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Fixed points of discrete nilpotent group actions on S^2

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FIXED POINTS OF DISCRETE NILPOTENT
GROUP ACTIONS ON S²

by S. DRUCK (*), F. FANG (**) and S. FIRMO (*)

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1. Introduction.

The classical Poincaré Theorem [13] asserts that a C^1 vector field on a closed surface Σ with nonzero Euler characteristic has a singularity. Another way to phrase this conclusion is to say that the flow tangent to the vector field must have a stationary point. In [9], [10], [11] Lima proved that pairwisely commuting vector fields on the surface Σ have a common singularity. This result has been generalized by Plante [12] for connected nilpotent Lie group actions on Σ. The same result does not hold for solvable Lie group actions, as pointed out in the work of Lima.

From the Lefschetz Fixed Point Theorem any diffeomorphism of the surface Σ homotopic to the identity map has a fixed point. This strenghthened the Poincaré Theorem. A discrete analogue of Lima's Theorem was proved by Bonatti [1], [2] asserting that commuting diffeomorphisms of Σ have a common fixed point, provided they are sufficiently C^1-close to the identity map.

In this paper we will prove a fixed point theorem for discrete nilpotent group actions on the 2-sphere, extending Bonatti's Theorem [1]. This may be considered as a discrete version of Plante's Theorem for S^2.

For a group G, its lower central series is

$$G = G_{(0)} \supset G_{(1)} \supset \cdots \supset G_{(k)} \supset G_{(k+1)} \supset \cdots$$

where $G_{(k+1)} = [G, G_{(k)}]$. The first $k \geq 1$ such that $G_{(k)}$ is trivial is called the *nilpotency length* of G. If such k exists then G is called a *nilpotent group* with nilpotency length k, or shortly, a *k-nilpotent group*.

The group of C^1 diffeomorphisms of S^2 endowed with the C^1 topology is denoted by $\mathrm{Diff}^1(S^2)$. For each subset $\mathcal{H}$ of $\mathrm{Diff}^1(S^2)$, we denote by $\mathrm{Fix}(\mathcal{H})$ the fixed point set of $\mathcal{H}$, i.e.,

$$\mathrm{Fix}(\mathcal{H}) = \big\{ x \in S^2 \,;\, f(x) = x, \ \forall f \in \mathcal{H} \big\}.$$

When $\mathcal{H}$ is a subgroup of $\mathrm{Diff}^1(S^2)$, the fixed points of $\mathcal{H}$ will be called the fixed points of the *natural $\mathcal{H}$-action on S^2* (shortly $\mathcal{H}$-action).

Our main results are as follows:

THEOREM 1.1. — *There exists a decreasing nested sequence $\{\mathcal{V}_k\}_{k \geq 1}$ of open neighborhoods of the identity map, in $\mathrm{Diff}^1(S^2)$, such that: if G is a k-nilpotent subgroup of $\mathrm{Diff}^1(S^2)$ finitely generated by elements in $\mathcal{V}_k$ then the G-action has a fixed point.*

We refer to Section 5 for the precise definition of the open neighborhood $\mathcal{V}_k$, which is independent of the number of generators for the group G.

Theorem 1.1 also holds for a k-nilpotent group G generated by any family Γ of elements in $\mathcal{V}_k$ for the following reasoning (cf. Lima [11]). Since G is a k-nilpotent group then any subgroup of G generated by a finite subset of Γ is also nilpotent with nilpotency length at most k. By Theorem 1.1 the fixed point set $\mathrm{Fix}(\gamma)$ is nonempty for any finite subset γ of Γ. Therefore the family of closed subsets $(\mathrm{Fix}(f))_{f \in \Gamma}$ of S^2 has the "finite intersection property" which implies that $\mathrm{Fix}(\Gamma) \neq \emptyset$. Consequently we get

COROLLARY 1.2. — *Let $\mathcal{V}_k$ be given by Theorem 1.1. If $G \subset \mathrm{Diff}^1(S^2)$ is a k-nilpotent subgroup generated by elements in $\mathcal{V}_k$ then the G-action has a fixed point.*

Theorem 1.1 is just Bonatti's Theorem when $k = 1$.

