) = \text{rank } v$ . Let

$$S_l = \{\text{tripotents of rank } l\},$$

$$\partial_l \Omega = \bigcup_{v \in S_l} \Omega(v) \quad (1 \leq l \leq r).$$

Then the decompositions of  $\bar{\Omega}$  and  $\partial\Omega$  into  $G$ -orbits are, respectively,

$$\bar{\Omega} = \Omega \cup \partial_1 \Omega \cup \dots \cup \partial_r \Omega$$

and

$$\partial\Omega = \partial_1\Omega \cup \dots \cup \partial_r\Omega.$$

The set  $\partial_r\Omega$  of all maximal tripotents coincides with the Shilov boundary  $\partial_e\Omega$ .

The following is the main result of [KS].

**THEOREM (Kaup-Sauter).** — *Let  $c \in \partial\Omega$ ,  $z \in \Omega$ . Then the following limits exist pointwise and locally uniformly:*

$$s_c(z) := \lim_{\Omega \ni a \rightarrow c} s_a(z), \quad g_c(z) := \lim_{\Omega \ni a \rightarrow c} g_a(z),$$

and the following limit exists in the operator norm topology:

$$B(c, c) = \lim_{\Omega \ni a \rightarrow c} B(a, a).$$

Moreover, the maps  $(z, a) \mapsto s_a(z)$  and  $(z, a) \mapsto g_a(z)$  extend to continuous maps  $(\overline{\Omega} \times \overline{\Omega}) \setminus (\partial\Omega \times \partial\Omega) \rightarrow \overline{\Omega}$ , and for  $(z, a) \in \Omega \times \overline{\Omega}$  they are still given by the formulas (4.1), and  $B(a, a)$  by (4.3). In particular, for  $v$  a tripotent and  $z \in \Omega$ ,

$$(4.4) \quad g_v(z) = v + z_0 - \{z_{1/2}, (v + z_1^*)^{-1}, z_{1/2}\}$$

where  $z = z_1 + z_{1/2} + z_0$  is the Peirce decomposition of  $z$  relative to  $v$ .

Since  $\phi_a = g_a s_0$ , we certainly have also the limit for  $c \in \partial\Omega$

$$\phi_c(z) := \lim_{\Omega \ni a \rightarrow c} \phi_a(z)$$

locally uniformly on  $\Omega$ , and

$$\phi_v(z) = v - z_0 - \{z_{1/2}, (v - z_1^*)^{-1}, z_{1/2}\} \quad (z \in \Omega).$$

**PROPOSITION 4.1.** — *Let  $v$  be a tripotent. Then  $s_v = \phi_v$ .*

*Proof.* — We know by [KS], Lemma 3.2, that  $m(z) = h^{-1}(z)$  and  $s_z = \phi_{h(z)}$ , where  $h(t) = \frac{2t}{1+t^2}$ ,  $h^{-1}(s) = \frac{s}{1+\sqrt{1-s^2}}$  and  $h(z), h^{-1}(z)$  are defined in the sense of the odd functional calculus. In particular, since  $h^{-1}(1) = 1 = h(1)$ , we have  $h^{-1}(v) = v = h(v)$ , and it follows that  $s_v = \phi_v$ . □

COROLLARY 4.2. — For every  $c = v + \alpha \in \partial\Omega$ , with  $v$  a tripotent and  $\alpha \in D_0(v)$ , we have

$$m(c) := \lim_{\Omega \ni a \rightarrow c} m(a) = h^{-1}(c) = v + h^{-1}(\alpha) = v + m(\alpha).$$

Consequently,

$$\phi_c = s_{m(c)} = s_{v+m(\alpha)}.$$

*Proof.* — The function  $a \mapsto h^{-1}(a)$  is continuous from  $\bar{\Omega}$  into itself. Hence  $\lim_{a \rightarrow c} m(a) = \lim_{a \rightarrow c} h^{-1}(a) = h^{-1}(c)$ . Now, if  $f$  is an odd continuous function, extended to  $Z$  via the functional calculus, then if  $z, w$  are orthogonal elements then  $f(z), f(w)$  are orthogonal and  $f(z + w) = f(z) + f(w)$ . It follows that  $h^{-1}(c) = h^{-1}(v + \alpha) = h^{-1}(v) + h^{-1}(\alpha)$ . As  $h^{-1}(v) = v$ , it follows that  $h^{-1}(c) = v + h^{-1}(\alpha) = v + m(\alpha)$ .  $\square$

LEMMA 4.3. — Let  $a, b \in \Omega$  be orthogonal. Then

$$g_a(b) = a + b = g_b(a).$$

*Proof.* — Recall that the quadratic vector field  $\xi_\alpha(z) := \alpha - \{z\alpha z\}$  is related to the transvections via

$$\exp(\xi_\alpha) = g_{\tanh(\alpha)} \quad \forall \alpha \in Z.$$

Here  $\tanh(\alpha)$  is defined via the odd functional calculus (in fact – by a convergent power series with odd powers). Let  $\alpha = \tanh^{-1}(a)$ ; then  $\alpha$  is also orthogonal to  $b$ . Let  $u_t = u_t(b) = \exp(t\xi_\alpha)(b) = g_{\tanh(t\alpha)}(b)$ . Then  $u_t$  is the unique solution of the initial value problem  $u_0 = b, \frac{d}{dt}u_t = \xi_\alpha(u_t)$ . Let  $v_t = \tanh(t\alpha) + b$ . Then  $v_0 = b$  and

$$\begin{aligned} \frac{d}{dt}v_t &= \frac{d}{dt} \tanh(t\alpha) = \alpha - \{\tanh(t\alpha), \alpha, \tanh(t\alpha)\} \\ &= \alpha - \{v_t, \alpha, v_t\} \quad (\text{since } \alpha \perp b) \\ &= \xi_\alpha(v_t). \end{aligned}$$

