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# SUBALGEBRAS TO A WIENER TYPE ALGEBRA OF PSEUDO-DIFFERENTIAL OPERATORS

by Joachim TOFT

## 0. Introduction.

In 1993 J. Sjöstrand introduced in [S1] without explicit reference to any derivatives, a normed space of symbols, denoted by $S_w$ in [S2], which contains the Hörmander class $S^0_{0,0}$ (smooth functions bounded together with all their derivatives), and such that the corresponding space of pseudo-differential operators of the type

$$\begin{aligned} a_t(x,D)f(x) &= \big(\mathrm{Op}_t(a)f\big)(x) \\ &\equiv (2\pi)^{-n}\iint a((1-t)x+ty,\xi)f(y)e^{i\langle \xi,x-y\rangle}\,dyd\xi, \quad f\in\mathcal{S}(\mathbb{R}^n), \end{aligned} \tag{0.1}$$

when $a\in S_w(\mathbb{R}^{2n})$, is stable under composition and is contained in the space of bounded operators on $L^2(\mathbb{R}^n)$. (Throughout the paper we are going to use the same notations as in [H] for the usual spaces of functions and distributions.) Here $0\leqslant t\leqslant 1$. He discussed also some invariant properties and proved for example that if $a\in S_w$ and $a_s(x,D)=b_t(x,D)$, for some choice of $s,t\in[0,1]$, then $b\in S_w$.

Some further developments and improvements have been made since [S1]. In [S2] J. Sjöstrand proved among other results that $\mathrm{Op}_t(S_w)$ is a

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Wiener algebra. Interesting results in the topic have also been presented by A. Boulkhemair in [B], who extended the results concerning $L^2$-continuity not only to pseudo-differential operators of the type

$$\text{(0.2)}\quad \operatorname{Op}(a)f(x) \equiv (2\pi)^{-n} \iint a(x,y,\xi)f(y)e^{i\langle x-y,\xi\rangle}\,dyd\xi, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

when $a \in S_w(\mathbb{R}^{3n})$, but also to Fourier integral operators, in which the factor $e^{i\langle x-y,\xi\rangle}$ in the integral in (0.2) may be replaced by $e^{i\varphi(x,y,\xi)}$, where the phase function $\varphi$ should satisfy some appropriate growth conditions.

In this paper we shall discuss some continuity properties for an increasing family $S_w^p$, $1 \leqslant p \leqslant \infty$, of symbols such that $S_w^\infty = S_w$. This is done essentially in the framework of the analysis which was used in [S1]. However, when proving the general results, valid for any $S_w^p$-space, we shall apply some additional convexity arguments which are frequently used in many other situations. Our discussions concerning $L^2$-continuity will also be different comparing to [S1]. In these considerations we use the definition of admissible partitions, a generalization of the concept admissible partition of unity, introduced in Section 2.4 in [T1]. Admissible partitions give rise to symbol classes, where the corresponding pseudo-differential operators are continuous on $L^2$, and we obtain $S_w$ by choosing the admissible partition in an appropriate way.

Since the results and the proofs for $S_w$, given in [S1] will be in the background of this paper, we start by briefly recall the definition for $S_w(\mathbb{R}^m)$, when $m \geqslant 1$ is an integer, and write down in Proposition 0.1 below, the results from [S1] which are important to us.

Let $\Lambda \subset \mathbb{R}^m$ be a lattice, and let $\chi \in C_0^\infty(\mathbb{R}^m)$ be non-negative such that if $\chi_x(y) = (\tau_x\chi)(y) \equiv \chi(y-x)$, then $\sum_{j\in\Lambda}\chi_j(y) = 1$. Then $S_w(\mathbb{R}^m)$ is the set of all $a \in \mathcal{S}'(\mathbb{R}^m)$ such that the function

$$\text{(0.3)}\qquad H_{a,\infty}(\xi) \equiv \sup_{j\in\Lambda} |\mathcal{F}(\chi_j a)(\xi)|$$

belongs to $L^1(\mathbb{R}^m)$. Here $\mathcal{F}$ denotes some Fourier transform. J. Sjöstrand then proved the following.

Proposition 0.1 (J. Sjöstrand).— *Let $a, b \in S_w(T^*\mathbb{R}^n)$. Then the following is true:*

(1) *$a_t(x,D)$ is continuous on $L^2(\mathbb{R}^n)$ for every $t \in [0,1]$;*

(2) *if $c \in \mathcal{S}'(T^*\mathbb{R}^n)$ satisfies $c_s(x,D) = a_{t_1}(x,D)b_{t_2}(x,D)$ for some $s, t_1, t_2 \in [0,1]$, then $c \in S_w(T^*\mathbb{R}^n)$;*

(3) *the Hörmander space* $S^0_{0,0}$ *is contained in* $S_w$*;*

(4) $S_w(T^*\mathbb{R}^n)$ *is independent of the choice of* $\chi$ *and the lattice* $\Lambda$*, as long as* $\sum_{j\in\Lambda}\chi_j = 1$*;*

(5) *if* $\Phi$ *is a real valued non-degenerate quadratic form on* $T^*\mathbb{R}^n$*, then the convolution mapping* $a \mapsto e^{i\Phi} * a$ *on* $\mathcal{S}(T^*\mathbb{R}^n)$ *may be uniquely continued to a continuous mapping on* $S_w(T^*\mathbb{R}^n)$.

J. Sjöstrand remarked also in [S1], that he got the ideas for these results by proving that if $a \in \mathcal{S}'(T^*\mathbb{R}^n)$ and that

$$H_{a,1}(\xi) \equiv \sum_{j\in\Lambda} |\mathcal{F}(\chi_j a)(\xi)| \tag{0.3$'$}$$

belongs to $L^1(\mathbb{R}^m)$, then $a_t(x,D)$ is a trace class operator. A strict proof of this result may be found in Chapter IX in [DS].

The functions in (0.3) and (0.3)$'$ are indeed related to each other. More precisely, if $d\mu$ is the measure $d\mu(x) = \sum_{j\in\Lambda}\delta_j(x)$ on $\mathbb{R}^m$, and $H_{a,p}$ is the function

$$H_{a,p}(\xi) = H_{a,p,\chi,d\mu}(\xi) \equiv \Big(\int |\mathcal{F}(\chi_x a)(\xi)|^p\, d\mu(x)\Big)^{1/p}, \tag{0.3$''$}$$

where $1 \leqslant p \leqslant \infty$ (with obvious interpretation when $p=\infty$), then it follows that $H_{a,\infty}$ and $H_{a,1}$ agree with the earlier definitions in (0.3) and (0.3)$'$. By letting $\mathcal{I}_1$ and $\mathcal{I}_\infty$ be the sets of trace-class operators and continuous operators respectively, it follows from Sjöstrand's results above that for any $t \in [0,1]$ and $p \in \{1,\infty\}$, then

$$H_{a,p} \in L^1(\mathbb{R}^{2n}) \qquad \text{implies} \qquad a_t(x,D) \in \mathcal{I}_p. \tag{0.4}$$

One of our goals is to present an extension of this result, where we prove that (0.4) is true for any $p \in [1,\infty]$, when $\mathcal{I}_p$ is the set of Schatten-von Neumann operators of order $p$ on $L^2(\mathbb{R}^n)$.

According to the previous discussions it follows that we are particularly concerned about the set $S^p_w(\mathbb{R}^m)$, where $p \in [1,\infty]$, which consists of all $a \in \mathcal{S}'(\mathbb{R}^m)$ such that $H_{a,p} \in L^1(\mathbb{R}^m)$. As before we observe that any $S^p_w$-space is defined without any reference to derivatives, and that $S^\infty_w$ coincides with $S_w$. In a large part of the paper we are concerned with extending and improving Proposition 0.1 to any $S^p_w$-space. In Theorem 1.5 and Proposition 1.6 we prove for example that (0.4) is true for any $p$, which extends Proposition 0.1 (1). The statement (2) of Proposition 0.1 is generalized into: if $a \in S^p_w(\mathbb{R}^{2n})$, $b \in S^q_w(\mathbb{R}^{2n})$ where $p, q \in [1, \infty]$, and $c \in \mathcal{S}'(\mathbb{R}^{2n})$ satisfies $c_s(x, D) = a_{t_1}(x, D)b_{t_2}(x, D)$ for some choice of $s, t_1, t_2 \in [0, 1]$, then $c \in S^r_w(\mathbb{R}^{2n})$, when $r \in [1, \infty]$ is chosen such that the Hölder condition $1/p + 1/q = 1/r$ is fulfilled. (Cf. Theorem 4.1.)

In order to discuss the invariance property (4) in Proposition 0.1, we observe that for any Borel measure $d\mu$ and any function $\chi$ on $\mathbb{R}^m$, then $H_{a,p,\chi,d\mu}$ in (0.3)$''$ makes sense, and we let $s_{p,\chi,d\mu}(\mathbb{R}^m)$ be the set of all $a \in \mathcal{S}'(\mathbb{R}^m)$ such that $H_{a,p,\chi,d\mu} \in L^1(\mathbb{R}^m)$. Then as a consequence of Theorem 2.7, it follows that if $d\mu$ is a positive periodic Borel measure, and $\chi \in \mathcal{S}(\mathbb{R}^m)$ satisfies

$$\int \chi(y - x)\, d\mu(x) \neq 0 \quad \text{for every } y \in \mathbb{R}^m, \tag{0.5}$$

then $s_{p,\chi,d\mu}(\mathbb{R}^m) = S^p_w(\mathbb{R}^m)$. In particular $s_{p,\chi,d\mu}(\mathbb{R}^m)$ is independent of the choice of test function $\chi$ and measure $d\mu$, which gives an extension and improvement of (4) in Proposition 0.1. Finally we prove in Proposition 2.14 that for any $\Phi$ in Proposition 0.1 (5), then the map $a \mapsto e^{i\Phi} * a$ is continuous on $S^p_w(\mathbb{R}^m)$, for every $p$.

Some other continuity results in the calculus are also discussed. We consider for example compositions of entire functions and prove that if $f$ is entire, then the map $a \mapsto f(a)$ is continuous on $S^\infty_w$. If in addition $f(0) = 0$, then $a \mapsto f(a)$ is continuous on $S^p_w$, for any $p \in [1, \infty]$. The proofs are based on a Hölder type relation, proved in Proposition 3.1, which asserts that $S^p_w \cdot S^q_w \subset S^r_w$ when $p, q, r \in [1, \infty]$ satisfy $1/p + 1/q = 1/r$.

We consider also pseudo-differential operators of the type (0.2), which makes obviously sense when $a \in \mathcal{S}(\mathbb{R}^{3n})$. In [B], one proves that it is possible to extend the definition of $\mathrm{Op}(a)$ in a unique way to any $a \in S^\infty_w(\mathbb{R}^{3n})$, and that one still has that $\mathrm{Op}(a)$ is continuous on $L^2(\mathbb{R}^n)$. Through the papers in [S2] and [B], it seems to be well-known also that for any $a \in S^\infty_w(\mathbb{R}^{3n})$ and $t \in [0, 1]$, then there exists a unique $b \in S^\infty_w(\mathbb{R}^{2n})$ such that $b_t(x, D) = \mathrm{Op}(a)$. In Proposition 4.6 we present a refinement of this result and prove that if $p \in [1, \infty]$ and $a \in S^p_w(\mathbb{R}^{3n})$, then $b \in S^p_w(\mathbb{R}^{2n})$.

In the last part of the paper we discuss a case of $d\mu$-admissible functions where $d\mu$ may not be necessarily periodic. The corresponding symbol classes which arise here, are in some sense related to the Hörmander classes $S(m, g)$ (see Section 18.4–18.6 in [H]), and in general quite different from the $S^p_w$-spaces above.

## 1. Preliminaries.

In this section we introduce the notion of admissible functions, an extension of the concept admissible partition of unity (see Section 2.4 in [T1]), and prove a result which indicate that admissible functions are interesting, when discussing $L^2$-continuity in the Weyl calculus. The considerations are based on an analysis of the Weyl calculus from the operator theoretical point of view, presented in Section 1.4 in [T1], Section 1 in [T2] and Section in [T3]. We start therefore the section by a short review of the results which we need from these papers.

We recall that for any $a \in \mathcal{S}(\mathbb{R}^{2n})$, then $a_t(x,D)$ in (0.1) is continuous on $\mathcal{S}(\mathbb{R}^n)$. The definition extends to any $a \in \mathcal{S}'(\mathbb{R}^{2n})$, for which one obtains a continuous operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. We remark also that the map $a \mapsto a_t(x,D)$ is injective, and that any continuous operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ may be written of the form in (0.1). Hence the map $a \mapsto a_t(x,D)$ is a homeomorphism from $\mathcal{S}'(\mathbb{R}^{2n})$ to the set of continuous operators from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. (See Chapter XVIII in [H] for more details.)

Next we shall consider the Weyl quantization and a certain type of symbols spaces. The Weyl quantization $a^w(x,D)$ for $a \in \mathcal{S}'(\mathbb{R}^{2n})$, is obtained by choosing $t = 1/2$ in (0.1), i.e., $a^w(x,D) = \mathrm{Op}_{1/2}(a)$. We let $s_p(\mathbb{R}^{2n})$ be the set of all $\mathcal{S}'(\mathbb{R}^{2n})$ such that $a^w(x,D) \in \mathcal{I}_p$, where $\mathcal{I}_p$ is the set of Schatten-von Neumann operators of order $p$ on $L^2(\mathbb{R}^n)$. We recall that an operator $T$ on $L^2(\mathbb{R}^n)$ is a Schatten-von Neumann operator of order $p \in [1,\infty]$, if and only if

$$\|T\|_{\mathcal{I}_p} \equiv \sup \Big( \sum_j |(Tf_j, g_j)|^p \Big)^{1/p}$$

is finite. Here the supremum should be taken over all $(f_j)_{j=1}^{\infty} \in \mathrm{ON}(\mathbb{R}^n)$ and $(g_j)_{j=1}^{\infty} \in \mathrm{ON}(\mathbb{R}^n)$, where $\mathrm{ON}(\mathbb{R}^n)$ is the set of orthonormal sequences on $L^2(\mathbb{R}^n)$. We observe that $\|\cdot\|_{\mathcal{I}_1}$, $\|\cdot\|_{\mathcal{I}_2}$ and $\|\cdot\|_{\mathcal{I}_\infty}$ are the trace norm, Hilbert-Schmidt norm and operator norm respectively. In particular, the definition here concerning $\mathcal{I}_1$ and $\mathcal{I}_\infty$ agree with the earlier definitions in the introduction.

We notice also that if $p < \infty$, then $T \in \mathcal{I}_p$, if and only if $T = \sum \lambda_j f_j \otimes g_j$, for some sequences $(\lambda_j)_1^\infty \in l^p$ and $(f_j)_1^\infty, (g_j)_1^\infty \in \mathrm{ON}(\mathbb{R}^n)$ such that $\lambda_j \geqslant 0$ for every $j$, and then $\|T\|_{\mathcal{I}_p} = \|(\lambda_j)_1^\infty\|_{l^p}$. Here and in what follows we identify operators with their corresponding Schwartz kernels. We refer to [Si] for more facts about the $\mathcal{I}_p$-spaces.

For any $a \in \mathcal{S}'(\mathbb{R}^{2n})$, we set $\|a\|_{s_p} = \|a^w(x,D)\|_{\mathcal{I}_p}$. It is then true that $a \in s_p(\mathbb{R}^{2n})$, if and only if $\|a\|_{s_p} < \infty$, and that $a \mapsto a^w(x,D)$ is an isometric homeomorphism from $s_p(\mathbb{R}^{2n})$ to $\mathcal{I}_p$. In particular, $s_p(\mathbb{R}^{2n})$ is a Banach space under the norm $\|\cdot\|_{s_p}$, for any $p$, since similar facts hold for the $\mathcal{I}_p$-spaces (see [Si]).