If the action in Theorem 1.1 has a finite but *nontrivial* orbit, i.e., the orbit contains at least two points, we have the following strengthened theorem.

THEOREM 1.3. — *Let G and $\mathcal{V}_k$ be as in Theorem 1.1. If the G-action has a finite nontrivial orbit then it has at least two fixed points.*

As an immediate corollary we have:

COROLLARY 1.4. — *Let G be as in Theorem 1.1 and let $Z(G)$ be the center of G. If there exists an element $h \in Z(G)$ such that the number of fixed points of h satisfies $2 \leq \#\big(\mathrm{Fix}(h)\big) < \infty$ then the G-action has at least two fixed points.*

The above theorem should be compared with a result of Ghys [6] which asserts that an analytic action of a nilpotent group on S^2 has always a finite orbit.

Using the universal covering $S^2 \to \mathbb{RP}^2$ it is easy to see that Theorems 1.1 and 1.3, and Corollaries 1.2 and 1.4 hold identically replacing S^2 by $\mathbb{RP}^2$.

A typical application of Theorem 1.1 to foliation theory is the following.

THEOREM 1.5. — *Let F be a closed manifold with nilpotent fundamental group. Then every foliation C^1-close to the trivial foliation defined by the fibration $F \to F \times \Sigma^2 \to \Sigma^2$ must have a compact leaf close to some fiber, where $\Sigma^2 = S^2$ or $\mathbb{RP}^2$.*

Bonatti's Theorem was generalized by Handel [8] to commuting homeomorphisms of S^2 which are C^0-close to the identity map. In view of Handel and Plante's Theorems, it is natural to ask the following questions:

(i) Do our theorems hold for nilpotent subgroups of homeomorphisms of S^2?

(ii) Does Plante's Theorem hold for discrete nilpotent group actions on surfaces with higher genus (at least 2)?

In a forthcoming paper [4] we will deal with the latter question, corresponding to Bonatti's work [2].

We conclude the introduction with the following remark.

An elementary result of Epstein and Thurston [5] asserts that finite dimensional connected nilpotent Lie subgroup of $\mathrm{Diff}^1(S^2)$ is metabelian, i.e., its commutator subgroup is abelian. By Ghys [6] any nilpotent subgroup

of the real analytic diffeomorphisms group of S^2 is also metabelian. It is not known if the same result holds for C^1 diffeomorphisms. On the other hand, there are examples (cf. [5]) of connected nilpotent metabelian Lie subgroups of $\mathrm{Diff}^1(\mathbb{R}^2)$ with arbitrarily large nilpotency length.

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2. An algebraic proposition.

In this section we prove an algebraic proposition concerning with the choice of a set of generators for the commutator subgroup of a nilpotent group. First we need the following two lemmas.

LEMMA 2.1. — *Let G be a $(k+1)$-nilpotent group where $k \geq 1$. Then*

1) $[f, h_1 h_2] = [f, h_1][f, h_2]$ *for all $f \in G$ and $h_1, h_2 \in G_{(k-1)}$;*

2) $[f_1 f_2, h] = [f_1, h][f_2, h]$ *for all $f_1, f_2 \in G$ and $h \in G_{(k-1)}$.*

In particular, $[f, h^{-1}] = [f, h]^{-1} = [f^{-1}, h]$ whenever $f \in G$ and $h \in G_{(k-1)}$.

Proof. — Note that $G_{(k)}$ is a subgroup of the center of G since G is $(k+1)$-nilpotent.

