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Evgueni DOUBTSOV & Artur NICOLAU

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SYMMETRIC AND ZYGMUND MEASURES IN SEVERAL VARIABLES

by E. DOUBTSOV and A. NICOLAU

1. Introduction.

A (signed) finite Borel measure μ on the unit circle $\mathbb{T}$ is called a Zygmund measure if there exists a constant $C = C(\mu) > 0$ such that

$$|\mu(I_+) - \mu(I_-)| \leqslant C|I_+|, \tag{1.1}$$

for any pair of adjacent arcs $I_+, I_- \subset \mathbb{T}$ of the same length $|I_+| = |I_-|$. If the constant C can be taken arbitrarily small as $|I_+|$ tends to 0, μ is called a small Zygmund measure.

Corresponding to the Zygmund measures, there are the Bloch functions, that is, analytic functions f in the unit disc for which there exists a positive constant $C = C(f) > 0$ such that

$$(1 - |z|^2)\,|f'(z)| \leqslant C\,, \quad |z| < 1.$$

The little Bloch space consists of those Bloch functions f satisfying

$$\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0.$$

P. Duren, H. Shapiro and A. Shields proved the following result.

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THEOREM A ([9]). — *Let μ be a finite measure on the unit circle and let $H(\mu)$ be its Herglotz transform,*

$$H(\mu)(z) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\mu(\xi)\,, \quad |z| < 1.$$

Then μ is a (small) Zygmund measure if and only if $H(\mu)$ is a (little) Bloch function.

A finite positive measure μ on the unit circle is called doubling if there exists a positive constant $C = C(\mu) > 0$ such that

$$C^{-1} \leqslant \frac{\mu(I_+)}{\mu(I_-)} \leqslant C, \tag{1.2}$$

for any pair of adjacent arcs of equal length. Symmetric measures are those doubling measures satisfying

$$\lim_{|I| \to 0} \frac{\mu(I_+)}{\mu(I_-)} = 1.$$

A classical result of Beurling and Ahlfors says that a homeomorphism from the unit circle onto itself extends quasiconformally to the whole complex plane if and only if its distributional derivative is a measure satisfying the doubling condition (1.2). Also, symmetric measures correspond to quasiconformal extensions whose conformal distortion tends to 0 at the unit circle ([11]).

The connection with the previous setting is that (1.2) is the "multiplicative" version of the "additive" condition (1.1), see [10].

In the spirit of Theorem A, the following description of symmetric measures has been obtained.

THEOREM B ([1]). — *Let μ be a positive measure in $\mathbb{T}$ and let $H(\mu)$ be its Herglotz transform,*

$$H(\mu)(z) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\mu(\xi)\,, \quad |z| < 1.$$

Then μ is symmetric if and only if

$$\frac{(1 - |z|^2)|H(\mu)'(z)|}{\mathrm{Re}\ H(\mu)(z)} \to 0 \quad \text{as} \quad |z| \to 1.$$

The nature of Theorems A and B is purely real but complex analysis techniques are used at certain steps of the corresponding proofs. The main aim of this note is to study symmetric and Zygmund measures in the Euclidean space.

We now fix some notations which will be used throughout the paper. The symbol m denotes Lebesgue measure. Given a set $E \subset \mathbb{R}^n$, $|E| = m(E)$. The upper-half space will be denoted by $\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n, y > 0\}$. Also, Q_+ and Q_- denote two adjacent cubes in $\mathbb{R}^n$ of equal volume. More precisely, $Q_+ = I_1 \times \cdots \times I_n$, $Q_- = J_1 \times \cdots \times J_n$ where I_i, J_k are intervals of equal length $\ell(Q_+) = \ell(Q_-)$ and $I_i = J_i$ for any i, $1 \leqslant i \leqslant n$, except for exactly one index s, $1 \leqslant s \leqslant n$, for which I_s and J_s are adjacent. A regular gauge function will be a positive non-decreasing, bounded function $\omega : (0,\infty) \to (0,\infty)$ such that for some positive number $\varepsilon > 0$, the function $\omega(t)/t^{1-\varepsilon}$ is decreasing. Observe that a regular gauge function ω satisfies that $\omega(t)/\omega(s)$ is uniformly bounded if $1/2 < t/s < 2$ and that $\lim_{t\to 0} \omega(t)t^{-1} = \infty$.

Given a finite measure μ in $\mathbb{R}^n$, let u be its harmonic extension, that is,

$$u(x,y) = \int_{\mathbb{R}^n} P(x-t,y)\, d\mu(t)\,, \quad (x,y) \in \mathbb{R}^{n+1}_+,$$

where $P(x,y)$ is the Poisson kernel of the upper-half space,

$$P(x,y) = P_y(x) = C_n \frac{y}{(||x||^2 + y^2)^{\frac{n+1}{2}}}\,, \quad x \in \mathbb{R}^n, \quad y > 0,$$

where C_n is chosen so that $\int_{\mathbb{R}}^n P(x,y) dm(x) = 1$.

Let ω be a regular gauge function. A (signed) Borel measure μ in $\mathbb{R}^n$ is called ω-Zygmund if there exists a positive constant C such that

$$|\mu(Q_+) - \mu(Q_-)| \leqslant C\omega(\ell(Q_+))|Q_+|,$$

for any pair $Q_+, Q_- \subset \mathbb{R}^n$ of adjacent cubes. Our first result is an Euclidean version of the Duren-Shapiro-Shields characterization.

THEOREM 1.1. — *Let μ be a (signed) finite Borel measure in $\mathbb{R}^n$ and let ω be a regular gauge function. Let u be the harmonic extension of μ. The following properties are equivalent:*

(a) *μ is an ω-Zygmund measure.*

(b) *There exists a positive constant $C > 0$ such that*

$$y|\nabla u(x,y)| \leqslant C\omega(y),$$

for any point $(x,y) \in \mathbb{R}^{n+1}_+$.

A positive Borel measure μ in $\mathbb{R}^n$ is called doubling if there exists a constant $C > 0$ such that

$$C^{-1} \leqslant \frac{\mu(Q_+)}{\mu(Q_-)} \leqslant C,$$

for any pair $Q_+, Q_- \subset \mathbb{R}^n$ of adjacent cubes. Clearly a doubling measure can not be finite. Similarly, a finite positive measure μ is called ω-symmetric if there exists a positive constant $C > 0$ such that

$$\left| \frac{\mu(Q_+)}{\mu(Q_-)} - 1 \right| \leqslant C\omega(\ell(Q_+)),$$

for any pair of adjacent cubes Q_+, Q_- with $\ell(Q_+) < 1$. So a ω-symmetric measure satisfies the doubling condition for cubes of sidelength smaller than 1.

Let u be a positive harmonic function in $\mathbb{R}^{n+1}_+$. Harnack's inequality asserts that

$$y|\nabla u(x,y)| \leqslant nu(x,y),$$

for any point $(x,y) \in \mathbb{R}^{n+1}_+$. This inequality is best possible as one can easily see by taking μ a Delta mass. Actually, if P denotes the Poisson kernel, one has $y\partial_y P(0,y) = -nP(0,y)$.

The analogue of Theorem B is the following result.

THEOREM 1.2. — *Let ω be a regular gauge function, $\omega(0^+) = 0$. Let μ be a finite, positive measure in $\mathbb{R}^n$ and let u be its harmonic extension. Then the following conditions are equivalent:*

(a) *μ is ω-symmetric.*

(b) *There exists a constant $C = C(\mu) > 0$ such that*

$$y\frac{|\nabla u(x,y)|}{u(x,y)} \leqslant C\omega(y),$$

for any point $(x,y) \in \mathbb{R}^{n+1}_+$.

The situation for doubling measures is not so nice.

THEOREM 1.3. — *Let μ be a positive measure in $\mathbb{R}^n$. Assume that $\int_{\mathbb{R}^n}(1+|x|)^{-n-1}d\mu(x) < \infty$ and let u be its harmonic extension.*

(a) *If μ is doubling, then there exists a constant $p < n$ such that*

$$y|\partial_y u(x,y)| \leqslant \max\{1,p\}u(x,y),$$

for any $(x,y) \in \mathbb{R}^{n+1}_+$.

