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GENERICALLY STRONGLY q-CONVEX COMPLEX MANIFOLDS

by T. NAPIER* and M. RAMACHANDRAN**

0. Introduction.

In [DiO], Diederich and Ohsawa obtain the following:

THEOREM (Diederich-Ohsawa). — *Let Ω be a relatively compact pseudoconvex domain with connected smooth real analytic boundary in a complex manifold of dimension 2. Assume that the boundary of Ω is strongly pseudoconvex at some point. Then Ω is holomorphically convex. In fact, Ω admits a C^∞ exhaustion function ψ which is strictly plurisubharmonic on the complement of some compact subset (i.e. Ω is strongly 1-convex as a manifold).*

Remark. — Strong 1-convexity is equivalent to the existence of a Stein space Y (the Remmert reduction of Ω), a surjective proper holomorphic mapping $\Psi : \Omega \to Y$ with connected fibers, and a finite subset Z of Y such that the restriction of Ψ maps $\Omega \setminus \Psi^{-1}(Z)$ biholomorphically onto $Y \setminus Z$ (by the work of Grauert [G], Docquier and Grauert [DG], Cartan [Car], and Remmert [Re]).

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A function φ of class C^2 on a complex manifold X of dimension n is said to be *strongly q-convex* if the Levi form

$$\mathcal{L}(\varphi) = \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

has at most $q-1$ nonpositive eigenvalues at each point in X. Equivalently, for each point $p \in X$ there exists a germ of a complex submanifold of X of dimension at least $n-q+1$ at p such that the restriction of φ to the germ is strictly plurisubharmonic. The manifold X is called *strongly q-complete* (respectively, *strongly q-convex*) if X admits a C^∞ exhaustion function which is strongly q-convex (respectively, strongly q-convex on the complement of some compact subset). The main goal of this paper is to obtain analogous versions of the above theorem of Diederich and Ohsawa for q-convex domains in higher dimensional manifolds. The hypotheses will be strengthened in the sense that we will assume that there is a global defining function for Ω on X with the appropriate properties. On the other hand, a version in which the defining function is only locally a maximum of real analytic functions will also be obtained. Thus the regularity of the boundary may be weakened in this sense. If X is Kähler and each point of some nonempty open subset lies in an irreducible compact analytic set of dimension $q-1$, then well known facts from the theory of Barlet spaces ([Ba1], [Ba2]) immediately give holomorphic convexity. The precise statements of the main results are given below.

THEOREM 0.1. — *Let X be a connected complex manifold of dimension n and suppose φ is a real analytic plurisubharmonic exhaustion function which is strongly $(n-1)$-convex at some point (i.e. the Levi form $\mathcal{L}(\varphi)$ has at least 2 positive eigenvalues at some point). Then, for almost every sufficiently large (regular) value a of φ, the sublevel*

$$\Omega \equiv \{\, x \in X \mid \varphi(x) < a \,\}$$

admits a C^∞ exhaustion function which is strongly $(n-1)$-convex on the complement of some compact subset of Ω (i.e. Ω is strongly $(n-1)$-convex as a complex manifold).

COROLLARY 0.2. — *Let X be a connected Kähler manifold of dimension n which admits a real analytic plurisubharmonic exhaustion function φ which is strongly $(n-1)$-convex at some point. Assume that there is a nonempty open subset V of X such that each point of V lies in some* *irreducible compact analytic subset of X of dimension at least $n-2$. Then X is holomorphically convex (with Remmert reduction of dimension 2).*

Remarks. — 1) When $n = 2$, the existence of V is automatic and, moreover, the Kähler condition is not needed.

2) If φ is a C^∞ plurisubharmonic function and, for some point $p \in X$ and for some tangent vector $v \in T_p^{1,0}X$, we have

$$(*) \qquad (d\varphi)_p \neq 0, \quad (\partial\varphi)_p(v) = 0, \quad \text{and} \quad \mathcal{L}(\varphi)(v,v) > 0,$$

then, since

$$\mathcal{L}(e^\varphi)(w,w) = e^\varphi(\mathcal{L}(\varphi)(w,w) + |\partial\varphi(w)|^2) \qquad \forall\, w \in T_p^{1,0}X,$$

the function e^φ is strongly $(n-1)$-convex at p. Thus, in place of strong $(n-1)$-convexity, one need only assume that φ satisfies $(*)$ at some point.

3) The results and proofs are related to the work of Huckleberry in [Hu].

4) We will obtain versions of Theorem 0.1 and Corollary 0.2 in the context of strongly q-convex functions as well as versions for φ a function which is real analytic with corners. For example, these versions, when applied with $q = 1$, will have as a consequence the following version of the theorem of Diederich and Ohsawa:

THEOREM 0.3. — *Let X be a connected noncompact complex manifold of dimension n. Assume that there exist a continuous plurisubharmonic exhaustion function φ on X, a compact subset K of X, and, for each point $p \in X \setminus K$, a finite collection $\mathcal{A}$ of real analytic plurisubharmonic functions on a neighborhood U of p in X such that*

$$\varphi(x) = \max_{\alpha \in \mathcal{A}} \alpha(x) \quad \forall\, x \in U$$

and such that, for each $\alpha \in \mathcal{A}$, the real analytic set of points in U at which α is not strictly plurisubharmonic is of (real) dimension at most 3. Then X is holomorphically convex with Remmert reduction of dimension n.

For $n = 2$, this theorem becomes

COROLLARY 0.4. — *Let X be a connected noncompact complex manifold of dimension 2. Assume that there exist a continuous plurisubharmonic exhaustion function φ on X, a compact subset K of X, and, for each* *point $p \in X \setminus K$, a finite collection $\mathcal{A}$ of real analytic plurisubharmonic functions on a connected neighborhood U of p in X such that*

$$\varphi(x) = \max_{\alpha \in \mathcal{A}} \alpha(x) \quad \forall\, x \in U$$

and such that each function in $\mathcal{A}$ is strictly plurisubharmonic at some point in U. Then X is holomorphically convex with Remmert reduction of dimension 2.

For X Kähler and $q \geqslant 1$, the following version of Theorem 0.1 will be obtained:

THEOREM 0.5. — *Let X be a connected noncompact Kähler manifold of dimension n, let q be a positive integer, and let φ be a continuous plurisubharmonic exhaustion function on X such that, on the complement of some compact subset K of X, φ is locally equal to the maximum of a finite collection of real analytic plurisubharmonic functions for which the real analytic set of points at which at least one of the functions is not strongly q-convex is of (real) dimension at most $2q+1$. Then, for almost every sufficiently large positive real number a, the sublevel*

$$\Omega \equiv \{\, x \in X \mid \varphi(x) < a \,\}$$

admits a C^∞ exhaustion function which is strongly q-convex on the complement of some compact subset of Ω.

Again, well known facts from the theory of Barlet spaces then give the following:

COROLLARY 0.6. — *Let X be a connected noncompact complex manifold of dimension n and let q be a positive integer. Assume that X admits*

(i) *a Kähler metric;*

(ii) *a continuous plurisubharmonic exhaustion function φ such that, on the complement of some compact subset K of X, φ is locally equal to the maximum of a finite collection of real analytic plurisubharmonic functions for which the real analytic set of points at which at least one of the functions is not strongly q-convex is of (real) dimension at most $2q+1$; and*

(iii) *a nonempty open subset V such that each point of V lies in some irreducible compact complex analytic subset of X of dimension at least $q-1$.*

Then X is holomorphically convex with Remmert reduction of dimension $n-q+1$.

Remark. — As indicated in Theorem 0.3, for $q = 1$ (i.e. $2q + 1 = 3$), the conclusion holds even if X is not Kähler (and, clearly, the condition (iii) holds automatically).

A description of the approach to the proofs of the above results will now be given. We begin with a brief sketch of Diederich and Ohsawa's proof of their theorem. By an observation of Diederich and Fornaess [DiF2], the set A of points at which there exists a germ of a 1-dimensional complex submanifold of X contained in $\partial\Omega$ is itself a compact complex submanifold of dimension 1. Other results of Diederich and Fornaess in [DiF1] provide, on a neighborhood of $\partial\Omega$, a C^∞ plurisubharmonic function which is strictly plurisubharmonic on the complement of an arbitrarily small neighborhood of A. Finally, Diederich and Ohsawa produce a function on Ω which is strictly plurisubharmonic near A (they also produce an example for which $A \neq \varnothing$).

In the situation of Theorem 0.1 (in which a global real analytic plurisubharmonic defining function exists), we may choose an arbitrarily large sublevel $\Omega = \{ x \in X \mid \varphi(x) < a \}$ so that the corresponding subset A of $\partial\Omega$ is empty; thus eliminating the need for the last step in Diederich and Ohsawa's proof. This is done by applying the theory of Barlet spaces [Ba1] as follows. Let G be the graph over the Barlet space $\mathcal{C}_{n-1}(X)$ of compact analytic $(n-1)$-cycles in X. Then the existence of φ implies that the image of G in X has $(2n-1)$-dimensional Hausdorff measure 0. Hence, for generic a, $\Psi(G) \cap \partial\Omega$ is a set of $(2n-2)$-dimensional Hausdorff measure 0. Thus $\partial\Omega$ cannot contain a compact complex analytic set of dimension $n-1$. A standard construction, analogous to the work of Richberg [Ri] and Demailly [De2] (see also Siu [Si1] and Coltoiu [Co]), now gives a strongly $(n-1)$-convex function on a neighborhood of the (real analytic) set of points in $\partial\Omega$ at which $\mathcal{L}(\varphi)$ has rank at most 1. Cutting off and adding a suitable multiple to $\varphi - \log(a - \varphi)$, one gets the required function on Ω.

Theorem 0.1 and the following well known consequence of the work of Barlet ([Ba1], [Ba2]) together immediately give Corollary 0.2:

THEOREM 0.7 (Barlet). — *Let X be a connected noncompact complex manifold of dimension n. Assume that, for some positive integer q, there exist*

(i) *a Kähler metric on X;*

(ii) *a continuous plurisubharmonic exhaustion function on X;*

(iii) *a sequence* $\{\Omega_\nu\}$ *of* C^∞ *relatively compact strongly* q*-convex domains in* X *such that* $X = \bigcup_\nu \Omega_\nu$ *and such that, for each* ν, $\Omega_\nu \subset \Omega_{\nu+1}$*; and*

(iv) *a nonempty open subset* V *of* X *such that each point of* V *lies in some irreducible compact analytic subset of* X *of dimension at least* $q-1$.

Then X *is holomorphically convex with Remmert reduction of dimension* $n-q+1$.

The main point is that, by a result of Barlet [Ba2], the Barlet space $\mathcal{C}_{q-1}(\Omega)$ of a strongly q-convex Kähler manifold Ω is holomorphically convex. In the situation of Theorem 0.7, properness of the projection from the graph over a suitable subset of $\mathcal{C}_{q-1}(\Omega_\nu)$ into a sublevel of the plurisubharmonic exhaustion function then gives holomorphic convexity of the sublevel and, therefore, of X. For the convenience of the reader, a sketch of the proof of the above well known theorem (assuming holomorphic convexity of the corresponding Barlet spaces) is included in Section 5.

The main result of [NR1] is that a connected complete Kähler manifold which has at least three ends and which is weakly 1-complete or has bounded geometry admits a proper holomorphic mapping onto a Riemann surface. The following is obtained as a consequence:

THEOREM ([NR1, Theorem 4.6]). — *Let* X *be a connected noncompact Kähler manifold (completeness is not required). Assume that*

(i) X *admits a* C^∞ *plurisubharmonic exhaustion function* φ *such that, for every sufficiently large regular value* a *of* φ *and for* $M = \varphi^{-1}(a)$, $\mathcal{L}(\varphi) = 0$ *on* $T^{1,0}M$*;*

(ii) *For every compact subset* C *of* X*, there is a holomorphic automorphism* γ *of* X *such that* $\gamma(C) \cap C = \varnothing$*; and*

(iii) X *does not have exactly two ends.*

Then X *admits a proper holomorphic mapping onto a Riemann surface.*

The above theorem and Corollary 0.2 together give the following:

COROLLARY 0.8. — *Let* X *be a connected noncompact Kähler manifold of dimension* n*. Assume that*

(i) X *admits a real analytic plurisubharmonic exhaustion function* φ*;*

(ii) *for every compact subset* C *of* X*, there is a holomorphic automorphism* γ *of* X *such that* $\gamma(C) \cap C = \varnothing$*;*

(iii) *X does not have exactly two ends; and*

(iv) *there is a nonempty open subset V of X such that each point of V lies in some irreducible compact analytic subset of dimension at least $n-2$ in X.*

Then X is holomorphically convex.

Proof. — Suppose that there exist an arbitrarily large regular value a of φ, a point $p \in M = \varphi^{-1}(a)$, and a tangent vector $v \in T_p^{1,0}M$ such that $\mathcal{L}(\varphi)(v,v) > 0$. Then the plurisubharmonic function e^φ is strongly $(n-1)$-convex in a neighborhood of p (for the zero eigenspace for $\mathcal{L}(e^\varphi)_p = e^{\varphi(p)}(\mathcal{L}(\varphi)_p + |(\partial\varphi)_p|^2)$ is contained in $T_p^{1,0}M$ and hence must be a subspace of dimension at most $n-2$). Hence Corollary 0.2 gives holomorphic convexity in this case. Otherwise, X is holomorphically convex by above theorem. □

Remarks. — 1) This corollary was stated in [NR1, Theorem 4.8] for the case $n = 2$ (in which the condition (iv) holds automatically) and was obtained as a consequence of the Levi flat case and the theorem of Diederich and Ohsawa. So the proof given here is, in a sense, simpler than that given in [NR1].