The spaces $s_1(\mathbb{R}^{2n})$ and $s_2(\mathbb{R}^{2n})$ consist of all symbols whose corresponding Weyl operators are of trace class type and Hilbert-Schmidt respectively. Parseval's formula shows that $\|a\|_{s_2} = (2\pi)^{-n/2}\|a\|_{L^2}$, since the Hilbert-Schmidt norm for an operator is equal to the $L^2$-norm for the Schwartz kernel of the operator. In particular, $s_2(\mathbb{R}^{2n}) = L^2(\mathbb{R}^{2n})$.

In Proposition 1.1 below we list some general facts which we need concerning the $s_p$-spaces, and refer to [T1], [T2] and [T3] for other facts about these spaces. First we recall a few facts concerning the Fourier transform. The Fourier transform which we shall mainly use is defined by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) \equiv \pi^{-m/2} \int f(x) e^{-2i\langle x,\xi\rangle}\, dx, \quad f \in \mathcal{S}(\mathbb{R}^m).$$

In the case when $m = 2n$ is even, we shall use also the symplectic Fourier transform, $\mathcal{F}_\sigma$, defined by

$$(\mathcal{F}_\sigma a)(X) = \widehat{a}(X) \equiv \pi^{-n} \int a(Y) e^{2i\sigma(X,Y)}\, dY.$$

Here $\sigma$ is the standard symplectic form defined by $\sigma(X,Y) = \langle y,\xi\rangle - \langle x,\eta\rangle$, where $X = (x,\xi) \in \mathbb{R}^{2n}$ and $Y = (y,\eta) \in \mathbb{R}^{2n}$, and $dY = dyd\eta$. The definitions for $\mathcal{F}$ and $\mathcal{F}_\sigma$ extend in a usual way to homeomorphisms on $\mathcal{S}'$ which are unitary on $L^2$. We observe also that

$$\mathcal{F}(f * g) = \pi^{m/2} \widehat{f}\, \widehat{g}, \quad \text{and} \quad \mathcal{F}(f\, g) = \pi^{-m/2} \widehat{f} * \widehat{g}, \tag{1.1}$$

where $*$ denotes the convolution, and that (1.1) holds also when $m = 2n$ and the Fourier transforms are replaced by the symplectic Fourier transform. An interesting property, of the symplectic Fourier transform is that $\mathcal{F}_\sigma^2$ is the identity operator. In the statement (2) of the following proposition we get a motivation for our choice of Fourier transform.

PROPOSITION 1.1. — *The following is true for the $s_p$-spaces:*

(1) *let $p_1, p_2, r \in [1,\infty]$ such that $1/p_1 + 1/p_2 = 1/r$. Then the Weyl composition # from $\mathcal{S}(\mathbb{R}^{2n}) \times \mathcal{S}(\mathbb{R}^{2n})$ to $\mathcal{S}(\mathbb{R}^{2n})$ extends to a continuous bilinear map from $s_{p_1}(\mathbb{R}^{2n}) \times s_{p_2}(\mathbb{R}^{2n})$ to $s_r(\mathbb{R}^{2n})$. One has*

$$\|a_1 \# a_2\|_{s_r} \leqslant \|a_1\|_{s_{p_1}} \|a_2\|_{s_{p_2}};$$

*(We recall that the Weyl product $a\#b$, between $a \in \mathcal{S}$ and $b \in \mathcal{S}$ is defined through the relation $(a\#b)^w(x,D) = a^w(x,D)b^w(x,D)$. Cf. Section 18.5 in [H].);*

(2) *the Fourier transformations $\mathcal{F}$ and $\mathcal{F}_\sigma$, the map $a \mapsto \check{a}$ and composition by any translation on $\mathbb{R}^{2n}$ are unitary transformations on $s_p(\mathbb{R}^{2n})$, for every $p \in [1,\infty]$;*

(3) *if $\mu$ is a measure with finite mass $\|\mu\|$, then the mapping $a \mapsto \mu * a$ on $\mathcal{S}(\mathbb{R}^{2n})$ extends to a continuous mapping on $s_p(\mathbb{R}^{2n})$, for every $p \in [1,\infty]$. One has that $\|\mu * u\|_{s_p} \leqslant \|\mu\|\,\|u\|_{s_p}$;*

(4) *the product $\langle a,b\rangle = \int a(x)b(x)\,dx$ on $\mathcal{S}(\mathbb{R}^{2n})$ extends to a duality between $s_p(\mathbb{R}^{2n})$ and $s_{p'}(\mathbb{R}^{2n})$, when $p \in [1,\infty]$. Here $p'$ is the conjugate exponent for $p$ satisfying $1/p + 1/p' = 1$. One has that $|\langle a,b\rangle| \leqslant \|a\|_{s_p}\|b\|_{s_{p'}}$, when $a \in s_p$ and $b \in s_{p'}$. On the other hand, if $a \in s_p$, then $\|a\|_{s_p} = \sup|\langle a,b\rangle|$, where the supremum is taken over all $b \in s_{p'}$ such that $\|b\|_{s_{p'}} \leqslant 1$;*

(5) *the set $C_0^\infty(\mathbb{R}^{2n})$ is dense in $s_p(\mathbb{R}^{2n})$ when $p < \infty$, and dense in the weak* topology when $p = \infty$.*

*Proof.* — We prove only the assertion concerning the Fourier transform $\mathcal{F}$ in (2), since the other results follow from Section 1.4 in [T1], Section 1 in [T2] or Section in [T3]. We observe that $\mathcal{F} = \mathcal{J} \circ \mathcal{F}_\sigma$, where $\mathcal{J}$ is the map on $\mathcal{S}'$, given by $(\mathcal{J}a)(x,\xi) = a(x,-\xi)$. The assertion follows therefore if we prove that $\mathcal{J}$ is unitary on $s_p(\mathbb{R}^{2n})$. But this is obviously true, since we have by some simple calculations, that if $K(x,y)$ is the Schwartz kernel for $a^w(x,y)$, then $K(y,x)$ is the Schwartz kernel for $(\mathcal{J}a)^w(x,D)$, which in turn implies that $\|a^w(x,D)\|_{\mathcal{I}_p} = \|(\mathcal{J}a)^w(x,D)\|_{\mathcal{I}_p}$. The proof is complete. $\square$

We shall need also the following Young type results.

Proposition 1.2. — *Assume that $p,q,r \in [1,\infty]$ satisfy the Young condition $1/p + 1/q = 1 + 1/r$. Then the convolution on $\mathcal{S}(\mathbb{R}^{2n})$ extends uniquely to continuous bilinear mappings $s_p(\mathbb{R}^{2n}) \times s_q(\mathbb{R}^{2n}) \to L^r(\mathbb{R}^{2n})$ and $L^p(\mathbb{R}^{2n}) \times s_q(\mathbb{R}^{2n}) \to s_r(\mathbb{R}^{2n})$ if one requires that $*$ is the ordinary convolution when one of the factors is in $\mathcal{S}$. One has the estimates*

$$\|a*b\|_{L^r} \leqslant (2\pi)^{n/r}\|a\|_{s_p}\|b\|_{s_q} \quad (\|a*b\|_{s_r} \leqslant (2\pi)^{n/p'}\|a\|_{L^p}\|b\|_{s_q}),$$

*when $a \in s_p(\mathbb{R}^{2n})$ ($a \in L^p(\mathbb{R}^{2n})$) and $b \in s_q(\mathbb{R}^{2n})$.*

*Proof.* — The result follows from Theorem 2.2.3 in [T1], Theorem 1.13 in [T2] or Theorem 2.1 in [T3]. □

PROPOSITION 1.3. — *Assume that $p_1, p_2, \dots, p_N, r \in [1, \infty]$ satisfy $1/p_1 + \dots + 1/p_N = N - 1 + 1/r$, and that $t_1, \dots, t_N \in \mathbb{R} \setminus 0$ are real numbers such that*

$$\pm t_1^{-2} \pm \dots \pm t_N^{-2} = 1, \quad \text{for some choice of } \pm \text{ at each place.}$$

*Then the mapping $(a_1, \dots, a_N) \mapsto a_1(t_1 \cdot) * \dots * a_N(t_N \cdot)$ on $\mathcal{S}(\mathbb{R}^{2n})$ extends uniquely to a continuous map from $s_{p_1}(\mathbb{R}^{2n}) \times \dots \times s_{p_N}(\mathbb{R}^{2n})$ to $s_r(\mathbb{R}^{2n})$. One has the estimate*

$$\|a_1(t_1 \cdot) * \dots * a_N(t_N \cdot)\|_{s_r} \leqslant C^{2n} \|a\|_{s_{p_1}} \cdots \|a_N\|_{s_{p_N}},$$

*where $C = (2\pi)^{(N-1)/4} |t_1|^{-1/p_1} \cdots |t_N|^{-1/p_N}$.*

*Proof.* — The result follows from Theorem 2.3.2′ in [T1] or Theorem 3.3′ in [T3]. □

We shall now discuss admissible functions, which are defined as follows.

DEFINITION 1.4. — *Assume that $m \geqslant 2$ is even, let $d\mu$ be a positive measure in a measurable set $\mathcal{M}$, and let $\chi$ be some function from $\mathcal{M} \times \mathbb{R}^m$ to $\mathbb{C}$. We say that $\chi$ is a $d\mu$-admissible function to the order $q$, $1 \leqslant q \leqslant \infty$, if there exists a function $\widetilde{\chi} : \mathcal{M} \times \mathbb{R}^m \to \mathbb{C}$ such that the following conditions are fulfilled:*

1. *the functions $\chi$ and $\widetilde{\chi}$ are measurable functions with respect to the measure $d\mu \otimes dy$;*
2. *if $x \in \mathcal{M}$, then the mappings $y \mapsto \chi_x(y) \equiv \chi(x, y)$ and $y \mapsto \widetilde{\chi}_x(y) \equiv \widetilde{\chi}(x, y)$ belong to $\mathcal{S}(\mathbb{R}^m)$;*
3. *if $y \in \mathbb{R}^m$, then the mappings $x \mapsto \chi(x, y)$ and $x \mapsto \widetilde{\chi}(x, y)$ are $d\mu$-measurable and*
$$\int \chi(x, y) \widetilde{\chi}(x, y)\, d\mu(x) = 1;$$
4. *there exists a constant $C$ such that for every $v \in s_q(\mathbb{R}^m)$ one has*

$$\Big( \int |\langle \widetilde{\chi}_x, v \rangle|^q \, d\mu(x) \Big)^{1/q} \leqslant C \|v\|_{s_q}. \tag{1.3}$$

We shall mainly consider the case when $\mathcal{M} \subset \mathbb{R}^m$ and that $\chi(x, y) = \chi_0(y - x)$, for some $\chi_0 \in \mathcal{S}(\mathbb{R}^m)$. Then we say that $\chi_0$ is $d\mu$-admissible when $\chi$ is $d\mu$-admissible. In Proposition 1.6 below we prove that $\chi_0$ is $d\mu$-admissible when $\mu$ is periodic and $\chi_0, d\mu$ satisfy (0.5).

The next results are motivated by [S1]. We recall from the introduction that for any measure $d\mu$ and function $\chi$ as in Definition 1.4, then $s_{p,\chi,d\mu}(\mathbb{R}^m)$ is the set of all $a \in \mathcal{S}'(\mathbb{R}^m)$ such that $\|a\|_{s_{p,\chi,d\mu}} \equiv \|H_{a,p,\chi,d\mu}\|_{L^1}$ is finite. Here $H_{a,p,\chi,d\mu}$ is given by (0.3)$''$, and we note that the Fourier transform $\mathcal{F}$ acts on the function $y \mapsto \chi_x(y)a(y)$, considering $x$ as a fixed parameter.

THEOREM 1.5.— *Assume that $m \geqslant 2$ is even, let $p \in [1, \infty]$, and let $p'$ be its conjugate exponent. Let $\chi$ be a $d\mu$-admissible function to the order $p'$. Then $s_{p,\chi,d\mu}(\mathbb{R}^m)$ is continuously embedded in $s_p(\mathbb{R}^m)$.*

*Proof.*— Assume that $a \in s_{p,\chi,d\mu}(\mathbb{R}^m)$. By Proposition 1.1 it is enough to prove that there is a positive constant $C$ such that $|\langle a, \phi\rangle| \leqslant C\|\phi\|_{s_{p'}}$ when $\phi \in C_0^\infty(\mathbb{R}^m)$ and $a \in s_{p,\chi,d\mu}(\mathbb{R}^m)$. We have

$$(1.4) \qquad |\langle a, \phi\rangle| \leqslant \int |\langle \chi_x a, \widetilde{\chi}_x \phi\rangle| \, d\mu(x) = \int |\langle \mathcal{F}(\chi_x a), \mathcal{F}\psi_x\rangle| \, d\mu(x),$$

where $\psi_x(-y) = \widetilde{\chi}_x(y)\phi(y)$. It follows that

$$(\mathcal{F}\psi_x)(\xi) = \pi^{-m/2}\langle \widetilde{\chi}_x, \mathcal{F}\Phi_\xi\rangle,$$

where $\Phi_\xi(\eta) = \widehat{\phi}(\xi + \eta)$. By Proposition 1.1 we have that $\|\Phi_\xi\|_{s_{p'}} = \|\phi\|_{s_{p'}}$. Hence there is a constant $C$, which depends on $\widetilde{\chi}$ only such that

$$\Big(\int |(\mathcal{F}\psi_x)(\xi)|^{p'} \, d\mu(x)\Big)^{1/p'} \leqslant C\|\phi\|_{s_{p'}}.$$

It follows that the right-hand side of (1.4) can be estimated from above by

$$\int \Big(\int |\mathcal{F}(\chi_x a)(\xi)|^p \, d\mu(x)\Big)^{1/p} \Big(\int |(\mathcal{F}\psi_x)(\xi)|^{p'} \, d\mu(x)\Big)^{1/p'} \, d\xi$$
$$\leqslant C\|\phi\|_{s_{p'}} \int \Big(\int |\mathcal{F}(\chi_x a)(\xi)|^p \, d\mu(x)\Big)^{1/p} \, d\xi \leqslant C'\|\phi\|_{s_{p'}},$$

where $C'$ is a positive constant which is independent of $\phi$. □

PROPOSITION 1.6.— *Let $m \geqslant 2$ be even and let $d\mu$ be a positive periodic Borel measure on $\mathbb{R}^m$. If $\chi \in \mathcal{S}(\mathbb{R}^m)$ satisfies (0.5), then $\chi$ is a $d\mu$-admissible function to the order $q$, for every $q \in [1, \infty]$.*

In Section 5 we present a generalization of this result. We need some preparations for the proof. We start by discussing some translation invariant subspaces of $L^p(d\mu)$. Let $m \geqslant 1$ be an integer and let $\mathcal{S}'_T(\mathbb{R}^m)$ be the set of all $f \in \mathcal{S}'(\mathbb{R}^m)$ such that $x \mapsto f(\alpha x + y)$ is $d\mu$-measurable when $\alpha \in \{-1, 1\}$ and $y \in \mathbb{R}^m$. Then for any $p \in [1, \infty]$ we let $\|\cdot\|_{L^p_T}$ be the norm

$$\|f\|_{L^p_T} \equiv \sup \Big( \int |f(\alpha x + y)|^p \, d\mu(x) \Big)^{1/p}, \tag{1.5}$$

where the supremum is taken over all $\alpha \in \{-1, 1\}$ and $y \in \mathbb{R}^m$. If $p < \infty$, then we let $L^p_T(d\mu)$ be the completion of $\mathcal{S}(\mathbb{R}^m)$ under the topology defined by $\|\cdot\|_{L^p_T}$, and if $p = \infty$, then we let $L^\infty_T(d\mu)$ be the set of all $f \in \mathcal{S}'_T(\mathbb{R}^m)$ such that $\|f\|_{L^\infty_T} < \infty$.

Since $d\mu$ is periodic, we note that it is enough to take the supremum in (1.5) over a ball in $\mathbb{R}^m$, provided that the radius for the ball is chosen large enough.