(b) *If there exists a constant $p < 1$, such that*

$$y|\partial_y u(x,y)| \leqslant pu(x,y),$$

for any $(x,y) \in \mathbb{R}^{n+1}_+$, then μ is doubling.

Remark. — It is convenient to mention the following phenomena in dimension one. Let μ be a positive measure in the real line and let u be its harmonic extension.

1. Assume that μ is a doubling measure and its doubling constant is sufficiently close to 1, that is, assume that there exists $\varepsilon > 0$ such that

$$1 - \varepsilon < \frac{\mu(I_+)}{\mu(I_-)} < 1 + \varepsilon,$$

for all intervals I of the real line, then there exists a constant $p = p(\varepsilon) < 1$ if ε is sufficiently small, such that

$$y \frac{|\partial_y u(x,y)|}{u(x,y)} < p < 1 \quad \text{for any } (x,y) \in \mathbb{R}^2_+.$$

2. On the other hand, as part (b) of Theorem 1.3 asserts, if

$$y \frac{|\partial_y u(x,y)|}{u(x,y)} < p < 1 \quad \text{for any } (x,y) \in \mathbb{R}^2_+,$$

then μ is doubling.

The proofs of Theorems 1.1, 1.2 and 1.3 are given in Section 2, but the main idea can be explained as follows. Integrating by parts, one may relate the size of $y|\nabla u(x,y)|$ with cancellation properties of the measure μ, namely with the size of $|\mu(L_+) - \mu(L_-)|$ where $L_+, L_- \subset \mathbb{R}^n$ are adjacent parallelepipeds. In the one-dimensional case, L_+ and L_- are intervals and one uses directly the doubling condition (1.2) or the condition (1.1). However when $n > 1$, we need to transfer the information we have in cubes to parallelepipeds. This is the main technical difficulty in the paper. The idea is to consider partitions of L_+ and L_- into pairwise disjoint dyadic cubes, $L_+ = \cup Q_k^+$, $L_- = \cup Q_k^-$, where Q_k^+ and Q_k^- are symmetric with respect the common side of L_+ and L_-. Hence, Q_k^+ and Q_k^- are not adjacent and to estimate $|\mu(Q_k^+) - \mu(Q_k^-)|$ we need to take into account the distance from Q_k^+ to Q_k^-. This leads to some technical difficulties which require a detailed analysis of the Zygmund and the doubling conditions. A precise statement is proved in Section 4.

Classical constructions provide examples of positive singular symmetric (small Zygmund) measures. See [5], [9], [12], [15], [16]. It is also well known that in the one-dimensional case, the quadratic condition

$$\int_0^1 \frac{\omega^2(t)}{t}\, dt < \infty, \tag{1.3}$$

governs the existence of singular ω-Zygmund (ω-symmetric) measures. See [5], [17].

Our next results tell that the quadratic condition (1.3) also governs the situation in higher dimensions.

THEOREM 1.4. — *Let ω be a regular gauge function.*

(a) *Assume that*

$$\int_0 \frac{\omega^2(t)}{t}\, dt < \infty.$$

Then any ω-Zygmund measure μ is absolutely continuous, that is, $\mu = f dm$. Moreover for any cube $Q \subset \mathbb{R}^n$ and any $A > 0$, one has

$$\int_Q \exp(A|f|^2) < \infty.$$

(b) *Assume that*

$$\int_0 \frac{\omega^2(t)}{t}\, dt = \infty,$$

then there exists a positive, finite, singular ω-Zygmund measure on $\mathbb{R}^n$.

The corresponding result for symmetric measures is the following.

THEOREM 1.5. — *Let ω be a regular gauge function, $\omega(0^+) = 0$.*

(a) *Assume that*

$$\int_0 \frac{\omega^2(t)}{t}\, dt < \infty.$$

Then any ω-symmetric measure μ is absolutely continuous, that is, $\mu = f dm$. Moreover for any cube Q in $\mathbb{R}^n$ and any $A > 0$, one has

$$\int_Q \exp(A|\log f|^2) < \infty.$$

(b) *Assume that*

$$\int_0 \frac{\omega^2(t)}{t}\, dt = \infty,$$

then there exists a positive, finite, singular ω-symmetric measure on $\mathbb{R}^n$.

The proof of these results is given in Section 3. In both cases, part (b) follows easily from the one-dimensional constructions. In part (a) of Theorem 1.4, to show that μ is absolutely continuous, one considers its harmonic extension u. As it is well known, Green's formula relates the measure μ with a truncated version of the area function of u, which can be estimated using Theorem 1.1 and the integral condition in the assumption. Similar considerations applied to $\log u$, also give the absolute continuity of the measure μ in the part (a) of Theorem 1.5. In both results, the statement on the integrability of the density of the measure μ is deeper and uses martingale techniques. Namely, let $\mathcal{F}_k$ be the σ-algebra generated by the dyadic cubes of sidelenght 2^{-k}, $k \geqslant 0$. In Theorem 1.4, one considers the dyadic martingale $M = (M_k, \mathcal{F}_k)$ corresponding to the measure $\mu = f dm$ defined as $M_k\,|_Q = \mu(Q)/|Q|$, where Q is a dyadic cube of $\mathcal{F}_k$, $k \geqslant 0$. Also, consider the martingale difference function $\Delta M_k = M_k - M_{k-1}$ and the quadratic variation of the martingale, $SM = \sum_{k=1}^{\infty} (\Delta M_k)^2$. The fact that μ is a ω-Zygmund measure and the integral condition in the assumption tell that the quadratic variation SM is uniformly bounded and our result follows from a Theorem of Chang, Wilson and Wolff ([7]). Part (a) of Theorem 1.5 is proved by applying similar considerations to the logarithmic transform $\{N_k\}$ of the martingale $\{M_k\}$. A similar idea was used in [Br]. It turns out that μ be ω-symmetric translates to $\{N_k\}$ have increments bounded by ω and the considerations above can be applied to this martingale.

The letter C will denote an absolute constant which may depend on the dimension whose value may change from line to line. Also, $C(a)$ will denote a constant depending on the parameter a.

2. Harmonic extensions.

In many different situations, the normal derivative $\partial_y u$ of a harmonic function u in the upper-half space controls the whole gradient of u. See for instance Chapter V of [17]. In our case, we have the following result.

LEMMA 2.1. — *Let ω be a regular gauge function. Let u be a harmonic function in the upper-half space $\mathbb{R}^{n+1}_+$. Assume*

$$A = \sup_{y>1}\{|u(x,y)| \, : \, x \in \mathbb{R}^n\} < \infty.$$

(a) The following two conditions are equivalent.

(a.1) There exists a constant $C_1 > 0$ such that

$$y\left|\frac{\partial u}{\partial y}(x,y)\right| \leqslant C_1\omega(y),$$

for any $(x,y) \in \mathbb{R}^{n+1}_+$.

(a.2) There exists a constant $C_2 > 0$ such that

$$y\left|\frac{\partial u}{\partial x_i}(x,y)\right| \leqslant C_2\omega(y), \quad i = 1,\dots,n,$$

for any $(x,y) \in \mathbb{R}^{n+1}_+$.

(b) Assume u is positive and there exists a constant $C_3 > 0$ such that

$$y\left|\frac{\partial u}{\partial y}(x,y)\right| \leqslant C_3\omega(y)u(x,y),$$

for any $(x,y) \in \mathbb{R}^{n+1}_+$. Then there exists a constant $C_4 > 0$ such that

$$y\left|\frac{\partial u}{\partial x_i}(x,y)\right| \leqslant C_4\omega(y)u(x,y), \quad i = 1,\dots,n,$$

for any $(x,y) \in \mathbb{R}^{n+1}_+$.