It follows that  $u_t = v_t, \forall t \in \mathbb{R}$ . In particular, at  $t = 1$  we obtain

$$g_a(b) = g_{\tanh(\alpha)}(b) = u_1 = v_1 = \tanh(\alpha) + b = a + b.$$

This completes the proof.  $\square$

*Remark.* — For  $a, b \in \Omega$  not necessarily orthogonal, we have only the following weaker result:

$$\text{there is a } k \in K \text{ such that } g_a(b) = kg_b(a).$$

Indeed, as  $g_b g_a(0) = g_b(a)$  we conclude that there is a  $k_1 \in K$  such that  $g_b g_a = g_{g_b(a)} k_1$ . Since  $g_c^{-1} = s_0 g_c s_0$ , we obtain by inversion  $k_1^{-1} s_0 g_{g_b(a)} s_0 = s_0 g_a g_b s_0$ , and substituting  $z = 0$  we get  $k_1^{-1} s_0(g_b(a)) = s_0 g_a(b)$ , hence  $g_a(b) = k g_b(a)$  with  $k = s_0 k_1^{-1} s_0 = k_1^{-1}$ .  $\square$

LEMMA 4.4. — *Let  $a, b \in Z$  be orthogonal. Then*

$$B(a + b, a + b) = B(a, a)B(b, b) = B(b, b)B(a, a).$$

*Proof.* — Let  $\{e_j\}_{j=1}^r$  be a frame of orthogonal minimal tripotents so that  $a = \sum_1^k \alpha_j e_j$ ,  $b = \sum_{k+1}^r \beta_j e_j$ , and let  $c = a + b$  and  $\gamma_j = \alpha_j$  ( $1 \leq j \leq k$ ),  $\gamma_j = \beta_j$  ( $k < j \leq r$ ). Then  $c = \sum_{j=1}^r \gamma_j e_j$ , and thus by (4.3)

$$B(c, c) = \sum_{0 \leq i \leq j \leq k} (1 - |\alpha_i|^2)(1 - |\alpha_j|^2)P_{ij} + \sum_{\substack{0 \leq i \leq k \\ k < j \leq r}} (1 - |\alpha_i|^2)(1 - |\beta_j|^2)P_{ij} \\ + \sum_{k < i \leq j \leq r} (1 - |\beta_i|^2)(1 - |\beta_j|^2)P_{ij}.$$

(Here again  $\alpha_0 := 0, \beta_0 := 0$ .) On the other hand, letting  $v = \sum_{i=1}^k e_i$  we get

$$B(a, a) = \sum_{0 \leq i \leq j \leq k} (1 - |\alpha_i|^2)(1 - |\alpha_j|^2)P_{ij} + \sum_{\substack{0 \leq i \leq k \\ k < j \leq r}} (1 - |\alpha_i|^2)P_{ij} + P_0(v), \\ B(b, b) = P_1(v) + \sum_{\substack{0 \leq i \leq k \\ k < j \leq r}} (1 - |\beta_j|^2)P_{ij} + \sum_{k < i \leq j \leq r} (1 - |\beta_i|^2)(1 - |\beta_j|^2)P_{ij}.$$

Thus  $[B(a, a), B(b, b)] = 0$  and

$$B(a, a)B(b, b) = \sum_{0 \leq i \leq j \leq k} (1 - |\alpha_i|^2)(1 - |\alpha_j|^2)P_{ij} + \sum_{\substack{0 \leq i \leq k \\ k < j \leq r}} (1 - |\alpha_i|^2)(1 - |\beta_j|^2)P_{ij} \\ + \sum_{k < i \leq j \leq r} (1 - |\beta_i|^2)(1 - |\beta_j|^2)P_{ij} \\ = B(c, c). \quad \square$$

COROLLARY 4.5. — *Let  $a, b$  be orthogonal elements in  $\overline{\Omega} = \{z; \|z\| \leq 1\}$ . Then*

$$B(a + b, a + b)^{1/2} = B(a, a)^{1/2}B(b, b)^{1/2} = B(b, b)^{1/2}B(a, a)^{1/2}.$$

Remember that if  $\|a\| < 1, \|b\| < 1$  and  $a, b$  are orthogonal, then each of  $B(a, a), B(b, b), B(a + b, a + b)$  is invertible.