Let $f, f_1, f_2 \in G$ and $h, h_1, h_2 \in G_{(k-1)}$. By definition $[f, h_1]$ and $[f, h_2]$ are elements of $G_{(k)}$ and so they are in the center of G. Therefore,

$$[f, h_1 h_2] = f(h_1 h_2) f^{-1} (h_1 h_2)^{-1} = f h_1 f^{-1} h_1^{-1} h_1 f h_2 f^{-1} h_2^{-1} h_1^{-1}$$
$$= [f, h_1] h_1 [f, h_2] h_1^{-1} \quad = [f, h_1][f, h_2].$$

Similarly,

$$[f_1 f_2, h] = (f_1 f_2) h (f_1 f_2)^{-1} h^{-1} = f_1 [f_2, h] h f_1^{-1} h^{-1}$$
$$= f_1 h f_1^{-1} h^{-1} [f_2, h] \quad = [f_1, h][f_2, h].$$

The proof is finished.   □

For each nonempty subset $\mathcal{S}$ of a group G we set

$$\mathcal{S}_{(0)} = \mathcal{S} \quad \text{and} \quad \mathcal{S}_{(i+1)} = \big\{ [a, b] \, ; \, a \in \mathcal{S} \text{ and } b \in \mathcal{S}_{(i)} \big\}, \ i \geq 0.$$

We shall denote by $\langle \mathcal{H} \rangle$ the subgroup generated by the subset $\mathcal{H}$ of a group.

LEMMA 2.2. — *If G is a k-nilpotent group generated by S then we have $G_{(k-1)} = \langle S_{(k-1)} \rangle$.*

Proof. — We prove this lemma by induction on k.

The lemma is trivial for $k = 1$ since in this case we have

$$G_{(0)} = G = \langle S \rangle = \langle S_{(0)} \rangle.$$

Suppose the lemma is true for some $k \geq 1$. We need to prove it for $k + 1$. For this, let G be a $(k + 1)$-nilpotent group and let S be a set of generators for G. Consider the lower central series

$$(2.2.1) \quad G \supsetneqq G_{(1)} \supsetneqq \cdots \supsetneqq G_{(k-1)} \supsetneqq G_{(k)} \supsetneqq G_{(k+1)} = [G, G_{(k)}] = \{e\}.$$

Since $G_{(k)}$ is a subgroup of the center of G, we get

$$\frac{G_{(i)}}{G_{(k)}} = \left(\frac{G}{G_{(k)}}\right)_{(i)} \quad \text{for all} \quad 0 \leq i \leq k.$$

By 2.2.1 we have

$$\frac{G}{G_{(k)}} \supsetneqq \left(\frac{G}{G_{(k)}}\right)_{(1)} \supsetneqq \cdots \supsetneqq \left(\frac{G}{G_{(k)}}\right)_{(k-1)} \supsetneqq \left(\frac{G}{G_{(k)}}\right)_{(k)} = \{e\}.$$

Therefore $G/G_{(k)}$ is a k-nilpotent group generated by the projection $\bar{S}$ of S onto $G/G_{(k)}$ by the quotient map. By the induction assumption we have that

$$\frac{G_{(k-1)}}{G_{(k)}} = \left(\frac{G}{G_{(k)}}\right)_{(k-1)} = \langle \bar{S}_{(k-1)} \rangle$$

where $\bar{S}_{(k-1)}$ denotes the projection of $S_{(k-1)}$ onto $G/G_{(k)}$ by the quotient map. Therefore for each $h \in G_{(k-1)}$ there exist

$$h_1, \ldots, h_n \in S_{(k-1)} \quad \text{and} \quad \epsilon_1, \ldots, \epsilon_n \in \{1, -1\}$$

such that

$$\bar{h} = \left(\bar{h}_1\right)^{\epsilon_1} \cdots \left(\bar{h}_n\right)^{\epsilon_n}.$$

Consequently, there exists a $\xi \in G_{(k)}$ such that

$$(2.2.2) \qquad\qquad h = h_1^{\epsilon_1} \cdots h_n^{\epsilon_n} \, \xi.$$

Note that each $f \in G$ can be written as $f = f_1^{\delta_1} \cdots f_m^{\delta_m}$ where $f_1, \ldots, f_m \in \mathcal{S}$ and $\delta_1, \ldots, \delta_m \in \{1, -1\}$. From $[G, G_{(k)}] = \{e\}$, the identity (2.2.2) and Lemma 2.1 we get that

$$[f, h] = \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} [f_i, h_j]^{\delta_i \epsilon_j} [f_i, \xi]^{\delta_i} = \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} [f_i, h_j]^{\delta_i \epsilon_j}.$$