Proof. — Recall that the letter C denotes an absolute constant which may depend on the dimension whose value may change from line to line. Observe that the mean value property gives that for any $x \in \mathbb{R}^n$ and $y \geqslant 3/2$ one has

$$y|\nabla u(x,y)| < CA.$$

So in the proof of (a) we may assume that $0 < y < 3/2$. By Harnack's inequality we may also assume $0 < y < 3/2$ in the proof of (b).

Assume (a.1) holds. Fixed $i = 1,\dots,n$ and $0 < y < 3/2$, one has

$$\frac{\partial u}{\partial x_i}(x,y) - \frac{\partial u}{\partial x_i}(x,2) = \int_2^y \frac{\partial^2 u}{\partial y \partial x_i}(x,t)\, dt.$$

Let B be the ball centered at (x,t) of radius $2^{-1}t$, the mean value property and the fact that ω is regular give

$$\left|\frac{\partial^2 u}{\partial y \partial x_i}(x,t)\right| \leqslant \frac{C}{t}\frac{1}{|B|}\int_B \left|\frac{\partial u}{\partial y}\right| \leqslant C\frac{\omega(t)}{t^2}.$$

Thus, for $0 < y < 3/2$, one has

$$\left|\frac{\partial u}{\partial x_i}(x,y)\right| \leqslant CA + C\int_y^2 \frac{\omega(t)}{t^2}.$$

Since $\omega(t)/t^{1-\varepsilon}$ is decreasing, one has

$$\int_y^2 \frac{\omega(t)}{t^2}\,dt = \int_y^2 \frac{\omega(t)}{t^{1-\varepsilon}t^{1+\varepsilon}}\,dt \leqslant \frac{\omega(y)}{y^{1-\varepsilon}}\frac{y^{-\varepsilon}}{\varepsilon} = \frac{\omega(y)}{\varepsilon y}$$

and (a.2) holds because $\lim_{y\to 0}\omega(y)/y = \infty$.

Conversely, assume (a.2) holds. Let $B \subset \mathbb{R}^{n+1}_+$ be the ball centered at (x,t) of radius $t/2$. The mean value property and the fact that ω is regular give

$$\left|\frac{\partial^2 u}{\partial x_i^2}(x,t)\right| \leqslant \frac{C}{t|B|}\int_B \left|\frac{\partial u}{\partial x_i}\right| \leqslant C\frac{\omega(t)}{t^2}.$$

Thus, the harmonicity of u yields

$$\left|\frac{\partial^2 u}{\partial y^2}(x,t)\right| \leqslant C\frac{\omega(t)}{t^2}.$$

As before, we deduce that for $0 < y < 3/2$, one has

$$\left|\frac{\partial u}{\partial y}(x,y)\right| \leqslant CA + C\int_y^2 \frac{\omega(t)}{t^2}\,dt \leqslant C(\varepsilon)\frac{\omega(y)}{y}.$$

The proof of (b) is along the same lines and we only sketch it. One may assume $\omega(0^+) = 0$, since otherwise, one applies Harnack's inequality. Hence, given $\varepsilon > 0$, one has

$$\lim_{y\to 0} u(x,y)y^{-\varepsilon} = \infty. \tag{2.1}$$

Indeed, choose $\eta > 0$ such that $2^\varepsilon(1-\eta) > 1$. Since $\omega(0^+) = 0$, one has $u(x,y/2) > (1-\eta)u(x,y)$ for all $y > 0$ small enough. Therefore $u(x,y/2)(y/2)^{-\varepsilon} > 2^\varepsilon(1-\eta)u(x,y)y^{-\varepsilon}$ and iterating this inequality one obtains (2.1).

As in the Zygmund case, the mean value property gives

$$\left|D^2u(x,y)\right| \leqslant C\omega(y)u(x,y)/y^2,$$

where D^2 denotes any second derivative of u. By (2.1), $\omega(y)u(x,y) \geqslant Cy$ if $0 < y < 2$, so to prove (b) it is sufficient to show the following estimate:

$$\int_y^2 \omega(t)u(x,t)dt/t^2 \leqslant C\omega(y)u(x,y)/y.$$

We bound the integral by a fixed multiple of

$$\sum_0^{\log_2(1/y)} \frac{\omega(2^k y)u(x,2^k y)}{2^k y}.$$

Now, one only has to observe that

$$\omega(2^k y)/2^{k(1-\varepsilon)} \leqslant \omega(y),$$

and since $\omega(0^+) = 0$, one may assume

$$u(x,2^k y) \leqslant (1+\varepsilon^2)^k u(x,y)$$

when $2^k y$ is sufficiently small. To estimate the rest of the sum, one uses that $\omega(y)u(x,y)/y$ is bounded below when y is small. $\square$

One can also write estimates equivalent to (a.1) and (a.2) using derivatives of higher order. For instance, under the hypothesis of Lemma 2.1, let $k > 1$, the estimate

$$y^k \left|\frac{\partial^k u}{\partial y^k}(x,y)\right| \leqslant C_k\omega(y),$$

for any point $(x,y) \in \mathbb{R}_+^{n+1}$, is equivalent to (a.1) and (a.2). Similarly,

$$y^k \left|\frac{\partial^k u}{\partial y^k}(x,y)\right| \leqslant C_k\omega(y)u(x,y),$$

for any point $(x,y) \in \mathbb{R}_+^{n+1}$, is equivalent to the condition in (b).

In the proof of Theorems 1.1 and 1.2 an integration by parts argument is used. For convenience, it is collected in the following result.

For $x \in \mathbb{R}^n$, $t \in (\mathbb{R}_+)^n$, define $L_t(x) = (x_1, x_1+t_1)\times\ldots\times(x_n, x_n+t_n)$ (with obvious modifications for arbitrary $t \in \mathbb{R}^n$). We say that $L_t(x)$ is a parallelepiped. Usually we write L_t in place of $L_t(0)$.

LEMMA 2.2.— *Let μ be a signed measure on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n}(1+|x|)^{-n-1}d\mu(x)<\infty$. Assume $\mu(\partial L)=0$ for any parallelepiped $L\subset\mathbb{R}^n$. Let u be the harmonic extension of μ. Then for any point $(x,y)\in\mathbb{R}^{n+1}_+$, one has*

$$\frac{\partial u}{\partial x_j}(x,y) = \frac{(-1)^{n+1}}{2}\int_{\mathbb{R}^n}\frac{\partial^{n+1}P_y(t)}{\partial t_j\partial t_1\dots\partial t_n}\operatorname{sign}(t)(\mu(L_t(x))-\mu(L_{-t}(x)))dm(t),$$

and

$$\frac{\partial u}{\partial y}(x,y)=\frac{(-1)^n}{2}\int_{\mathbb{R}^n}\frac{\partial^{n+1}P_y(t)}{\partial y\partial t_1\dots\partial t_n}\operatorname{sign}(t)(\mu(L_t(x))-\mu(L_{-t}(x)))dm(t).$$

Proof.— The two formulas are proved in a similar way. So we only show the first one. One only has to show that

$$\int_{\mathbb{R}^n}\frac{\partial P_y(t-x)}{\partial t_j}\,d\mu(t)=(-1)^n\int_{\mathbb{R}^n}\frac{\partial^{n+1}P_y(t)}{\partial t_j\partial t_1\dots\partial t_n}\operatorname{sign}(t)\mu(L_t(x))dm(t),$$

because the formula will follow after a change of variables.