PROPOSITION 4.6. — *Let  $a, b \in \Omega$  be orthogonal. Then*

$$g_{a+b} = g_a g_b = g_b g_a.$$

*Proof.* — We have  $g_{a+b}(0) = a + b$  and  $g_a(g_b(0)) = g_a(b) = a + b$ , by Lemma 4.3. It follows that there is a  $k \in K$  for which

$$g_{a+b}k = g_a g_b.$$

Differentiating at the origin we obtain

$$g'_{a+b}(0)k = g'_a(b)g'_b(0).$$

Now recall that, quite generally, for  $z, w \in \Omega$ ,

$$g'_z(w) = B(z, z)^{1/2}B(-w, z)^{-1}.$$

Thus we get

$$\begin{aligned} B(a + b, a + b)^{1/2}k &= B(a, a)^{1/2}B(-b, a)^{-1}B(b, b)^{1/2} \\ &= B(a, a)^{1/2}B(b, b)^{1/2} \end{aligned}$$

(since the orthogonality of  $a$  and  $b$  implies  $B(-b, a) = I$ ). The invertibility of  $B(a, a)^{1/2}, B(b, b)^{1/2}$  and  $B(a + b, a + b)^{1/2}$  and Corollary 4.5 imply that  $k = I$ . Thus  $g_{a+b} = g_a g_b$ . The formula  $g_{a+b} = g_b g_a$  is proved similarly.  $\square$

COROLLARY 4.7. — *Let  $v$  be a tripotent and  $\alpha \in D_0(v)$ . Then*

$$g_{v+\alpha} = g_v g_\alpha = g_\alpha g_v.$$

*Proof.* — Use the last proposition with  $a = \alpha$  and  $b = tv$  with  $t < 1$ , and let  $t \nearrow 1$ .  $\square$

*Remark.* — It is possible to give another proof of Proposition 4.6 along the following lines. Let  $\alpha, \beta$  be orthogonal elements of  $Z$  and set  $a = \tanh(\alpha), b = \tanh(\beta)$ . Then  $a, b \in \Omega$  and they are orthogonal. Let  $\xi_\alpha(z) = \alpha - \{z\alpha z\}$ . Then  $g_a = \exp(\xi_\alpha)$ , and similarly  $g_b = \exp(\xi_\beta)$ . Also  $\xi_{\alpha+\beta} = \xi_\alpha + \xi_\beta, \tanh(\alpha + \beta) = \tanh(\alpha) + \tanh(\beta) = a + b$ . Thus  $g_{a+b} = \exp(\xi_{\alpha+\beta}) = \exp(\xi_\alpha + \xi_\beta)$ . Now

$$[\xi_\alpha, \xi_\beta] = 2(D(\beta, \alpha) - D(\alpha, \beta)) = 0$$



since  $\alpha, \beta$  are orthogonal. Consequently (cf. [He]), we can conclude that

$$\exp(\xi_\alpha + \xi_\beta) = \exp(\xi_\alpha) \exp(\xi_\beta) = \exp(\xi_\beta) \exp(\xi_\alpha),$$

that is,

$$g_{a+b} = g_a g_b = g_b g_a$$

as desired. □

We actually have a stronger result.

PROPOSITION 4.8. — *Let  $v$  be a tripotent and let  $a, c \in D_1(v)$  and  $b, d \in D_0(v)$ . Then*

$$g_{a+b}(c + d) = g_a(c) + g_b(d).$$

*Proof.* — Since  $g_{a+b} = g_a g_b = g_b g_a$  by Proposition 4.6, it suffices to prove the assertion in the case when one of  $a, b$  is zero. For definiteness we shall prove  $g_a(c + d) = g_a(c) + d$ . For fixed  $d$  and variable  $c$  in  $D_1(v)$  consider the holomorphic map  $F : D_1(v) \rightarrow \Omega$  defined by

$$F(c) = g_a(c + d) - g_a(c) - d.$$

Then  $F(0) = g_a(d) - a - d = 0$  from Lemma 4.3. On the other hand,

$$F'(c) = B(a, a)^{1/2} [B(-c - d, a)^{-1} - B(-c, a)^{-1}] = 0$$

since  $D(c + d, a) = D(c, a)$  and  $Q(c + d)Q(a) = Q(c)Q(a)$  by the orthogonality of  $d$  and  $a$ . Hence  $F$  is the constant map  $F(c) \equiv F(0) = 0$ . □

*Remark.* — Again, another proof of the equality  $g_a(c + d) = g_a(c) + d$  can be given along the lines of the proof of Lemma 4.3. Namely, let  $\alpha \in Z_1(v)$  be such that  $\tanh(\alpha) = a$ , and define

$$v_t = d + \exp \xi_{t\alpha}(c).$$

Then  $v_0 = d + c$  and, as before,

$$\frac{d}{dt} v_t = \xi_\alpha(\exp \xi_{t\alpha}(c)) = \xi_\alpha(v_t).$$

Hence  $v_t = \exp(\xi_{t\alpha})(d + c)$  by the uniqueness of the solution to an initial value problem. In particular, at  $t = 1$  we get  $g_a(d + c) = v_1 = d + g_a(c)$ . □

For a tripotent  $v$ , introduce the mapping

$$(4.5) \quad \rho_v(z) = z_0 - \{z_{1/2}, (v + z_1^*)^{-1}, z_{1/2}\}$$

where  $z = z_1 + z_{1/2} + z_0$  is the Peirce decomposition of  $z$  relative to  $v$ . We know by (4.4) that

$$(4.6) \quad g_v(z) = v + \rho_v(z).$$

From (4.5) it is immediate that

$$\rho_v \circ \rho_v = \rho_v$$

and

$$\rho_v(\Omega) = D_0(v),$$

namely, that  $\rho_v$  is a holomorphic retraction of  $\Omega$  onto  $D_0(v)$ .