Since $[f_i, h_j] \in \mathcal{S}_{(k)}$ this proves that $G_{(k)}$ is generated by $\mathcal{S}_{(k)}$ and the proof is finished. $\qquad \square$

PROPOSITION 2.3. — If G is a k-nilpotent group generated by $\mathcal{S}$ then $G_{(1)} = \langle \mathcal{S}_{(1)}, \ldots, \mathcal{S}_{(k)} \rangle$.

Proof. — We use induction on k once again in the proof.

The case $k = 1$ is trivial. The case $k = 2$ follows from Lemma 2.2.

Suppose the lemma is true for some $k \geq 2$. Let G be a $(k+1)$-nilpotent group and let $\mathcal{S}$ be a set of generators for G. By the proof of the last lemma we have

$$\frac{G_{(i)}}{G_{(k)}} = \left(\frac{G}{G_{(k)}} \right)_{(i)} \quad \text{for all} \quad 0 \leq i \leq k$$

and $G/G_{(k)}$ is k-nilpotent.

By the induction assumption we get

$$\frac{G_{(1)}}{G_{(k)}} = \langle \bar{\mathcal{S}}_{(1)}, \ldots, \bar{\mathcal{S}}_{(k-1)} \rangle.$$

Consequently, for each $h \in G_{(1)}$ its image $\bar{h}$ in the quotient group $G_{(1)}/G_{(k)}$ can be written as

$$\bar{h} = \left(\bar{h}_1 \right)^{\epsilon_1} \cdots \left(\bar{h}_n \right)^{\epsilon_n}$$

where

$$h_1, \ldots, h_n \in \bigcup_{i=1}^{k-1} \mathcal{S}_{(i)} \quad \text{and} \quad \epsilon_1, \ldots, \epsilon_n \in \{1, -1\}.$$

It is easy to see that there exists a $\xi \in G_{(k)}$ so that

$$h = h_1^{\epsilon_1} \cdots h_n^{\epsilon_n} \, \xi.$$

On the other hand, by Lemma 2.2 there exist

$$\xi_1, \ldots, \xi_m \in \mathcal{S}_{(k)} \quad \text{and} \quad \delta_1, \ldots, \delta_m \in \{1, -1\}$$

so that

$$\xi = \xi_1^{\delta_1} \cdots \xi_m^{\delta_m}.$$

Therefore $G_{(1)}$ is generated by $\bigcup_{1 \leq i \leq k} \mathcal{S}_{(i)}$ and the proof is finished. $\qquad \square$

3. Invariance and recurrence.

If f, g are commuting diffeomorphisms then $\mathrm{Fix}(g)$ is f-invariant. In this section we show that the fixed point set of the commutator subgroup, $\mathrm{Fix}(G_{(1)})$ with $G \subset \mathrm{Diff}^1(S^2)$, has invariance and recurrence properties even without the assumption of commutativity. These properties play an important role in this paper.

Let $f \in \mathrm{Diff}^1(S^2)$. The *positive semi-orbit* of a point $p \in S^2$ is the set

$$\mathcal{O}_p^+(f) = \{f^i(p) \,;\, i \in \mathbb{N}\}.$$

Its closure will be denoted by $\overline{\mathcal{O}_p^+(f)}$.

We say that $p \in S^2$ is ω-*recurrent* for f if p is the limit of some subsequence of $(f^n(p))_{n \in \mathbb{N}}$.

PROPOSITION 3.1. — *Let $G \subset \mathrm{Diff}^1(S^2)$ be a subgroup and let*

$$f, f_1, \ldots, f_n \in G.$$

Then $\mathrm{Fix}(G_{(1)}, f_1, \ldots, f_n)$ *is f-invariant. Moreover, it contains ω-recurrent points for f lying in* $\overline{\mathcal{O}_p^+(f)}$, *for all* $p \in \mathrm{Fix}(G_{(1)}, f_1, \ldots, f_n)$.