To simplify the notation, assume $x=0$. Fix $\alpha>0$ and put $f(t)=f_\alpha(t)=u(t,\alpha)$, $t\in\mathbb{R}^n$. We use the notation $t=(t_1,t')$, where $t_1\in\mathbb{R}^{n-1}$ and $t'\in\mathbb{R}$ We have $f(\cdot,t')\in L^1(\mathbb{R})$ for all $t'\in\mathbb{R}^{n-1}$ and $\partial P_y(t)/\partial t_j\to 0$ as $|t_1|\to\infty$. Hence, integration by parts gives

$$\int_{\mathbb{R}}\frac{\partial P_y(t)}{\partial t_j}f(t_1,t')\,dt_1=-\int_{\mathbb{R}}\frac{\partial^2 P_y(t)}{\partial t_j\partial t_1}\left(\int_0^{t_1}f(\tau_1,t')\,d\tau_1\right)dt_1.$$

Since $f\in L^1(\mathbb{R}^n)$, we may apply Fubini's theorem and repeat the integration by parts. Repeating it n times we get

$$\int_{\mathbb{R}^n}\frac{\partial P_y(t)}{\partial t_j}f(t)dm(t)=(-1)^n\int_{\mathbb{R}^n}\frac{\partial^{n+1}P_y(t)}{\partial t_j\partial t_1\dots\partial t_n}\operatorname{sign}(t)(fm)(L_t)\,dm(t).$$

Now, we claim that the lemma follows as $\alpha\to 0$. Indeed $f_\alpha dm\to d\mu$ weakly* as $\alpha\to 0$, so the convergence for the left hand sides holds. On the other hand, if $z\in L_t$ ($z\notin\overline{L_t}$), then $\int_{L_t}P_\alpha(\tau-z)\,dm(\tau)\nearrow 1$ (respectively $\searrow 0$) as $\alpha\to 0$. Hence

$$\int_{L_t}u(\tau,\alpha)\,dm(\tau)=\int_{\mathbb{R}^n}\int_{L_t}P_\alpha(\tau-z)\,dm(\tau)\,d\mu(z)\to\mu(L_t)\quad\text{as }\alpha\to 0$$

since $\mu(\partial L_t)=0$. Now, we apply Lebesgue's theorem. □

Remark. — If μ is symmetric or Zygmund, then $\mu(\partial L) = 0$ for any parallelepiped L.

When applying Lemma 2.2, one needs to understand the cancellation properties of the measure on parallelepipeds.

LEMMA 2.3. — *Let $L^+ = (0,\ell_1) \times \ldots \times (0,\ell_n)$ and $L^- = (-\ell_1,0) \times \ldots \times (-\ell_n,0)$, $\ell_j > 0$. Put $\ell = \max_{1\leqslant j\leqslant n} \ell_j$.*

(a) *Assume that μ is an ω-Zygmund measure on $\mathbb{R}^n$. Then*

$$|\mu(L^+) - \mu(L^-)| \leqslant C(n)\omega(\ell)\ell^n.$$

(b) *Assume that μ is an ω-symmetric measure on $\mathbb{R}^n$. Then,*

$$|\mu(L^+) - \mu(L^-)| \leqslant C(n)\omega(\ell)\left(\mu(L^+) + \mu(L^-)\right).$$

The proof of Lemma 2.3 is very technical and is given in Section 4.

Proof of Theorem 1.1

(a) $\Longrightarrow$ (b). By Lemma 2.2, we have

$$y\frac{\partial u(x,y)}{\partial x_j} = y\frac{(-1)^{n+1}}{2}$$
$$\int_{\mathbb{R}^n} \frac{\partial^{n+1}P_y(t)}{\partial t_j\partial t_1\ldots\partial t_n}\operatorname{sign}(t)(\mu(L_t(x)) - \mu(L_{-t}(x)))\,dm(t) = I + II,$$

where (I) is the last integral over the ball $Q_y(0) = \{t \in \mathbb{R}^n : ||t|| < y\}$ and (II) is the integral over $\mathbb{R}^n \setminus Q_y(0)$. The estimate

$$\left|\frac{\partial^{n+1}P_y(t)}{\partial t_j\partial t_1\ldots\partial t_n}\right| \leqslant c(n)/y^{2n+1} \quad \text{if} \quad \|t\| \leqslant y, \tag{2.2}$$

and Lemma 2.3 yield

$$|I| \leqslant c_1(n)y\int_{Q_y(0)} \frac{\omega(y)y^n}{y^{2n+1}}\,dm(t) \leqslant c_1(n)\omega(y).$$

On the other hand, $\omega(t)t^{\varepsilon-1}$ is decreasing, thus $2^{-k}\omega(2^k y) \leqslant 2^{-k\varepsilon}\omega(y)$, $k \in \mathbb{N}$. Therefore, the estimate

$$\left|\frac{\partial^{n+1}P_y(t)}{\partial t_j\partial t_1\ldots\partial t_n}\right| \leqslant \frac{c(n)}{2^{2k(n+1)}y^{2n+1}} \quad \text{if} \quad 2^{k-1}y \leqslant \|t\| \leqslant 2^k y,\ k \in \mathbb{N}, \tag{2.3}$$

and Lemma 2.3 give

$$|II| \leqslant y \sum_{k=1}^{\infty} \frac{c_2(n)}{2^{2k(n+1)} y^{2n+1}} \left(2^k y\right)^{2n} \omega(2^k y)$$

$$\leqslant c_2(n)\omega(y) \sum_{k=1}^{\infty} 2^{-k\varepsilon - k} \leqslant c_3(n)\omega(y).$$

In other words, we obtain

$$\left| y \frac{\partial u(x,y)}{\partial x_j} \right| \leqslant C\omega(y), \tag{2.4}$$

and Lemma 2.1 finishes the proof.

(b) $\Longrightarrow$ (a). This implication is essentially known (compare with [13], where a similar problem for general Lipschitz domains is investigated). Indeed, fix $\ell > 0$ and consider adjacent cubes $Q_\pm \subset \mathbb{R}^n$ of sidelength ℓ and with centers $x_\pm \in \mathbb{R}^n$. Put $x_0 = (x_+ + x_-)/2$ and $v(x,y) = u(x,y) - u(x_0, \ell)$. Then, Green's formula on $Q_\pm \times [\varepsilon, \ell]$ applied to the functions v and y and the estimate (b) yield

$$\left| \int_{Q_\pm} v(x,\varepsilon) dx \right| \leqslant \left| \int_{Q_\pm} v(x,\ell) dx \right| + \left| \int_{\partial Q_\pm \times [\varepsilon,\ell]} y \frac{\partial v}{\partial n}(x,y) d\sigma(x,y) \right|$$
$$+ \left| \int_{Q_\pm} \ell \frac{\partial v}{\partial n}(x,\ell) dx \right| + \left| \int_{Q_\pm} \varepsilon \frac{\partial v}{\partial n}(x,\varepsilon) dx \right|.$$

Since $|v(x,\ell)| < C\omega(\ell)$ if $x \in Q_\pm$, the estimate (b) gives that

$$\left| \int_{Q_\pm} v(x,\varepsilon) dx \right| \leqslant C\omega(\ell) |Q_\pm|$$

since ω is increasing. Hence, the triangular inequality gives

$$\left| \int_{Q_+} u(x,\varepsilon) dx - \int_{Q_-} u(x,\varepsilon) dx \right| \leqslant C\omega(\ell) |Q_\pm|.$$

The latter estimate yields $\mu(\partial Q) = 0$ for all cubes Q. Therefore, tending $\varepsilon \to 0$, we obtain $|\mu(Q_+) - \mu(Q_-)| \leqslant C\omega(\ell)|Q_\pm|$. □

Proof of Theorem 1.2.— Assume that (a) holds. As in the proof of Theorem 1.1, one uses Lemma 2.2 to write

$$\frac{\partial u}{\partial y}(x,y) = \frac{(-1)^n}{2} \int_{\mathbb{R}^n} \frac{\partial^{n+1} P_y(t)}{\partial y \partial t_1 \ldots \partial t_n} \operatorname{sign}(t) (\mu(L_t(x)) - \mu(L_{-t}(x))) dm(t).$$