LEMMA 1.7. — *Let $m \geqslant 1$, let $d\mu$ be a positive periodic Borel measure on $\mathbb{R}^m$, and assume that $\phi \in L^1_T(d\mu) \cap L^1(\mathbb{R}^m)$. Then for every $q \in [1, \infty]$, and $v \in L^q(\mathbb{R}^m)$ we have*

$$\|\phi * v\|_{L^q_T} \leqslant C \|v\|_{L^q}.$$

*Proof.* — Assume first that $q = 1$. Then we have for $\alpha \in \{-1, 1\}$ that

$$\int |(\phi * v)(\alpha x + y)| \, d\mu(x) \leqslant \iint |\phi(\alpha x + y - z) v(z)| \, d\mu(x) dz \leqslant \|\phi\|_{L^1_T} \|v\|_{L^1}.$$

Hence $\|\phi * v\|_{L^1_T} \leqslant \|\phi\|_{L^1_T} \|v\|_{L^1}$ and the assertion holds for $q = 1$.

The result follows now for general $q$ by interpolation, since the statement is obviously true in the case $q = \infty$. □

LEMMA 1.8. — *Let $m \geqslant 1$, let $d\mu$ be as in Lemma 1.7, and assume that $\chi \in \mathcal{S}(\mathbb{R}^m)$ satisfies (0.5). Then for some $\widetilde{\chi}(x, y) = \chi'(y - x) F(y)$, where $F \in C^\infty(\mathbb{R}^m)$ is periodic and $\chi' \in C_0^\infty(\mathbb{R}^m)$, we have*

$$\int \chi_x(y) \widetilde{\chi}(x, y) \, d\mu(x) = 1, \qquad \text{when } y \in \mathbb{R}^m.$$

*In particular $\widehat{F}$ is a bounded measure.*

*Proof.* — First we prove that it exists a function $\chi' \in C_0^\infty(\mathbb{R}^m)$ such that $0 \leqslant \chi' \leqslant 1$, $\chi'(0) = 1$ and

$$\int \chi_x(y) \chi'_x(y) \, d\mu(x) \neq 0, \quad \text{when } y \in \mathbb{R}^m, \tag{1.6}$$

where $\chi'_x(y) = \chi'(y-x)$. In fact, since $d\mu$ is periodic, it follows for any choice of $\chi' \in C_0^\infty$ that the left-hand side of (1.6) is a periodic $C^\infty$-function, with the same period as $d\mu$. Hence, if $K \subset \mathbb{R}^m$ is a fix compact set, containing a whole period for $d\mu$, then it suffices to prove that (1.6) is true only for every $y \in K$. From (0.5), it follows now easily, using some simple arguments of approximation, that (1.6) holds when $y \in K$, for some choice of largely supported $\chi' \in C_0^\infty$.

The assertion follows now if we let $F(y)^{-1}$ be equal to the left-hand side of (1.6). It is then obvious that $F \in C^\infty$ and has the same period as $d\mu$. From this fact one obtains also that for some lattice $\Lambda$ in $\mathbb{R}^m$, then $F(x) = \sum_{\lambda\in\Lambda} c_\lambda e^{2i\langle x,\lambda\rangle}$, where $|c_\lambda|$ is rapidly decreasing to zero as $|\lambda| \to \infty$. This in turn implies that $|\widehat{F}| \leqslant \pi^{m/2}\sum_\lambda |c_\lambda|\delta_\lambda$ is a bounded measure on $\mathbb{R}^m$. The proof is complete. □

*Remark 1.9.* — Let $\mu$ be a positive periodic Borel measure on $\mathbb{R}^m$ and define the $\mu$-convolution $*_\mu$ by the formula

$$a *_\mu b(x) = \int a(x-y)b(y)\,d\mu(y),$$

when $a, b \in \mathcal{S}(\mathbb{R}^m)$. Then it follows easily, by similar computations as in the proof of Young's inequality in [RS], that the Young's inequality is true, when the $L^p$-spaces and the usual convolution are replaced by the $L^p_T(d\mu)$-spaces and the $\mu$-convolution.

*Proof of Proposition 1.6.* — Let $\chi'$, $\widetilde{\chi}$ and $F$ be as in Lemma 1.8. Then the conditions (1)–(3) in Definition 1.4 are fulfilled. It remains to prove that (4) in Definition 1.4 holds.

Let $v \in C_0^\infty$. Then we have

$$\langle v, \widetilde{\chi}_x\rangle = \int \chi'(y-x)F(y)v(y)\,dy = \big(\phi * (Fv)\big)(x), \tag{1.7}$$

where $\phi(x) = \chi'(-x) \in C_0^\infty(\mathbb{R}^m)$. If $h$ is the Fourier transform of $x \mapsto (1+|x|^2)^{-m}$, then it follows that $h \in L^1(\mathbb{R}^m) \cap s_1(\mathbb{R}^m)$. (See for example Theorem 2.2.7 in [T1] or Theorem 2.6 in [T3].) This gives

$$\phi = (1-\Delta)^{-m}(1-\Delta)^m\phi = \pi^{-m/2}h * \phi_m,$$

where $\phi_m = (1-\Delta)^m\phi \in C_0^\infty(\mathbb{R}^m) \subset L^1_T(d\mu) \cap L^1(\mathbb{R}^m)$. Inserting this into (1.7) and using Lemma 1.7 we get

$$\begin{aligned}\Big(\int |\langle\widetilde{\chi}_x, v\rangle|^q\,d\mu(x)\Big)^{1/q} &\leqslant C_1\Big(\int |\phi_N * h * (Fv)(x)|^q\,d\mu(x)\Big)^{1/q}\\ &\leqslant C_2\|h*(Fv)\|_{L^q} \leqslant C_2\|h\|_{s_1}\|Fv\|_{s_q},\end{aligned} \tag{1.8}$$

where the last inequality follows from Proposition 1.2. Since $\widehat{F}$ is a bounded measure by Lemma 1.8, it follows from (1.1) and Proposition 1.1 that

$$\|Fv\|_{s_q} = \pi^{-m/2}\|\widehat{F} * \widehat{v}\|_{s_q} \leqslant C\|\widehat{v}\|_{s_q} = C\|v\|_{s_q}. \tag{1.9}$$

Combining (1.8) and (1.9), we obtain (1.3). The proof is complete. □

Theorem 1.5 and Proposition 1.6 motivate us to study more carefully the spaces $s_{p,\chi,d\mu}$, when $d\mu$ is periodic. This will be done in the next section, where we discuss some continuity properties for the $s_{p,\chi,d\mu}$-spaces.

## 2. Some basic continuity properties.

In this section we shall discuss some basic continuity properties for $s_{p,\chi,d\mu}(\mathbb{R}^m)$, when $d\mu$ is some positive periodic Borel measure on $\mathbb{R}^m$, and $\chi \in \mathcal{S}(\mathbb{R}^m)$. We discuss invariant properties and prove that $s_{p,\chi,d\mu}(\mathbb{R}^m) = S^p_w(\mathbb{R}^m)$, when $\mu$ is a positive periodic Borel measure on $\mathbb{R}^m$, and $\chi \in \mathcal{S}(\mathbb{R}^m)$ satisfies (0.5). In particular it follows that $s_{p,\chi,d\mu}$ is independent of any such positive periodic Borel measure $\mu$ and any such function $\chi$.

In many situations we need to approximate elements in $s_{p,\chi,d\mu}$ by elements in $\mathcal{S}$. It was remarked already in [S1] that $\mathcal{S}$ is not dense in $S^\infty_w$ with respect to the usual topology. For this reason, the narrow convergence was introduced in that paper. Here we shall use a similar technique where we modify the definition of narrow convergence, in order to obtain similar approximation possibilities for any $s_{p,\chi,d\mu}$-space.

DEFINITION 2.1 (cf. [S1]).— *Assume that $a, a_j \in s_{p,\chi,d\mu}(\mathbb{R}^m)$, $j = 1,2,\dots$ . We say that $a_j$ converges narrowly to $a$ (with respect to $p, \chi, d\mu$), if the following conditions are satisfied:*

(1) *$a_j \to a$ in $\mathcal{S}'(\mathbb{R}^m)$ as $j \to \infty$;*

(2) *$H_{a_j,p}(\xi) \to H_{a,p}(\xi)$ in $L^1(\mathbb{R}^m)$ as $j \to \infty$.*

*Remark 2.2.* — Assume that $a, a_1, a_2, \dots \in \mathcal{S}'(\mathbb{R}^m)$ satisfies (1) in Definition 2.1, and assume that $\xi \in \mathbb{R}^m$. Then it follows from Fatou's lemma that

$$\liminf_{j\to\infty} H_{a_j,p}(\xi) \geqslant H_{a,p}(\xi) \quad \text{and} \quad \liminf_{j\to\infty} \|a_j\|_{s_{p,\chi,d\mu}} \geqslant \|a\|_{s_{p,\chi,d\mu}}.$$

We have now the following proposition which guarantees that we may approximate elements in $s_{p,\chi,d\mu}$ with elements in $\mathcal{S}$.

PROPOSITION 2.3. — *Assume that $d\mu$ is a positive periodic Borel measure on $\mathbb{R}^m$, and that $\chi \in \mathcal{S}(\mathbb{R}^m)$ satisfies* (0.5). *Then for every $p \in [1,\infty]$ one has that $C_0^\infty(\mathbb{R}^m)$ is dense in $s_{p,\chi,d\mu}(\mathbb{R}^m)$ with respect to the narrow convergence.*

In the proof of Proposition 2.3 and in many other situations we need to apply Minkowski's inequality, in a somewhat general form. We recall that for a $d\nu$-measurable function $f$ with values in the Banach space $B$ with norm $\|\cdot\|$, then Minkowski's inequality states that $\|\int f\,d\nu\| \leqslant \int \|f\|\,d\nu$. In our applications one has that $B$ is equal to $L^p(d\mu)$, for some $p \in [1,\infty]$, and then Minkowski's inequality takes the form

$$\Big(\int \Big|\int f(x,y)\,d\nu(y)\Big|^p\,d\mu(x)\Big)^{1/p} \leqslant \int \Big(\int |f(x,y)|^p\,d\mu(x)\Big)^{1/p}\,d\nu(y).$$

LEMMA 2.4. — *Assume that $a_{j,k}, a_j, a \in s_{p,\chi,d\mu}(\mathbb{R}^m)$ for every $j,k \geqslant 1$, and that $a_{j,1}, a_{j,2},\ldots$ converges narrowly to $a_j$ for every $j \geqslant 1$, and that $a_1, a_2,\ldots$ converges narrowly to $a$. Then for some increasing sequence $j_1, j_2,\ldots$, the sequence $a_{k_1,1}, a_{k_2,2},\ldots$ converges narrowly to $a$, for any choice of the sequence $k_1, k_2,\ldots$ such that $j_r \leqslant k_r$ for every $r \geqslant 1$.*

*Proof.* — The result follows easily by an application of Cantor's diagonal principle. □

LEMMA 2.5. — *Assume that $p \in [1,\infty]$, and that $a \in s_{p,\chi,d\mu}(\mathbb{R}^m) \cap \mathcal{E}'(\mathbb{R}^m)$, where $\chi$ and $d\mu$ are as in Proposition 2.3. Then $a \in C_0(\mathbb{R}^m) \cap \mathcal{F}L^1(\mathbb{R}^m)$.*

*Proof.* — Let $\Omega$ be a fix and bounded open neighbourhood of $\operatorname{supp} a$. Since (0.5) holds, we may find a function $\varphi \in C_0^\infty(\mathbb{R}^m)$ such that

$$F_0(y) \equiv \int \chi_x(y)\varphi(x)\,d\mu(x)$$

is non-zero in $\Omega$. We note that $F_0 \in \mathcal{S}$ and that for some choice of $\psi \in C_0^\infty(\mathbb{R}^m)$ we have $\psi F_0 = 1$ in $\Omega$. From the fact that $a \in \mathcal{E}'$, it follows also that $\widehat{a} \in C^\infty \cap \mathcal{S}'$. We claim that

$$\widehat{a} = G * \widehat{\psi}, \quad \text{where} \quad G(\xi) = (2\pi)^{-m/2}\int \mathcal{F}(\chi_x a)(\xi)\,\varphi(x)\,d\mu(x). \tag{2.1}$$

In fact, let $*_2$ be the partial convolution multiplication for functions on $\mathbb{R}^m \times \mathbb{R}^m$, defined by $(u *_2 v)(x,\xi) = \int u(x,\xi-\eta)v(x,\eta)\,d\eta$, and set

$M(x,\xi) = \varphi(x)\widehat{\chi}(\xi)e^{2i\langle x,\xi\rangle}$. Then

$$\begin{aligned}G * \widehat{\psi} &= \pi^{-m} \int \big(M *_2 (1 \otimes \widehat{a})\big)(x,\cdot)\, d\mu(x) * \widehat{\psi}\\ &= \pi^{-m} \int M(x,\cdot)\, d\mu(x) * \widehat{a} * \widehat{\psi} = \pi^{-m}(\widehat{F}_0 * \widehat{a}) * \widehat{\psi}\\ &= \mathcal{F}(F_0\psi a) = \widehat{a}.\end{aligned}$$

This proves (2.1). By Hölder's inequality we have

$$|G(\xi)| \leqslant CH_{a,p}(\xi) \in L^1,$$

where $C = \|\varphi\|_{L^{p'}(d\mu)} < \infty$. This gives $|\widehat{a}| \leqslant |G| * |\widehat{\psi}| \in L^1$, in view of (2.1). Hence $a \in \mathcal{F}L^1$, which in turns implies that $a \in C(\mathbb{R}^m)$. Since $a$ is compactly supported, we may therefore conclude that $a \in \mathcal{F}L^1 \cap C_0$. The proof is complete. □

*Proof of Proposition 2.3.* — It follows from Lemma 2.4 that it is enough to first prove that $\mathcal{E}'(\mathbb{R}^m) \cap s_{p,\chi,d\mu}(\mathbb{R}^m)$ is narrowly dense in $s_{p,\chi,d\mu}(\mathbb{R}^m)$, and then that $C_0^\infty(\mathbb{R}^m)$ is narrowly dense in $\mathcal{E}'(\mathbb{R}^m) \cap s_{p,\chi,d\mu}(\mathbb{R}^m)$.

Assume therefore first that $a \in s_{p,\chi,d\mu}(\mathbb{R}^m)$, and choose a function $\phi \in C_0^\infty(\mathbb{R}^m)$ such that $\phi$ and $\widehat{\phi}$ are non-negative functions which satisfies $\phi(0) = 1$. Set $a_j = a\phi_j$, where $\phi_j(x) = \phi(\varepsilon_j x)$ and $\varepsilon_j \searrow 0$ as $j \to \infty$. We shall prove that $a_j$ converges to $a$ narrowly.

It is clear that $a_j \to a$ in $\mathcal{S}'$ as $j \to \infty$, and we shall therefore prove that $H_{a_j,p} \to H_{a,p}$ in $L^1$ as $j \to \infty$. It is clear that $\mathcal{F}(\chi_x a_j)(\xi) = \pi^{-m/2}\int \mathcal{F}(\chi_x a)(\xi - \varepsilon_j\eta)\widehat{\phi}(\eta)\, d\eta$. Hence Minkowski's inequality gives

$$\begin{aligned}H_{a_j,p}(\xi) &= \pi^{-m/2}\Big(\int \Big| \int (\mathcal{F}(\chi_x a)(\xi - \varepsilon_j\eta)\widehat{\phi}(\eta)\, d\eta\Big|^p d\mu(x)\Big)^{1/p}\\ &\leqslant \pi^{-m/2}\int \Big(\int |(\mathcal{F}(\chi_x a)(\xi - \varepsilon_j\eta)|^p d\mu(x)\Big)^{1/p}\widehat{\phi}(\eta)\, d\eta = U_j(\xi),\end{aligned}$$

where $U_j(\xi) = \pi^{-m/2}\int H_{a,p}(\xi - \varepsilon_j\eta)\widehat{\phi}(\eta)\, d\eta$. Since $\pi^{-m/2}\int \widehat{\phi}\, d\xi = \phi(0) = 1$ and $H_{a,p} \in L^1$, it follows that $U_j \to H_{a,p}$ in $L^1$ as $j \to \infty$. Hence $\|U_j - U_k\|_{L^1} \to 0$ as $j,k \to \infty$, and since $H_{a_j,p} \leqslant U_j$ one has

$$\limsup_{j\to\infty} H_{a_j,p}(\xi) \leqslant H_{a,p}(\xi), \quad \text{a. e.}$$

It follows now that $H_{a_j,p} \to H_{a,p}$ in $L^1$ as $j \to \infty$ from these facts, Remark 2.2 and a generalization of Lebesgue's theorem which asserts that if $f_j \to f$

a. e. as $j \to \infty$ and if there exists a sequence $g_j \in L^1$ such that $|f_j| \leqslant g_j$ and $\|g_j - g_k\|_{L^1} \to 0$ as $j, k \to \infty$, then $\|f - f_j\|_{L^1} \to 0$ as $j \to 0$. Hence we have proved that $s_{p,\chi,d\mu} \cap \mathcal{E}'$ is narrowly dense in $s_{p,\chi,d\mu}$.