PROPOSITION 4.9. — *Let  $v$  be a tripotent and  $\alpha \in D_0(v)$ . Then for all  $z \in \Omega$ ,*

$$g_{v+\alpha}(z) = v + g_\alpha(\rho_v(z)) = v + \rho_v(g_\alpha(z)).$$

*In particular,*

$$g_\alpha(\rho_v(z)) = \rho_v(g_\alpha(z)).$$

*Proof.* — This follows by Corollary 4.7, (4.6) and Proposition 4.8.  $\square$

Denoting also

$$\rho^v(z) = s_0 \rho_v s_0(z) = z_0 + \{z_{1/2}, (v - z_1^*)^{-1}, z_{1/2}\}$$

we get from  $\phi_a = g_a s_0$ ,  $\phi_a = s_{m(a)}$  the analog of Proposition 4.9 for  $s_a$  and  $\phi_a$ .

PROPOSITION 4.10. — *Let  $v$  be a tripotent. Then  $\rho^v \circ \rho^v = \rho^v$  and  $\rho^v(\Omega) = D_0(v)$ , i.e.  $\rho^v$  is a holomorphic retraction of  $\Omega$  onto  $D_0(v)$ . Further, for  $\alpha \in D_0(v)$ ,*

$$\begin{aligned} s_{v+\alpha}(z) &= v + s_\alpha(\rho^v(z)), \\ \phi_{v+\alpha}(z) &= v + \phi_\alpha(\rho^v(z)). \end{aligned}$$

*Proof.* — Let  $\beta = m^{-1}(\alpha)$ , i.e.  $\alpha = m(\beta)$ . Then (cf. Corollary 4.2)

$$\begin{aligned} s_{v+\alpha}(z) &= s_{v+m(\beta)}(z) = s_{m(v+\beta)}(z) = \phi_{v+\beta}(z) \\ &= g_{v+\beta}(s_0(z)) = v + g_\beta(\rho_v(-z)) \\ &= v + \phi_\beta(s_0 \rho_v s_0(z)) = v + \phi_\beta(\rho^v(z)) \\ &= v + s_\alpha(\rho^v(z)). \end{aligned} \quad \square$$

### 5. Convolution operators on Cartan domains: behavior on the boundary.

As before let  $\Omega = G/K$  be a Cartan domain,  $\mu$  a  $K$ -invariant probability measure on  $\Omega$  absolutely continuous with respect to the Lebesgue measure, and  $B_\mu f := f * \mu$  the convolution operator (3.1). The results of the preceding section have an immediate consequence about the operator  $B_\mu$ .

PROPOSITION 5.1. — *Let  $v$  be a tripotent and  $f$  a function in  $L^\infty(\Omega)$  which extends continuously to  $v + D_0(v)$ . Then  $B_\mu f$  has the same property. Further, the restriction  $B_\mu f|_{v+D_0(v)}$  depends only on  $f|_{v+D_0(v)}$ .*

*In particular,  $B_\mu$  maps  $C(\overline{\Omega})$  into itself.*

*Proof.* — By the Lebesgue Dominated Convergence Theorem,  $\lim_{a \rightarrow v+\beta} \phi_a = \phi_{v+\beta}$  implies

$$(5.1) \quad \lim_{a \rightarrow v+\beta} B_\mu f(a) = \int_{\Omega} f(\phi_{v+\beta}(z)) d\mu(z).$$

As  $\phi_{v+\beta}$  maps  $\Omega$  onto  $v + D_0(v)$ , the assertion follows. □

We will denote the last limit simply by  $B_\mu f(v + \beta)$ .

For a function  $f \in C(\overline{\Omega})$  and  $v$  a tripotent, denote by

$$(5.2) \quad f|_{v+D_0(v)}(\zeta) := f(v + \zeta), \quad \zeta \in D_0(v),$$

the “restriction”<sup>1</sup> of  $f$  to the boundary face  $v + D_0(v)$ . Recall that  $D_0(v)$  is itself a bounded symmetric domain (of type  $(r - \text{rank}(v), a, b)$ ) and denote by  $K_0(v)$  its corresponding  $K$ -group. It can be shown that the restriction map  $k \mapsto k|_{D_0(v)}$  induces an isomorphism  $K_0(v) \simeq \{k \in K : kv = v\} / \{k \in K : k|_{v+D_0(v)} = \text{id}\}$ . In particular, every element of  $K_0(v)$  is a restriction of some  $k \in K$  which fixes  $v$ .

THEOREM 5.2. — *Let  $v$  be a tripotent. Then there exists a  $K_0(v)$ -invariant probability measure  $\mu_v$  on  $D_0(v)$ , absolutely continuous with respect to the Lebesgue measure on  $D_0(v)$ , such that*

$$(5.3) \quad (B_\mu f)|_{v+D_0(v)} = f|_{v+D_0(v)} * \mu_v,$$

where the convolution is taken in  $D_0(v)$ .