2) One may also state versions in the context of q-convex functions. But when $q < n-1$, even if e^φ is not strongly q-convex at p (for a plurisubharmonic function φ and some point p over a regular value a), the Levi form need not vanish on $T_p^{1,0}M$ (where $M = \varphi^{-1}(a)$). So, at least based on what is currently known, one must assume that either the conditions of [NR1, Theorem 4.6] hold or the conditions of Corollary 0.6 hold. Thus the statements are not edifying.

In Section 1, two (mostly well known) facts are recalled. In the proof of Theorem 0.1, which appears in Section 2, for a suitable sublevel provided by Barlet space arguments, these two facts give a strongly q-convex function on a neighborhood of the set of points in the boundary at which the exhaustion function is not strongly q-convex. In Section 3, basic facts concerning q-plurisubharmonic functions are recalled. Immediate generalizations of Theorem 0.1 and Corollary 0.2, in which the real analytic exhaustion function is only assumed to be q-plurisubharmonic with respect to some real analytic metric (not necessarily plurisubharmonic), are then stated. Finally, in Section 4, generalizations for exhaustion functions which are real analytic with corners are proved. Section 5 contains a brief sketch of the proof of Theorem 0.7.

1. Strongly q-convex extensions and complex analytic components.

In this section we recall two (mostly well known) facts. The first is that any C^∞ function on a weakly stratified set whose complex tangent spaces are of dimension $< q$ admits an approximate strongly q-convex extension to a neighborhood. This an (easy) analogue of Demailly's approximate extension theorem for strongly q-convex functions [De2]. The second fact, which is due to Diederich and Fornaess [DiF3], is that the set of q-dimensional complex analytic germs in a real analytic set of real dimension $2q$ is a (properly embedded) complex analytic set. Throughout this paper, all submanifolds and all real analytic and complex analytic subsets are assumed to be properly embedded in their ambient spaces (except, of course, when germs are being considered or when otherwise indicated). We first recall the following weak version of the notion of stratification:

DEFINITION 1.1. — *A C^∞* weak stratification *of a closed subset N of a C^∞ manifold M is a countable locally finite collection of sets $\{N_j\}_{j=1}^\infty$ such that*

$$N = \bigcup_{j=1}^{\infty} N_j$$

and such that, for each $j = 1, 2, 3, \ldots$, either $N_j = \varnothing$ or N_j is a (properly embedded) C^∞ submanifold of the open set

$$M \setminus \bigcup_{i=1}^{j} N_i.$$

If such a C^∞ weak stratification of $N \subset M$ exists, then the dimension *of N is given by*

$$\dim_{\mathbb{R}} N = \dim N = \sup_j \dim N_j \leqslant \infty$$

(which is clearly equal to the Hausdorff dimension of N). A real analytic weak stratification *of a closed subset of a real analytic manifold and a* complex analytic *(or* holomorphic *)* weak stratification *of a closed subset of a complex manifold are defined analogously.*

Of course, a complex analytic set in a complex manifold admits a finite complex analytic weak stratification. We also have the following well known examples:

LEMMA 1.2. — *Let N be a closed subset of a C^∞ manifold M.*

(a) *If N admits local C^∞ weak stratifications, then N admits a (global) C^∞ weak stratification.*

(b) *If M is a real analytic manifold and N admits local real analytic weak stratifications, then N admits a (global) real analytic weak stratification. In particular, any real analytic subset of M admits a real analytic weak stratification.*

Proof. — For the proof of (a), we may choose a locally finite collection of relatively compact open subsets $\{U_\nu\}_{\nu=1}^\infty$ of M such that $N \subset \bigcup_{\nu=1}^\infty U_\nu$ and such that, for each ν, there exists a finite C^∞ weak stratification

$$N \cap U_\nu = N_1^\nu \cup N_2^\nu \cup \cdots \cup N_{r_\nu}^\nu.$$

We may also choose a collection of open subsets $\{V_\nu\}_{\nu=1}^\infty$ of M such that $N \subset \bigcup_{\nu=1}^\infty V_\nu$ and such that, for each $\nu = 1, 2, 3, \dots$ and each $j = 1, \dots, r_\nu$, $V_\nu \subset\subset U_\nu$ and $N_j^\nu \cap \partial V_\nu$ is the empty set or is a C^∞ submanifold of (pure) dimension $\dim N_j^\nu - 1$ in N_j^ν. For example, we may take $V_\nu = \{\, x \in U_\nu \mid \psi(x) < a \,\}$, where ψ is a C^∞ exhaustion function on U_ν and a is a large regular value for the functions $\psi|_{N_1^\nu}, \dots, \psi|_{N_{r_\nu}^\nu}$. The following sequence then gives a C^∞ weak stratification for N:

$$\begin{aligned}
&N_1^1 \cap \partial V_1, N_2^1 \cap \partial V_1, \dots, N_{r_1}^1 \cap \partial V_1,\\
&N_1^1 \cap V_1, N_2^1 \cap V_1, \dots, N_{r_1}^1 \cap V_1,\\
&(N_1^2 \cap \partial V_2) \setminus \overline{V_1}, (N_2^2 \cap \partial V_2) \setminus \overline{V_1}, \dots, (N_{r_2}^2 \cap \partial V_2) \setminus \overline{V_1},\\
&(N_1^2 \cap V_2) \setminus \overline{V_1}, (N_2^2 \cap V_2) \setminus \overline{V_1}, \dots, (N_{r_2}^2 \cap V_2) \setminus \overline{V_1}, \dots,\\
&(N_1^\nu \cap \partial V_\nu) \setminus (\overline{V_1} \cup \cdots \cup \overline{V_{\nu-1}}), \dots, (N_{r_\nu}^\nu \cap \partial V_\nu) \setminus (\overline{V_1} \cup \cdots \cup \overline{V_{\nu-1}}),\\
&(N_1^\nu \cap V_\nu) \setminus (\overline{V_1} \cup \cdots \cup \overline{V_{\nu-1}}), \dots, (N_{r_\nu}^\nu \cap V_\nu) \setminus (\overline{V_1} \cup \cdots \cup \overline{V_{\nu-1}}), \dots.
\end{aligned}$$

Thus (a) is proved.

The proof of the first part of (b) is similar to the proof of (a) (one chooses V_ν to have real analytic boundary for each ν). For the proof of the second part, one need only recall the well known fact, due to Bruhat and Cartan [BrC1], [BrC2] and Bruhat and Whitney [BrW, Proposition 13], that if N is a real analytic set of dimension m, then each point in N has a neighborhood U in which

$$N \cap U = N_0 \cup N_1$$

where N_0 is a real analytic subset of dimension strictly less than m in U and N_1 is a real analytic submanifold of (pure) dimension m in $U \setminus N_0$. □

If M is a C^∞ submanifold of (real) dimension m in a complex manifold X of (complex) dimension n and J is the complex structure on X, then, for each point $p \in M$, $T_p^{1,0}M$ is the image of the complex subspace $T_pM \cap J(T_pM)$ of (T_pX, J) in $T_p^{1,0}X$ under the complex linear isomorphism $(T_pX, J) \to T_p^{1,0}X$ given by $u \mapsto \frac{1}{2}(u - \sqrt{-1}Ju)$. If φ is a real-valued C^∞ function on X and $u \in T_pX$, then

$$\partial\varphi\left(\frac{1}{2}(u - \sqrt{-1}Ju)\right) = d\varphi\left(\frac{1}{2}(u - \sqrt{-1}Ju)\right) = \frac{1}{2}d\varphi(u) - \frac{1}{2}\sqrt{-1}d\varphi(Ju).$$

Hence $\partial\varphi(\frac{1}{2}(u - \sqrt{-1}Ju)) = 0$ if and only if $d\varphi(u) = d\varphi(Ju) = 0$. It follows that if $\varphi_1, \dots, \varphi_{2n-m}$ are C^∞ functions on a neighborhood U of p such that

$$M \cap U = \{\, x \in U \mid \varphi_1(x) = \cdots = \varphi_{2n-m}(x) = 0 \,\} \qquad \text{and} \qquad (d\varphi_1 \wedge \cdots \wedge d\varphi_{2n-m})_p \neq 0,$$

then

$$T_p^{1,0}M = \bigcap_{j=1}^{2n-m} \ker(\partial\varphi_j)_p.$$

The following lemma is a simple analogue of [De2, Theorem 4]:

Lemma 1.3. — *Let X be a complex manifold, let K be a closed subset of X, let N be a closed subset of $X \setminus K$, and let α be a C^∞ real-valued function on X which is strongly q-convex on a neighborhood U of K in X. Suppose there exists a C^∞ weak stratification $\{N_j\}_{j=1}^\infty$ of N in $X \setminus K$ such that, for each $j = 1, 2, 3, \dots$ and each point $p \in N_j$,*

$$\dim_{\mathbb{C}} T_p^{1,0}N_j < q \qquad (\text{i.e.} \dim_{\mathbb{R}}\left[T_pN_j \cap J(T_pN_j)\right] < 2q).$$

Then, for every positive continuous function δ on X, there exists a C^∞ function β on X such that β is strongly q-convex on a neighborhood of the closed set $K \cup N$ in X, $\beta = \alpha$ on $K \cup N_1$, and $\alpha \leqslant \beta \leqslant \alpha + \delta$ on X.

Proof. — We will construct an extension of $\alpha|_{K\cup N_1}$ to a function which is strongly q-convex on a neighborhood of $K \cup N_1$ and then proceed inductively. Let $n = \dim_{\mathbb{C}} X$. Because N_1 is a (properly embedded) C^∞ submanifold of $X\setminus K$ and U is a neighborhood of K, there exists a sequence of C^∞ compactly supported real-valued functions $\gamma = \{\gamma_m\}_{m=1}^\infty$ on X such that

(i) each γ_m vanishes on $K \cup N_1$ (i.e. $\gamma \equiv 0$ on $K \cup N_1$);

(ii) the collection $\{\operatorname{supp}\gamma_m\}_{m\in\mathbb{N}}$ is locally finite in X; and

(iii) for each point $p \in N_1 \setminus U$, we have

$$T_pN_1 = \ker(d\gamma)_p = \bigcap_{m=1}^{\infty} \ker(d\gamma_m)_p = \bigcap_{m=1}^{k} \ker(d\gamma_m)_p \qquad \text{for } k = k(p) \gg 0$$

(i.e. $(d\gamma)_p = \{(d\gamma_m)_p\}_{m=1}^\infty$ has rank $2n - \dim N_1$).

The function

$$|\gamma|^2 \equiv \sum_{m=1}^{\infty} \gamma_m^2$$

is of class C^∞, because the sum is locally finite. Moreover, for each point $p \in K \cup N_1$ and each tangent vector $v \in T_p^{1,0}X$, we have

$$\mathcal{L}(|\gamma|^2)(v,v) = 2|\partial\gamma(v)|^2 = 2\sum_{m=1}^{\infty} |\partial\gamma_m(v)|^2 \geqslant 0.$$

In particular, if $p \in N_1\setminus U$, then, since $\dim_{\mathbb{C}} T_p^{1,0}N_1 < q$, the condition (iii) implies that $\mathcal{L}(|\gamma|^2)_p$ has at least $n - q + 1$ positive eigenvalues (and no negative eigenvalues) on $T_p^{1,0}X$.

Now let $\lambda : X \to (0,\infty)$ and $\eta : X \to [0,1]$ be C^∞ functions with $\eta \equiv 1$ on a neighborhood of $K \cup N_1$ in X. Then the function

$$\alpha_1 \equiv \alpha + \eta \cdot \lambda \cdot |\gamma|^2$$

is equal to α on $K \cup N_1$ and, for each point $p \in K \cup N_1$, we have

$$\mathcal{L}(\alpha_1)_p = \mathcal{L}(\alpha)_p + \lambda(p)\mathcal{L}(|\gamma|^2)_p \geqslant \mathcal{L}(\alpha)_p$$

(because $|\gamma(p)|^2 = 0$ and $(d|\gamma|^2)_p = 0$). Moreover, if we choose λ so that $\lambda(x) \to \infty$ sufficiently fast as $x \to \infty$, then, since α is strongly q-convex on U, the Levi form $\mathcal{L}(\alpha_1)_p$ will have at least $n-q+1$ positive eigenvalues for each point $p \in K \cup N_1$. It follows that, if we choose η to be supported on a sufficiently small neighborhood of $K \cup N_1$ in X, then, for a sufficiently small neighborhood V_1 of $K \cup N_1$ in X and a sufficiently small neighborhood U_1 of the closure $K_1 \equiv \overline{V_1}$, α_1 will be strongly q-convex on U_1 and we will have

$$\alpha_1 = \alpha \text{ on } K \cup N_1 \qquad \text{and} \qquad \alpha \leqslant \alpha_1 \leqslant \alpha + \frac{1}{2}\delta \text{ on } X.$$

Setting $\alpha_0 = \alpha$ and $K_0 = K$ and proceeding as above inductively with pairs

$$(K, N_1) = (K_0, N_1 \setminus K_0), (K_1, N_2 \setminus K_1), (K_2, N_3 \setminus K_2), \dots,$$

we get a sequence of C^∞ functions $\{\alpha_j\}$ and sequences of open sets $\{V_j\}$ and $\{U_j\}$ and closed sets $\{K_j\}$ such that, for each $j = 1, 2, 3, \dots$, we have

$$K_{j-1} \cup N_j = K_{j-1} \cup (N_j \setminus K_{j-1}) \subset V_j \subset \overline{V_j} = K_j \subset U_j,$$

the function α_j is strongly q-convex on U_j, $\alpha_j = \alpha_{j-1}$ on $K_{j-1} \cup N_j$, and $\alpha_{j-1} \leqslant \alpha_j \leqslant \alpha_{j-1} + 2^{-j}\delta$ on X. We now get a C^∞ strongly q-convex function α_∞ on the neighborhood $V \equiv \bigcup_{j=1}^\infty V_j$ of $K \cup N$ by setting

$$\alpha_\infty|_{V_j} = \alpha_j|_{V_j} \qquad \forall j = 1, 2, 3, \dots.$$

Clearly, $\alpha_\infty = \alpha_0 = \alpha$ on $K \cap N_1$. By induction, we also have, for each j,

$$\alpha = \alpha_0 \leqslant \alpha_j = \alpha_\infty \leqslant \alpha_0 + 2^{-1}\delta + 2^{-2}\delta + \cdots + 2^{-j}\delta \leqslant \alpha_0 + \delta \text{ on } V_j.$$

Hence

$$\alpha \leqslant \alpha_\infty \leqslant \alpha + \delta \text{ on } V.$$

Thus if we fix a C^∞ function $\tau : X \to [0, 1]$ such that $\tau \equiv 1$ on a neighborhood of the closed set $K \cup N$ and such that $\operatorname{supp} \tau \subset V$, then the C^∞ function

$$\beta \equiv (1 - \tau)\alpha + \tau\alpha_\infty : X \to \mathbb{R}$$

has the required properties. □

The following fact is contained implicitly in the work of Diederich and Fornaess [DiF3]. A sketch of the proof is included here for the convenience of the reader.