Proof. — The first assertion is an immediate consequence of next lemma. To prove the second assertion let $p \in \mathrm{Fix}(G_{(1)}, f_1, \ldots, f_n)$. Since $\mathrm{Fix}(G_{(1)}, f_1, \ldots, f_n)$ is f-invariant it follows that

$$\overline{\mathcal{O}_p^+(f)} \subset \mathrm{Fix}(G_{(1)}, f_1, \ldots, f_n).$$

By Zorn's Lemma f has a minimal set in $\overline{\mathcal{O}_p^+(f)}$. The points of $\overline{\mathcal{O}_p^+(f)}$ in a minimal set of f are ω-recurrent points for f. This completes the proof. $\square$

LEMMA 3.2. — *Let G be as in Proposition 3.1. Then $\mathrm{Fix}(G_{(i)}, g)$ is f-invariant for all $f \in G$ and $g \in G_{(i-1)}$ where $i \geq 1$.*

Proof. — Let $p \in \mathrm{Fix}(G_{(i)}, g)$. As $p \in \mathrm{Fix}(G_{(i)})$ we have that $p \in \mathrm{Fix}(G_{(i+1)})$. For $h \in G_{(i)}, g \in G_{(i-1)}$ and $f \in G$ we have

$$[f^{-1}, h] \in G_{(i+1)} \quad \text{and} \quad [f^{-1}, g] \in G_{(i)}.$$

Thus

$$p = [f^{-1}, h](p) = f^{-1}hfh^{-1}(p) = f^{-1}hf(p)$$

and

$$p = [f^{-1}, g](p) = f^{-1}gfg^{-1}(p) = f^{-1}gf(p).$$

Therefore

$$hf(p) = f(p) \quad \text{and} \quad gf(p) = f(p)$$

which prove that $\mathrm{Fix}(G_{(i)}, g)$ is f-invariant. $\square$

Remark. — The results of this section hold for any group of homemorphisms of an n-dimensional manifold M where M needs to be compact for the second assertion in Proposition 3.1.

4. A character curve.

In this section we generalize Lemma 4.1 of [1] by proving it without the commutativity hypothesis on f and g (cf. Lemmas 4.3 and 4.5 below). This generalization will be an important tool in the proofs of our theorems.

First, let us fix some notations and definitions.

Let S^2 denote the unit 2-sphere in $\mathbb{R}^3$ endowed with the standard norm, denoted by $\|\cdot\|$. If $\varphi : S^2 \to \mathbb{R}^3$ is a C^1 map, we define

$$\|\varphi\|_1 = \sup_{x \in S^2} \left\{ \|\varphi(x)\| + \sup_{\substack{v \in T_x S^2 \\ \|v\|=1}} \|D\varphi(x) \cdot v\| \right\}.$$

For $a, b \in S^2$ where $a \neq -b$, let $[a, b]$ denote the oriented minimal geodesic segment from a to b, and let $d(a, b)$ denote its length.

For $f \in \mathrm{Diff}^1(S^2)$ without antipodal points and $p \in S^2$ let

$$\gamma_f^p : [0, \mu) \longrightarrow S^2$$

be the curve obtained joining the oriented minimal geodesic segments $[f^i(p), f^{i+1}(p)]$ where $i \in \mathbb{N}$ and $0 < \mu \leq \infty$. Note that $\mu = \infty$ if p is a nonfixed ω-recurrent point for f.

Now let us fix the following neighborhood of the identity map of S^2:

$$\mathcal{V}_1 = \left\{ f \in \mathrm{Diff}^1(S^2) \, ; \, \|f - \mathrm{Id}\|_1 < \tfrac{1}{60} \right\}.$$

We recall the following two results of Bonatti [1] concerning the curve γ_f^p which will be used in the proofs of our results.

PROPOSITION 4.1 (Bonatti). — Let $f \in \mathcal{V}_1$.