---

<sup>1</sup> Strictly speaking, a genuine restriction would require  $\zeta + v$  instead of  $\zeta$  on the left-hand side; the reason for the different choice is that we prefer  $f|_{v+D_0(v)}$  to be defined again on a bounded symmetric domain centered at the origin, i.e.  $D_0(v)$ , and not on its translate  $v + D_0(v)$  in  $\mathbb{C}^n$ .

In other words, the “restriction to a boundary face” of the convolution operator  $f \mapsto f * \mu$  is again an operator of this form.

*Proof.* — Define the measure  $\mu_v$  on  $D_0(v)$  by

$$\int_{D_0(v)} F(\zeta) d\mu_v(\zeta) := \int_{\Omega} F(\rho_v(z)) d\mu(z).$$

Then by (5.1) one has for any  $\beta \in D_0(v)$ ,

$$\begin{aligned} B_\mu f(v + \beta) &= \int_{\Omega} f(g_{v+\beta}(z)) d\mu(z) \\ &= \int_{\Omega} f(v + g_\beta(\rho_v(z))) d\mu(z) \\ &= \int_{D_0(v)} f(v + g_\beta(\zeta)) d\mu_v(\zeta) \\ &= \int_{D_0(v)} f|_{v+D_0(v)}(g_\beta(\zeta)) d\mu_v(\zeta). \end{aligned}$$

Here we have used, in turn, (3.3); Proposition 4.9; the definition of  $\mu_v$ ; and (5.2). Finally, in view of (4.1) the transvection with respect to  $\beta$  in  $D_0(v)$  coincides with the restriction of  $g_\beta$  to  $D_0(v)$ , and thus (5.3) follows.

The assertion concerning absolute continuity follows from the holomorphy of the map  $z \mapsto \rho_v(z)$ . (In fact, it is immediate from (4.5) that the latter map is even rational and its Jacobi matrix is of full rank at every point  $z \in \Omega$ .)

It remains to prove  $K_0(v)$ -invariance. In view of the remark preceding this theorem, it suffices to show that  $d\mu_v(k(\zeta)) = d\mu_v(\zeta)$  for any  $k \in K$  which fixes  $v$ . However, by the definition of  $\mu_v$ ,

$$\begin{aligned} \int_{D_0(v)} F(k(\zeta)) d\mu_v(\zeta) &= \int_{\Omega} F(k\rho_v(z)) d\mu(z) \\ &= \int_{\Omega} F(\rho_{kv}(kz)) d\mu(z) \\ &= \int_{\Omega} F(\rho_v(kz)) d\mu(z) \\ &= \int_{\Omega} F(\rho_v(z)) d\mu(z) = \int_{D_0(v)} F(\zeta) d\mu_v(\zeta), \end{aligned}$$

if  $kv = v$  and  $\mu$  is  $K$ -invariant. This completes the proof. □

Analogues of the last two theorems can also be proved for the Poisson transform  $P$ . This result is probably well known, but we were unable to locate a reference in the literature.

**THEOREM 5.3.** — *Let  $F$  be a continuous function on the Shilov boundary  $\partial_e\Omega$  and  $f = PF$  its Poisson extension given by (3.4). Then*

- (a)  $f \in C(\overline{\Omega})$ , and
- (b) *for each nonzero tripotent  $v$ , the restriction of  $f$  to the boundary face  $\Omega(v) = v + D_0(v)$  coincides with the Poisson extension, taken in the bounded symmetric domain  $D_0(v)$ , of the restriction of  $F$  to  $(v + \overline{D_0(v)}) \cap \partial_e\Omega$ , the Shilov boundary of  $v + D_0(v)$ :*

$$f|_{v+D_0(v)} = P[F|_{v+\partial_e(D_0(v))}],$$

where the  $P$  on the right-hand side is taken in  $D_0(v)$ , and

$$F|_{v+\partial_e(D_0(v))}(\zeta) := F(v + \zeta), \quad \zeta \in \overline{D_0(v)} \cap \partial_e\Omega = \partial_e(D_0(v)).$$

The proof uses a lemma. As before,  $d\sigma$  stands for the unique  $K$ -invariant probability measure on  $\partial_e\Omega$ .

**LEMMA 5.4.** — *Let  $b \in \partial\Omega$ . Then for  $d\sigma$ -almost all  $z \in \partial_e\Omega$ , the limit*

$$\lim_{\Omega \ni a \rightarrow b} g_a(z) =: g_b(z)$$

*exists and, further,  $g_b(z)$  lies on the Shilov boundary of the boundary face containing  $b$ .*

*Proof of Lemma 5.4.* — We know from the Kaup-Sauter theorem that the mapping  $a \mapsto B(a, a)^{1/2}$  extends continuously to  $\partial\Omega$ . By the formula (4.1) it therefore follows that the mapping  $(z, a) \mapsto g_a(z)$  extends continuously to the set

$$\mathcal{M} := \{(z, a) \in \overline{\Omega} \times \overline{\Omega} : I + D(z, a) \text{ is an invertible operator on } Z\}.$$