Assume next that $a \in s_{p,\chi,d\mu} \cap \mathcal{E}'$ and let $a_j = \phi_{\varepsilon_j} * a$, where $\phi_\varepsilon = \varepsilon^{-n}\phi(\cdot/\varepsilon)$, $\phi \in C_0^\infty$ is nonnegative, $\int \phi\, dx = 1$ and $\varepsilon_j \searrow 0$ as $j \to \infty$. The proposition will follow if we prove that $a_j \in C_0^\infty$ converges to $a$ narrowly.

By an application of Parseval's formula we have

$$
\begin{aligned}
|\mathcal{F}\big((\tau_x\chi)\, a\big)(\xi)| &= \pi^{-m/2}|\big((\check{\chi}e^{2i\langle\cdot,\xi\rangle}) * a\big)(x)| \\
&= |\mathcal{F}^{-1}\big((\tau_\xi(\mathcal{F}\check{\chi}))\, \widehat{a}\big)(x)|.
\end{aligned}
\tag{2.2}
$$

If we replace $a$ by $a_j$, then we get

$$
|\mathcal{F}\big((\tau_x\chi)a_j\big)(\xi)| = \pi^{-m/2}|\big((\check{\chi}e^{2i\langle\cdot,\xi\rangle} * a * \phi_{\varepsilon_j}\big)(x)|.
$$

This gives

$$
\begin{aligned}
&H_{a_j,p}(\xi) \\
&= \pi^{-m/2}\Big(\int\Big|\int \big((\check{\chi}e^{2i\langle\cdot,\xi\rangle}) * a\big)(x-y)\phi_{\varepsilon_j}(y)\, dy\Big|^p d\mu(x)\Big)^{1/p} \\
&= \pi^{-m/2}\Big(\int\Big|\int \big((\check{\chi}e^{2i\langle\cdot,\xi\rangle}) * a\big)(x-\varepsilon_j y)\phi(y)\, dy\Big|^p d\mu(x)\Big)^{1/p}.
\end{aligned}
\tag{2.3}
$$

By Minkowski's inequality and that $\phi$ is non-negative it follows

$$
H_{a_j,p}(\xi) \leqslant \pi^{-m/2}\int\Big(\int|\big((\check{\chi}e^{2i\langle\cdot,\xi\rangle}) * a\big)(x-\varepsilon_j y)|^p d\mu(x)\Big)^{1/p}\phi(y)\, dy.
$$

Since $a \in C_0$ by Lemma 2.5, it follows from the last inequality that

$$
\begin{aligned}
\limsup_{j\to\infty} H_{a_j,p}(\xi) \leqslant \pi^{-m/2}\Big(\int|\big((\check{\chi}e^{2i\langle\cdot,\xi\rangle}) * a\big)(x)|^p d\mu(x)\Big)^{1/p}\int\phi(y)\, dy \\
= H_{a,p}(\xi).
\end{aligned}
$$

Here we have used (2.2) and that $\int \phi\, dy = 1$. Hence $H_{a_j,p} \to H_{a,p}$ a. e. by Remark 2.2.

In order to prove that $H_{a_j,p} \to H_{a,p}$ in $L^1$ as $j \to \infty$, it suffices now to find a majoring $L^1$-function to the $H_{a_j,p}$-functions. Take a function $\psi \in C_0^\infty(\mathbb{R}^m)$ such that $\psi = 1$ in the supports of $a$ and $a_j$ for every $j$. Then it follows that $\psi a_j = a_j$. By (2.2) and (2.3) one obtains

$$
H_{a_j,p}(\xi) = \Big(\int\Big|\int \mathcal{F}\big(\chi_x\psi\,(\tau_{\varepsilon_j y}a)\big)(\xi)\phi(y)\, dy\Big|^p\, d\mu(x)\Big)^{1/p}.
$$

From the fact that $|\mathcal{F}(\tau_{\varepsilon_j y}a)| = |\widehat{a}|$, it follows from Minkowski's inequality that

$$H_{a_j,p}(\xi) \leqslant \pi^{-m/2}\Big(\int ((|\mathcal{F}(\chi_x\psi)| * |\widehat{a}|)(\xi))^p \, d\mu(x)\Big)^{1/p} \leqslant (g * |\widehat{a}|)(\xi),$$

where

$$g(\xi) = \pi^{-m/2}\Big(\int |\mathcal{F}(\chi_x\psi)(\xi)|^p \, d\mu(x)\Big)^{1/p}.$$

Since the mapping $(x,\xi) \mapsto \mathcal{F}(\chi_x\psi)(\xi)$ belongs to $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$ (it might be considered as a partial Fourier transform acting on the tempered function $(x,y) \mapsto \chi(y-x)\psi(y))$, it follows that $g \in L^1$. Hence $g * |\widehat{a}| \in L^1$, since $\widehat{a} \in L^1$ by Lemma 2.5. We have therefore found the desired majoring $L^1$-function to the $H_{a_j,p}$-functions, and the result follows. □

*Remark 2.6.* — Assume that $\mu_1, \mu_2$ are positive periodic Borel measures on $\mathbb{R}^m$, and that $\chi_1, \chi_2 \in \mathcal{S}(\mathbb{R}^m)$ satisfy $\int \chi_k(y-x)\, d\mu_k(x) \neq 0$ when $k \in \{1,2\}$ and $y \in \mathbb{R}^m$. Then it follows from Proposition 2.3 and its proof that for any $a \in \mathcal{S}'(\mathbb{R}^m)$, we may find a sequence $(a_j)$ in $C_0^\infty(\mathbb{R}^m)$ such that $a_j \to a$ in $\mathcal{S}'$, and $\|a_j\|_{s_{p,\chi_1,d\mu_1}} \to \|a\|_{s_{p,\chi_1,d\mu_1}}$ and $\|a_j\|_{s_{p,\chi_2,d\mu_2}} \to \|a\|_{s_{p,\chi_2,d\mu_2}}$ as $j \to \infty$.

We shall next prove that $s_{p,\chi,d\mu}(\mathbb{R}^m)$ is independent of the choice of the periodic measure $d\mu$ and function $\chi \in \mathcal{S}$ satisfying (0.5).

THEOREM 2.7. — *Let $d\mu$ and $d\nu$ be positive periodic Borel measures on $\mathbb{R}^m$, $\chi, \psi \in \mathcal{S}(\mathbb{R}^m)$, and assume that $\chi, \mu$ satisfy* (0.5). *Then for every $p \in [1,\infty]$ one has $s_{p,\chi,d\mu}(\mathbb{R}^m) \subset s_{p,\psi,d\nu}(\mathbb{R}^m)$, and $\|a\|_{s_{p,\psi,d\nu}} \leqslant C\|a\|_{s_{p,\chi,d\mu}}$, where $C$ is independent on $a$ and $p$.*

*If in addition, $\int \psi(y-x)\, d\nu(x) \neq 0$, for every $y \in \mathbb{R}^m$, then $s_{p,\chi,d\mu}(\mathbb{R}^m)$ is equal to $s_{p,\psi,d\nu}(\mathbb{R}^m)$, with equivalent norms.*

We need some preparations for the proof. In what follows we use the notation $L^{1,p}(d\mu)$ for the completion of $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$ under the norm

$$\|a\|_{L^{1,p}} = \|a\|_{L^{1,p}(d\mu)} = \|a\|_{L^1(\mathbb{R}^m;L^p(d\mu))} \equiv \int \Big(\int |a(x,\xi)|^p d\mu(x)\Big)^{1/p} d\xi.$$

Then $L^{1,p}(d\mu)$ is continuously embedded in $L^{1,*}(d\mu)$, the completion of the set $L^{1,1}(d\mu) + L^{1,\infty}(d\mu)$ under the norm $\|a\|_{L^{1,*}(d\mu)} \equiv \inf\big(\|a_1\|_{L^{1,1}} + \|a_2\|_{L^{1,\infty}}\big)$, where the infimum is taken over all $a_1 \in L^{1,1}(d\mu)$ and $a_2 \in L^{1,\infty}(d\mu)$ such that $a = a_1 + a_2$.

In the following lemma we use the notation $\mathcal{F}_2$ for the partial Fourier transform on $\mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^m)$ with respect to the second variable.

LEMMA 2.8. — *Assume that $\chi$, $\widetilde{\chi}$, $\psi$, $d\mu$ and $d\nu$ are as in Theorem 2.7 and Lemma 1.8. Let $T$ be the operator from $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$ to $\mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^m)$, defined by $T = \mathcal{F}_2 \circ S$, where*

$$(Su)(x, y) = \psi_x(y) \int \widetilde{\chi}(z, y)(\mathcal{F}_2^{-1}u)(z, y)\, d\mu(z).$$

*Then the following is true:*

(1) *$T$ is continuous on $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$;*

(2) *if $a \in \mathcal{S}(\mathbb{R}^m)$ and $u(x, \xi) = \mathcal{F}(\chi_x a)(\xi)$, then $(Tu)(x, \xi) = \mathcal{F}(\psi_x a)(\xi)$;*

(3) *$T$ may be extended to a continuous operator from $L^{1,p}(d\mu)$ to $L^{1,p}(d\nu)$ for every $p \in [1, \infty]$. In particular*

$$\|Tu\|_{L^{1,p}(d\nu)} \leqslant C\|u\|_{L^{1,p}(d\mu)}, \tag{2.4}$$

*for some constant $C$ independent on $u$ and $p$.*

*Here we have identified subspaces of $\mathcal{D}'(\mathbb{R}^m; \mathcal{D}'(\mathbb{R}^m; \mathbb{C}))$ with the corresponding subspaces of $\mathcal{D}'(\mathbb{R}^m \times \mathbb{R}^m; \mathbb{C})$.*

*Proof.* — Since (2) is a consequence of some straightforward computations, using Fourier's inversion formula, we prove only (1) and (3). Let $F$ be as in Lemma 1.8, and assume that $u_y(\xi) = u(y, \xi) \in \mathcal{S}(\mathbb{R}^{2m})$. A straightforward computation shows that $Tu = \pi^{-m/2}(\delta_0 \otimes \widehat{F}) * U$ where

$$U(x, \xi) = \iint e^{-2i\langle \xi - \eta, y\rangle} K(x - y, \xi - \eta)\, u_y(\eta)\, d\mu(y) d\eta, \tag{2.5}$$

and

$$K(x, \xi) = K_x(\xi) = \pi^{-m/2} \int \chi'(z)\psi(z - x)e^{-2i\langle z, \xi\rangle}\, dz.$$

Here $\chi'$ is the same function as in Lemma 1.8. It follows that $K \in \mathcal{S}$, since it is essentially a partial Fourier transform of a Schwartz function. This implies that $U$ is rapidly decreasing for every $u \in \mathcal{S}$. The same is true for any derivative of $U$, since any $U^{(\alpha)}$ may be written as a finite sum $\sum_j U_j$, where each $U_j$ has the same form as $U$ in (2.5), with $K$ replaced by some derivatives of $K$, and $u_y(\eta)$ by $P(y)u_y(\eta)$, for some polynomial $P$ on $\mathbb{R}^m$. In particular, the map $u \mapsto U$ is continuous on $\mathcal{S}$. Since $F \in C^\infty$ is periodic, $T$ is continuous on $\mathcal{S}$, which proves (1).

In order to prove (3) we observe that from the above we have for any $u \in \mathcal{S}(\mathbb{R}^{2m})$ that

$$|(Tu)(x,\xi)| \leqslant \pi^{-m/2} \int \big(|\widehat{F}| * |K_{x-y}| * |u_y|\big)(\xi)\, d\mu(y). \tag{2.6}$$

Here the convolutions on $K_{x-y}(\xi)$, $u_y(\xi)$ and $\widehat{F}(\xi)$ in the integral act on the $\xi$-variable only. From this inequality it follows that

$$\begin{aligned}
\|Tu\|_{L^{1,1}(d\nu)} &= \iint |(Tu)(x,\xi)|\, d\nu(x)d\xi \\
&\leqslant \iint \Big(\int \big(|\widehat{F}| * |K_{x-y}| * |u_y|\big)(\xi)\, d\xi\Big)\, d\mu(y)d\nu(x) \\
&= \|\widehat{F}\| \int \Big(\iint |K(x-y,\xi)| d\nu(x)d\xi\Big)\Big(\int |u(y,\xi)|\, d\xi\Big)\, d\mu(y) \\
&\leqslant C \iint |u(y,\xi)|\, d\xi d\mu(y) = C\|u\|_{L^{1,1}(d\mu)},
\end{aligned}$$

where $C = \|\widehat{F}\| \sup_y \iint |K(x-y,\xi)| d\nu(x)d\xi$. Here $\|\widehat{F}\|$ is the total mass norm of $\widehat{F}$. Since $K$ is rapidly decreasing, $C < \infty$, and we have proved that (2.4) holds when $p = 1$.

Next we prove that (2.4) holds when $p = \infty$ and $u \in \mathcal{S}$. Set $U(\xi) = \sup_{y \in \mathrm{supp}(d\mu)} |u(y,\xi)|$. Then (2.6) gives

$$\begin{aligned}
\|Tu\|_{L^{1,\infty}(d\nu)} &= \int \sup_{x \in \mathrm{supp} d\nu} |(Tu)(x,\xi)|\, dx \\
&\leqslant C \int \Big(\sup_x \int \big(|\widehat{F}| * |K_{x-y}| * |u_y|\big)(\xi)\, d\mu(y)\Big)\, d\xi \\
&\leqslant C \int \Big(\sup_x \int \big(|\widehat{F}| * |K_{x-y}| * |U|\big)(\xi)\, d\mu(y)\Big)\, d\xi.
\end{aligned} \tag{2.7}$$

If we let

$$\widetilde{K}(\xi) = \sup_x \int |K(\xi, x-y)|\, d\mu(y),$$

then $\widetilde{K} \in L^1(\mathbb{R}^m)$ since $K$ is rapidly decreasing. By Minkowski's inequality it follows that the right-hand side of (2.7) is less than or equal to

$$C \int \big(|\widehat{F}| * |\widetilde{K}| * |U|\big)(\xi)\, d\xi = C\|\widehat{F}\|\, \|\widetilde{K}\|_{L^1} \|U\|_{L^1} = C_2 \|u\|_{L^{1,\infty}(d\mu)},$$

where $C' = C\|\widehat{F}\|\, \|\widetilde{K}\|_{L^1} < \infty$. Hence (2.4) holds when $p \in \{1,\infty\}$.

Since we have already proved the result for $p = 1$, it follows that $T$ extends to a continuous operator from $L^{1,*}(d\mu)$ to $L^{1,*}(d\nu)$.