Using Lemma 2.3 and (2.2) one can estimate the integral over $\{t \in \mathbb{R}^n : ||t|| \leqslant y\}$ by

$$C\frac{\omega(y)}{y}\frac{\mu\{t \in \mathbb{R}^n : ||t-x|| < y\}}{y^n} \leqslant C\frac{\omega(y)}{y}u(x,y),$$

where the last inequality follows from the estimate $P_y(t) > C_1 y^{-n}$ if $||t|| < y$. Using the estimate

$$\left|\frac{\partial^{n+1}P_y(t)}{\partial y \partial t_1 \ldots \partial t_n}\right| \leqslant \frac{c(n)}{2^{2k(n+1)}y^{2n+1}} \quad \text{if} \quad 2^{k-1}y \leqslant ||t|| \leqslant 2^k y,\ k \in \mathbb{N},$$

and Lemma 2.3, one bounds the integral over $\{t \in \mathbb{R}^n : ||t|| > y\}$ by

$$C\sum_{k=1}^{\infty}\frac{\omega(2^k y)}{2^k}\frac{\mu(Q_{2^k y}(x))}{(2^k y)^{n+1}},$$

where $Q_{2^k y}(x) = \{t \in \mathbb{R}^n : ||t-x|| < 2^k y\}$. Since

$$\omega(2^k y)/2^{k(1-\varepsilon)} \leqslant \omega(y), \quad k = 1, 2, \ldots$$

and

$$P_y(t) > Cy/(2^k y)^{n+1}, ||t|| < 2^k y,$$

one deduces

$$\sum_{k=1}^{\infty}\frac{\omega(2^k y)}{2^k}\frac{\mu(Q_{2^k y}(x))}{(2^k y)^{n+1}} \leqslant C\frac{\omega(y)}{y}u(x,y).$$

Lemma 2.1 finishes the argument.

Conversely, assume that (b) holds with $C = 1$.

First, as in the Zygmund case, we apply Green's formula to the functions u and y. Namely, let $\ell > 0$ and $Q_\pm \subset \mathbb{R}^n$ be adjacent cubes of sidelength ℓ with centers $x_\pm \in \mathbb{R}^n$. Then

$$\begin{aligned}(2.5)\quad \left|\int_{Q_\pm} u(x,\varepsilon)dx - \int_{Q_\pm} u(x,\ell)dx\right| &\leqslant \left|\int_{\partial Q_\pm \times [\varepsilon,\ell]} y\frac{\partial u}{\partial n}(x,y)d\sigma(x,y)\right| \\ &+ \left|\int_{Q_\pm} \ell\frac{\partial u}{\partial n}(x,\ell)dx\right| + \left|\int_{Q_\pm} \varepsilon\frac{\partial u}{\partial n}(x,\varepsilon)dx\right|,\end{aligned}$$

where $d\sigma$ is surface measure. Secondly, for $x \in Q_\pm$ and $0 < y < \ell$, we have

$$|\log u(x,y) - \log u(x,\ell)| \leqslant \int_y^\ell \frac{|\nabla u(x,t)|}{u(x,t)}\,dt \leqslant \omega(\ell)\log\frac{\ell}{y}.$$

Put $x_0 = (x_+ + x_-)/2$. Note that Harnack's inequality gives that there exists a constant $C > 0$ such that $u(x,\ell) \leqslant Cu(x_0,\ell)$ for all $x \in Q_\pm$. When checking the symmnetry condition on the cubes $Q_\pm$ we may assume that ℓ is so small that $\omega(\ell) \leqslant 1/2$. Then

$$u(x,y) \leqslant u(x,\ell)\left(\frac{\ell}{y}\right)^{\omega(\ell)} \leqslant u(x,\ell)\left(\frac{\ell}{y}\right)^{1/2}.$$

Hence

$$\begin{aligned}\int_\varepsilon^\ell |y\nabla u(x,y)|\,dy \leqslant \int_\varepsilon^\ell \omega(y)u(x,y)\,dy \leqslant \omega(\ell)\int_\varepsilon^\ell u(x,\ell)\left(\frac{\ell}{y}\right)^{1/2} dy \\ \leqslant 2\omega(\ell)\ell u(x,\ell) \leqslant C\omega(\ell)\ell u(x_0,\ell).\end{aligned}$$

Therefore, we obtain

$$\left|\int_{\partial Q_\pm \times [0,\ell]} y\frac{\partial u}{\partial n}(x,y)d\sigma(x,y)\right| \leqslant C\omega(\ell)|Q_\pm|u(x_0,\ell).$$

Also

$$\left|\int_{Q_\pm} \ell\frac{\partial u}{\partial n}(x,\ell)dx\right| \leqslant C\omega(\ell)|Q_\pm|u(x_0,\ell)$$

and

$$\left|\int_{Q_\pm} \varepsilon\frac{\partial u}{\partial n}(x,\varepsilon)dx\right| \leqslant C\omega(\varepsilon)\int_{Q_\pm} u(x,\varepsilon)dx.$$

So, tending $\varepsilon \to 0$ in (2.5), one gets

$$\left|\mu(Q_\pm) - \int_{Q_\pm} u(x,\ell)dx\right| \leqslant C\omega(\ell)|Q_\pm|u(x_0,\ell),$$

and we deduce $u(x_0,\ell)|Q_\pm| \leqslant C\mu(Q_\pm)$. Clearly (b) implies that

$$\left|\int_{Q_+} u(x,\ell)dx - \int_{Q_-} u(x,\ell)dx\right| \leqslant C\omega(\ell)|Q_\pm|u(x_0,\ell),$$

and hence

$$\begin{aligned}|\mu(Q_+) - \mu(Q_-)| &\leqslant C\omega(\ell)\max\{\mu(Q_+),\mu(Q_-)\} \\ &\leqslant C_1\omega(\ell)\min\{\mu(Q_+),\mu(Q_-)\}. \quad \square\end{aligned}$$

Proof of Theorem 1.3.

(a) By Harnack's inequality we may assume $n \geqslant 2$. A simple calculation shows

$$y\partial_y u(x,y) = \int_{\mathbb{R}^n} P_y(x-t)\,\frac{||x-t||^2 - ny^2}{||x-t||^2 + y^2}\,d\mu(t).$$

Denote by

$$K = K(x,t,y) = \frac{||x-t||^2 - ny^2}{||x-t||^2 + y^2}.$$

Observe that $|K| \leqslant n$ and $|K| \leqslant \max\{1, n/(1+\varepsilon^2)\}$ at any point $t \in \mathbb{R}^n$ where $||x-t|| \geqslant \varepsilon y$. Since the doubling condition (1.2) holds, the measure μ can not be too concentrated in $\{t \in \mathbb{R}^n \; : \; ||x-t|| < \varepsilon y\}$. Actually,

$$\mu\{t \in \mathbb{R}^n \; : \; \varepsilon y \leqslant ||x-t|| \leqslant 2\varepsilon y\} \geqslant C\mu\{t \in \mathbb{R}^n \; : \; ||x-t|| < \varepsilon y\}.$$

Hence,

$$\begin{aligned} y|\partial_y u(x,y)| \leqslant n \int_{\{t\in\mathbb{R}^n : ||x-t|| \leqslant \varepsilon y\}} & P_y(x-t)d\mu(t) \\ & + n(1+\varepsilon^2)^{-1} \int_{\{t\in\mathbb{R}^n : ||x-t|| > \varepsilon y\}} P_y(x-t)d\mu(t) \leqslant pu(x,y) \end{aligned}$$

where $p < n$ depends on the doubling constant of μ.