LEMMA 1.4 (Diederich-Fornaess). — *Let S be a (properly embedded) real analytic set of real dimension at most $2q$ in a complex manifold X. Then the set Y of points in S at which there exists a germ of a q-dimensional* *complex analytic subset of X contained in S is a (properly embedded) complex analytic subset of X.*

Remarks. — 1) Clearly, either $Y = \varnothing$ or Y is of pure dimension q.

2) In [DiF2], Diederich and Fornaess prove that if M is a connected smooth real analytic hypersurface in a complex manifold X of dimension n and Y is the set of points in M at which there exists a germ of an $(n-1)$-dimensional complex analytic subset of X contained in M, then either $Y = \varnothing$, $Y = M$, or Y is a (properly embedded) complex submanifold of X (of dimension $n-1$ contained in M).

Sketch of the proof of Lemma 1.4. — The statement is local, so we may assume that X is an open set in $\mathbb{C}^n$ and that S is the zero set of a real analytic function $\rho : X \to \mathbb{R}$. Given $p = (p_1, \dots, p_n) \in S$, there exists an $r > 0$ such that the polydisk

$$\Delta^n(p;r) = \{\, (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_j - p_j| < r \,\forall\, j = 1, \dots, n \,\}$$

is contained in X and such that the Taylor expansion

$$\rho(z, \bar{z}) = \sum_{I,J} a_{I,J} (z-p)^I (\bar{z} - \bar{p})^J$$

for ρ centered at p converges in $\Delta^n(p;r)$. The main point is the following observation (see [DiF3, pp. 383–384] for the proof):

Every germ of a q-dimensional complex submanifold of X at p which is contained in S is contained in some (properly embedded) complex analytic subset Z of dimension q in $\Delta^n(p;r)$ which is contained in S.

We may assume that

$$S = S_0 \cup S_1 \cup \cdots \cup S_{2q}$$

where S_0 is a finite set and, for each $j = 1, \dots, 2q$, S_j is a (properly embedded) real analytic submanifold of (pure) dimension j in $X \setminus (S_0 \cup S_1 \cup \cdots \cup S_{j-1})$. Let $N = S_0 \cup S_1 \cup \cdots \cup S_{2q-1}$ and let $Y' = Y \cap S_{2q} = Y \setminus N$. Then Y' is dense in Y because N cannot contain a germ of a complex manifold of dimension q. Moreover, Y' is both open and closed in S_{2q}. For if $p \in Y'$, then there is a neighborhood U of p and a purely q-dimensional complex analytic subset A of U such that $p \in A \subset S_{2q} \cap U$. Since $\dim_{\mathbb{R}} S_{2q} = 2q = \dim_{\mathbb{R}} A$, A must be open relative to S_{2q}. Thus Y' is open in S_{2q}. According to the theorem of Levi-Civita [L] (see also Freeman [Fr]), a C^∞ submanifold M of a complex manifold is a complex submanifold if and only if $JTM = TM$. Since the condition $JT_zS_{2q} = T_zS_{2q}$ is real analytic on $z \in S_{2q}$ and since $z \in Y'$ if and only if a neighborhood of z in S_{2q} is a complex submanifold of an open subset of X, it follows that Y' is also closed in S_{2q}. In particular, $Y' = Y \setminus N$ is a complex submanifold of dimension q in $X \setminus N$.

To complete the proof of the lemma, we fix a point $p_0 \in \overline{Y} \cap N = \overline{Y'} \cap N$ and a constant $r > 0$ such that the Taylor expansion for $\rho(z, \bar{z})$ centered at p_0 converges in the polydisk $\Delta^n(p_0; 8r) \subset X$. By applying Lemma 1.3 to the set $N = S_0 \cup \cdots \cup S_{2q-1} \subset X$ and a suitable C^∞ function α on X, one gets a C^∞ function β on X such that $\beta > 1$ on $\Delta^n(p_0; r)$, $0 < \beta < 1$ on $X \setminus \Delta^n(p_0; 2r)$, and β is strongly q-convex on a neighborhood V of N in X. For each point $p \in Y' \cap \Delta^n(p_0; r) \cap V$, we have

$$\Delta^n(p_0; 3r) \subset\subset \Delta^n(p; 4r) \subset\subset \Delta^n(p_0; 5r),$$

and hence the Taylor expansion for ρ centered at p converges in the polydisk $\Delta^n(p; 4r)$. Therefore, by the claim, there exists an irreducible complex analytic subset $Z(p)$ of dimension q in $\Delta^n(p; 4r)$ with

$$p \in Z(p) \subset S \cap \Delta^n(p; 4r).$$

In particular, $Z(p) \subset Y$ and $Z(p) \setminus N$ is an open subset of the complex manifold Y'. Let $Z'(p)$ be the irreducible component of $Z(p) \cap \Delta^n(p_0; 3r)$ containing p. The restriction of a strongly q-convex function φ to a complex analytic subset of pure dimension q cannot attain a local maximum at a point z, because the restriction of φ to some germ of an $(n - q + 1)$-dimensional complex submanifold at z is strictly plurisubharmonic and the intersection of the germ and the analytic subset is of positive dimension. On the other hand, the restriction $\beta|_{Z'(p)}$ attains its maximum at some point $z(p) \in Z'(p) \cap \Delta^n(p_0; 2r)$ because $\beta(p) > 1$ while $0 < \beta < 1$ on $X \setminus \Delta^n(p_0; 2r)$. Therefore, since β is strongly q-convex on V, we have

$$z(p) \in Z'(p) \cap \Delta^n(p_0; 2r) \setminus V \subset\subset Y' \cap \Delta^n(p_0; 3r).$$

Thus $z(p)$ lies in some (unique) connected component $W(p)$ of the complex submanifold $Y' \cap \Delta^n(p_0; 3r)$ of $\Delta^n(p_0; 3r) \setminus N$. We have $W(p) \subset Z'(p)$ because $z(p) \in W(p) \cap Z'(p)$ and the subset $Z'(p) \setminus N$ is both open and closed in $Y' \cap \Delta^n(p_0; 3r)$. Only finitely many connected components of $Y' \cap \Delta^n(p_0; 3r)$ can meet the compact subset $\overline{\Delta^n(p_0; 2r)} \setminus V$ of $\Delta^n(p_0; 3r) \setminus N$. Hence the collection $\{\, W(p) \mid p \in Y' \cap \Delta^n(p_0; r) \cap V \,\}$ is finite and is therefore equal to $\{W(p_1), \dots, W(p_k)\}$, for some choice of finitely many points $p_1, \dots, p_k \in Y' \cap \Delta^n(p_0; r) \cap V$. In particular, we have

$$z(p) \in W(p) \subset W(p_1) \cup \cdots \cup W(p_k) \quad \forall p \in Y' \cap \Delta^n(p_0; r) \cap V.$$

But we also have, for each $i = 1, \dots, k$,

$$z(p_i) \in W(p_i) \cap \Delta^n(p_0; 2r) \subset S$$
$$\text{and} \quad \Delta^n(p_0; 3r) \subset\subset \Delta^n(z(p_i); 5r) \subset\subset \Delta^n(p_0; 8r).$$

Thus by applying the claim to the germ of $W(p_i)$ at $z(p_i)$ and to the polydisk $\Delta^n(z(p_i), 5r)$, one gets an irreducible complex analytic subset Z_i of dimension q in $\Delta^n(z(p_i), 5r)$ such that

$$W(p_i) \subset Z_i \subset S.$$

As above, $Z_i \subset Y$ and $Z_i \setminus N$ is open and closed in $Y' \cap \Delta^n(z(p_i), 5r)$. Moreover, the irreducible component Z_i' of $Z_i \cap \Delta^n(p_0, 3r)$ containing $W(p_i)$ must equal the irreducible analytic subset $Z'(p)$ of $\Delta^n(p_0, 3r)$ whenever $z(p) \in W(p_i)$, because $W(p_i) = W(p)$ is then a neighborhood of $z(p)$ in both Z_i' and $Z'(p)$. Therefore, since $p \in Z'(p)$ for each point $p \in Y' \cap \Delta^n(p_0; r) \cap V$, we have

$$\begin{aligned} Y' \cap \Delta^n(p_0; r) \cap V \subset & \bigcup_{p \in Y' \cap \Delta^n(p_0; r) \cap V} Z'(p) \cap \Delta^n(p_0; r) \cap V \\ & \subset (Z_1' \cup \cdots \cup Z_k') \cap \Delta^n(p_0; r) \cap V \\ & \subset Y \cap \Delta^n(p_0; r) \cap V. \end{aligned}$$

Passing to closures in the neighborhood $\Delta^n(p_0; r) \cap V$ of p_0, we get

$$Y \cap \Delta^n(p_0; r) \cap V = (Z_1' \cup \cdots \cup Z_k') \cap \Delta^n(p_0; r) \cap V.$$

It follows that Y is a complex analytic subset of pure dimension q in X. $\square$

2. The smooth case.

Theorem 0.1 is an immediate consequence of the following version (for a generically q-convex plurisubharmonic exhaustion function) which will be proved in this section:

THEOREM 2.1. — *Let X be a connected noncompact complex manifold of dimension n, let φ be a C^∞ plurisubharmonic exhaustion function* *which is real analytic on the complement $X \setminus K$ of some compact subset K of X, let q be a positive integer, and let S be the real analytic set of points $p \in X \setminus K$ at which the Levi form $\mathcal{L}(\varphi)_p$ has rank at most $n - q$. Assume that*

$$\dim_{\mathbb{R}} S \leqslant 2q + 1.$$

Then, for almost every sufficiently large (regular) value a of φ, the sublevel

$$\Omega \equiv \{ x \in X \mid \varphi(x) < a \}$$

admits a C^∞ exhaustion function which is strongly q-convex on the complement of some compact subset of Ω (i.e. Ω is strongly q-convex as a complex manifold).

Remarks. — 1) In place of the set S, one need only consider the smaller set S_0 of points $p \in X \setminus K$ at which

$$(d\varphi)_p \neq 0 \qquad \text{and} \qquad \operatorname{rank} \mathcal{L}(\varphi)|_{\ker(\partial\varphi)_p} < n - q;$$

a real analytic subset of $S \setminus C$, where $C = \{ p \in X \setminus K \mid (d\varphi)_p = 0 \}$. For if b is a sufficiently large regular value of φ, X' is a connected component of $\{ x \in X \mid \varphi(x) < b \}$, K' is the compact subset $(K \cup \overline{C}) \cap X'$, and $\varphi' = -\log(e^b - e^\varphi)$, then φ' is a plurisubharmonic exhaustion function on X' which is real analytic on $X' \setminus K'$. Moreover, for each point $p \in X' \setminus K'$, we have

$$(d\varphi')_p \neq 0$$
$$\text{and} \quad \operatorname{rank} \mathcal{L}(\varphi')_p = \operatorname{rank} \mathcal{L}(\varphi')|_{\ker(\partial\varphi')_p} + 1 = \operatorname{rank} \mathcal{L}(\varphi)|_{\ker(\partial\varphi)_p} + 1,$$

because the 0-eigenspace for $\mathcal{L}(\varphi')_p$ is contained in $\ker(\partial\varphi)_p = \ker(\partial\varphi')_p$ and hence must agree with that of $\mathcal{L}(\varphi')|_{\ker(\partial\varphi')_p} = e^{\varphi(p)}(e^b - e^{\varphi(p)})^{-1}$ $\mathcal{L}(\varphi)|_{\ker(\partial\varphi)_p}$. Thus

$$S' \equiv \{ p \in X' \setminus K' \mid \operatorname{rank} \mathcal{L}(\varphi')_p \leqslant n - q \} = S_0 \cap (X' \setminus K')$$

and we may, therefore, replace X by X', K by K', and φ by φ'.

2) For the case $q = n - 1$ (Theorem 0.1), we need only assume that each connected component of U of $X \setminus K$ contains a point p such that

$$(d\varphi)_p \neq 0 \qquad \text{and} \qquad \operatorname{rank} \mathcal{L}(\varphi)|_{\ker(\partial\varphi)_p} > 0$$

(i.e. $U \setminus C \not\subset S_0$). For by replacing φ by e^φ, we get $\dim_{\mathbb{R}} S \leqslant 2n - 1 = 2q + 1$.