Fix a maximal tripotent  $e$ . Since  $K$  acts transitively on  $\partial_e\Omega$  and  $d\sigma$  is just the image under the projection map  $k \mapsto ke$  of the normalized Haar measure  $dk$  on  $K$ , to prove the first part of the lemma it thus suffices to show that the set  $\{k \in K : I + D(ke, b) \text{ is not invertible}\}$  is of  $dk$ -measure zero. Now  $I + D(ke, b)$  is a linear operator on  $Z \simeq \mathbb{C}^n$ , hence it is invertible if and only if its determinant  $f(k) := \det(I + D(ke, b))$  is nonzero. As  $D(z, b)$  is linear in  $z$ ,  $f$  is a real-analytic function on  $K$ , hence

it either vanishes identically or its zero-set is a real-analytic submanifold of  $K$  of lower dimension. The former is clearly not the case (just take  $k$  so that  $ke$  is a maximal tripotent dominating  $b$ ). As the Haar measure  $dk$  is absolutely continuous with respect to the Lebesgue measure in any local coordinate chart, a lower-dimensional submanifold of  $K$  must have  $dk$ -measure zero. Hence the zero set of  $f$  is of  $dk$ -measure zero, which is the required assertion.

For  $a \in \Omega$ ,  $g_a$  is holomorphic in a neighborhood of  $\bar{\Omega}$ , hence it must map the Shilov boundary  $\partial_e\Omega$  into itself. Letting  $a \rightarrow b$  it follows by the compactness of  $\partial_e\Omega$  that  $g_b(z) \in \partial_e\Omega$  whenever  $z \in \partial_e\Omega$  and the limit defining  $g_b(z)$  exists. Further, for  $(z, b) \in \mathcal{M}$  we also have  $g_b(z) = \lim_{\Omega \ni x \rightarrow z} g_b(x) \in \overline{g_b(\Omega)}$  = the closure of the boundary face containing  $b$ , by (4.6). Since the Shilov boundary of a face is precisely the intersection of the closure of the face with  $\partial_e\Omega$ , the second assertion of the lemma follows. □

*Proof of Theorem 5.3.* — Applying the Dominated Convergence Theorem to the integral

$$PF(a) = \int_{\partial_e\Omega} F(g_a(z)) d\sigma(z), \quad \Omega \ni a \rightarrow b \in \partial\Omega,$$

and using the first part of the lemma, we obtain (a). For (b), let  $v$  be a tripotent and  $\beta \in D_0(v)$ , and define a probability measure  $d\sigma_v$  by the recipe

$$\int F(\zeta) d\sigma_v(\zeta) := \int_{\partial_e\Omega} F(g_v(z)) d\sigma(z) = PF(v).$$

By the second part of the lemma,  $d\sigma_v$  is supported on  $v + \partial_e(D_0(v))$ . The same argument as in the proof of Theorem 5.2 shows that  $d\sigma_v$  is  $K_0(v)$ -invariant. Hence it must coincide with the unique  $K_0(v)$ -invariant probability measure on  $v + \partial_e(D_0(v))$ , and by Corollary 4.7,

$$\begin{aligned} PF(v + \beta) &= \int_{\partial_e\Omega} F(g_{v+\beta}(z)) d\sigma(z) \\ &= \int_{\partial_e\Omega} F(g_\beta(g_v(z))) d\sigma(z) \\ &= \int_{v+\partial_e(D_0(v))} F(g_\beta(\zeta)) d\sigma_v(\zeta), \end{aligned}$$

as claimed. □

We can now state our final result about the convergence of the iterates  $B_\mu^k f$  of the convolution operator (3.2).

**THEOREM 5.5.** — *Let  $f \in C(\overline{\Omega})$ . Then as  $k \rightarrow \infty$ ,  $B_\mu^k f$  tends uniformly on  $\overline{\Omega}$  to the Poisson extension (3.4) of  $f|_{\partial_e \Omega}$ .*

*Proof.* — As, in view of Theorem 5.3(a),  $P[f|_{\partial_e \Omega}] \in C(\overline{\Omega})$ , we can replace  $f$  by  $f - P[f|_{\partial_e \Omega}]$  and proceed as in the proof of Theorem 1.6 to reduce the problem to showing that the function  $h(z)$ , defined by (3.6), satisfies  $B_\mu^k h \rightrightarrows 0$  on  $\overline{\Omega}$ . Owing to Proposition 5.1 we now have  $B_\mu^k h \searrow g$  not only on  $\Omega$  but even on  $\overline{\Omega}$ , with some nonnegative (but possibly discontinuous) function  $g$  on  $\overline{\Omega}$ . By the Lebesgue Monotone Convergence Theorem and Theorem 5.2,  $g = g * \mu$  on  $\Omega$  and also  $g|_{v+D_0(v)} = g|_{v+D_0(v)} * \mu_v$  for every tripotent  $v$ . Therefore by Fürstenberg's theorem,  $g|_{v+D_0(v)}$  is harmonic for every  $v$  (we set  $v + D_0(v) = \Omega$  for  $v = 0$ ). Since  $h$  vanishes on the Shilov boundary,  $0 \leq g \leq h$  on  $\overline{\Omega}$ , and  $\partial_e(v + D_0(v)) \subset \partial_e \Omega$ , it then follows from the Poisson formula (3.4) applied to  $D_0(v)$  that  $g|_{v+D_0(v)} \equiv 0$ , for every  $v$ , i.e.,  $g \equiv 0$  on  $\overline{\Omega}$ . Thus  $B_\mu^k h \searrow 0$  on  $\overline{\Omega}$ , and the uniformity of convergence follows by Dini's theorem.  $\square$