The result follows now for general $p$ by interpolation, using the Theorems 4.1.2, 5.1.1 and 5.1.2 in [BL]. The proof is complete. □

*Proof of Theorem 2.7.* — The result is obviously true when $d\nu = 0$. We assume therefore that $d\nu \neq 0$, and consider first the case $\int \psi(y-x)\,d\nu(x) \neq 0$ for every $y \in \mathbb{R}^m$. Let $a \in s_{p,\chi,d\mu}(\mathbb{R}^m)$. By Proposition 2.3 and Remark 2.4 we may assume that $a \in \mathcal{S}(\mathbb{R}^m)$.

Set $u(x,\xi) = \mathcal{F}(\chi_x a)(\xi)$. Then Lemma 2.8 gives for any $p \in [1,\infty]$ that

$$\int \Big( \int |\mathcal{F}(\psi_x a)(\xi)|^p \,d\nu(x)\Big)^{1/p} d\xi \leqslant C \int \Big( \int |\mathcal{F}(\chi_x a)(\xi)|^p \,d\mu(x)\Big)^{1/p} d\xi,$$

for some constant $C$. Hence $\|a\|_{s_{p,\psi,d\nu}} \leqslant C\|a\|_{s_{p,\chi,d\mu}}$. Since an opposite estimate is obtained by interchanging the roles for $(\psi, d\nu)$ and $(\chi, d\mu)$, it follows that $s_{p,\psi,d\nu}(\mathbb{R}^m) = s_{p,\chi,d\mu}(\mathbb{R}^m)$ for every $p \in [1,\infty]$, and we have proved the assertion when $\int \psi(y-x)\,d\nu(x) \neq 0$ for every $y$.

Assume next that $\psi \in \mathcal{S}$ is arbitrary. Then we may find a function $\varphi \in \mathcal{S}$ such that if $\psi_1 = \psi + \varphi$ and $\psi_2 = \varphi$, then $\int \psi_j(y-x)\,d\nu(x) \neq 0$, $j = 1,2$, for every $y \in \mathbb{R}^m$. From the first part of the proof it follows that

$$\|a\|_{s_{p,\psi,d\nu}} \leqslant \|a\|_{s_{p,\psi_1,d\nu}} + \|a\|_{s_{p,\psi_2,d\nu}} \leqslant C\|a\|_{s_{p,\chi,d\mu}},$$

and the proof is complete. □

DEFINITION 2.9. — *Assume that $d\mu$ is a positive periodic measure on $\mathbb{R}^m$, that $\chi \in \mathcal{S}(\mathbb{R}^m)$ such that (0.5) holds, and that $p \in [1,\infty]$. Then the set $S^p_w(\mathbb{R}^m) \equiv s_{p,\chi,d\mu}(\mathbb{R}^m)$ is called the Boulkhemair-Sjöstrand space, or $BS$-space, of order $p$.*

It follows from Theorem 2.7 that $S^p_w(\mathbb{R}^m)$ is independent on the choice of $d\mu$ and $\chi$ in Definition 2.9.

PROPOSITION 2.10. — *Assume that $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$. Then one has the following inclusion relations:*

$$\mathcal{S}(\mathbb{R}^m) \subset S^{p_1}_w(\mathbb{R}^m) \subset S^{p_2}_w(\mathbb{R}^m) \subset C(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m). \tag{2.8}$$

*Moreover, if $1 \leqslant p \leqslant 2$, then $S^p_w(\mathbb{R}^m) \subset L^2(\mathbb{R}^m)$.*

*Proof.* — Since $S^p_w$ is independent of the choice of $d\mu$ and $\chi$ in Definition 2.9 we obtain $S^{p_1}_w \subset S^{p_2}_w$ as $p_1 \leqslant p_2$, if we choose $d\mu(x) =$

$\sum_{j\in\Lambda}\delta(x-j)$, where $\Lambda$ is some lattice. If instead $d\mu(x) = dx$, then the other inclusions in (2.8) become obvious.

The last part is an immediate consequence of Proposition 1.1, Theorem 1.5 and Proposition 1.6. □

In most of our considerations we have $d\mu(x) = dx$, and in these cases we use the more short notation $\|\cdot\|_{s_{p,\chi}}$ instead of $\|\cdot\|_{s_{p,\chi,dx}}$.

We shall next prove that the Fourier transform is continuous on $S^1_w(\mathbb{R}^m)$.

PROPOSITION 2.11. — *Assume that $\chi \in \mathcal{S}(\mathbb{R}^m)$ has non-vanishing integral such that $\widehat{\chi} = \check{\chi}$. Then $\|a\|_{s_{1,\chi}} = \|\widehat{a}\|_{s_{1,\chi}}$. In particular it follows that $\mathcal{F}$ is a homeomorphism on $S^1_w$.*

*Proof.* — By (2.2) we get

$$\|\widehat{a}\|_{s_{1,\chi}} = \iint |\mathcal{F}(\chi_x\,\widehat{a})(\xi)|\,dxd\xi = \iint |\mathcal{F}(\chi_\xi\,a)(-x)|\,dxd\xi = \|a\|_{s_{1,\chi}},$$

and the result follows. □

We have also the following property for $S^p_w(\mathbb{R}^m)$, which we shall use later on.

PROPOSITION 2.12. — *Assume that $V \subset \mathbb{R}^m$ is a vector space with Euclidean structure inherited from $\mathbb{R}^m$. Then the mapping $a \mapsto a\big|_V$ is continuous from $S^p_w(\mathbb{R}^m)$ to $S^p_w(V)$, for every $p \in [1,\infty]$. Here $a\big|_V$ denotes the restriction of $a$ to $V$.*

For the proof we need the following lemma.

LEMMA 2.13. — *Let $T$ be an automorphism on $\mathbb{R}^m$. Then the map $a \mapsto T^*a = a \circ T$ is continuous on $S^p_w(\mathbb{R}^m)$, also in the narrow convergence with respect to the Lebesgue measure.*

*Proof.* — The result follows immediately from the definitions. □

*Proof of Proposition 2.12.* — It follows from Lemma 2.13 that we may assume that $V = \mathbb{R}^k$ for some $k$. Assume that $a \in S^p_w(\mathbb{R}^m)$, and set $b(x_1) = a(x_1, 0) \in \mathcal{S}'(\mathbb{R}^k) \cap C(\mathbb{R}^k)$. Let $\psi \in \mathcal{S}(\mathbb{R}^k)$ such that $\int \psi\,dx_1 \neq 0$, and take a function $\varphi \in C_0^\infty(\mathbb{R}^{m-k})$ such that $\varphi(0) = 1$. We shall prove that $\|b\|_{s_{p,\psi}} \leqslant C\|a\|_{s_{p,\chi}}$, for some $\chi \in \mathcal{S}(\mathbb{R}^m)$.

It follows that $b(x_1) = c(x_1, 0)$, if $c(x) = a(x)\varphi(x_2)$. By Fourier's inversion formula we get

$$\begin{aligned}\|b\|_{s_{p,\psi}} &= \int \Big( \int |\mathcal{F}(b\psi_{x_1})(\xi_1)|^p \, dx_1 \Big)^{1/p} d\xi_1 \\ &= \pi^{-m} \int \Big( \int \Big| \iiint a(y)\varphi(y_2)\psi_{x_1}(y_1) e^{-2i\langle y,\xi\rangle} \, dy_1 dy_2 d\xi_2 \Big|^p dx_1 \Big)^{1/p} d\xi_1 \\ &= \pi^{-m} \int \Big( \int \Big| \iiint a(y)\varphi(y_2)\chi_x(y) e^{-2i\langle y,\xi\rangle} \, dy dx_2 d\xi_2 \Big|^p dx_1 \Big)^{1/p} d\xi_1 .\end{aligned}$$

In the last expression we let $\chi = \psi \otimes \kappa$, where $\kappa \in C_0^\infty(\mathbb{R}^{m-k})$ satisfies $\int \kappa \, dx_2 = 1$. By Minkowski's inequality we obtain

$$\pi^{-m}\|b\|_{s_{p,\psi}} \leqslant \iint \Big( \int \Big| \int a(y)\varphi(y_2)\chi_x(y) e^{-2i\langle y,\xi\rangle} \, dy \Big|^p dx_1 \Big)^{1/p} d\xi dx_2 .$$

Now we let $C_p = \pi^{-m} (\int \langle x_2 \rangle^{-2mp'} \, dx_2)^{1/p'} < \infty$, where $\langle x_2 \rangle = (1 + |x_2|^2)^{1/2}$ and $1/p + 1/p' = 1$ as before. Then Hölder's inequality with respect to the $y_2$-variable implies that $\|b\|_{s_{p,\psi}}$ may be estimated by

$$\begin{aligned}&\pi^{-m} \iint \Big( \int \Big| \int a(y)\varphi(y_2)\chi_x(y)\langle x_2\rangle^{2m} e^{-2i\langle y,\xi\rangle} \, dy \Big|^p dx_1 \Big)^{1/p} \langle x_2 \rangle^{-2m} d\xi dx_2 \\ &\qquad \leqslant C_p \int \Big( \iint \Big| \int a(y)\varphi(y_2)\chi_x(y)\langle x_2\rangle^{2m} e^{-2i\langle y,\xi\rangle} \, dy \Big|^p dx_1 dx_2 \Big)^{1/p} d\xi .\end{aligned}$$

Next we observe that for some constants $c_{\alpha,\beta}$, depending on $m$, $\alpha$ and $\beta$ only, we have

$$\langle x_2 \rangle^{2m} \varphi(y_2)\chi_x(y) = \sum_{|\alpha+\beta| \leqslant 2m} c_{\alpha,\beta}\varphi_\alpha(y_2)\chi_{\beta,x}(y).$$

Here $\varphi_\alpha(y_2) = y_2^\alpha \varphi(y_2)$ and $\chi_{\beta,x}(y) = \chi_\beta(y - x)$, where $\chi_\beta(y) = y_2^\beta \chi(y)$. This implies that

$$\|b\|_{s_{p,\psi}} \leqslant C \sum_{|\alpha+\beta| \leqslant 2m} I_{\alpha,\beta}, \tag{2.9}$$

for some constant $C$, where

$$I_{\alpha,\beta} = \pi^{-m} \int \Big( \int \Big| \int a(y)\varphi_\alpha(y_2)\chi_{\beta,x}(y) e^{-2i\langle y,\xi\rangle} \, dy \Big|^p dx \Big)^{1/p} d\xi .$$

If we let $c_\alpha(x) = c_\alpha(x_1, x_2) = a(x)\varphi_\alpha(x_2)$, then it follows that the last expression is equal to $\|c_\alpha\|_{s_{p,\chi_\beta}}$, i.e., $I_{\alpha,\beta} = \|c_\alpha\|_{s_{p,\chi_\beta}}$. Since $\varphi_\alpha \in \mathcal{S}$, it follows easily that $\|a\varphi_\alpha\|_{s_{p,\chi}} \leqslant C_\alpha\|a\|_{s_{p,\chi}}$, for some constant $C_\alpha$. Hence Theorem 2.7 implies that for any choice of $\alpha$ and $\beta$, then $I_{\alpha,\beta} \leqslant C_{\alpha,\beta}\|a\|_{s_{p,\chi}}$, for some constant $C_{\alpha,\beta}$. Hence $\|b\|_{s_{p,\psi}} \leqslant C'\|a\|_{s_{p,\chi}}$ by (2.9). This completes the proof of the proposition. □

Next we shall consider a similar type of convolution operators acting on the $BS$-spaces, as in (5) in Proposition 0.1. Assume that $V_1$ and $V_2$ are vector spaces such that $V_1 \oplus V_2 = \mathbb{R}^m$ and $V_1 \perp V_2$, and define for every non-degenerate real quadratic form $\Phi$ on $V_1$, the operator $T_\Phi$ on $\mathcal{S}$, by the formula

$$T_\Phi a = \pi^{-m_1/2}|\det A_\Phi|^{1/2}(e^{i\Phi}\otimes\delta) * a.$$

Here $A_\Phi$ is the matrix for $\Phi$, satisfying $\Phi(x_1) = \langle A_\Phi x_1, x_1\rangle$, $m_1 = \dim V_1$ and $\delta$ is the Dirac measure on $V_2$. By a simple application of Fourier's inversion formula, it follows that $T_\Phi$ is a homeomorphism on $\mathcal{S}$, with inverse

$$(T_\Phi)^{-1} = T_\Phi^{\,t} = T_{(-\Phi)}.$$

This allows us to extend $T_\Phi$ to a continuous operator on $\mathcal{S}'$. We also observe that the Fourier transform of $|\det A_\Phi|^{1/2}e^{i\Phi}$ is equal to $c\,e^{-i\langle A_\Phi^{-1}\xi_1,\xi_1\rangle}$, for some complex constant $c$ such that $|c| = 1$. We have now the following result.

Proposition 2.14. — *Assume that $V_1, V_2 \subset \mathbb{R}^m$ are vector spaces such that $V_1 \oplus V_2 = \mathbb{R}^m$ and $V_1 \perp V_2$, and let $\psi = T_\Phi^{-1}\chi$, where $\chi \in \mathcal{S}(\mathbb{R}^m)$ has non-vanishing integral. Then*

1. *$\psi \in \mathcal{S}(\mathbb{R}^m)$, and that $\int \psi\,dx = c\int\chi\,dx \neq 0$, for some complex constant $c$ such that $|c| = 1$;*
2. *$H_{T_\Phi a,p,\psi} = H_{a,p,\chi}$. In particular $T_\Phi$ is homeomorphic on $S^p_w(\mathbb{R}^m)$ for every $p \in [1,\infty]$, and $\|T_\Phi a\|_{s_{p,\psi}} = \|a\|_{s_{p,\chi}}$.*

*Moreover, if $a_j \to a$ narrowly with respect to $p$ and $\chi$, then $T_\Phi a_j \to T_\Phi a$ narrowly with respect to $p$ and $\psi$.*

*Proof.* — The assertion (1) follows easily by a straightforward computation. It suffices therefore to prove that $H_{T_\Phi a,p,\psi} = H_{a,p,\chi}$.

If we write $\xi = (\xi_1, \xi_2)$, where $\xi_j \in V_j$, then it follows from (2.2) and from the above that

$$\begin{aligned}|\mathcal{F}(\psi_x T_\Phi a)(\xi)| &= |\mathcal{F}^{-1}\big((\mathcal{F}\check{\psi})_\xi\,\mathcal{F}(T_\Phi a)\big)(x)|\\ &= |\mathcal{F}^{-1}\big((\mathcal{F}\check{\chi})_\xi\,e^{-2i\langle(A_\Phi^{-1}\xi_1,0),\cdot\rangle}\widehat{a}\big)(x)| = |\mathcal{F}(\chi_{x-y_0}\,a)(\xi)|,\end{aligned}$$

where $y_0 = (A_\Phi^{-1}\xi_1, 0)$. This gives

$$H_{T_\Phi a,p,\psi}(\xi) = \Big( \int |\mathcal{F}(\chi_{x-y_0}\, a)(\xi)|^p \, dx \Big)^{1/p} = H_{a,p,\chi}(\xi),$$

where we in the last equality have taken $x - y_0$ as new variables of integration. This proves the proposition. □

We finish this section with the following remark, which gives an alternative explanation of the $S^p_w$-spaces. In the case $p = \infty$, the result was presented by Professor A. Melin at the Lund University in the end of 1999, and is similar to the localization result Corollary 1.3 in [B].

*Remark 2.15.* — Assume that $p \in [1, \infty]$ and that $a \in \mathcal{S}'(\mathbb{R}^m)$. Then $a \in S^p_w(\mathbb{R}^m)$, if and only if for some $b \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$ one has that

$$a(x) = \pi^{-m/2} \int e^{2i\langle x,\xi\rangle} b(x,\xi)\, d\xi, \quad \text{and} \tag{2.10}$$

$$|b|_{p,\alpha} \equiv \int \Big( \int |\partial_x^\alpha b(x,\xi)|^p \, dx \Big)^{1/p} d\xi < \infty \quad \forall\ \alpha. \tag{2.11}$$

In fact, assume first that $a \in S^p_w(\mathbb{R}^m)$, and let $\psi, \chi \in C_0^\infty(\mathbb{R}^m)$ be chosen such that $\int \chi\, dx = 1$ and $\psi = 1$ in the support of $\chi$. Let also $b$ be the smooth function given by

$$b(x,\xi) = \pi^{-m/2} \int \mathcal{F}(\chi_z a)(\xi)\psi(x-z)\, dz.$$

By Fourier's inversion formula it follows that (2.10) holds. Since $a \in S^p_w$ and $\psi \in C_0^\infty$, an application of Minkowski's inequality proves that $|b|_{p,\alpha} \leqslant \|a\|_{S_{p,\chi}} \|\psi^{(\alpha)}\|_{L^1} < \infty$, and (2.11) follows.