For the proof of Theorem 2.1, we first recall some of the basic facts concerning Barlet spaces. A *compact analytic q-cycle* on a reduced complex space X is a formal finite linear combination

$$c = \sum_i m_i Y_i$$

of distinct compact irreducible analytic subsets $\{Y_i\}$ of dimension q in X with positive integer coefficients $\{m_i\}$. The *support* of c is the analytic subset

$$|c| = \bigcup_i Y_i.$$

By a theorem of Barlet [Ba1], the space $\mathcal{C}_q(X)$ of compact analytic q-cycles in X is a (second countable) reduced complex space. Moreover, we have holomorphic mappings

$$\begin{array}{ccc} G & \xrightarrow{\Psi} & X \\ {\scriptstyle\Phi}\big\downarrow & & \\ \mathcal{C}_q(X) & & \end{array}$$

where $G = \{\,(x,c) \in X \times \mathcal{C}_q(X) \mid x \in |c|\,\}$ is the graph and Φ and Ψ are the associated projections. The map Φ is always proper and surjective, but the map Ψ is not proper or surjective in general. However, as is well known (see, for example, Campana [Cam] or Fujiki [Fu]), if X is a Kähler manifold which is weakly 1-complete or which has bounded geometry, then the restriction of Ψ to the graph over a suitable irreducible component of $\mathcal{C}_q(X)$ is proper.

Proof of Theorem 2.1. — We first show that, for almost every sufficiently large regular value a of φ, the fiber $M = \varphi^{-1}(a)$ does *not* contain a q-dimensional compact analytic subset of X. For this, we consider the graph and holomorphic projections

$$\begin{array}{ccc} G & \xrightarrow{\Psi} & X \\ {\scriptstyle\Phi}\big\downarrow & & \\ \mathcal{C}_q(X) & & \end{array}$$

over $\mathcal{C}_q(X)$. The image $\Psi(x,c) = x$ of each point $(x,c) \in G$ lies in a (purely) q-dimensional compact (complex) analytic set $|c|$ and, therefore, in the set $K \cup S$, because φ is locally constant on $|c|$. By hypothesis, $\dim_{\mathbb{R}} S \leqslant 2q+1$. Therefore, since the image $\Psi(G)$ of G is equal to a countable union of local complex analytic sets in X which are, of course, of even real dimension, the set $\Psi(G) \setminus K$ must be of $(2q+1)$-dimensional Hausdorff measure 0. Hence $\varphi^{-1}(a) \cap \Psi(G)$ is a set of $2q$-dimensional Hausdorff measure 0 for almost every $a \in \mathbb{R}$ with $a > \max_K \varphi$. On the other hand, any compact complex analytic set of pure dimension q in X is contained in $\Psi(G)$ and has positive $2q$-dimensional Hausdorff measure. Therefore, for almost every regular value $a \in \mathbb{R}$ with $a > \max_K \varphi$, the fiber $M = \varphi^{-1}(a) \subset X \setminus K$ does *not* contain a compact complex analytic subset of dimension q in X.

Similarly, for almost every regular value $a \in \mathbb{R}$ with $a > \max_K \varphi$, $S \cap M$ is a compact real analytic set of dimension at most $2q$. Hence, since any germ of a q-dimensional complex analytic subset of X which is contained in M must be contained in $S \cap M$, Lemma 1.4 implies that the set of points in M at which such a germ exists is a purely q-dimensional compact complex analytic subset of X which is contained in $S \cap M$.

Combining the above observations, one sees that, for almost every sufficiently large regular value a of φ, the fiber $M = \varphi^{-1}(a)$ is nonempty, the compact real analytic set $S \cap M$ is of (real) dimension at most $2q$, and M does *not* contain any q-dimensional germs of complex analytic subsets of X. According to Lemma 1.2, $S \cap M$ admits a finite real analytic weak stratification

$$S \cap M = S_1 \cup S_2 \cup \cdots \cup S_r.$$

For each $j = 1, \ldots, r$, the set

$$B_j = \{\, x \in S_j \mid \dim_{\mathbb{R}}(T_x S_j \cap J T_x S_j) = 2q \,\}$$

is a nowhere dense real analytic subset of S_j. For if B_j contains an open subset of S_j, then the open set determines a q-dimensional germ of a complex analytic subset of X which is contained in $S \cap M$. This would contradict the choice of the regular value a.

We now apply Lemma 1.3 inductively. Set $B_0 = S_0 = \varnothing$ and $\alpha_0 \equiv 0$. Given j with $1 \leqslant j \leqslant r$ and a C^∞ function α_{j-1} on X which is strongly q-convex on a neighborhood of the compact set

$$L_{j-1} = S_0 \cup S_1 \cup \cdots \cup S_{j-1},$$

we may apply Lemma 1.3 to the real analytic subset B_j of $X \setminus L_{j-1}$ to get a C^∞ function α'_j on X which is strongly q-convex on a neighborhood of the compact set $L'_j = L_{j-1} \cup B_j$. Applying Lemma 1.3 again, one gets a C^∞ function α_j on X which is strongly q-convex on a neighborhood of

$L_j = L_{j-1} \cup S_j = L'_j \cup (S_j \setminus L'_j)$. Proceeding inductively for $j = 1, \dots, r$, one gets a C^∞ function $\beta = \alpha_r$ on X which is strongly q-convex on a neighborhood of $S \cap M$ in X.

Finally, since φ is plurisubharmonic on X and strongly q-convex on $X \setminus (K \cup S)$, for a sufficiently small positive constant ϵ, the function

$$\psi \equiv -\log(a - \varphi) + \varphi|_\Omega + \epsilon\beta|_\Omega$$

will be a C^∞ exhaustion function on the set $\Omega = \{\, x \in X \mid \varphi(x) < a \,\}$, and ψ will be strongly q-convex near $\partial\Omega = M$. □

Theorem 2.1 and Theorem 0.7 together immediately give Corollary 0.2 as well as the following more general version:

COROLLARY 2.2. — *Let X be a connected noncompact Kähler manifold of dimension n on which there exists a C^∞ plurisubharmonic exhaustion function φ which is real analytic on the complement $X \setminus K$ of some compact subset K of X, let q be a positive integer, and let S be the real analytic set of points $p \in X \setminus K$ at which the Levi form $\mathcal{L}(\varphi)_p$ has rank at most $n - q$. Assume that*

$$\dim_{\mathbb{R}} S \leqslant 2q + 1$$

and that there is a nonempty open subset V of X such that each point of V lies in some irreducible compact analytic subset of X of dimension at least $q - 1$. Then X is holomorphically convex with Remmert reduction of dimension $n - q + 1$.

Similarly, one gets the following slightly more general version of Corollary 0.8 (the proof is identical to that of Corollary 0.8 appearing in the introduction):

COROLLARY 2.3. — *Let X be a connected noncompact Kähler manifold of dimension n. Assume that*

(i) *X admits a plurisubharmonic exhaustion function φ which is real analytic on the complement $X \setminus K$ of some compact subset K of X;*

(ii) *for every compact subset C of X, there is a holomorphic automorphism γ of X such that $\gamma(C) \cap C = \varnothing$;*

(iii) *X does not have exactly two ends; and*

(iv) *there is a nonempty open subset V of X such that each point of V lies in some irreducible compact analytic subset of dimension at least $n-2$ in X.*

Then X is holomorphically convex.

3. Generalizations for q-plurisubharmonic functions.

Let (X, g) be a Hermitian manifold of dimension n and let q be a positive integer. A function $\varphi : X \to \mathbb{R}$ of class C^2 on X is said to be *q-plurisubharmonic* (*strictly q-plurisubharmonic*) if, for each point $p \in X$, the trace of the restriction of the Levi form $\mathcal{L}(\varphi)$ to any complex vector subspace of $T^{1,0}_p X$ of dimension q is nonnegative (respectively, positive).

Remarks. — 1) For a C^2 function φ on X, the following are equivalent:

(i) The function φ is q-plurisubharmonic (strictly q-plurisubharmonic).

(ii) For each point $p \in X$ and every choice of orthonormal vectors $e_1, \dots, e_q$ in $T^{1,0}_p X$, we have

$$\sum_{i=1}^{q} \mathcal{L}(\varphi)(e_i, e_i) \geqslant 0 \quad (> 0).$$

(iii) The eigenvalues $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ for $\mathcal{L}(\varphi)$ at each point satisfy

$$\lambda_1 + \cdots + \lambda_q \geqslant 0 \quad (> 0).$$

2) The *Laplace operator* on X is the elliptic operator given by the trace of the Levi form:

$$\Delta_g = \sum g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$$

where $(g^{i\bar{j}}) = \overline{(g_{i\bar{j}})^{-1}}$. This operator is equal to the usual Laplace operator if g is Kähler. A C^2 function φ is called *subharmonic* (*strictly subharmonic*) if $\Delta_g \varphi \geqslant 0$ (respectively, $\Delta_g \varphi > 0$). Clearly, φ is q-plurisubharmonic (strictly q-plurisubharmonic) if and only if the restriction of φ to any q-dimensional germ of a complex submanifold of X is subharmonic (respectively, strictly subharmonic) with respect to the restriction of g.

3) If φ is a q-plurisubharmonic (strictly q-plurisubharmonic) function, then φ is also $(q+1)$-plurisubharmonic (strictly $(q+1)$-plurisubharmonic).

4) A C^2 function φ is plurisubharmonic (strictly plurisubharmonic) if and only if φ is 1-plurisubharmonic (strictly 1-plurisubharmonic).

5) If φ is strictly q-plurisubharmonic (and of class C^2), then φ is strongly q-convex.

6) The sum of two q-plurisubharmonic functions is q-plurisubharmonic and the sum of a q-plurisubharmonic function and a strictly q-plurisubharmonic function is strictly q-plurisubharmonic. This is one of the main advantages of working with q-plurisubharmonic functions in place of strongly q-convex functions.

7) If $\varphi : X \to \mathbb{R}$ and $\chi : \mathbb{R} \to \mathbb{R}$ are C^2 functions, then

$$\mathcal{L}(\chi(\varphi))(v,v) = \chi'(\varphi)\mathcal{L}(\varphi)(v,v) + \chi''(\varphi)|\partial\varphi(v)|^2 \quad \forall v \in T^{1,0}X.$$

Hence if φ is q-plurisubharmonic (strictly q-plurisubharmonic) and $\chi', \chi'' \geqslant 0$ (respectively, $\chi' > 0, \chi'' \geqslant 0$), then $\chi(\varphi)$ is q-plurisubharmonic (respectively, strictly q-plurisubharmonic).

8) *Maximum principle.* The restriction of a strongly q-convex function to a complex analytic subset Y of pure dimension q has no local maximum points because, at each point $p \in Y$, there is an $(n-q+1)$-dimensional germ of a complex submanifold Z of X for which the restriction to Z is strictly plurisubharmonic. Since $\dim_p(Y \cap Z) > 0$, the restriction to $Y \cap Z$ cannot have a local maximum at p. Similarly, if φ is a C^2 q-plurisubharmonic function and the restriction of φ to some connected complex analytic set Y of pure dimension q attains its maximum value at some point, then $\varphi|_Y$ is constant. A proof for g Kähler (and φ continuous) appears, for example, in [NR2, Proposition 1.5], and the proof also works when g is not Kähler, provided φ is of class C^2.

9) *Continuous q-plurisubharmonic functions on a Kähler manifold.* By the work of Wu [Wu], if g is Kähler, then one may also consider *continuous* q-plurisubharmonic functions and still retain most of the above properties. A real-valued continuous function φ on a Kähler manifold (X, g) is called *strictly q-plurisubharmonic* if φ is an element of the class $\Psi(q)$ defined by Wu [Wu]. We will call φ *q-plurisubharmonic* if the function $\varphi + \psi$ is

strictly q-plurisubharmonic for every function $\psi \in \Psi(q)$. According to [Wu, Proposition 1], the smooth elements of $\Psi(q)$ form a dense subset in the sense that, if $\varphi \in \Psi(q)$ and δ is a positive continuous function, then there exists a C^∞ element $\psi \in \Psi(q)$ such that $|\varphi - \psi| < \delta$ on X. In particular, it follows that the restriction of a continuous q-plurisubharmonic (strictly q-plurisubharmonic) function to a q-dimensional germ of a complex submanifold is subharmonic (respectively, strictly subharmonic). If φ and φ' are two continuous q-plurisubharmonic functions on X, then $\varphi + \varphi'$, $\max(\varphi, \varphi')$, and the composition $\chi(\varphi)$ of any nondecreasing convex function χ with φ, are all q-plurisubharmonic. Finally, the weak maximum principle (strong maximum principle) holds for the restriction of a continuous q-plurisubharmonic (respectively, strictly q-plurisubharmonic) function to a complex analytic subset of pure dimension q (see, for example, [NR2, Proposition 1.5]).

10) The notion of q-plurisubharmonicity studied by Hunt and Murray [HM] and the notion studied by Stehlé [Ste] are different from the notion considered in this paper.

After some modifications (which will be described in this section), the arguments of Section 2 give the following version of Theorem 0.1 and Theorem 2.1:

THEOREM 3.1. — *Let X be a connected noncompact complex manifold of dimension n, let g be a C^∞ Hermitian metric on X, let φ be a C^∞ exhaustion function on X which is real analytic on the complement $X \setminus K$ of some compact subset K of X, and let q be a positive integer. Assume that φ is q-plurisubharmonic with respect to g and that there is a real analytic subset S of $X \setminus K$ such that φ is strictly q-plurisubharmonic with respect to g on $X \setminus (K \cup S)$ and such that*

$$\dim_{\mathbb{R}} S \leqslant 2q + 1.$$

Then, for almost every sufficiently large (regular) value a of φ, the sublevel

$$\Omega \equiv \{\, x \in X \mid \varphi(x) < a \,\}$$

admits a C^∞ q-plurisubharmonic (with respect to g) exhaustion function ψ which is strictly q-plurisubharmonic on the complement of some compact subset of Ω.

Remarks.— 1) If φ is plurisubharmonic, then the set of points in $X \setminus K$ at which φ is *not* strictly q-plurisubharmonic is precisely the real analytic set of points at which φ is *not* strongly q-convex. Thus Theorem 3.1 is a direct generalization of Theorem 0.1 and Theorem 2.1.