## 6. Concluding remarks.

For unbounded functions, the iterates  $B^k f$  can in general diverge and even blow up to infinity. The simplest example occurs when  $B$  is the Berezin transform on the unit disc  $\mathbb{D}$  with respect to the normalized Lebesgue measure  $dm$  mentioned in the Introduction (the operator of convolution with  $dm$  on  $\mathbb{D}$ ) and  $f(x) = \log(1 - |x|^2)$ . One can then show that  $Bf = f + 1$ , and thus  $B^k f$  is uniformly divergent to infinity. Note that  $f \in L^p(\mathbb{D}, dm)$  for all  $0 < p < \infty$ .

Another family of examples, for  $B$  the convolution operator (3.1) with a  $K$ -invariant probability measure  $\mu$  on a Cartan domain  $\Omega$ , is furnished by the spherical functions  $\phi_\lambda$  ( $\lambda \in \mathfrak{a}^* \simeq \mathbb{C}^r$ , where  $\mathfrak{a}$  is the Lie algebra of  $A$ ,  $*$  stands for the dual and  $r$  is the rank of  $\Omega$ ). It is known that  $\phi_\lambda$  are eigenfunctions of the operator  $B_\mu$ :

$$(6.1) \quad B_\mu \phi_\lambda = \hat{\mu}(\lambda) \phi_\lambda,$$

where  $\hat{\mu}(\lambda)$ , the spherical transform of  $\mu$ , is a holomorphic function of  $\lambda$ . One has  $\hat{\mu}(\lambda) = 1$  if  $\lambda \in W\mathfrak{p}$ , the orbit under the Weyl group  $W$  of the half-sum of positive roots  $\mathfrak{p}$ , since for these  $\lambda$   $\phi_\lambda = \mathbf{1}$ . Further,  $\phi_\lambda \in L^\infty$  if and

only if  $\operatorname{Re} \lambda \in \operatorname{co}(W\rho)$ , the convex hull of  $W\rho$ , while for any  $p < \infty$ , there is an open convex set  $\mathcal{U}$  containing  $W\rho$  such that  $\phi_\lambda \in L^p(\Omega, dm)$  whenever  $\operatorname{Re} \lambda \in \mathcal{U}$ . See [AZ]. By the open mapping theorem for holomorphic functions, in any neighborhood of  $\rho$  there exists a  $\lambda$  for which  $|\hat{\mu}(\lambda)| > 1$ . Since

$$B_\mu^k \phi_\lambda = \hat{\mu}(\lambda)^k \phi_\lambda,$$

it follows that for any  $p < \infty$ , there exists a spherical function  $\phi_\lambda \in L^p(\Omega, dm)$  for which  $B_\mu^k \phi_\lambda$  diverges (blows up to infinity). Similar argument also works when  $L^p(\Omega, dm)$  is replaced by some other Banach lattices.

There also exist bounded functions  $f$  for which  $B_\mu^k f$  does not converge pointwise. (Even bounded continuous ones — just replace  $f$  by  $B_\mu f$  and recall that  $B_\mu$  maps  $L^\infty$  into  $BC$ .) For  $B_\mu$  the convolution with the Lebesgue measure on the unit ball in  $\mathbb{C}^n$ , this was shown by Lee [Le]. We present a simpler version of his proof, which works for any Cartan domain and some other measures in place of  $\mu$ , and even yields the slightly stronger assertion about Cesàro means.

Let  $\nu$  be the invariant measure on  $\Omega$ . It is known that the operator  $B_\mu$  is formally self-adjoint with respect to  $\nu$ , that is,

$$(6.2) \quad \int_\Omega B_\mu f \cdot g \, d\nu = \int_\Omega f \cdot B_\mu g \, d\nu \quad \forall f \in L^\infty(\Omega), \forall g \in L^1(\Omega, d\nu).$$

(It is enough to prove this for  $f, g$  in  $L^2(d\nu)$ , and then it follows from (6.1) and the Plancherel theorem.)

**THEOREM 6.1.** — *Let  $\Omega$  be a Cartan domain,  $\mu$  a  $K$ -invariant probability measure on  $\Omega$  absolutely continuous with respect to the Lebesgue measure, and  $B_\mu$  the convolution operator (3.1). Assume that there is  $C > 0$  such that  $\mu \leq C\nu$ . Then there exists  $f \in BC(\Omega)$  for which the Cesàro means  $C_k f = \frac{1}{k} \sum_{j=0}^{k-1} B_\mu^j f$  do not converge pointwise.*

*Proof.* — It is enough to produce  $f$  in  $L^\infty$  with this property (then just replace  $f$  by  $B_\mu f$ ). Assume that, to the contrary,  $\lim_{k \rightarrow \infty} C_k f$  exists pointwise for every  $f \in L^\infty$ . Let  $g \in L^1(\Omega, d\nu)$ . Then by the Dominated Convergence Theorem and (6.2),

$$\lim_{k \rightarrow \infty} \int_\Omega C_k g \cdot f \, d\nu \quad \text{exists} \quad \forall f \in L^\infty.$$