On the other hand, assume that $a$ satisfies (2.10) and (2.11), for some $b \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$. Then

$$\mathcal{F}(\chi_x a)(\xi) = \pi^{-m} \iint \chi(y-x) b(y,\eta) e^{2i\langle y,\eta-\xi\rangle}\, dy d\eta. \tag{2.12}$$

Now we may replace $e^{2i\langle y,\eta-\xi\rangle}$ in (2.12) by $\langle \eta-\xi\rangle^{-2m}(1-\Delta_y/4)^m e^{2i\langle y,\eta-\xi\rangle}$. Then an integration by parts $2m$ times gives

$$\mathcal{F}(\chi_x a)(\xi) = \sum_{|\alpha+\beta|\leqslant 2m} C_{\alpha,\beta} \iint \chi^{(\beta)}(y-x)\partial_y^\alpha b(y,\eta)\langle \eta-\xi\rangle^{-2m} e^{2i\langle y,\eta-\xi\rangle}\, dy d\eta,$$

for some constants $C_{\alpha,\beta}$. If we let $C'_{\alpha,\beta}=|C_{\alpha,\beta}|\int\langle\eta\rangle^{-2m}\,d\eta<\infty$, then an application of Minkowski's inequality finally gives

$$\|a\|_{s_p,\chi}=\int\Big(\int|\mathcal{F}(\chi_x a)(\xi)|^p\,dx\Big)^{1/p}d\xi\leqslant\sum C'_{\alpha,\beta}\|\chi^{(\beta)}\|_{L^1}|b|_{p,\alpha}<\infty.$$

Hence $a\in S^p_w$, and the assertion is proved.

*Remark 2.16.* — If $m$ is even, then $S^p_w(\mathbb{R}^m)\subset s_p(\mathbb{R}^m)$, by Theorem 1.5 and Proposition 1.6. This is also a consequence of Remark 2.15 and some inclusion relations between $s_p$ and Sobolev spaces, proved in [T1] or [T3].

In fact, assume that $a\in S^p_w(\mathbb{R}^m)$, and let $b\in C^\infty$ be chosen such that (2.10) and (2.11) are fulfilled. By Theorem 2.2.7 in [T1] or Theorem 2.6 in [T3] it follows that

$$\int\|b(\cdot,\xi)\|_{s_p}\,d\xi<\infty.$$

Hence $\|a\|_{s_p}<\infty$, by (2.10) since the mapping $a\mapsto e^{i\langle\cdot,\xi\rangle}a$ is unitary on $s_p$ for every $\xi\in\mathbb{R}^m$. (Cf. Proposition 1.1 (2).)

More generally, assume that $\mathcal{F}\subset s_p(\mathbb{R}^m)$ is a Fréchet space, and that $a$ is given by (2.10) for some $b\in L^1(\mathbb{R}^m;\mathcal{F})$. Then $a\in s_p(\mathbb{R}^m)$.

## 3. Convolutions, multiplications and compositions by holomorphic functions for elements in $S^p_w$.

In this section we start by discussing multiplication and convolution properties for the $S^p_w$-spaces, and prove a Hölder type inequality for multiplication and a Young type inequality for convolutions. We may then apply the results to prove that if $f$ is an entire function on $\mathbb{C}$ such that $f(0)=0$, then $a\mapsto f(a)$ is continuous on $S^p_w$ for any $p\in[1,\infty]$. Before starting the discussion, we remark that the result of this section is not used in the forthcoming sections, where continuity properties for pseudo-differential operators are discussed, and may therefore be skipped by the reader who is interested only in these questions.

We start with the following result.

PROPOSITION 3.1. — *Assume that $p_1,\dots,p_N,q_1,\dots,q_N,r\in[1,\infty]$ satisfies the conditions*

$$1/p_1+\cdots+1/p_N=1/r\quad\text{and}\quad 1/q_1+\cdots+1/q_N=N-1+1/r,$$

*and assume that $a_j \in S_w^{p_j}(\mathbb{R}^m)$ for every $j = 1, \dots, N$. Then $a_1 a_2 \cdots a_N \in S_w^r(\mathbb{R}^m)$ and we have that*

$$\|a_1 a_2 \cdots a_N\|_{s_{r,\psi}} \leqslant \pi^{-(N-1)m/2} \prod_1^N \|a_j\|_{s_{p_j,\chi}}, \tag{3.1}$$

*where $\psi = \chi^N$.*

*The convolution $*$ on $\mathcal{S}(\mathbb{R}^m)$ may be uniquely continued to a continuous mapping from $S_w^{q_1}(\mathbb{R}^m) \times \cdots \times S_w^{q_N}(\mathbb{R}^m)$ to $S_w^r(\mathbb{R}^m)$, and if instead $a_j \in S_w^{q_j}(\mathbb{R}^m)$ for every $j = 1, \dots, N$, then*

$$\|a_1 * a_2 * \cdots * a_N\|_{s_{r,\psi}} \leqslant \pi^{-(N-1)m/2} \prod_1^N \|a_j\|_{s_{q_j,\chi}}, \tag{3.2}$$

*where $\psi(x) = \chi^N(x/2)$.*

*Proof.* — We prove the proposition in the case $N = 2$, and leave the details for general $N$ to the reader.

Assume that $a_j \in S_w^{p_j}(\mathbb{R}^m)$, $j = 1, 2$. By (1.1) and Minkowski's inequality we have

$$\begin{aligned}
\|a_1 a_2\|_{s_{r,\psi}} &= \int \Big( \int |\mathcal{F}(\chi_x^2\, a_1 a_2)(\xi)|^r \, dx \Big)^{1/r} d\xi \\
&\leqslant \pi^{-m/2} \int \Big( \int \Big( \int |(\mathcal{F}(\chi_x\, a_1))(\xi - \eta) (\mathcal{F}(\chi_x a_2))(\eta)| \, d\eta \Big)^r dx \Big)^{1/r} d\xi \\
&\leqslant \pi^{-m/2} \iint \Big( \int |(\mathcal{F}(\chi_x\, a_1))(\xi - \eta)(\mathcal{F}(\chi_x a_2))(\eta)|^r \, dx \Big)^{1/r} d\xi d\eta.
\end{aligned}$$

Applying Hölder's inequality to the inner integral in the last expression, we obtain

$$\begin{aligned}
\|a_1 a_2\|_{s_{r,\psi}} \leqslant \pi^{-m/2} \iint H_{p_1,a_1,\chi}(\xi - \eta) H_{p_2,a_2,\chi}(\eta) \, d\xi d\eta \\
= \|a_1\|_{s_{p_1,\chi}} \|a_2\|_{s_{p_2,\chi}},
\end{aligned}$$

and the first part of the assertion is proved.

In order to prove the second part, we observe first that it is enough to prove the assertion when $m$ is even, since the case when $m$ is odd may be transmitted to the case that $m$ should be even using the relations that $a \in S_w^p(\mathbb{R}^m)$, if and only if $a \otimes a \in S_w^p(\mathbb{R}^{2m})$, and $\|a\|_{s_{p,\chi}}^2 = \|a \otimes a\|_{s_{p,\chi\otimes\chi}}$.

Since $m$ is even, it follows that $a_1 * a_2$ makes sense as an element in $s_r(\mathbb{R}^m)$, by Proposition 1.3, Theorem 1.5, Proposition 1.6 and Lemma 2.13.

Let $\psi(x) = \chi^2(x/2)$. Then it follows by some straightforward computations that

$$
\begin{aligned}
&|\mathcal{F}(\psi_x a_1 * a_2)(\xi/2)| \\
&\qquad = 2^m \pi^{-m/2} \Big| \iint \big(\chi_{x/2}(y) a_1(y-z) e^{-2i\langle y,\xi\rangle}\big)\big(\chi_{x/2}(y) a_2(y+z)\big)\, dy dz \Big|.
\end{aligned}
$$

By applying Parseval's formula on the integral involving the $y$-variable, we obtain

$$
\begin{aligned}
&|\mathcal{F}(\psi_x a_1 * a_2)(\xi/2)| \\
&= (4/\pi)^{m/2} \Big| \iint \mathcal{F}\big(\chi_{x/2} a_1(\cdot - z) e^{-2i\langle \cdot,\xi\rangle}\big)(\eta)\, \mathcal{F}\big(\chi_{x/2} a_2(\cdot + z)\big)(-\eta)\, d\eta dz \Big| \\
&\qquad\qquad \leqslant (4/\pi)^{m/2} \iint |\mathcal{F}(\chi_{x/2+z} a_1)(\xi-\eta)\, \mathcal{F}(\chi_{x/2-z} a_2)(\eta)|\, d\eta dz.
\end{aligned}
$$

If we combine the last estimate with $H_{a_1 * a_2, r, \psi}(\xi/2) = \left(\int |\mathcal{F}(\psi_x a_1 * a_2)(\xi/2)|^r\, dx\right)^{1/r}$, we get

$$
\begin{aligned}
&H_{a_1 * a_2, r, \psi}(\xi/2) \\
&\leqslant (4/\pi)^{m/2} \Big( \int \Big( \iint |\mathcal{F}(\chi_{x/2+z} a_1)(\xi-\eta) \mathcal{F}(\chi_{x/2-z} a_2)(\eta)|\, d\eta dz \Big)^r dx \Big)^{1/r} \\
&\leqslant (4/\pi)^{m/2} \int \Big( \int \Big( \int |\mathcal{F}(\chi_{x/2+z} a_1)(\xi-\eta) \mathcal{F}(\chi_{x/2-z} a_2)(\eta)|\, dz \Big)^r dx \Big)^{1/r} d\eta\,.
\end{aligned}
$$

To establish the last inequality we have used Minkowski's inequality. We note that we may consider the integral in the $x$-variable as the $L^r$-norm of a convolution product. An application of Young's inequality therefore gives

$$
\begin{aligned}
&H_{a_1 * a_2, r, \psi}(\xi/2) \\
&\leqslant (4/\pi)^{m/2} \int \Big( \int |\mathcal{F}(\chi_x a_1)(\xi-\eta)|^{q_1}\, dx \Big)^{1/q_1} \Big( \int |\mathcal{F}(\chi_x a_2)(\eta)|^{q_2}\, dx \Big)^{1/q_2} d\eta \\
&\qquad\qquad = (4/\pi)^{m/2} \big( H_{a_1, q_1, \chi} * H_{a_2, q_2, \chi} \big)(\xi).
\end{aligned}
$$

If we integrate the $\xi$-variable we finally obtain

$$
\|a_1 * a_2\|_{s_r,\psi} \leqslant \pi^{-m/2} \|a_1\|_{s_{p_1,\chi}} \|a_2\|_{s_{p_2,\chi}},
$$

and the proof follows. □

We shall now reformulate Proposition 3.1 in such a way that the admissible functions which occurs in the estimates (3.1) and (3.2) should be the same, i.e., for some constants $C_N$ and $C'_N$, we will try to obtain estimates of the type

$$\|a_1 a_2 \cdots a_N\|_{s_{r,\chi}} \leqslant C_N \prod_1^N \|a_j\|_{s_{p_j,\chi}}, \tag{3.1'}$$

and

$$\|a_1 * a_2 * \cdots * a_N\|_{s_{r,\chi}} \leqslant C'_N \|a_j\|_{s_{q_j,\chi}}. \tag{3.2'}$$

This seems to be a difficult task for general $\chi$, if one desires that the constants $C_N$ and $C'_N$ should be simple expressions. However, if we consider functions $\chi = \chi_\lambda$ on $\mathbb{R}^m$ of the type

$$\chi_\lambda(x) = (2\lambda/\pi)^{m/4} e^{-\lambda|x|^2}, \tag{3.3}$$

(i. e. $\chi$ is Gauss function) we shall now prove that (3.1)′ and (3.2)′ are true for some simple choice of the constant $C_N$. We start with the following lemma.

LEMMA 3.2.— *Let $\chi_\lambda$ be as in* (3.3), *and let $a \in \mathcal{S}'(\mathbb{R}^m)$. Then for any $p \in [1,\infty]$ and any $\beta \geqslant 1$, one has that*

$$\|a\|_{s_{p,(\chi_\lambda)^\beta}} \leqslant (2\lambda/\pi)^{(\beta-1)m/4} \|a\|_{s_{p,\chi_\lambda}} \leqslant \beta^{m/2} \|a\|_{s_{p,(\chi_\lambda)^\beta}}. \tag{3.4}$$

*Proof.* — The assertion is obviously true for $\beta = 1$. We may therefore assume that $\beta > 1$, and start to prove the first inequality in (3.4). Since

$$\begin{aligned} \chi_{\lambda\beta} &= (2\lambda/\pi)^{-(\beta-1)m/4} \beta^{m/4} (\chi_\lambda)^\beta \quad \text{and} \\ \chi_{\lambda_1+\lambda_2} &= \big(\pi(\lambda_1+\lambda_2)/(2\lambda_1\lambda_2)\big)^{m/4} \chi_{\lambda_1}\chi_{\lambda_2} \end{aligned} \tag{3.5}$$

by the definitions, it follows that we have to prove that $\|a\|_{s_{p,\chi_{\lambda\beta}}} \leqslant \beta^{m/4}\|a\|_{s_{p,\chi_\lambda}}$. From the definitions we have

$$\|a\|_{s_{p,\chi_{\lambda\beta}}} = \int \Big( \int |\mathcal{F}\big((\tau_x\chi_{\lambda\beta})a\big)(\xi)|^p \, dx \Big)^{1/p} d\xi.$$

By (1.1) and (3.5) we get

$$\begin{aligned} \|a\|_{s_{p,\chi_{\lambda\beta}}} &= \Big( \frac{\beta\pi}{2(\beta-1)\lambda} \Big)^{m/4} \int \Big( \int |\mathcal{F}\big((\tau_x\chi_{\lambda(\beta-1)})(\tau_x\chi_\lambda)a\big)(\xi)|^p \, dx \Big)^{1/p} d\xi \\ &\leqslant \Big( \frac{\beta}{2\pi(\beta-1)\lambda} \Big)^{m/4} \int \Big( \int \Big( |\mathcal{F}(\tau_x\chi_{\lambda(\beta-1)})| * |\mathcal{F}\big((\tau_x\chi_\lambda)a\big)|(\xi) \Big)^p dx \Big)^{1/p} d\xi \\ &\leqslant \Big( \frac{\beta}{2\pi(\beta-1)\lambda} \Big)^{m/4} \int \Big( \int \Big( \chi_{1/(\lambda(\beta-1))} * |\mathcal{F}\big((\tau_x\chi_\lambda)a\big)|(\xi) \Big)^p dx \Big)^{1/p} d\xi. \end{aligned}$$

Here the convolution product is taken with respect to the $\xi$-variable, considering $x$ as fixed parameter. An application of Minkowski's inequality together with the fact that $\mathcal{F}(\chi_\lambda) = \chi_{1/\lambda}$, one gets

$$\begin{aligned}\|a\|_{s_{p,\chi_{\lambda\beta}}} &\leqslant (2\pi\lambda(\beta-1)/\beta)^{-m/4}\int (\chi_{1/(\lambda(\beta-1))} * H_{a,p,\chi_\lambda})(\xi)\,d\xi,\\ &= (2\pi\lambda(\beta-1)/\beta)^{-m/4}\|\chi_{1/(\lambda(\beta-1))}\|_{L^1}\|a\|_{s_{p,\chi_\lambda}}.\end{aligned}$$

Here $H_{a,p,\chi_\lambda}$ is given by $(0.3)''$ with $d\mu(x) = dx$. The desired estimate follows now by evaluating $\|\chi_{1/(\lambda(\beta-1))}\|_{L^1}$.