(b) We first observe that given $N > N_0(n,p)$ there exists a constant $C = C(N,p) > 0$ such that

$$\int_{NQ_y(x)} P_y(x-t)d\mu(t) \geqslant C \int_{\mathbb{R}^n} P_y(x-t)d\mu(t) \tag{2.6}$$

for any point $(x,y) \in \mathbb{R}^{n+1}_+$. Here $NQ_y(x)$ denotes the ball $\{t \in \mathbb{R}^n \; : \; ||t-x|| \leqslant Ny\}$. Indeed, since $|K| \leqslant n$, one has

$$\begin{aligned} & y|\partial_y u(x,y)| \\ & \qquad \geqslant \left| \int_{\mathbb{R}^n \setminus NQ_y(x)} P_y(x-t)K(x,t,y)d\mu(t) \right| - n \int_{NQ_y(x)} P_y(x-t)d\mu(t). \end{aligned}$$

Given $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that $K(x,t,y) \geqslant 1-\varepsilon$ for any $t \in \mathbb{R}^n \setminus NQ_y(x)$. Hence, if (2.6) does not hold, one would deduce that for any $\varepsilon > 0$ there exist points $(x,y) \in \mathbb{R}^{n+1}_+$ such that

$$y|\partial_y u(x,y)| \geqslant (1-\varepsilon)u(x,y).$$

So (2.6) is proved. Now, an easy estimate of the Poisson kernel and (2.6) give that

$$u(x,y) < C(N)C^{-1}\frac{\mu(NQ_y(x))}{|Q_y(x)|} .$$

So,

$$Cu(x,y) \geqslant \frac{\mu(NQ_y(x))}{|Q_y(x)|} \geqslant C^{-1}u(x,y),$$

for any point $(x,y) \in \mathbb{R}^{n+1}_+$, where C is a constant depending on N. Now Harnack's inequality gives

$$C^{-1} \leqslant \frac{u(x+(0,\dots,y,\dots,0),y)}{u(x,y)} \leqslant C, \quad (x,y) \in \mathbb{R}^{n+1}_+$$

and one deduces that μ is doubling. □

It is convenient to mention that the bound $p < 1$ in part (b) can not be replaced by $p < n$ if $n \geqslant 2$. Actually let σ be a doubling measure and let μ be its restriction to $\{x \in \mathbb{R}^n \,:\, ||x|| > 1\}$. So, μ is not doubling. Observe that for any point $(x,y) \in \mathbb{R}^{n+1}_+$ and any $\varepsilon > 0$ one has

$$\mu\{t \in \mathbb{R}^n \,:\, \varepsilon y \leqslant ||x-t|| \leqslant 2\varepsilon y\} \geqslant C\mu\{t \in \mathbb{R}^n \,:\, ||x-t|| < \varepsilon y\}.$$

Let u be the harmonic extension of μ. Arguing as in (a), one deduces

$$y|\partial_y u(x,y)| \leqslant pu(x,y),$$

for any $(x,y) \in \mathbb{R}^{n+1}_+$, where $p < n$ is a constant only depending on the doubling constant of σ.

3. Best decays.

This section is devoted to the proofs of Theorems 1.4 and 1.5.

Proof of Theorem 1.4.— Part (b). Assume (1.3) does not hold. Let $\nu \in M(\mathbb{R})$ be an ω-Zygmund positive singular measure with compact support (see e.g. [12]). Fix a compactly supported Hölder continuous function $h \in \mathrm{Lip}_1(\mathbb{R}^{n-1})$ and consider the measure $\mu = \nu \times hm_{n-1}$. We now check that μ is ω-Zygmund. The only non-trivial case arises when $Q_\pm = I \times J_\pm \subset \mathbb{R}^n$ are adjacent cubes of side $\ell < 1$ (i.e., $I \subset \mathbb{R}$ is an interval, $|I| = \ell$, and $J_\pm \subset \mathbb{R}^{n-1}$ are adjacent cubes). Then there exists

$x_\pm \in J_\pm$ such that $|\mu(Q_+) - \mu(Q_-)| = \nu(I)\ell^{n-1}|h(x_+) - h(x_-)| \leqslant C\nu(I)\ell^n$ since $h \in \mathrm{Lip}_1$. Now, applying the Zygmund condition one can check that $\nu(I) \leqslant C|I|\log|I|^{-1}$. Thus $\nu(I) \leqslant C|I|^{1-\varepsilon} \leqslant C_1\omega(\ell)$, because $\omega(t)/t^{1-\varepsilon}$ decreases.

Part (a). We assume that ω satisfies (1.3). Let μ be a ω-Zygmund measure. We claim that μ is absolutely continuous. Indeed, fix a cube $Q \subset \mathbb{R}^n$, $\ell(Q) = \ell \leqslant 1$. Since $\Delta\left(u^2\right) = 2\left|\nabla u\right|^2$, Green's formula yields

$$\int_{Q\times[\varepsilon,\ell]} y\left|\nabla u(x,y)\right|^2 dm(x)dy = \int_{\partial(Q\times[\varepsilon,\ell])} \left(2yu\frac{\partial u}{\partial n} - u^2\frac{\partial y}{\partial n}\right)d\sigma, \tag{3.1}$$

where $d\sigma$ is surface measure. First, by Theorem 1.1, $y\left|\nabla u(x,y)\right|^2 \leqslant C\omega^2(y)/y$, hence

$$\int_{Q\times[\varepsilon,\ell]} y\left|\nabla u(x,y)\right|^2 dm(x)dy \leqslant C|Q|\int_0^\ell \frac{\omega^2(y)}{y}\,dy.$$

Secondly, by Theorem 1.1 and Fubini, for $y \in (0,\ell]$, we have

$$\int_Q y\left|u(x,y)\frac{\partial u}{\partial n}(x,y)\right| dm(x) \leqslant C\omega(y)\int_Q |u(x,y)|\,dm(x) \leqslant C(\ell)|\mu|(Q).$$

Finally, for $y \in [\varepsilon,\ell]$, choose $t(y) \in [y,\ell]$ such that

$$|u(x,y)| \leqslant |u(x,\ell)| + \ell\left|\frac{\partial u}{\partial y}(x,t(y))\right|.$$

Note that $|\nabla u(x,t(y))| \leqslant C\omega(t(y))/t(y) \leqslant C\omega(y)/y$ since $\omega(t)/t$ is decreasing. Thus

$$\begin{aligned}\int_{\partial Q\times[\varepsilon,\ell]} y\left|u(x,y)\frac{\partial u(x,y)}{\partial n}\right| d\sigma&\\ \leqslant C(\ell) + C\ell\int_{\partial Q\times[\varepsilon,\ell]}\frac{\omega^2(y)}{y} &\leqslant C(\ell) + C|Q|\int_0^\ell\frac{\omega^2(y)}{y}\,dy.\end{aligned}$$

Therefore, (3.1) provides

$$\begin{aligned}\int_Q u^2(x,\varepsilon)dm(x) \leqslant \int_Q u^2(x,\ell)dm(x)&\\ +C(\ell) + C|Q|\int_0^\ell \frac{\omega^2(y)}{y}\,dy + C(\ell)|\mu|(Q) &\leqslant C_1(\ell).\end{aligned}$$

Recall that $u(\cdot,\varepsilon)dm \to d\mu$ weakly* as $\varepsilon \to 0$, so the latter estimate gives $d\mu\,|_Q = f dm\,|_Q$ with $f \in L^2(Q)$.

Now we investigate the local integrability properties of f. As in [7], [13] and [14], we consider dyadic martingales on a cube $Q_0 = [0, \ell]^n \subset \mathbb{R}^n$, $\ell \in (0, 1]$. Namely, let $\mathcal{F}_k$ be the σ-algebra generated by the dyadic subcubes of Q_0 of sidelenght $2^{-k}\ell$, $k \geqslant 0$. The martingale $M = (M_k, \mathcal{F}_k)$ corresponding to the measure $\mu = f dm$, is defined as $M_k |_{Q_k} = \mu(Q_k)/|Q_k|$, where Q_k is a dyadic cube of rank k, $k \geqslant 0$.