As  $L^1$  is weakly complete, this means that  $C_k g$  converges weakly to  $h \in L^1$ , say. By continuity,  $B_\mu h = h$ . For  $z \in \Omega$ , the hypothesis that  $\mu$  be dominated



by a multiple of  $\nu$  implies

$$|h(z)| = |B_\mu h(z)| \leq \int_\Omega |h \circ g_z| d\mu \leq C \int_\Omega |h \circ g_z| d\nu = C \int_\Omega |h| d\nu = C \|h\|_1$$

by the invariance of  $\nu$ . Thus  $h \in L^\infty$ , so by Fürstenberg's theorem  $h$  is harmonic. As the only harmonic function in  $L^1(d\nu)$  is the constant zero, we thus see that

$$C_k g \rightarrow 0 \text{ weakly } \forall g \in L^1(d\nu).$$

Using again (6.2) with  $f = \mathbf{1}$  we thus get

$$\int_\Omega g d\nu = \int_\Omega C_k \mathbf{1} \cdot g d\nu = \int_\Omega \mathbf{1} \cdot C_k g d\nu \rightarrow 0 \quad \forall g \in L^1(d\nu),$$

a contradiction. This completes the proof.  $\square$

We finish with a remark concerning the measures  $\mu_v$  occurring in Theorem 5.2, that is, the measures defined, for  $\mu$  an absolutely continuous  $K$ -invariant probability measure on a Cartan domain  $\Omega$  and  $v \in \partial\Omega$  a tripotent, by the formula

$$(f *_{\Omega} \mu)|_{v+D_0(v)} = (f|_{v+D_0(v)}) *_{D_0(v)} \mu_v \quad \forall f \in C(\bar{\Omega}),$$

where the subscript at the convolution signs refers to the domain where the convolution is being taken.

PROBLEM. — Give an explicit formula for  $\mu_v$  in terms of  $\mu$ .

In particular, we expect that if  $\mu$  is one of the standard measures  $d\mu(z) = c_\nu h(z, z)^{\nu-p} dm(z)$  (with  $h$  the Jordan triple determinant,  $p$  the genus,  $\nu > p - 1$  and  $c_\nu$  the normalizing constant), then the measure  $\mu_v$  will also be of this type, possibly with a different  $\nu$ .

## BIBLIOGRAPHY

- [Ar] J. ARAZY, A survey of invariant Hilbert spaces of analytic functions on bounded symmetric domains, *Multivariable operator theory* (R.E. Curto, R.G. Douglas, J.D. Pincus, N. Salinas, eds.), *Contemporary Mathematics*, vol. 185, Amer. Math. Soc., Providence, 1995, 7–65.
- [AZ] J. ARAZY, G. ZHANG, Invariant mean value and harmonicity in Cartan and Siegel domains, *Interaction between functional analysis, harmonic analysis, and probability* (Columbia, MO, 1994), *Lecture Notes in Pure and Appl. Math.*, vol. 175, Dekker, New York, 1996, 19–40.
- [AL] S. AXLER, J. LECH, Fixed points of the Berezin transform on multiply connected domains, preprint (1997).

- [FK] J. FARAUT, A. KORÁNYI, Function spaces and reproducing kernels on bounded symmetric domains, *J. Funct. Anal.*, 88 (1990), 64–89.
- [Fü] H. FÜRSTENBERG, Poisson formula for semisimple Lie groups, *Ann. Math.*, 77 (1963), 335–386.
- [Ga] T.W. GAMELIN, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, 1969.
- [Gd] R. GODEMENT, Une généralisation des représentations de la moyenne pour les fonctions harmoniques, *C. R. Acad. Sci. Paris*, 234 (1952), 2137–2139.
- [Go] G.M. GOLUZIN, *Geometric theory of functions of a complex variable*, Nauka, Moscow, 1966.
- [He] S. HELGASON, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
- [KS] W. KAUP, J. SAUTER, Boundary structure of bounded symmetric domains, *Manuscripta Math.*, 101 (2000), 351–360.
- [Kr] S.G. KRANTZ, *Function theory of several complex variables*, Wadsworth & Brooks/Cole, Pacific Grove, 1992.
- [Le] J. LEE, Properties of the Berezin transform of bounded functions, *Bull. Austral. Math. Soc.*, 59 (1999), 21–31.
- [Lo] O. LOOS, *Bounded symmetric domains and Jordan pairs*, University of California, Irvine, 1977.
- [Zh] K. ZHU, A limit property of the Berezin transform, preprint (1999).

Manuscrit reçu le 24 août 2000,  
accepté le 24 janvier 2001.

Jonathan ARAZY,  
University of Haifa  
Department of Mathematics  
Haifa 31905 (Israel)  
jarazy@math.haifa.ac.il  
and  
Miroslav ENGLIŠ,  
Academy of Sciences  
Mathematics Institute  
Žitná 25  
11567 Prague 1 (Czech Republic).  
englis@math.cas.cz