The second inequality of (3.3) follows from (3.5), and by letting $q = p$ in the following lemma. □

LEMMA 3.3. — *Let $0 < t < 1$ and that $a \in \mathcal{S}'(\mathbb{R}^m)$, and assume that $p, q \in [1,\infty]$ satisfy $p \leqslant q$. Then for any $\beta > 1$ we have*

$$\|a\|_{s_{q,\chi_\lambda}} \leqslant \Big(\frac{\lambda^2\beta^3}{\pi^2(\beta-1)^2}\Big)^{m/4}\Big(\frac{\pi(\beta-1)}{\lambda\beta p_0}\Big)^{m/(2p_0)}\|a\|_{s_{p,\chi_{\beta\lambda}}}, \tag{3.6}$$

*where $p_0 \in [1,\infty]$ satisfies $1/p_0 = 1 + 1/q - 1/p$. Here $\chi_\lambda$ is the same as in Lemma 3.2.*

*Proof.* — We shall proceed in a similar way as in the proof of the first inequality of (3.3). Let $t = 1/\beta$, and set $s = 1 - t$. Then it follows that $s, t > 0$. From the fact that $\mathcal{F}(\chi_\lambda) = \chi_{1/\lambda}$, an application of (2.2) and (3.5) gives

$$\begin{aligned}\|a\|_{s_{q,\chi_\lambda}} &= \int\Big(\int |\mathcal{F}^{-1}\big((\tau_\xi\chi_{1/\lambda})\widehat{a}\big)(x)|^q\,dx\Big)^{1/q}d\xi\\ &= \Big(\frac{\pi\lambda}{2st}\Big)^{m/4}\int\Big(\int |\mathcal{F}^{-1}\big((\tau_\xi\chi_{s/\lambda})(\tau_\xi\chi_{t/\lambda})\widehat{a}\big)(x)|^q\,dx\Big)^{1/q}d\xi\\ &= \Big(\frac{\lambda}{2\pi st}\Big)^{m/4}\int\Big(\int |(\mathcal{F}^{-1}(\tau_\xi\chi_{s/\lambda}) * \mathcal{F}^{-1}\big((\tau_\xi\chi_{t/\lambda})\widehat{a}\big)\big)(x)|^q\,dx\Big)^{1/q}d\xi\end{aligned}$$

where the last equality follows from (1.1). Here the convolution should be taken with respect to the $x$-variables, considering the $\xi$-variable as parameter.

The inner integral in the last expression is the $L^q$-norm of a convolution. From the fact that $\|\mathcal{F}^{-1}(\tau_\xi\chi_{s/\lambda})\|_{L^{p_0}} = \|\chi_{\lambda/s}\|_{L^{p_0}}$, an application of Young's inequality therefore gives

$$\begin{aligned}\|a\|_{s_{q,\chi_\lambda}} &\leqslant \Big(\frac{\lambda}{2\pi st}\Big)^{m/4}\|\chi_{\lambda/s}\|_{L^{p_0}}\int\Big(\int |\mathcal{F}^{-1}\big((\tau_\xi\chi_{t/\lambda})\widehat{a}\big)(x)|^p\,dx\Big)^{1/p}d\xi\\ &= \Big(\frac{\lambda}{2\pi st}\Big)^{m/4}\|\chi_{\lambda/s}\|_{L^{p_0}}\|a\|_{s_{p,\chi_{\lambda/t}}}.\end{aligned}$$

By evaluating $\|\chi_{\lambda/s}\|_{L^{p_0}}$, we obtain (3.6) from the last inequality. The proof is complete. $\square$

We have now the following alternative version of Proposition 3.1.

PROPOSITION 3.1′. — *Assume that the hypothesis in Proposition 3.1 is fulfilled and let* $\chi(x) = \chi_\lambda(x) = (2\lambda/\pi)^{m/4}e^{-\lambda|x|^2}$, *where* $\lambda > 0$. *Then* (3.1)′ *and* (3.2)′ *are true with* $C_N = (2\pi\lambda)^{-(N-1)/4}N^{m/2}$ *and* $C'_N = (2\pi/\lambda)^{m/2}(2\pi\lambda)^{-(N-1)m/4}$.

*Proof.* — If we let $\beta = N$, then (3.1)′ is a consequence of (3.1), (3.4) and (3.5). By similar reasons we obtain (3.2)′, if we let $\beta = N/4$, since $\big(\chi_\lambda(\cdot/2)\big)^N = (2\lambda/\pi)^{Nm/4}\big(2\pi/(N\lambda)\big)^{m/4}\chi_{N\lambda/4}$. $\square$

We shall next use Proposition 3.1′, to investigate compositions of elements in $S^p_w$ with analytic functions. First we observe that if we let $a = a_1 = a_2 = \ldots = a_N$, $p_1 = p$, and $p_2 = p_3 = \ldots = p_N = \infty$ in Proposition 3.1′, then we get

$$\|a^N\|_{s_{p,\chi_\lambda}} \leqslant (2\pi\lambda)^{-(N-1)m/4}N^{m/4}\|a\|_{s_{p,\chi_\lambda}}\big(\|a\|_{s_{\infty,\chi_\lambda}}\big)^{N-1}. \tag{3.7}$$

In order to estimate $\|a\|_{s_{\infty,\chi_\lambda}}$, in terms of $\|a\|_{s_{p,\chi_\lambda}}$, we have the following lemma.

LEMMA 3.4. — *Let* $\chi_\lambda$ *be as before and assume that* $a \in \mathcal{S}'(\mathbb{R}^m)$. *Let also* $h : [0,1] \to [1,4]$ *be the continuous and increasing function defined by* $h(0) = 1$, $h(1) = 4$ *and* $h(\alpha) = \alpha^{-\alpha}(1-\alpha)^{1-\alpha}(1+\alpha)^{1+\alpha}$ *when* $0 < \alpha < 1$. *Then for any* $p, q \in [1,\infty]$ *such that* $p \leqslant q$, *one has that*

$$\|a\|_{s_{q,\chi_\lambda}} \leqslant (\lambda/\pi)^{m(p^{-1}-q^{-1})}h(p^{-1}-q^{-1})^{m/2}\|a\|_{s_{p,\chi_\lambda}}. \tag{3.8}$$

*Proof.* — We observe that for any $\beta > 0$, then (3.6) holds. If we set apply Lemma 2.13 and (3.3), then we get

$$\|a\|_{s_{q,\chi_\lambda}} \leqslant \big(\lambda\beta^2/(\pi(\beta-1))\big)\big(\pi(\beta-1)/(p_0\lambda\beta)\big)^{m/(2p_0)}\|a\|_{s_{p,\chi_\lambda}}. \tag{3.9}$$

Here $p_0$ is given by $1/p_0 + 1/p = 1 + 1/q$. The estimate (3.9) is valid for any $\beta > 1$. By a straightforward minimalization treatment, one obtains that the minimum of the right-hand side is attained for $\beta = 1 + p^{-1} - q^{-1}$. If we insert this into (3.9), then we obtain (3.8), which completes the proof. $\square$

*Remark 3.5.* — We may obtain stronger estimates by using the best constants in Young's inequality and Haussdorf-Young's inequality in the proofs. (See [L].)

If we let $q = \infty$ in Lemma 3.4, then we get

$$\|a\|_{s_{\infty,\chi_\lambda}} \leqslant \big( (\lambda/\pi)^{1/p} h(1/p)\big)^{m/2} \|a\|_{s_{p,\chi_\lambda}}.$$

Inserting this into (3.8) one gets

$$\|a^N\|_{s_{p,\chi_\lambda}} \leqslant N^{m/2} C_{p,\lambda,m}^{N-1} \|a\|_{s_{p,\chi_\lambda}}^N, \tag{3.10}$$

where

$$C_{p,\lambda,m} = (2\pi)^{-m/4}\big( (\lambda/\pi)^{1/p} h(p^{-1})\big)^{m/2}, \tag{3.11}$$

where $h$ is the function in Lemma 3.4. It follows now from (3.10) and (3.11) that the following must be true.

PROPOSITION 3.6.— *Assume that $R > 0$, $C_{p,\lambda,m}$ is given by* (3.11), *and that $f$ is an analytic function in $\{ z\,;\, |z| < R\, C_{p,\lambda,m} \}$ such that $f(0) = 0$, with expansion $f(z) = \sum_0^\infty a_k z^k$, and let $g(z) = \sum_0^\infty k^{m/2} |a_k| z^k$. If $\|a\|_{s_{p,\chi_\lambda}} < R$, then $f(a) \in S_w^p(\mathbb{R}^m)$, and one has the estimate $\|f(a)\|_{s_{p,\chi_\lambda}} \leqslant g(C_{p,\lambda,m}\|a\|_{s_{p,\chi_\lambda}})$. If the condition $f(0) = 0$ is removed, then the assertion is still true for $p = \infty$.*

## 4. Some applications in the theory for pseudo-differential operators.

In this section we shall apply the results in the previous sections, in order to prove some general continuity properties for pseudo-differential operators. The section starts by discussing compositions between pseudo-differential operators, when the symbols belong to some $BS$-spaces. The last part of the section is devoted to some considerations of the operator $\mathrm{Op}(a)$ in (0.2), where we prove that if $a \in S_w^p(\mathbb{R}^{3n})$, and $b \in \mathcal{S}'(\mathbb{R}^{2n})$ satisfies $b_t(x, D) = \mathrm{Op}(a)$, for some $t \in [0, 1]$, then $b \in S_w^p(\mathbb{R}^{2n})$.

Before discussing composition properties for pseudo-differential operators, we recall that the condition that $H_{a,p,\chi,d\mu} \in L^1(\mathbb{R}^{2n})$ is invariant under the choice of Fourier transform, and in this section we shall use the symplectic Fourier transform $\mathcal{F}_\sigma$ (cf. (1.2)) for distributions on $\mathbb{R}^{2n}$. The reason for this is that we shall mainly deal with the Weyl calculus, and in this topic the symplectic Fourier transform is in many situations handsome to use. (See for example (4.1) below.)

THEOREM 4.1.— *Assume that* $s, t_1, t_2 \in [0,1]$, *and that* $p_1, p_2, r \in [1,\infty]$ *satisfies* $1/p_1 + 1/p_2 = 1/r$, *and let* $c \in \mathcal{S}(\mathbb{R}^{2n})$ *be defined through the formula* $\mathrm{Op}_s(c) = \mathrm{Op}_{t_1}(a_1)\mathrm{Op}_{t_2}(a_2)$, *as* $a_1, a_2 \in \mathcal{S}(\mathbb{R}^{2n})$. *Then the bilinear map* $(a_1, a_2) \mapsto c$ *may be uniquely extended to a continuous mapping from* $S_w^{p_1}(\mathbb{R}^{2n}) \times S_w^{p_2}(\mathbb{R}^{2n})$ *to* $S_w^r(\mathbb{R}^{2n})$.

*Moreover, if* $\chi = \pi^n \chi_2 \# \chi_1$, *where* $\chi_1, \chi_2 \in \mathcal{S}(\mathbb{R}^{2n})$ *have non-vanishing integrals, then*

$$\|a_1 \# a_2\|_{s_{r,\chi}} \leqslant \|a_1\|_{s_{p_1,\chi_1}} \|a_2\|_{s_{p_2,\chi_2}}, \quad a_1 \in S_w^{p_1}(\mathbb{R}^{2n}),\ a_2 \in S_w^{p_2}(\mathbb{R}^{2n}).$$

We observe that $a_1 \# a_2$ is well-defined as an element in $s_r(\mathbb{R}^{2n})$ when $a_j \in s_{p_j,\chi_j}(\mathbb{R}^{2n})$, $j = 1,2$, by Proposition 1.1, since $S_w^{p_j}(\mathbb{R}^{2n}) \subset s_{p_j}(\mathbb{R}^{2n})$.

Before the proof we introduce some notations and discuss some preparatory results. First we note that we may write the Weyl product $a \# b$ as

$$(4.1)\quad (a\#b)(X) = \pi^{-n} \int \widehat{a}(Y) b(Y+X) e^{2i\sigma(X,Y)}\, dY = \mathcal{F}_\sigma\big(\widehat{a}\,(\tau_{-X} b)\big)(X),$$

when $a, b \in \mathcal{S}(\mathbb{R}^{2n})$. (See Section 1.2 in [T2].) Here $\tau_X$, $X \in \mathbb{R}^{2n}$, is the translation operator $\tau_X a(Y) = a(Y - X)$ as before.

We omit the proofs of the following two lemmas, since they are simple consequences of the definitions.

LEMMA 4.2.— *Let* $a, b \in s_\infty(\mathbb{R}^{2n})$ *and* $c \in s_1(\mathbb{R}^{2n})$. *Then the following equalities are true:*

(1) $\mathcal{F}_\sigma(a\#b) = (\mathcal{F}_\sigma a)\#b = \check{a}\#\mathcal{F}_\sigma b$;

(2) $\langle a\#b, c\rangle = \langle a, b\#c\rangle$;

(3) $\langle \mathcal{F}_\sigma a, c\rangle = \langle \check{a}, \mathcal{F}_\sigma c\rangle$.

LEMMA 4.3.— *Let* $a, b \in s_\infty(\mathbb{R}^{2n})$ *and set* $(\widehat{\tau}_X a)(Y) = e^{2i\sigma(Y,X)} a(Y)$, *for every* $X, Y \in \mathbb{R}^{2n}$. *Then the following is true:*

(1) $\tau_X \circ \tau_Y = \tau_{X+Y}$ *and* $\tau_X \circ \widehat{\tau}_Y = e^{2i\sigma(Y,X)} \widehat{\tau}_Y \circ \tau_X$;

(2) $\mathcal{F}_\sigma \circ \tau_X = \widehat{\tau}_X \circ \mathcal{F}_\sigma$;

(3) $\tau_X(a\#b) = (\tau_X a)\#(\tau_X b)$ *and* $\widehat{\tau}_X(a\#b) = (\widehat{\tau}_X a)\#(\tau_X b)$.