Define $\Delta M_k = M_k - M_{k-1}$ and $SM = \sum_{k=1}^{\infty} (\Delta M_k)^2$ (the martingale difference functions and the square function of the martingale). If μ is an ω-Zygmund measure, then

$$|\Delta M_k(x)| \leqslant C\omega(2^{-k}\ell), \quad x \in Q_0. \tag{3.2}$$

Put $M_\infty(x) = \lim_{k\to\infty} M_k(x)$. Recall that $\mu = f dm$, so Lebesgue's differentiation theorem yields $M_\infty(x) = f(x)$ at m-almost every point $x \in \mathbb{R}^n$. On the other hand, (3.2) gives

$$SM \leqslant C \sum_{k=1}^{\infty} \omega^2(2^{-k}\ell) \leqslant C \int_0^\ell \frac{\omega^2(t)}{t}\, dt < \infty.$$

So put $N_k = M_k - \mu(Q_0)/|Q_0|$ and apply the following result of Chang, Wilson and Wolff (see [7]):

THEOREM C.— *Let $(N_k, \mathcal{F}_k)$ be a real dyadic martingale on Q_0, $N_0 = 0$, and $\|SN\|_\infty < (2A)^{-1} < \infty$. Then $N_\infty(x)$ exists a.e. and $\int_{Q_0} \exp(AN_\infty^2(x))dx < \infty$.* □

Proof of Theorem 1.5.— Part (a). Assume ω does not satisfy (1.3) and consider a singular ω-symmetric measure μ on $\mathbb{T}$ (see e.g. [1]). Identify μ and its "periodic" counterpart on $\mathbb{R}$. Then $e^{-|t|}\, d\mu(t)$ is a finite ω-symmetric measure on $\mathbb{R}$. Now, it is sufficient to observe that $\mu_1 \times \mu_2$ is ω-symmetric on $\mathbb{R}^2$ if μ_1 and μ_2 are ω-symmetric on $\mathbb{R}$.

Part (b). Assume ω satisfies (1.3). Fix a cube $Q \subset \mathbb{R}^n$ with center x_0, $\ell(Q) = \ell \leqslant 1$. The identity $\Delta(\log u) = -|\nabla u|^2/u^2$ and Green's formula give

$$-\int_{Q\times[\varepsilon,\ell]} y \frac{|\nabla u(x,y)|^2}{u^2(x,y)} dm(x)dy = \int_{\partial(Q\times[\varepsilon,\ell])} (y \frac{\partial(\log u)}{\partial n} - (\log u)\frac{\partial y}{\partial n})d\sigma.$$

Then Theorem 1.2 yields

$$\int_{Q\times[\varepsilon,\ell]} y \frac{|\nabla u(x,y)|^2}{u^2(x,y)} dm(x)dy \leqslant C \int_{Q\times[\varepsilon,\ell]} \frac{\omega^2(y)}{y} \leqslant C|Q| \int_0^\ell \frac{\omega^2(y)}{y}\, dy.$$

On the other hand, we have

$$\int_{\partial(Q\times[\varepsilon,\ell])} y\,\frac{|\nabla u(x,y)|}{u(x,y)}\,d\sigma(x,y) \leqslant C|Q|\omega(\ell).$$

Therefore

$$\int_Q \log u(x,\varepsilon)dx - \int_Q \log u(x,\ell)dx = \mathrm{o}(|Q|),$$

where $\mathrm{o}(A)$ denotes a quantity such that $\mathrm{o}(A)/A$ tends to 0 when $|Q|$ tends to 0. Observe that $|\log u(x,\ell) - \log u(x_0,\ell)| \leqslant C\omega(\ell)$ for all $x \in Q$. Thus

$$u(x_0,\ell)\exp\left(-\frac{1}{|Q|}\int_Q \log u(x,\varepsilon)dx\right) = \exp\left(\mathrm{o}(1)\right).$$

The property

$$\left|u(x_0,\ell) - \frac{1}{|Q|}\int_Q u(x,\varepsilon)dx\right| = \mathrm{o}(1)u(x_0,\ell)$$

is established in the proof of Theorem 1.2 (part (b) $\Longrightarrow$ (a)). Hence

$$\left(\frac{1}{|Q|}\int_Q u(x,\varepsilon)dx\right)\exp\left(-\frac{1}{|Q|}\int_Q \log u(x,\varepsilon)dx\right) = \exp\left(\mathrm{o}(1)\right).$$

It is well-known (see e.g. [8], Chapter 4) that the latter estimate provides the following "local" A_∞-property: There exist absolute constants $\delta > 0$, $C > 0$ independent of ε such that

$$\int_E u(x,\varepsilon)dm(x) \leqslant C\left(\frac{|E|}{|Q|}\right)^\delta \int_Q u(x,\varepsilon)dm(x) \tag{3.3}$$

for any measurable subset E of a cube $Q \subset \mathbb{R}^n$, $\ell(Q) \leqslant 1$. Now it follows easily that μ is absolutely continuous, $\mu = f dm$. See for instance [8], Chapter 4 .

To investigate the integrals over cubes, we apply an argument from [14], p. 34. So we consider dyadic martingales on a cube $Q_0 = [0,\ell]^n$, $\ell \in (0,1]$. Namely, put

$$Z_k\,|_{Q_k} = \frac{\mu(Q_k)|Q_0|}{|Q_k|\mu(Q_0)},$$

where $Q_k \subset Q_0$ is a dyadic cube of sidelenght $2^{-k}\ell$, $k \geqslant 0$. Since μ is ω-symmetric, we have $|\Delta Z_k|/Z_{k-1} \leqslant C\omega(\ell 2^{-k})$. Now we consider the logarithmic transform M_k of the martingale Z_k. The identities $M_0 = \log Z_0 = 0$ and $\Delta M_k = \operatorname{arth}\left(\frac{\Delta Z_k}{Z_{k-1}}\right)$, $k \geqslant 1$, define the corresponding real *martingale* with uniformly bounded differences. More precisely, $|\Delta M_k| \leqslant C\omega(\ell 2^{-k})$. Therefore, as in the Zygmund case, since $||SM||_\infty < \infty$, given $A > 0$, we obtain $\int_{Q_0} \exp\left(AM_\infty^2(x)\right) < \infty$. So $\int_{Q_0} \exp\left(A\left|\log Z_\infty(x)\right|^2\right) < \infty$ since $Z_k \asymp \exp\left(M_k - 1/2S_k(M)\right)$. The definition of Z_k and Lebesgue's differentiation theorem give the equality $Z_\infty(x) = f(x)|Q_0|/\mu(Q_0)$ m-a.e. In other words, we have

$$\int_{Q_0} \exp\left(A\left|\log f(x) - \log\frac{\mu(Q_0)}{|Q_0|}\right|^2\right) < \infty. \qquad \square$$

4. Cubes and parallelepipeds.

This section is devoted to the proof of Lemma 2.3.

Proof of Lemma 2.3. — We represent the parallelepiped L^+ as a union of *dyadic cubes*. More precisely, let U_1 be the collection of dyadic cubes of length $\ell/2$ which are contained in L^+. On the step k we choose the cubes "of generation k"

$$\prod_{j=1}^{n}\left(m_j\ell 2^{-k}, (m_j+1)\ell 2^{-k}\right), \quad 0 \leqslant m_j < 2^k,$$

which are contained in $L^+ \setminus U_{k-1}$ (here U_{k-1} is the union of the cubes selected on the previous steps).

Let μ be a ω-Zygmund measure with constant $C = 1$. By induction, we will compare dyadic cubes contained in $L^\pm$. In what follows, we use the "$\pm$" notation for the dyadic sets symmetric with respect to the origin. In particular, $Q^+ = (0,\ell)^n$ and $Q^- = (-\ell,0)^n$. We will show by induction that

$$(4.1) \qquad |\mu(Q_k^+) - \mu(Q_k^-)| \leqslant Cn|Q_k^\pm|\left(\omega(\ell) + \sum_{j=1}^{k}\omega(\ell 2^{-j})\right)$$

for any dyadic cube $Q_k^+ \in U_k$.