2) Every complex manifold X admits a real analytic Hermitian metric g. For example, one may take g to be the Hermitian metric corresponding to the real analytic Riemannian metric obtained from a real analytic embedding into $\mathbb{R}^N$ (for some N). Furthermore, if g is a real analytic Hermitian metric and φ is a real analytic function which is q-plurisubharmonic with respect to g on X, then the set S of points at which φ is not strictly q-plurisubharmonic is a real analytic subset of X. For if $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ are the eigenvalues for $\mathcal{L}(\varphi)_p$ at a point $p \in X$ (with respect to g), then $p \in S$ if and only if

$$\Pi_{1 \leqslant i_1 < \cdots < i_q \leqslant n}(\lambda_{i_1} + \cdots + \lambda_{i_q}) = 0.$$

As a symmetric polynomial in $\lambda_1, \ldots, \lambda_n$, the above expression determines a polynomial in the coefficients of the characteristic polynomial of $\mathcal{L}(\varphi)$ (with respect local real analytic orthonormal frames for $T^{1,0}X$). So, in fact, S is the zero set of a (globally defined) real analytic function on X.

3) In the theorem, one need only assume that S contains the set S_0 of points $p \in X \setminus K$ at which $d\varphi \neq 0$ and the trace of the restriction of $\mathcal{L}(\varphi)$ to some q-dimensional subspace of $\ker(\partial\varphi)_p$ (in $T_p^{1,0}X$) is 0 (if g is real analytic, then S_0 is itself a real analytic subset outside the zero set of $d\varphi$). To see this, we let b, X', K', and φ' be as in the first remark following the statement of Theorem 2.1 and observe that

$$S' \equiv \{p \in X' \setminus K' \mid \varphi' \text{ is } \textit{not} \text{ strictly } q\text{-plurisubharmonic at } p\} = S_0 \cap (X' \setminus K').$$

For if $p \in S'$ and V is a subspace of dimension q in $T_p^{1,0}X = T_p^{1,0}X'$, then

$$\operatorname{tr} \mathcal{L}(\varphi')|_V \geqslant e^{2\varphi(p)}(e^b - e^{\varphi(p)})^{-2} \operatorname{tr} \left|\partial\varphi|_V\right|^2 \geqslant 0.$$

Thus, if the left-hand side is zero, then $V \subset \ker(\partial\varphi)_p$ and hence $p \in S_0$. So we may replace X by X', K by K', and φ by φ'.

4) If g is real analytic and $q = n-1$, then we need only assume that each connected component of $X \setminus K$ contains a point p such that

$$(d\varphi)_p \neq 0 \qquad \text{and} \qquad \operatorname{tr}\left(\mathcal{L}(\varphi)|_{\ker(\partial\varphi)_p}\right) > 0$$

(see the second remark following Theorem 2.1). For, replacing φ by e^{φ}, we get $\dim_{\mathbb{R}} S \leqslant 2n-1 = 2q+1$, where S is the set of points in $X \setminus K$ at which φ is not strictly $(n-1)$-plurisubharmonic with respect to g.

The proof of Theorem 3.1, which is similar to that of Theorem 2.1, will only be sketched. The first observation is that the proof of Lemma 1.3 also gives the following version:

LEMMA 3.2. — *Let (X,g) be a Hermitian manifold, let K be a closed subset of X, let N be a closed subset of $X\setminus K$, and let α be a C^∞ real-valued function on X which is strictly q-plurisubharmonic on a neighborhood U of K in X. Suppose there exists a C^∞ weak stratification $\{N_j\}_{j=1}^{\infty}$ of N in $X \setminus K$ such that, for each $j = 1,2,3,\ldots$ and each point $p \in N_j$,*

$$\dim_{\mathbb{C}} T_p^{1,0} N_j < q \qquad (\text{i.e.} \dim_{\mathbb{R}} \big[T_p N_j \cap J(T_p N_j)\big] < 2q).$$

Then, for every positive continuous function δ on X, there exists a C^∞ function β on X such that β is strictly q-plurisubharmonic on a neighborhood of the closed set $K \cup N$ in X, $\beta = \alpha$ on $K \cup N_1$, and $\alpha \leqslant \beta \leqslant \alpha + \delta$ on X.

Sketch of the proof of Theorem 3.1. — By the maximum principle, φ is constant on any connected compact complex analytic subset of pure dimension q in X. Thus we may apply the argument in the proof of Theorem 2.1 (with Lemma 3.2 in place of Lemma 1.3) to get a C^∞ function β on X which is strictly q-plurisubharmonic on a neighborhood of $S \cap M$, where $M = \varphi^{-1}(a)$ for a suitable regular value a of φ. If ϵ is a sufficiently small positive constant, then the function

$$\psi_1 \equiv -\log(a-\varphi) + \varphi + \epsilon\beta$$

will be a C^∞ exhaustion function on the sublevel $\Omega = \{\, x \in X \mid \varphi(x) < a \,\}$, and ψ_1 will be strictly q-plurisubharmonic near $\partial\Omega = M$. Finally, by taking a sufficiently large positive constant r and a C^∞ function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi', \chi'' \geqslant 0$, $\chi(t) = 0$ if $t < r$, $\chi'(t) > 0$ if $t > r$, and $\chi(t) \to \infty$ as $t \to \infty$, we get the required q-plurisubharmonic function by setting $\psi = \chi(\psi_1)$. $\square$

Similarly, Theorem 3.1 and Theorem 0.7 together immediately give the following version of Corollary 0.2 and Corollary 2.2:

COROLLARY 3.3. — *Let X be a connected noncompact complex manifold of dimension n and let q be a positive integer. Assume that X admits*

(i) *a C^∞ Hermitian metric g, a C^∞ q-plurisubharmonic (with respect to g) exhaustion function φ, and a compact subset K such that φ is real analytic on $X \setminus K$ and φ is strictly q-plurisubharmonic on $X \setminus (K \cup S)$ for some real analytic subset S of $X \setminus K$ of dimension at most $2q+1$;*

(ii) *a C^∞ Kähler metric g';*

(iii) *a continuous plurisubharmonic exhaustion function; and*

(iv) *a nonempty open subset V such that each point of V lies in some irreducible compact analytic subset of X of dimension at least $q-1$.*

Then X is holomorphically convex with Remmert reduction of dimension $n-q+1$.

We may also prove the following version of Corollary 0.8 and Corollary 2.3 in which the exhaustion function is only assumed to be $(n-1)$-plurisubharmonic, not necessarily plurisubharmonic (as in the plurisubharmonic case, q-plurisubharmonic versions for $q < n-1$ are not edifying):

COROLLARY 3.4. — *Let X be a connected noncompact complex manifold of dimension n. Assume that*

(i) *X admits a C^∞ Kähler metric g and a C^∞ $(n-1)$-plurisubharmonic (with respect to g) exhaustion function φ, both of which are real analytic on the complement $X \setminus K$ of some compact subset K of X;*

(ii) *there exists a continuous plurisubharmonic exhaustion function on X;*

(iii) *for every compact subset C of X, there is a holomorphic isometry γ of X such that $\gamma(C) \cap C = \varnothing$;*

(iv) *X does not have exactly two ends; and*

(v) *there is a nonempty open subset V of X such that each point of V lies in some irreducible compact analytic subset of dimension at least $n-2$ in X.*

Then X is holomorphically convex.

The proof will require $(n-1)$-plurisubharmonic versions of the results of [NR1]. These versions are contained implicitly in [NR1] and [NR2], but the proofs are sketched here for completeness. The statements below are also slightly more general than will be required.

PROPOSITION 3.5 (cf. [NR1, Theorem 1]). — *Let X be a connected noncompact Kähler manifold.*

(a) *If Ω is a C^∞ relatively compact domain in X which has at least three boundary components and which admits a C^∞ defining function ρ such that $d\rho \neq 0$ near $\partial\Omega$ and*

$$\operatorname{tr}\left(\mathcal{L}(\rho)|_{T^{1,0}(\partial\Omega)}\right) \geqslant 0,$$

then Ω admits a proper holomorphic mapping onto a Riemann surface.

(b) *If X has at least three ends and there exist a C^∞ exhaustion function φ on X and arbitrarily large regular values a of φ such that, for $M = \varphi^{-1}(a)$,*

$$\operatorname{tr}\left(\mathcal{L}(\rho)|_{T^{1,0}M}\right) \geqslant 0,$$

then X admits a proper holomorphic mapping onto a Riemann surface.

Sketch of the proof. — For (a), we fix a boundary component C for Ω and we let u be the harmonic function on Ω with boundary values 1 on C and 0 on $(\partial\Omega) \setminus C$. Then u is C^∞ at $\partial\Omega$ and, by a result of Grauert and Riemenschneider [GR] and of Siu [Si2], u is pluriharmonic (this fact is stated and proved in [NR2, Theorem 1.6] for ρ an $(n-1)$-plurisubharmonic function, but the proof under the weaker hypothesis in (a) is the same). Therefore, the function $\psi = -\log(1-u) - \log u$ is a C^∞ plurisubharmonic exhaustion function on Ω. Applying a theorem of Nakano [Na] (see also Demailly [De1]), one gets a complete Kähler metric on Ω and therefore, by [NR1, Theorem 1], Ω admits a proper holomorphic mapping to a Riemann surface.

For (b), we may apply (a) to get a sequence of C^∞ domains $\{\Omega_\nu\}$ in X such that $X = \cup_\nu \Omega_\nu$ and such that, for each ν, $\Omega_\nu \subset\subset \Omega_{\nu+1}$ and Ω_ν admits a proper holomorphic mapping Φ_ν onto a Riemann surface Y_ν. By Stein factorization, we may assume the fibers of Φ_ν are connected (i.e. Φ_ν is the Remmert reduction mapping). Clearly, the fibers of the restriction of $\Phi_{\nu+1}$ to Ω_ν are precisely the fibers of Φ_ν, so the restriction descends to a biholomorphic mapping of Y_ν onto a region in $Y_{\nu+1}$. Passing to the corresponding limit as $\nu \to \infty$, we get a proper holomorphic mapping of X onto a Riemann surface. □

Proposition 3.6 (cf. [NR1, Theorem 4.6]). — *Let X be a connected noncompact Kähler manifold. Assume that*

(i) *there exist a C^∞ exhaustion function φ and arbitrarily large regular values a of φ such that, for $M = \varphi^{-1}(a)$,*

$$\operatorname{tr}\left(\mathcal{L}(\varphi)|_{T^{1,0}M}\right) = 0;$$

(ii) *for every compact subset C of X, there exists a holomorphic isometry γ of X such that $\gamma(C) \cap C = \varnothing$; and*

(iii) *X has exactly one end.*

Then X admits a proper holomorphic mapping onto a Riemann surface.

Remark. — The condition (ii) is satisfied (for example) by an infinite Galois covering of a compact Kähler manifold. Note that we need holomorphic isometries, not just holomorphic automorphisms, because the mappings must preserve the condition (i) on the trace of the Levi form with respect to the Kähler metric.

Sketch of the proof of Proposition 3.6. — Following the proof of [NR1, Theorem 4.6], we fix a regular value a of φ for which the condition (i) holds and for which $\{ x \in X \mid \varphi(x) < a \}$ has a nonempty connected component Ω_1. Applying (ii) to the compact set $\overline{\Omega}_1$, we get a holomorphic isometry γ of X such that

$$\Omega_2 = \gamma(\Omega_1) \subset\subset X \setminus \overline{\Omega}_1.$$

Since X has only one end, we may choose a regular value b of φ such that φ satisfies the condition (i) on $M = \varphi^{-1}(b)$, some connected component Ω of $\{ x \in X \mid \varphi(x) < b \}$ contains $\overline{\Omega}_1 \cup \overline{\Omega}_2$, and the set $\Omega_3 = \Omega \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ is connected. Applying part (a) of Proposition 3.5 to Ω_3, we get a proper holomorphic mapping Φ of Ω_3 onto a Riemann surface Y. Removing two distinct fibers F_1 and F_2 from Ω_3, we get a connected Kähler manifold $X \setminus (F_1 \cup F_2)$ which satisfies the condition (b) of Proposition 3.5 and hence admits a proper holomorphic mapping Ψ onto a Riemann surface Z. We may assume that the fibers of Φ and Ψ are connected and, therefore, Φ and Ψ determine a proper holomorphic mapping of X onto the quotient Riemann surface

$$Y \cup Z / \Phi(x) \sim \Psi(x) \, \forall \, x \in \Omega_3 \setminus (F_1 \cup F_2). \qquad \square$$

Proof of Corollary 3.4. — If X has at least three ends, then, by [NR1, Theorem 1] (or by part (b) of Proposition 3.5), X admits a proper holomorphic mapping onto a Riemann surface. If X has exactly one end, then we may choose K so that $X \setminus K$ is connected. If $\operatorname{tr}(\mathcal{L}(\varphi)|_{T^{1,0}_p M}) > 0$ for some regular value $a > \max_K \varphi$ and some point $p \in M = \varphi^{-1}(a)$, then Corollary 3.3 gives holomorphic convexity (see the fourth remark following Theorem 3.1). Otherwise, X is holomorphically convex by Proposition 3.6. $\square$

4. The real analytic with corners case.

The main goal of this section is the following version of Theorem 0.1 (and Theorem 2.1 and Theorem 3.1) for which, in particular, Theorem 0.3 and Theorem 0.5 are direct consequences:

THEOREM 4.1.— *Let X be a connected noncompact complex manifold of dimension n, let g be a C^∞ Hermitian metric on X, let φ be a continuous exhaustion function on X, and let q be a positive integer. Suppose that, for some compact subset K of X and for each point $p \in X \setminus K$, there is a connected neighborhood U of p and a finite collection $\mathcal{A}$ of C^∞ q-plurisubharmonic (with respect to g) functions on U such that*

$$\varphi(x) = \max_{\alpha\in\mathcal{A}} \alpha(x) \quad \forall\, x \in U$$

and such that, for each function $\alpha \in \mathcal{A}$, either α is strictly q-plurisubharmonic on U or α is real analytic on U and α is strictly q-plurisubharmonic on the complement $U\setminus S(\alpha)$ of some real analytic subset $S(\alpha)$ of U satisfying

$$\dim_{\mathbb{R}} S(\alpha) \leqslant 2q + 1.$$

Then, for almost every sufficiently large value a of φ, the sublevel

$$\Omega \equiv \{\, x \in X \mid \varphi(x) < a \,\}$$

admits a exhaustion function ψ which, on the complement of some compact subset H of Ω, is locally equal to the maximum of a finite collection of C^∞ strictly q-plurisubharmonic functions. Moreover, if g is Kähler or $q = 1$ or φ is of class C^∞, then there exists a C^∞ q-plurisubharmonic exhaustion function ψ' on Ω which is strictly q-plurisubharmonic on $\Omega \setminus H$.