We may now prove the following lemma:

LEMMA 4.4.— *Assume that the hypothesis in Theorem 4.1 is fulfilled,* $a \in S_w^{p_1}(\mathbb{R}^{2n})$, $b \in S_w^{p_2}(\mathbb{R}^{2n})$, *and that* $c$ *is equal to* $a\#b$. *If* $H_1$ *and* $H_2$ *are* *given by the formulas*

$$H_1(X,Z)=\mathcal{F}_\sigma\big((\tau_X\chi_1)a\big)(Z),\quad H_2(Y,Z)=\mathcal{F}_\sigma\big((\tau_Y\chi_2)b\big)(Z),$$

*then*

$$\mathcal{F}_\sigma\big((\tau_Y\chi)c\big)(X)=\int e^{2i\sigma(X,Z)}H_1(Y-X-Z,-Z)H_2(Y-Z,Z+X)\,dZ.$$

*Proof.* — We shall frequently apply Lemma 4.2 and Lemma 4.3. By some straightforward computations we get

$$\begin{aligned}\mathcal{F}_\sigma((\tau_Y\chi)c)(X)&=\pi^{-n}\int\chi(Z-Y)c(Z)e^{2i\sigma(X,Z)}\,dZ\\&=\pi^{-n}e^{2i\sigma(X,Y)}\langle\tau_Y(\widehat{\tau}_{-X}\chi),c\rangle.\end{aligned}$$

Since $\chi=\pi^n\chi_2\#\chi_1$, it follows from Lemma 4.3 (1) and (3) that

$$\begin{aligned}\mathcal{F}_\sigma\big((\tau_Y\chi)c\big)(X)&=\pi^{-n}e^{2i\sigma(X,Y)}\langle\tau_Y(\widehat{\tau}_{-X}\chi),a\#b\rangle\\&=\pi^{-n}e^{2i\sigma(X,Y)}\langle\widehat{\tau}_{-X}\chi,(\tau_{-Y}a)\#(\tau_{-Y}b)\rangle\\&=e^{2i\sigma(X,Y)}\langle(\widehat{\tau}_{-X}\chi_2)\#(\tau_{-X}\chi_1),(\tau_{-Y}a)\#(\tau_{-Y}b)\rangle.\end{aligned}$$

By applying Lemma 4.2 (2) on the last expression we get

$$\mathcal{F}_\sigma\big((\tau_Y\chi)c\big)(X)=e^{2i\sigma(X,Y)}\langle(\tau_{-Y}b)\#(\widehat{\tau}_{-X}\chi_2),(\tau_{-X}\chi_1)\#(\tau_{-Y}a)\rangle.$$

Let $J$ be the operator $Ju(X)=\check{u}(X)=u(-X)$. Then Lemma 4.2 (2) implies that $\langle Ja,c\rangle=\langle\widehat{a},\widehat{c}\rangle$. A combination of Lemma 4.3 and Lemma 4.2 (1) now gives

$$\mathcal{F}_\sigma\big((\tau_Y\chi)c\big)(X)=e^{2i\sigma(X,Y)}\langle J\big((\widehat{\tau}_{-X}\widehat{\chi}_1)\#(\tau_{-Y}a)\big),(\widehat{\tau}_{-Y}\widehat{b})\#(\widehat{\tau}_{-X}\chi_2)\rangle.$$

Hence

$$\mathcal{F}_\sigma\big((\tau_Y\chi)c\big)(X)=e^{2i\sigma(X,Y)}\int K_1(X,Y,-Z)K_2(X,Y,Z)\,dZ,\tag{4.2}$$

where

$$\begin{aligned}K_1(X,Y,Z)&=(\widehat{\tau}_{-X}\widehat{\chi}_1)\#(\tau_{-Y}a)(Z),\\K_2(X,Y,Z)&=(\widehat{\tau}_{-Y}\widehat{b})\#(\widehat{\tau}_{-X}\chi_2)(Z).\end{aligned}$$

We have to examine $K_1$ and $K_2$ more briefly. By (4.1), Lemma 4.3 and the fact that $\mathcal{F}_\sigma^2$ is the identity operator we have

$$\begin{aligned}K_1(X,Y,Z)&=\mathcal{F}_\sigma\big((\tau_{-X}\chi_1)\tau_{-Z}(\tau_{-Y}a)\big)(Z)\\&=\big(\widehat{\tau}_{-(Y+Z)}(\mathcal{F}_\sigma\big((\tau_{Y-X+Z}\chi_1)a\big))\big)(Z)=e^{2i\sigma(Y,Z)}\mathcal{F}_\sigma\big((\tau_{Y-X+Z}\chi_1)a\big)(Z).\end{aligned}$$

In the last equality we have used also that $\sigma(Z,Z) = 0$, which is a consequence of the anti-commutativity of $\sigma$. In the same way we obtain

$$\begin{aligned}K_2(X,Y,Z) &= \mathcal{F}_\sigma\big((\tau_{-Y}b)\tau_{-Z}(\widehat{\tau}_{-X}\chi_2)\big)(Z)\\ &= e^{2i\sigma(Y,Z)}\mathcal{F}_\sigma\big(b\,\tau_{Y-Z}(\widehat{\tau}_{-X}\chi_2)\big)(Z)\\ &= e^{2i\sigma(Y,Z)}e^{2i\sigma(X,Z-Y)}\mathcal{F}_\sigma\big(b\,\widehat{\tau}_{-X}\tau_{Y-Z}\chi_2\big)(Z)\\ &= e^{2i\sigma(Y,Z)}e^{2i\sigma(X,Z-Y)}\mathcal{F}_\sigma\big(b\,\tau_{Y-Z}\chi_2\big)(Z+X).\end{aligned}$$

The assertion now follows if we insert these expressions for $K_1$ and $K_2$ into (4.2). The proof is complete. □

We have finally the following lemma:

LEMMA 4.5. — *Assume that $a \in S^p_w(\mathbb{R}^{2n})$, and let $b \in \mathcal{S}'(\mathbb{R}^{2n})$ be chosen such that $a_t(x,D) = b^w(x,D)$, for some $t \in [0,1]$. Then $b \in S^p_w(\mathbb{R}^{2n})$. In particular, $\mathrm{Op}_t(S^p_w) \subset \mathcal{I}_p$.*

*Proof.* — One has that $b = e^{i(1/2-t)\langle D_\xi,D_x\rangle}a$. (Cf. Section 18.5 in [H].) Since it is possible to find a real non-degenerate quadratic form $\Phi$ on $\mathbb{R}^{2n}$ such that $e^{i(1/2-t)\langle D_\xi,D_x\rangle}a = Ce^{i\Phi} * a$, for some constant $C$, the assertion follows from Theorem 1.5, Proposition 1.6 and Proposition 2.14. □

*Proof of Theorem 4.1.* — It follows from Lemma 4.5 that we may assume that $s = t_1 = t_2 = 1/2$.

Assume that $a \in S^{p_1}_w(\mathbb{R}^{2n})$ and $b \in S^{p_2}_w(\mathbb{R}^{2n})$ and let $c = a\#b$. From Lemma 4.4 we obtain

$$\begin{aligned}\|a\#b\|_{s_{r,\chi}} &= \int\Big(\int|\mathcal{F}_\sigma((\tau_Y\chi)c)(X)|^r\,dY\Big)^{1/r}dX\\ &\leqslant \int\Big(\int\Big(\int|H_1(Y-X-Z,-Z)H_2(Y-Z,Z+X)|\,dZ\Big)^r dY\Big)^{1/r}dX\\ &\leqslant \iint\Big(\int|H_1(Y-X-Z,-Z)H_2(Y-Z,Z+X)|^r\,dY\Big)^{1/r}dXdZ,\end{aligned}$$

where the last inequality follows from Minkowski's inequality. Hölder's inequality applied to the inner integral in the last expresion gives

$$\begin{aligned}\|a\#b\|_{s_{r,\chi}} &\leqslant \iint\|H_1(\cdot-X-Z,-Z)\|_{L^{p_1}}\|H_2(\cdot-Z,Z+X)\|_{L^{p_2}}\,dXdZ\\ &= \iint\|H_1(\cdot,Z)\|_{L^{p_1}}\|H_2(\cdot,X)\|_{L^{p_2}}\,dXdZ = \|a\|_{s_{p_1,\chi_1}}\|b\|_{s_{p_2,\chi_2}}.\end{aligned}$$

Here we have used that $\|a\|_{s_{p_1,\chi_1}} = \int\|H_1(\cdot,Z)\|_{L^{p_1}}\,dZ$ and $\|b\|_{s_{p_2,\chi_2}} = \int\|H_2(\cdot,X)\|_{L^{p_2}}\,dX$. The proof is complete. □

If we have $\chi_1 = \chi_2 = \pi^{-n}\psi \equiv \chi$, where $\psi^w(x,D)$ is an orthonormal projection, it follows that $\chi = \pi^n\chi_1 \# \chi_2$, and $\int \chi(X)\,dX = C\int \chi_1\,dX \int \chi_2\,dX \neq 0$, for some non-zero constant $C$. Hence Theorem 4.1 gives us in this case

$$\|a_1 \# a_2\|_{s_{r,\chi}} \leqslant \|a_1\|_{s_{p_1,\chi}} \|a_2\|_{s_{p_2,\chi}}.$$

These conditions are obtained if for example $\chi_1(x,\xi) = \chi_2(x,\xi) = \chi(x,\xi) = \pi^{-n}e^{-(|x|^2+|\xi|^2)}$.

We shall end this section by considering the operator $\mathrm{Op}(a)$ in (0.2) when $a \in \mathcal{S}(\mathbb{R}^{3n})$. One has that $\mathrm{Op}(a) = \mathrm{Op}_t(b)$, when $t = 0$ and $b \in \mathcal{S}(\mathbb{R}^{2n})$ is uniquely determined and is given by

$$b(x,\xi) = e^{i\langle D_y, D_\xi\rangle} a(x,y,\xi)\Big|_{y=x}. \tag{4.3}$$

Since we have that $e^{i\langle D_y, D_\xi\rangle}a = H * a$, where $H(x,y,\xi) = (2\pi)^{-n}\delta(x)e^{-i\langle y,\xi\rangle}$, it follows from Lemma 2.13, Proposition 2.12 and Proposition 2.14 that (3.8) extends uniquely to a continuous map from $S^p_w(\mathbb{R}^{3n})$ to $S^p_w(\mathbb{R}^{2n})$. In particular we have the following result. (Cf. Lemma 4.5.)

PROPOSITION 4.6. — *The mapping $a \mapsto \mathrm{Op}(a)$ from $\mathcal{S}(\mathbb{R}^{3n})$ to $\mathcal{I}_\infty$ may be uniquely continued to a continuous mapping from $S^\infty_w(\mathbb{R}^{3n})$ to $\mathcal{I}_\infty$. If $a \in S^p_w(\mathbb{R}^{3n})$ where $p \in [1,\infty]$, and $t \in [0,1]$, then $\mathrm{Op}(a) = \mathrm{Op}_t(b)$ for some $b \in S^p_w(\mathbb{R}^{2n})$. In particular it follows that $\mathrm{Op}(a) \in \mathcal{I}_p$.*

## 5. Some further extensions.

In this section we shall discuss a case of $d\mu$-admissible functions, where the measure $d\mu$ is not necessarily periodic, and the symbol classes in some sense are related to the symbol classes $S(m,g)$, introduced by Hörmander in Section 18.4–18.5 in [H]. As in Section 1, we consider here only continuity properties in the Weyl calculus. Since these questions may be discussed independently of the choice of symplectic coordinates, we may formulate the results in such a way that the symbols are functions or distributions on the symplectic vector space $W = T^*V = V \oplus V'$, with symplectic form $\sigma$. Here $V$ is a vector space of dimension $n < \infty$, $V'$ its dual, and $\sigma(X,Y) = \langle y,\xi\rangle - \langle x,\eta\rangle$, where $X = (x,\xi) \in T^*V$ and $Y = (y,\eta) \in T^*V$.

The metric $g$ above should be slowly varying and $\sigma$-temperature on $W$, and we let $h_g(X) = (\sup g_X/g_X^\sigma)^{1/2}$ be the Plank's constant. (See Section 18.4–18.6 in [H].) By Lemma 18.6.4 in [H], there are symplectic coordinates $(y,\eta)$ on $W$, such that $g_X(y,\eta) = \sum_1^n \lambda_j(y_j^2 + \eta_j^2)$, where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n > 0$ are uniquely determined by $g_X$. In particular, $\kappa_g(X) \equiv \lambda_1\lambda_2\cdots\lambda_n$ is invariantly defined.

Some more notation will be needed in order to formulate the result. If $u$ is a smooth function on $W$ then as in Section 18.5 in [H] we norm the $k$:th differential of $u$ at $X \in W$ by

$$|u|_k^g(X) = \sup_{Y_1,\ldots,Y_k \in W\setminus 0} |u^{(k)}(X;Y_1,\ldots,Y_k)|/\prod_1^k g_X(Y_j)^{1/2},$$

and we set

$$\|u\|_N^g = \sum_{k\leqslant N} \sup_X |u|_k^g(X).$$

We have now the following (cf. Proposition 5.1.5 in [T1]).

Proposition 5.1. — *Let $d\mu$ be the measure on the index set $J \subset W$ defined by $\int \varphi(X)\,d\mu(X) = \sum_{j\in J}\varphi(j)$, let $g$ be a $\sigma$-temperate metric on $W$ with $h_g \leqslant 1$, and let $(\phi_j)_{j\in J}$ be a sequence of elements in $C_0^\infty(W)$ with the following properties:*

(1) *there is a bound for the number of overlapping supports of the $\phi_j$, and $\sum \phi_j \geqslant c$, for some $c$;*

(2) *there is a positive number $R$ and for each $j$ an element $X_j \in W$ such that*

$$\operatorname{supp}\phi_j \subset \{\, X \in W\,;\, g_{X_j}(X - X_j) < R^2 \,\};$$

(3) *there are constants $C_N$ when $N = 0,1,2,\ldots$ such that $\|\phi_j\|_N^{g_j} \leqslant C_N$, $j = 1,2,\ldots$, where $g_j = g_{X_j}$ and the norm is defined as in Section 4.1.*

*If $q \in [1,\infty]$ and $\chi_{j,q'} = \phi_j/\kappa_j^{1/q'}$, where $\kappa_j = \kappa(X_j)$, then $(\chi_{j,q'})_1^\infty$ is a $d\mu$-admissible partition to the order $q$, provided $R$ is small enough.*

We observe that $d\mu = \sum_j \delta_{X_j}$ is not necessarily periodic. In view of Theorem 2.7 it follows therefore that Proposition 5.1 in some sense generalizes Proposition 1.6.

*Proof.* — If $R$ is small enough, then we may choose a sequence $(\phi_j^0)$ such that $\phi_j^0 \in C_0^\infty(\mathbb{R}^{2n})$, $\phi_j^0 = 1$ in $\operatorname{supp} \phi_j$ and that (1)–(3) are fulfilled when the $\phi_j$ is replaced by $\phi_j^0$. This is in fact a consequence of Lemma 18.4.4 in [H]. Set $\widetilde{\chi}_{j,q'} = \widetilde{\phi}_j \kappa_j^{1/q'}$, where $\widetilde{\phi}_j = \phi_j^0 / \sum \phi_j$. It is then easily seen that the conditions (1)–(3) in Definition 1.4 are fulfilled for these choices of $\chi$ and $\widetilde{\chi}$. It remains to prove that (4) in Definition 1.4 holds.

We start by consider the case $q = 1$. Let $t = (t_j) \in l^\infty$ and set $a_t(X) = \sum t_j \chi_j(X)$. It follows from Theorem 18.6.3 in [H] that there is a constant $C$, which is independent of $t$ such that $\|a_t\|_{s_\infty} \leqslant C\|t\|_{l^\infty}$. Hence

$$|\sum t_j \langle \chi_{j,\infty}, \varphi \rangle| \leqslant C\|t\|_{l^\infty} \|\varphi\|_{s_1},$$

when $\varphi \in s_1(\mathbb{R}^{2n})$, and it follows that $\sum |\langle \chi_{j,\infty}, \varphi \rangle| \leqslant C\|\varphi\|_{s_1}$, which proves the proposition in this case.

Next we consider the case $q = \infty$. By Section 2.2 in [T1] or Section 2 in [T3] it follows that $\|\phi_j\|_{s_1} \leqslant C/\kappa_j$, for some constant $C$ independent on $j$. This implies that $\sup_j |\langle \widetilde{\chi}_{j,1}, \varphi \rangle| \leqslant C\|\varphi\|_{s_\infty}$, which implies that the condition (4) in Definition 1.4, and the proposition follows in this case.

For general $q$ we let $\theta \in [0,1]$, and consider the mapping $T_\theta : s_q(\mathbb{R}^{2n}) \to l^q$, defined by the formula

$$T_\theta(\varphi) = (\langle \varphi, \widetilde{\phi}_j \rangle / \kappa_j^\theta)_{j=1}^{N_0} = (\langle \varphi, \widetilde{\chi}_{j,\theta} \rangle)_{j=1}^{N_0},$$

where $N_0$ is an arbitrary positive integer. From above we have that

$$\|T_0\|_{s_1 \to l^1} \leqslant C \quad \text{and} \quad \|T_1\|_{s_\infty \to l^\infty} \leqslant C$$

for some constant $C < \infty$ independent on $N_0$. It follows now from Proposition 8 and Theorem IX.20 in Appendix to IX.4 in [RS] that $\|T_{1/q'}\|_{s_q \to l^q} \leqslant C$. Since $N_0$ was chosen arbitrary we conclude that

$$\|(\langle \varphi, \widetilde{\chi}_{j,q'} \rangle)\|_{l^q} \leqslant C\|\varphi\|_{s_q},$$

and the proof is complete. □

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