Step 1. Let $Q_1^{\pm} \subset Q^{\pm}$ be dyadic cubes of generation 1. Without loss of generality, we assume that $Q_1^+ = (0, \ell/2)^n$. Put $L_j^+ = (0,\ell)^j \times (0,\ell/2)^{n-j}$, $0 < j < n$. Thus

$$\begin{aligned}
|2\mu(Q_1^{\pm}) - \mu(L_1^{\pm})| &\leqslant C\omega(\ell/2)|Q_1^{\pm}| = \omega(\ell/2)|Q^{\pm}|2^{-n},\\
|2\mu(L_1^{\pm}) - \mu(L_2^{\pm})| &\leqslant C\omega(\ell/2)|Q^{\pm}|2^{1-n},\\
&\dots,\\
|2\mu(L_{n-2}^{\pm}) - \mu(L_{n-1}^{\pm})| &\leqslant C\omega(\ell/2)|Q^{\pm}|2^{-2},\\
|2\mu(L_{n-1}^{\pm}) - \mu(Q^{\pm})| &\leqslant C\omega(\ell/2)|Q^{\pm}|2^{-1};
\end{aligned}$$

hence

$$|2^n\mu(Q_1^{\pm}) - \mu(Q^{\pm})| \leqslant Cn\omega(\ell/2)|Q^{\pm}|/2.$$

On the other hand, $|\mu(Q^+) - \mu(Q^-)| \leqslant Cn\omega(\ell)|Q^{\pm}|$, therefore

$$|\mu(Q_1^+) - \mu(Q_1^-)| \leqslant Cn|Q_1^{\pm}|(\omega(\ell) + \omega(\ell/2)).$$

Step $k+1$. Repeating the argument in step 1 we have

$$|2^n\mu(Q_{k+1}^{\pm}) - \mu(Q_k^{\pm})| \leqslant Cn\omega(\ell 2^{-k-1})|Q_k^{\pm}|/2.$$

Hence

$$\begin{aligned}
\left|\mu(Q_{k+1}^+) - \mu(Q_{k+1}^-)\right| &\leqslant \left|\mu(Q_{k+1}^+) - 2^{-n}\mu(Q_k^+)\right| + 2^{-n}\left|\mu(Q_k^+) - \mu(Q_k^-)\right|\\
&\quad + \left|2^{-n}\mu(Q_k^-) - \mu(Q_{k+1}^-)\right|\\
&\leqslant Cn\omega(\ell 2^{-k-1})|Q_{k+1}^{\pm}| + 2^{-n}\left|\mu(Q_k^+) - \mu(Q_k^-)\right|.
\end{aligned}$$

Therefore, by induction (4.1) holds.

Finally, observe that there are at most $n2^{k(n-1)}$ cubes of generation k in the decomposition of L^+. Thus (4.1) gives

$$\begin{aligned}
|\mu(L^+) - \mu(L^-)| &\leqslant \sum_{k=1}^{\infty} \sum_{Q_k^+ \subset L^+} \left|\mu(Q_k^+) - \mu(Q_k^-)\right|\\
&\leqslant C\sum_{k=1}^{\infty} n2^{k(n-1)}\omega(\ell)\frac{n\ell^n}{2^{kn}}(k+1)\\
&\leqslant C(n)\omega(\ell)\ell^n.
\end{aligned}$$

The proof is completed in the Zygmund case. Now, we consider symmetric measures. Let μ be a ω-symmetric measure and L a parallelepiped with $\ell < 1$. For $k = 1, 2, \ldots$, let Q_k denote a cube of the family U_k. We will show

$$(4.2) \qquad \mu(Q^{\pm}_{k+1}) \leqslant C2^{-n}\mu(Q^{\pm}_k)(1 + C\omega(\ell 2^{-k-1})n(1 + C\omega(\ell 2^{-k-1}))^n);$$

and

$$(4.3) \quad |\mu(Q^{+}_{k+1}) - \mu(Q^{-}_{k+1})|$$
$$\leqslant 2^{-n}C(\mu(Q^{+}_k) + \mu(Q^{-}_k)\omega(\ell/2)n(1 + C\omega(\ell/2))^n + 2^{-n}|\mu(Q^{+}_k) - \mu(Q^{-}_k)|.$$

Let us prove them for $k = 0$. We have

$$|2^n\mu(Q^{\pm}_1) - 2^{n-1}\mu(L^{\pm}_1)| \leqslant C2^{n-1}\omega(\ell/2)\mu(L^{\pm}_1)/2,$$
$$\ldots,$$
$$|2^2\mu(L^{\pm}_{n-2}) - 2\mu(L^{\pm}_{n-1})| \leqslant C2\omega(\ell/2)\mu(L^{\pm}_{n-1})/2,$$
$$|2\mu(L^{\pm}_{n-1}) - \mu(Q^{\pm})| \leqslant C\omega(\ell/2)\mu(Q^{\pm})/2;$$

Hence $\mu(L^{\pm}_{n-k}) \leqslant (1 + C\omega(\ell/2))\mu(L^{\pm}_{n-k+1})/2$. We deduce

$$\mu(L^{\pm}_{n-k}) \leqslant \Big(\frac{1 + C\omega(\ell/2)}{2}\Big)^k \frac{\mu(Q^{\pm})}{2}.$$

So adding the chain of estimates above we get

$$|2^n\mu(Q^{\pm}_1) - \mu(Q^{\pm})| \leqslant C\mu(Q^{\pm})\omega(\ell/2)n(1 + C\omega(\ell/2))^n/2.$$

Therefore

$$2^n|\mu(Q^{+}_1) - \mu(Q^{-}_1)| \leqslant (\mu(Q^+) + \mu(Q^-))$$
$$C\omega(\ell/2)n(1 + C\omega(\ell/2))^n + |\mu(Q^+) - \mu(Q^-)|.$$

Note also that

$$\mu(Q^{\pm}_1) \leqslant 2^{-n}\mu(Q^{\pm})(1 + C\omega(\ell/2)n(1 + C\omega(\ell/2))^n).$$

So, we have proved (4.2) and (4.3) for $k = 0$. When $k > 0$, the proof proceeds by induction using the same idea.

Finally, we have

$$|\mu(L^+) - \mu(L^-)| \leqslant \sum_{k=0}^{\infty} \sum_{Q \in U_k} |\mu(Q^+) - \mu(Q^-)|.$$

Using (4.3), we proceed recurrently and estimate the latter double sum. Namely, let A_k be the first term in the right half side of (4.3), that is $A_k = 2^{-n-1}\omega(\ell/2)n(1 + C\omega(\ell/2))^n(\mu(Q^{+}_k) + \mu(Q^{-}_k))$.

Fix $k+1 \in \mathbb{N}$. First of all, there exist at most $n2^{(k+1)(n-1)}$ different cubes $Q_{k+1}^{+} \subset L^{+}$ of generation $k+1$. Respectively, the coefficient which corresponds to A_{k+1} and "originates" on the step $k+m+1$, $m \in \mathbb{N}$, is at most $n2^{-nm}2^{(k+m+1)(n-1)} = n2^{-m}2^{(k+1)(n-1)}$. Hence

$$|\mu(L^{+}) - \mu(L^{-})| \leqslant 2n \sum_{k=0}^{\infty} 2^{(k+1)(n-1)} A_{k+1}.$$

To finish the argument, one only has to prove

$$\sum_{k=0}^{\infty} 2^{(k+1)(n-1)} \mu(Q_k^{+}) \leqslant C(\omega)\mu(Q^{+}).$$

Indeed, choose $j = j(\omega) \in \mathbb{N}$ so large that

$$1 + C\omega(2^{-j-1})n(1+\omega(2^{-j-1}))^n = q(j) < 2.$$

Then, by (4.2)

$$\sum_{k \geqslant j} 2^{(k+1)(n-1)} \mu(Q_k^{+}) \leqslant \mu(Q^{+}) \sum_{k \geqslant j} 2^{(k+1)(n-1)} 2^{-n(k+1)} q(j)^{k+1} \leqslant C_1 \mu(Q^{+}).$$

This yields the estimate in question and ends the proof of the lemma. $\square$

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Evgueni DOUBTSOV,
St Petersburg State University
Department of Mathematical Analysis
Bibliotechnaya pl. 2
Staryi Petergof
198904 St Petersburg (Russia).
ed@ED8307.spb.edu
&
Artur NICOLAU,
Universitat Autònoma de Barcelona
Departament de Matemàtiques
08193 Bellaterra, Barcelona (Spain).
artur@manwe.mat.uab.es