Remarks.— 1) In fact, we will construct the function so that $\psi = -\log(a-\varphi) + \varphi|_\Omega + \eta|_\Omega$, where η is a C^∞ function on X and, near $\partial\Omega$ in X, φ is locally equal to the maximum of a finite collection of $C^\infty q$-plurisubharmonic functions α such that $\alpha + \eta$ is strictly q-plurisubharmonic. In particular, if φ is of class C^∞, then, by standard arguments, φ is everywhere q-plurisubharmonic and $\varphi + \eta$ is strictly q-plurisubharmonic near $\partial\Omega$, so we may take $\psi' = \psi$.

2) If $q = 1$, then q-plurisubharmonicity (for any Hermitian metric) is equivalent to plurisubharmonicity. Hence the constructed continuous function ψ is a continuous plurisubharmonic exhaustion function on Ω

which is strictly plurisubharmonic on $\Omega \setminus H$. Thus X is holomorphically convex with Remmert reduction of dimension n and Theorem 0.3 follows. In particular, Ω is strongly pseudoconvex in this case and hence the existence of a C^∞ plurisubharmonic exhaustion function ψ' on Ω which is strictly plurisubharmonic on the complement of some compact subset follows.

3) If g is Kähler, then $\varphi|_{X\setminus K}$ is a continuous q-plurisubharmonic function and $\psi|_{\Omega\setminus H}$ is a continuous strictly q-plurisubharmonic function. Therefore, by applying Wu's approximation theorem [Wu, Proposition 1] (and replacing H by a slightly larger compact set), one gets, as claimed, a C^∞ q-plurisubharmonic exhaustion function ψ' on Ω which is strictly q-plurisubharmonic on $\Omega \setminus H$. In particular, Theorem 0.5 follows. Moreover, as will be clear from the proof, for g Kähler we need only assume (in Theorem 4.1) that, for each $\alpha \in \mathcal{A}$, either α is a *continuous* strictly q-plurisubharmonic function on U or α is real analytic (and q-plurisubharmonic) with $\dim_{\mathbb{R}} S(\alpha) \leqslant 2q+1$.

4) As in the smooth case, one need only assume that, for each $\alpha \in \mathcal{A}$, either α is strictly q-plurisubharmonic or α is real analytic and some real analytic set $S(\alpha)$ of dimension $\leqslant 2q+1$ contains the set $S_0(\alpha)$ of points $p \in U$ at which $d\alpha \neq 0$ and the trace of the restriction of $\mathcal{L}(\alpha)$ to some q-dimensional subspace of $\ker(\partial\alpha)_p$ (in $T^{1,0}_p X$) is 0 (if g is real analytic, then $S_0(\alpha)$ is itself a real analytic subset outside the zero set of $d\alpha$).

5) Also as in the smooth case, if g is real analytic and $q = n-1$, then one need only assume that, for each U and α, either α is strictly q-plurisubharmonic or α is real analytic and there is a point $p \in U$ such that

$$(d\alpha)_p \neq 0 \qquad \text{and} \qquad \operatorname{tr}\left(\mathcal{L}(\alpha)|_{\ker(\partial\alpha)_p}\right) > 0.$$

Theorem 4.1 and Theorem 0.7 together give the following version of Corollary 0.2 which is a direct generalization of Corollary 0.6:

COROLLARY 4.2. — *Let X be a connected noncompact complex manifold of dimension n and let q be a positive integer. Assume that X admits*

(i) *a C^∞ Kähler metric g, a continuous exhaustion function φ, and a compact subset K with the properties described in Theorem 4.1;*

(ii) *a continuous plurisubharmonic exhaustion function; and*

(iii) *a nonempty open subset V such that each point of V lies in some irreducible compact analytic subset of X of dimension at least $q-1$.*

Then X is holomorphically convex with Remmert reduction of dimension $n - q + 1$.

As in the proof of Theorem 2.1, for the proof of Theorem 4.1 one considers the projection mapping from the graph over a Barlet space in order to choose a sublevel Ω whose boundary $\partial\Omega$ does not contain any q-dimensional compact analytic sets. However, in the situation of Theorem 4.1, one cannot rule out the existence of *germs* of q-dimensional complex analytic sets contained in $\partial\Omega$. Thus one must construct a strictly q-plurisubharmonic function on a neighborhood of the union of such germs. By the work of Demailly [De2], Greene and Wu [GW], and Ohsawa [O], a strictly q-plurisubharmonic function exists on some neighborhood of any analytic subset with no compact irreducible components of dimension at least q (see [NR2, Theorem 1.2]). We will adapt their arguments to the situation of Theorem 4.1. The main idea in the construction of such functions is to produce a q-plurisubharmonic function on the analytic subset and to then extend this function to a neighborhood. For the extension, one applies the following lemma:

LEMMA 4.3 (Richberg [Ri]). — *If φ is a C^∞ strictly q-plurisubharmonic function on a complex submanifold Y of a Hermitian manifold X, then there exists a C^∞ strictly q-plurisubharmonic function ψ on a neighborhood of Y in X such that $\psi|_Y = \varphi$.*

Remark. — This is essentially a special case of a theorem of Richberg [Ri] (although Richberg only states it for $q = 1$). A proof appears in [NR2, Proposition 1.3] (for the Kähler case, but the proof works in the Hermitian case as well).

Lemma 4.3 will be applied to submanifolds Y of dimension q; i.e. to functions φ which are strictly subharmonic. The existence of strictly subharmonic exhaustion functions is due to Greene and Wu [GW]. Demailly [De2] produced a relatively simple construction on Hermitian manifolds. The following is the main step:

LEMMA 4.4 (Demailly [De2, Lemma 7]). — *Let V and W be open subsets of a Hermitian manifold X such that, for each connected component V_0 of V, there is a connected component W_0 of W such that*

$$W_0 \cap V_0 \neq \varnothing \qquad \text{and} \qquad W_0 \not\subset \overline{V}.$$

Then there exists a C^∞ real-valued function u on X such that $u \geqslant 0$ on X, $u \equiv 0$ on $X \setminus (V \cup W)$, and u is positive and strictly subharmonic on V.

Remark. — In the conclusion of [De2, Lemma 7], u is only taken to satisfy $\operatorname{supp} u \subset \overline{V} \cup \overline{W}$ in place of the property $u \equiv 0$ on $X \setminus (V \cup W)$. However, Demailly's construction actually gives a function satisfying the stronger condition.

Applying Demailly's arguments, we get the following:

LEMMA 4.5. — *Let X be a Hermitian manifold, let B and C be compact subsets with $B \subset C$, and let q be a positive integer. If there exists a C^∞ strictly q-plurisubharmonic function on a neighborhood of B in X and there exists a (properly embedded) complex submanifold N of dimension q in an open subset Q of $X \setminus B$ such that $C \setminus B \subset N$ and such that no connected component of N is contained in C, then there exists a C^∞ strictly q-plurisubharmonic function on a neighborhood of C in X.*

Proof. — By hypothesis, there is a C^∞ function β on X which is strictly q-plurisubharmonic on a neighborhood U of B in X. It suffices to show that, for each point $p \in C \setminus B$, there exists, on a neighborhood of C in X, a C^∞ q-plurisubharmonic function γ which is strictly q-plurisubharmonic on some neighborhood S of p in X. For we may then cover the compact set $C \setminus U$ by finitely many such open sets S and form the corresponding q-plurisubharmonic functions $\gamma_1, \dots, \gamma_m$ on a neighborhood of C in X. For a sufficiently large positive constant a, the C^∞ function

$$\beta + a(\gamma_1 + \cdots + \gamma_m)$$

will then be strictly q-plurisubharmonic on a neighborhood of C in X.

For the proof of the claim, suppose $p \in C \setminus B$. Then there exist connected open subsets V_0 and W_0 of the complex manifold N such that $p \in V_0 \subset\subset N$, $V_0 \cap W_0 \neq \varnothing$, $W_0 \not\subset \overline{V_0}$, and $W_0 \subset\subset N \setminus C$. By Lemma 4.4 (Demailly), there exists a C^∞ real-valued nonnegative function u on N such that $u \equiv 0$ on $N \setminus (V_0 \cup W_0)$ and u is positive and strictly subharmonic on V_0. Applying Lemma 4.3 (Richberg) to the restriction $u|_{V_0}$, one gets an open set V_1 in X and a C^∞ strictly q-plurisubharmonic function v on V_1 such that $V_1 \cap N = V_0$, $V_1 \subset\subset Q \subset X \setminus B$, and $v|_{V_0} = u|_{V_0}$. Fix ϵ with $0 < \epsilon < 2\epsilon < u(p) = v(p)$. Since u vanishes on $N \setminus (V_0 \cup W_0)$ and since $W_0 \subset\subset N \setminus C$, the compact subset $\{ x \in N \mid u(x) \geqslant \epsilon \}$ of $V_0 \cup W_0$ and the compact set $C \setminus V_0$ are disjoint. Hence we may choose the neighborhood V_1 of V_0 so thin that, for some neighborhood V_2 of $C \setminus V_0 = C \setminus V_1$ in X, we have

$$v < \epsilon \quad \text{on} \quad V_1 \cap V_2.$$

Now fix a C^∞ function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi'(t) \geqslant 0$ and $\chi''(t) \geqslant 0$ for all t, $\chi(t) = 0$ if $t \leqslant \epsilon$, and $\chi'(t) > 0$ if $t \geqslant 2\epsilon$. Then we get a well-defined C^∞ q-plurisubharmonic function γ on the neighborhood

$$V = V_1 \cup V_2$$

of C in X by setting

$$\gamma(x) = \begin{cases} \chi(v(x)) & \text{if } x \in V_1 \\ 0 & \text{if } x \in V_2 \end{cases}$$

and γ is strictly q-plurisubharmonic on a neighborhood of p in X. □

Proof of Theorem 4.1. — The proof will consist of four steps.

Step 1. Choice of a suitable sublevel of φ as in the proof of Theorem 2.1.

We will not consider the global set of points at which there exists a q-dimensional germ of a complex analytic subset of X contained in a fiber M of the function φ (as in the proof of Theorem 2.1). It will be easier to work with the local analytic sets (given by Lemma 1.4) in the levels of each of the functions α described in the statement of the theorem (although one can show that the global set is fairly nice).

By replacing K by $\{ x \in X \mid \varphi(x) \leqslant c \}$ and φ by $\max(\varphi - c, 0)$ for some constant $c > \max_K \varphi$, we may assume that $\varphi \geqslant 0$, $K = \{ x \in X \mid \varphi(x) = 0 \} \neq \varnothing$, and φ is everywhere in X (not just in $X \setminus K$) locally equal to the maximum of a finite collection of C^∞ q-plurisubharmonic functions. By hypothesis, we may fix a locally finite (in $X \setminus K$) covering $\{U_i\}_{i=1}^\infty$ of $X \setminus K$ by relatively compact connected open subsets of $X \setminus K$ such that, for each $i = 1, 2, 3, \ldots$,

$$\varphi|_{U_i} = \max_{\alpha \in \mathcal{A}_i} \alpha,$$

where $\mathcal{A}_i$ is a finite collection of C^∞ q-plurisubharmonic functions on U_i and, for each function $\alpha \in \mathcal{A}_i$, either α is strictly q-plurisubharmonic (in which case we set $S(\alpha) = \varnothing$) or α is real analytic on U_i and α is strictly q-plurisubharmonic on the complement $U_i \setminus S(\alpha)$ of some real analytic subset $S(\alpha)$ of dimension at most $2q + 1$ in U_i.

Every compact complex analytic subset Y of pure dimension q in X is contained in $K \cup S$, where

$$S \equiv \bigcup_{i=1}^\infty \bigcup_{\alpha \in \mathcal{A}_i} S(\alpha) \subset X \setminus K.$$

In fact, for each i, we have

$$Y \cap U_i \subset \bigcup_{\alpha \in \mathcal{A}_i} S(\alpha).$$

For the restriction of φ to any local complex analytic set of pure dimension q satisfies the maximum principle. Hence $\varphi|_Y$ is locally constant on Y. If $p \in Y \backslash K$, then $p \in U_i$ for some i and, choosing $\alpha \in \mathcal{A}_i$ with $\alpha(p) = \varphi(p)$, we get $\varphi \equiv \alpha \equiv \alpha(p)$ on the connected component of $Y \cap U_i$ containing p (since, on this connected component, $\alpha \leqslant \varphi \equiv \varphi(p) = \alpha(p)$ and $\alpha|_{(Y \cap U_i)}$ satisfies the maximum principle). Since Y is of pure dimension q, this component must be contained in $S(\alpha)$ and the claim follows.

Proceeding now as in the proof of Theorem 2.1, we let

$$\begin{array}{ccc} G & \xrightarrow{\Psi} & X \\ {\scriptstyle\Phi}\big\downarrow & & \\ \mathcal{C}_q(X) & & \end{array}$$

be the graph and holomorphic projections over the Barlet space of compact analytic q-cycles in X. The image $\Psi(G)$ of G under Ψ is equal to a countable union of local complex analytic sets in X which are contained in $K \cup S$. Since these complex analytic sets are of even real dimension and since S is equal to a countable (locally finite in $X \setminus K$) union of local real analytic sets of dimension at most $2q+1$ in X, $\Psi(G) \backslash K$ must be a set of $(2q+1)$-dimensional Hausdorff measure 0. Therefore, recalling that $K = \varphi^{-1}(0) \neq \varnothing$, we see that, for almost every positive real number a and for every function α in the collection

$$\mathcal{A} = \bigcup_i \mathcal{A}_i,$$

a is a regular value of α and the set

$$\alpha^{-1}(a) \cap \Psi(G)$$

is of $2q$-dimensional Hausdorff measure 0. It follows that, for every such value a, the (nonempty, compact) fiber $M = \varphi^{-1}(a)$ does not contain any complex analytic subsets of dimension q in X. For any such analytic subset would contain a compact irreducible analytic set Y of dimension q. In particular, Y would be a set of positive $2q$-dimensional Hausdorff measure contained in $M \cap \Psi(G)$. But this is impossible because

$$M \cap \Psi(G) \subset \bigcup_{\alpha \in \mathcal{A}} \alpha^{-1}(a) \cap \Psi(G)$$

and the right-hand side is a set of $2q$-dimensional Hausdorff measure 0.

Similarly, for almost every $a > 0$, for every $i = 1, 2, 3, \ldots$, and for every $\alpha \in \mathcal{A}_i$, a is a regular value of α and, if $S(\alpha) \neq \varnothing$, then $S(\alpha) \cap \alpha^{-1}(a)$ is a (properly embedded) real analytic subset of dimension at most $2q$ in the $(2n-1)$-dimensional real analytic submanifold $\alpha^{-1}(a)$ in U_i.

Fixing a (generic) positive real number a with all of the above properties, we will construct the desired function ψ on the sublevel

$$\Omega \equiv \{ x \in X \mid \varphi(x) < a \}.$$

Step 2. Construction of a strictly q-plurisubharmonic function on a neighborhood of the q-dimensional germs of complex analytic sets contained in the boundary.

For every i, each $\alpha \in \mathcal{A}_i$ is strictly q-plurisubharmonic on $U_i \setminus S(\alpha)$ and hence the set $Y(\alpha)$ of points in $M(\alpha) = \alpha^{-1}(a)$ at which there exists a q-dimensional germ of a complex analytic subset of X contained in $M(\alpha)$ must be contained in the real analytic set $S(\alpha) \cap M(\alpha)$. Since $\dim_{\mathbb{R}}(S(\alpha) \cap M(\alpha)) \leqslant 2q$, Lemma 1.4 (Diederich and Fornaess) implies that $Y(\alpha)$ is a (properly embedded) complex analytic subset of pure dimension q in U_i.

By passing to a refinement of the covering $\{U_i\}$ of $X \setminus K$ and reordering, we may assume that, for some positive integer m, we have

$$\partial\Omega \subset M \equiv \varphi^{-1}(a) \subset U_1 \cup \cdots \cup U_m \subset\subset X \setminus K$$

and

$$U_i \subset\subset X \setminus (M \cup K) \text{ for } i = m+1, m+2, m+3, \ldots.$$

(Of course, if we choose a outside the countable set of local minimum values for φ, then we will have $\partial\Omega = M$, but this is not necessary.) Fix a covering $\{C_i\}_{i=1}^m$ of M by compact subsets and fix finite collections of open subsets $\{V_i\}_{i=1}^m$ and $\{W_{i,j}\}_{i,j=1}^m$ of X such that, for all $i, j = 1, \ldots, m$,

$$C_i \subset V_i \cap M, \quad V_i \subset\subset U_i, \quad \overline{V_i} \cap \overline{V_j} \subset W_{i,j} \subset\subset U_i \cap U_j,$$

and $W_{i,j}$ is smooth with real analytic boundary (for example, one may take $W_{i,j}$ to be a sublevel for a large regular value of a real analytic exhaustion function on $U_i \cap U_j$). Setting

$$C \equiv \bigcup_{i=1}^{m} \bigcup_{\alpha \in \mathcal{A}_i} Y(\alpha) \cap C_i,$$

$$Z \equiv \bigcup_{i,j=1}^{m} \bigcup_{\alpha \in \mathcal{A}_i, \beta \in \mathcal{A}_j} \big[\big(Y(\alpha) \cup Y(\beta)\big) \cap U_i \cap U_j\big]_{\mathrm{sing}} \cap \overline{W_{i,j}}, \text{ and}$$

$$B \equiv Z \cap C,$$

we will apply Lemma 4.5 to the compact sets B and C. Note that, in the above definitions, $\big[\big(Y(\alpha) \cup Y(\beta)\big) \cap U_i \cap U_j\big]_{\mathrm{sing}}$ is the singular set of the (properly embedded) complex analytic subset $\big(Y(\alpha) \cup Y(\beta)\big) \cap U_i \cap U_j$ of pure dimension q in $U_i \cap U_j$ and that this set contains the complex analytic subset $\big(Y(\alpha)_{\mathrm{sing}} \cup Y(\beta)_{\mathrm{sing}}\big) \cap U_i \cap U_j$.

We first produce a C^∞ strictly q-plurisubharmonic function on a neighborhood of B in X. Forming a finite real analytic weak stratification of the compact real analytic subset

$$Z_0 \equiv \bigcup_{i,j=1}^{m} \bigcup_{\alpha \in \mathcal{A}_i, \beta \in \mathcal{A}_j} \big[\big(Y(\alpha) \cup Y(\beta)\big) \cap U_i \cap U_j\big]_{\mathrm{sing}} \cap \partial W_{i,j}$$

(of real dimension at most $2q-2$) in X and following with a finite complex analytic weak stratification of the complex analytic subset $Z \setminus Z_0$ (of real dimension at most $2q-2$) in $X \setminus Z_0$, we get a real analytic weak stratification of the compact subset Z of X of real dimension at most $2q-2$. Therefore, by Lemma 3.2, there exists a C^∞ strictly q-plurisubharmonic function on a neighborhood of Z in X; and hence on a neighborhood of B since $B \subset Z$.

To get the q-dimensional complex submanifold N of an open subset Q of $X \setminus B$ required in Lemma 4.5, we first set

$$Y \equiv \bigcup_{i=1}^{m} \bigcup_{\alpha \in \mathcal{A}_i} Y(\alpha) \cap V_i.$$

Clearly, given a point $p \in C \setminus B = C \setminus Z$, we have $p \in Y(\alpha) \cap V_i$ for some $i \in \{1, \dots, m\}$ and some $\alpha \in \mathcal{A}_i$. If $j \in \{1, \dots, m\}$ and $\beta \in \mathcal{A}_j$, then

$$\begin{aligned} p \notin Z \supset & \big[\big(Y(\alpha) \cup Y(\beta)\big) \cap U_i \cap U_j\big]_{\mathrm{sing}} \cap \overline{W_{i,j}} \\ & \supset \big[\big(Y(\alpha) \cup Y(\beta)\big) \cap U_i \cap U_j\big]_{\mathrm{sing}} \cap \overline{V_i} \cap \overline{V_j}. \end{aligned}$$

Hence, if $p \in Y(\beta) \cap \overline{V_j}$, then we may choose a relatively compact neighborhood $Q(p, j, \beta)$ of p in $V_i \cap U_j \setminus Z$ such that

$$Y(\alpha) \cap Q(p, j, \beta) = Y(\beta) \cap Q(p, j, \beta)$$

and such that this set is a (properly embedded) complex submanifold of dimension q in $Q(p, j, \beta)$. Otherwise, we may choose a relatively compact neighborhood $Q(p, j, \beta)$ of p in the open set $V_i \setminus [Z \cup (Y(\beta) \cap \overline{V_j})]$. Therefore, if Q is the neighborhood of $C \setminus B$ in $X \setminus Z$ ($\subset X \setminus B$) given by

$$Q = \bigcup_{p \in C \setminus B} \bigcap_{j=1}^{m} \bigcap_{\beta \in \mathcal{A}_j} Q(p, j, \beta),$$

then the set

$$N \equiv Y \cap Q$$

is a q-dimensional complex submanifold of Q which contains $C \setminus B$.

Furthermore, each connected component N_0 of N meets $X \setminus C$. For if N_0 were contained in C, then N_0 would be closed relative to $X \setminus Z$ and hence N_0 would be a complex submanifold of dimension q in $X \setminus Z$. On the other hand, the compact set Z is contained in a finite union of local complex analytic sets of dimension strictly less than q. Therefore, by the Remmert-Stein-Thullen theorem, the closure $\overline{N_0}$ of N_0 in X would be a compact analytic subset of dimension q. Since we would also have $\overline{N_0} \subset M$, this would contradict the choice of $M = \varphi^{-1}(a)$. Thus $N_0 \not\subset C$.

Therefore, by Lemma 4.5, for a sufficiently small relatively compact neighborhood R of C in X, there exists a C^∞ strictly q-plurisubharmonic function on a neighborhood of the closure $\overline{R}$ in X.

Step 3. Construction (as in the proof of Theorem 2.1) of a strictly q-plurisubharmonic function on a neighborhood of the entire set of boundary points near which φ fails to be the maximum of a collection of strictly q-plurisubharmonic functions.

For each $i = 1, \dots, m$, we have

$$C_i \setminus R \subset C_i \setminus C \subset V_i \setminus \bigcup_{\alpha \in \mathcal{A}_i} Y(\alpha).$$

Hence we may choose an open set D_i with smooth real analytic boundary in X such that

$$C_i \setminus R \subset D_i \subset\subset V_i \setminus \bigcup_{\alpha \in \mathcal{A}_i} Y(\alpha).$$

In particular,

$$M \setminus R = \bigcup_i C_i \setminus R \subset \bigcup_i D_i.$$

We may form a finite real analytic weak stratification of dimension $\leqslant 2q$ for the compact set

$$E \equiv \bigcup_{i=1}^{m} \bigcup_{\alpha \in \mathcal{A}_i} S(\alpha) \cap M(\alpha) \cap \overline{D_i}$$

as follows. We first form a finite real analytic weak stratification of the compact real analytic subset

$$F_0 \equiv \bigcup_{i=1}^{m} \bigcup_{\alpha \in \mathcal{A}_i} S(\alpha) \cap M(\alpha) \cap \partial D_i.$$

We will proceed inductively to form a finite real analytic weak stratification of each of the compact sets

$$F_l \equiv F_0 \cup \bigcup_{i=1}^{l} \bigcup_{\alpha \in \mathcal{A}_i} S(\alpha) \cap M(\alpha) \cap \overline{D_i} = F_{l-1} \cup \bigcup_{\alpha \in \mathcal{A}_l} S(\alpha) \cap M(\alpha) \cap \overline{D_l},$$

for $l = 1, \ldots, m$. Given such a weak stratification for F_{l-1}, we may choose a (locally finite) real analytic weak stratification of the real analytic subset

$$\bigcup_{\alpha \in \mathcal{A}_l} S(\alpha) \cap M(\alpha)$$

of U_l. Since only finitely many strata will meet the compact subset $\overline{D_l}$ of U_l and since

$$S(\alpha) \cap M(\alpha) \cap \partial D_l \subset F_0 \subset F_{l-1}$$

for every $\alpha \in \mathcal{A}_l$, the intersections of such strata with $D_l \setminus F_{l-1}$ together with (and preceded by) the weak stratification of F_{l-1} gives a finite real analytic weak stratification of F_l of dimension at most $2q$. Therefore, by induction, we get the desired weak stratification

$$E = E_1 \cup \cdots \cup E_k$$

of $E = F_m$.

We now proceed to construct a strictly q-plurisubharmonic function on a neighborhood of $\overline{R} \cup E$ as in the proof of Theorem 2.1 (in which a strongly q-convex function was constructed on a neighborhood of the set $S \cap M$). The first observation is that, for $j = 1, \ldots, k$, the set

$$B_j \equiv \{ x \in E_j \mid \dim_{\mathbb{R}}(T_x E_j \cap J T_x E_j) = 2q \}$$

is a real analytic subset of dimension strictly less than $2q$ in the real analytic submanifold E_j of $X \setminus (E_1 \cup \cdots \cup E_{j-1})$. For if B_j contains a nonempty connected set which is open relative to E_j, then (by the theorem of Levi-Civita [L]) for some open subset P of X, the set $A \equiv E_j \cap P = B_j \cap P$ is a complex submanifold of dimension q in P. Fixing a point $p \in A$, we may choose the neighborhood P so that, for each $i = 1, \ldots, m$, we have $P \subset\subset V_i$ if $p \in \overline{D_i}$ and $P \cap \overline{D_i} = \varnothing$ if $p \notin \overline{D_i}$. Thus $S(\alpha) \cap M(\alpha) \cap A$ is a (properly embedded) real analytic subset of A whenever $p \in \overline{D_i}$ and $\alpha \in \mathcal{A}_i$. Moreover, since $A \subset E$, we have

$$A = \bigcup_{1 \leqslant i \leqslant m,\, p \in \overline{D_i}} \; \bigcup_{\alpha \in \mathcal{A}_i} S(\alpha) \cap M(\alpha) \cap A$$

and, therefore, since A is a connected complex manifold, we get

$$A \subset S(\alpha) \cap M(\alpha) \cap V_i$$

for some $i \in \{1, \ldots, m\}$ and some $\alpha \in \mathcal{A}_i$ with $p \in \overline{D_i}$. But then A determines a q-dimensional germ of a complex submanifold of X at p which is contained in $M(\alpha)$, and hence $p \in \overline{D_i} \cap Y(\alpha)$; contradicting the choice of D_i. Thus $\dim_{\mathbb{R}} B_j < 2q$ for $j = 1, \ldots, k$.

Set $L_0 = \overline{R}$ and, for $j = 1, \ldots, k$, let L_j be the compact set given by

$$L_j \equiv L_0 \cup E_1 \cup \ldots \cup E_j.$$

Given a C^∞ function on X which is strictly q-plurisubharmonic on a neighborhood of L_{j-1} in X, we may apply Lemma 3.2 to L_{j-1} and the real analytic subset $B_j \setminus L_{j-1}$ of $X \setminus L_{j-1}$ to get a C^∞ function on X which is strictly q-plurisubharmonic on a neighborhood of the compact set

$$L'_j = L_{j-1} \cup B_j.$$

Applying Lemma 3.2 to L'_j and to the real analytic submanifold $E_j \setminus L'_j$ of $X \setminus L'_j$, one gets a C^∞ function on X which is strictly q-plurisubharmonic on a neighborhood of L_j. Proceeding inductively, one gets a C^∞ function β on X which is strictly q-plurisubharmonic on a neighborhood of $L_k = \overline{R} \cup E$.

Step 4. Construction of the required function on the entire sublevel.

Given a point $p \in M \setminus (\overline{R} \cup E) = \bigcup_i C_i \setminus (\overline{R} \cup E)$, we have

$$p \in C_i \setminus (\overline{R} \cup E) \subset D_i \setminus \bigcup_{\alpha \in \mathcal{A}_i} S(\alpha) \cap M(\alpha)$$

for some i. If $\alpha \in \mathcal{A}_i$ and $p \notin M(\alpha) = \alpha^{-1}(a)$, then, since $\varphi(p) = a$ and $\varphi|_{U_i} = \max_{\alpha \in \mathcal{A}_i} \alpha$, we have $\alpha < \varphi$ near p. If $p \in M(\alpha)$, then, clearly, $p \notin S(\alpha)$. Thus there is a neighborhood P of p in X on which φ is equal to the maximum of a finite collection of C^∞ strictly q-plurisubharmonic functions. It follows that, if ϵ is a sufficiently small positive constant, then, for any such point p, we may choose the neighborhood P so that the sum of each of the corresponding functions with $\epsilon\beta$ is strictly q-plurisubharmonic. Hence the function

$$\psi \equiv -\log(a - \varphi) + (\varphi + \epsilon\beta)|_\Omega$$

will be a continuous exhaustion function on Ω and, outside some compact subset of Ω, ψ will locally be equal to the maximum of a finite collection of C^∞ strictly q-plurisubharmonic functions. Thus the theorem is proved. $\square$

5. Barlet spaces and q-convexity.

This section contains a brief summary of some facts from the theory of Barlet spaces, including a sketch of the proof of the consequence Theorem 0.7.

Let X be a reduced complex space and consider the graph and (holomorphic) projections

$$\begin{array}{ccc} G & \xrightarrow{\Psi} & X \\ \Phi \big\downarrow & & \\ \mathcal{C}_q(X) & & \end{array}$$

over the Barlet space $\mathcal{C}_q(X)$. The map Φ is always proper and surjective, but Ψ need not be surjective and, if $\mathcal{C}_q(X) \neq \varnothing$, then Ψ is not proper. On the other hand, if X is a Kähler manifold which admits a continuous plurisubharmonic exhaustion function or which has bounded geometry and Z is an irreducible component of $\mathcal{C}_q(X)$ such that $|c|$ is connected for every $c \in Z$, then the projection of the graph over Z is proper (see Campana [Cam], Fujiki [Fu], Barlet [Ba1], [Ba2]). Moreover, such an irreducible component always exists (see Proposition 5.3 below). The following version will suffice for our purposes:

PROPOSITION 5.1. — *Let (X, g) be a Kähler manifold, let q be a positive integer, let Z be a connected closed subset of $\mathcal{C}_q(X)$, let*

$$\begin{array}{ccc} G & \xrightarrow{\Psi} & X \\ {\scriptstyle\Phi}\big\downarrow & & \\ Z & & \end{array}$$

be the associated graph (over Z) and projections, let Θ be a relatively compact open subset of X, let $G_0 \equiv \Psi^{-1}(\Theta)$, and let

$$W \equiv \{\, c \in Z \mid |c| \subset \Theta \,\} \text{ (a subset which is open relative to } Z).$$

Suppose that $G_0 = \Phi^{-1}(W)$; that is, for each $c \in Z$, we have

$$|c| \cap \Theta \neq \varnothing \iff |c| \subset \Theta.$$

Then the restriction $\Psi_0 \equiv \Psi|_{G_0} : G_0 \to \Theta$ is a proper mapping.

The proof, which is standard, consists of a theorem of Bishop [Bi] together with the following special case of a theorem of Stoll [Sto]:

LEMMA 5.2 (Stoll). — *If (X, g) is a Kähler manifold and q is a positive integer, then the function $v : \mathcal{C}_q(X) \to \mathbb{R}$ given by*

$$v(c) = \operatorname{vol}_g(c) \quad \text{(counting multiplicities)} \quad \forall\, c \in \mathcal{C}_q(X)$$

is locally constant.

For the proof, one applies Stokes' theorem to the pullback of the Kähler form to the graph over a (suitable) smooth path in $\mathcal{C}_q(X)$ joining a given pair of generic q-cycles.

Proof of Proposition 5.1. — By a theorem of Bishop [Bi], a subset S of $\mathcal{C}_q(X)$ is relatively compact if and only if the volume function is bounded on S and there exists a compact subset of X which contains $|c|$ for every $c \in S$. Here, the volume function is equal to a constant v on Z by Lemma 5.2 and, by hypothesis, for each compact subset K of Θ, the support of each element of the set $S \equiv \{c \in Z \mid |c| \cap K \neq \varnothing\}$ is contained in the relatively compact subset Θ of X. Hence the subset $\Psi^{-1}(K) = \Psi_0^{-1}(K)$ of $G_0 \cap (K \times S)$, which is closed relative to $X \times \mathcal{C}_q(X)$, is compact. □

Since a continuous plurisubharmonic function is locally constant on compact analytic subsets, the hypotheses of Proposition 5.1 hold for any relatively compact sublevel Θ and any connected closed subset Z of $\mathcal{C}_q(X)$ whose elements have connected support. An irreducible component Z with this property always exists:

PROPOSITION 5.3. — *If Y is an irreducible compact analytic subset of dimension q in a reduced complex space X and Z is an irreducible component of the Barlet space $\mathcal{C}_q(X)$ containing the q-cycle $c_0 = 1 \cdot Y$, then the support $|c|$ of each q-cycle $c \in Z$ is connected.*

Proof. — First observe that each q-cycle $c \in Z$ near $c_0 = 1 \cdot Y$ is also given by a single irreducible q-dimensional analytic set with multiplicity 1. For if $\{c_\nu\}$ is a sequence in Z converging to c_0, then, for each ν, we have

$$c_\nu = Y_\nu + c'_\nu$$

where Y_ν is an irreducible compact analytic subset of dimension q in X and $c'_\nu = 0$ or $c'_\nu \in \mathcal{C}_q(X)$. Given a neighborhood U of Y in X, we have

$$Y_\nu \subset |c_\nu| \subset U$$

for all ν sufficiently large. Hence $Y_\nu \to Y$ by the Remmert-Stein-Thullen theorem and, therefore, $c'_\nu \to 0$. Thus $c_\nu = 1 \cdot Y_\nu + 0 = Y_\nu$ for all sufficiently large ν. On the other hand, the set S of q-cycles $c \in Z$ for which $|c|$ is not connected is open relative to Z. Moreover, the closure $\overline{S}$ is an analytic subset of Z. For if $\Phi : G \to Z$ is the graph over Z and

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[dr, "\Phi"'] & Z' \arrow[d, "\beta"] \\ & Z \end{tikzcd}$$

is the Stein factorization of Φ, then $c \in S$ if and only if the fiber $\beta^{-1}(c)$ is not a singleton. Hence $\bar{S}$ is the image of the analytic set

$$S' = \overline{\{\, (w,z) \in Z' \times Z' \mid \beta(w) = \beta(z) \,\} \setminus \text{diag.}}$$

under the proper mapping $S' \ni (w,z) \to \beta(w)$. Therefore, since $c_0 \in Z \setminus \overline{S}$ and since S is open in Z, we must have $S = \varnothing$. $\square$

Finally, in order to get holomorphic convexity from a mapping from the graph over a subset of a Barlet space, we will need the following well known fact:

LEMMA 5.4. — *Let Y be a normal complex space and suppose that there exists a surjective proper holomorphic mapping $\Psi : X \to Y$ of a holomorphically convex complex space X onto Y. Then Y is holomorphically convex.*

Proof of Theorem 0.7. — According to the hypothesis (iv), there is a nonempty open subset V of X such that each point $x \in V$ lies in some irreducible compact analytic subset Y_x of dimension at least $q-1$ in X. Clearly, we may assume that V is connected and relatively compact in X. Fix a continuous plurisubharmonic exhaustion function φ on X (hypothesis (ii)) and a real number $a > \sup_V \varphi$, and let Θ be the connected component of $\{ x \in X \mid \varphi(x) < a \}$ containing V. In particular, since φ is constant on connected compact analytic subsets of X, we have $Y_x \subset \Theta$ for each point $x \in V$. It suffices to show that Θ is holomorphically convex. For by passing to Remmert reductions and applying a theorem of Narasimhan [Ns, Corollary 1], one gets an exhaustion of X by an increasing sequence of holomorphically convex domains each of which is Runge in any of the larger domains. It will then follow that X is holomorphically convex.

According to the hypothesis (iii), we may choose a relatively compact domain Ω in X which contains $\overline{\Theta}$ and which admits a C^∞ exhaustion function which is strongly q-convex on the complement of some compact subset of Ω. By a theorem of Barlet [Ba2], the Barlet space of compact analytic $(q-1)$-cycles in a strongly q-convex Kähler manifold is holomorphically convex. Hence the complex space

$$\mathcal{C} = \mathcal{C}_{q-1}(\Omega) \cup \mathcal{C}_q(\Omega) \cup \cdots \cup \mathcal{C}_n(\Omega)$$

is holomorphically convex, because a strongly q-convex function is also strongly $(q+1)$-convex. Let $\mathcal{Z}$ be the (countable) collection of irreducible components of $\mathcal{C}$ which contain at least one of the cycles

$$1 \cdot Y_x \quad (x \in V).$$

By Proposition 5.3, $|c|$ is connected for each $c \in Z \in \mathcal{Z}$. The union of the images under the projection mappings of the graphs over the elements of $\mathcal{Z}$ is equal to the union of a countable collection of local analytic sets in X. Since this union contains the open set V, the image of at least one of these projections must contain an open set which is also contained in V. Thus we get holomorphic mappings

$$\begin{array}{ccc} G & \xrightarrow{\Psi} & X \\ {\scriptstyle\Phi}\Big\downarrow & & \\ Z & & \end{array}$$

where Z is some irreducible component of $\mathcal{C}$ (a holomorphically convex complex space) containing only cycles with connected support, G is the graph over Z, and the projection Ψ maps G onto a set containing an open set which we may take to be V (possibly after shrinking V). Since Φ is a proper mapping, G is holomorphically convex . Therefore, since the open subset $G_0 \equiv \Psi^{-1}(\Theta)$ is closed as a subset of the sublevel $(\varphi\circ\Psi)^{-1}((-\infty, a))$ of the continuous plurisubharmonic function $\varphi \circ \Psi$, the theorem of Narasimhan [Ns, Corollary 1] (applied to Remmert reductions) implies that G_0 is also holomorphically convex. Moreover, by Proposition 5.1, the restriction $\Psi_0 \equiv \Psi|_{G_0} : G_0 \to \Theta$ is a proper mapping and, since the image contains the open subset V, this mapping is also surjective. Therefore, by Proposition 5.4, Θ is holomorphically convex.

Applying the theorem of Narasimhan as described in the beginning, one gets holomorphic convexity of X. It remains to check that the Remmert reduction $\Upsilon : X \to R$ is of dimension $n - q + 1$. First, the condition (iv) immediately implies that the dimension m of the generic fiber of Υ is at least $q - 1$ and hence $\dim R \leqslant n - q + 1$. For the reverse inequality, we choose a relatively compact domain Ω in X which admits a C^∞ exhaustion function ψ that is strongly q-convex on the complement of some compact subset of Ω. Since each point of some nonempty open subset of X (for example, the inverse image of the set of regular values of Υ) is contained in an irreducible compact analytic subset of dimension $m = \dim X - \dim R$, the above arguments imply that there is an irreducible component Z of $\mathcal{C}_m(X)$ such that each element of Z has connected support and the image of the graph G over Z in X contains a nonempty open subset of X. Since X is weakly 1-complete, Proposition 5.1 then implies that the mapping $G \to X$ is proper and surjective. The set

$$Q = \{\, c \in Z \mid |c| \subset \Omega \,\}$$

is then an open subset of Z and we may choose Ω so large that $Q \neq \varnothing$. We have $Q \neq Z$, because $\Omega \neq X$ while $\Psi(G) = X$. So we may fix a q-cycle $c_0 \in \overline{Q}\setminus Q$ and a sequence $\{c_\nu\}$ in Q converging to c_0. We may also fix a real number b such that ψ is strongly q-convex on $\{ x \in \Omega \mid \psi(x) > b \}$. Since the compact connected analytic subset $|c_0|$ of X is *not* contained in Ω but the sequence of connected compact analytic subsets $\{|c_\nu|\}$ of Ω converges to $|c_0|$, we have $\max_{|c_\nu|} \psi \to \infty$. Thus $|c_\nu|$ meets $\{ x \in \Omega \mid \psi(x) > b \}$ for ν sufficiently large, and hence the restriction of ψ to $|c_\nu|$ assumes its maximum at some point at which $\psi > b$. Since ψ is strongly q-convex at such a point, it follows that $m = \dim |c_\nu| < q$. Thus $\dim R = n - m \geqslant n - q + 1$ and we have equality as claimed. □

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