1) If $p \in S^2 - \text{Fix}(f)$ then f does not have any fixed point in the open ball $B(p, 4d(p, f(p)))$. In particular, f does not have fixed points along γ_f^p.

2) If $p \in S^2 - \text{Fix}(f)$ is an ω-recurrent point for f then there exists a simple closed curve γ contained in γ_f^p.

We will call the simple closed curve γ obtained in the above proposition the *character curve* of f at p where p is an ω-recurrent point for f.
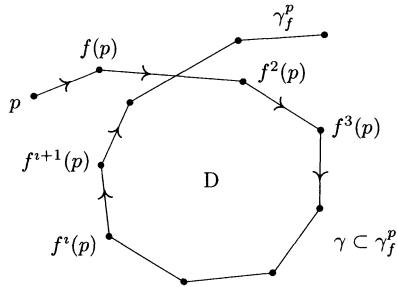
PROPOSITION 4.2 (Bonatti). — Let $f, h_1, \ldots, h_n \in \mathcal{V}_1$ be commuting diffeomorphisms and $n \in \mathbb{Z}^+$. Let $p \in \text{Fix}(h_1, \ldots, h_n) - \text{Fix}(f)$ be an ω-recurrent point for f and $\gamma \subset \gamma_f^p$ its character curve at p. If D is a disk enclosed by γ then $f, h_1, \ldots, h_n$ have a common fixed point in the interior of D.

Let $G \subset \text{Diff}^1(S^2)$ be a subgroup and $f, g, h_1, \ldots, h_n \in \mathcal{V}_1 \cap G$.

Suppose there exist two points $p, q \in S^2$ satisfying

• $p \in \text{Fix}(G_{(1)}, h_1, \ldots, h_n, g) - \text{Fix}(f)$;

• $q \in \text{Fix}(G_{(1)}, h_1, \ldots, h_n, f) - \text{Fix}(g)$.

With these assumptions we have the following three results.

LEMMA 4.3. — The curves γ_f^p and γ_g^q are disjoint.

Proof. — Suppose not. Then there exist $i, j \in \mathbb{N}$ so that

$$[g^i(q), g^{i+1}(q)] \cap [f^j(p), f^{j+1}(p)] \neq \emptyset.$$

By the triangle inequality we have

$$d(g^i(q), f^j(p)) \leq d(g^i(q), g^{i+1}(q)) + d(f^j(p), f^{j+1}(p))$$
$$\leq 2 \max \left(d(g^i(q), g^{i+1}(q)), d(f^j(p), f^{j+1}(p)) \right).$$

Consequently,

• either $f^j(p)$ is in the ball $B\big(g^i(q), 3d(g^i(q), g^{i+1}(q))\big)$, which is impossible by Proposition 3.1 and Proposition 4.1, since $f^j(p)$ is a fixed point of g;

• or $g^i(q)$ is in the ball $B(f^j(p), 3d(f^j(p), f^{j+1}(p)))$, which is impossible by Proposition 3.1 and Proposition 4.1 once again, since $g^i(q)$ is a fixed point of f.

This shows that γ_f^p and γ_g^q are disjoint curves.   □

COROLLARY 4.4. — *Suppose that q is an ω-recurrent point for g and $p \in \mathrm{Int}(D)$, where D is a disk enclosed by the character curve $\gamma \subset \gamma_g^q$. Then $\gamma_f^p \subset \mathrm{Int}(D)$. In particular, $\overline{\mathcal{O}_p^+(f)} \subset \mathrm{Int}(D)$.*

Proof. — By Lemma 4.3 we have that $\gamma_f^p \subset \mathrm{Int}(D)$ since $p \in \mathrm{Int}(D)$ and the curves γ_f^p and γ_g^q are disjoint. On the other hand, since

$$\overline{\mathcal{O}_p^+(f)} \subset \mathrm{Fix}(g)$$

and g does not have fixed points along γ_g^q then we get

$$\overline{\mathcal{O}_p^+(f)} \subset \mathrm{Int}(D).$$

This completes the proof.   □

LEMMA 4.5. — *Suppose there exists an $r > 0$ such that*

$$d\big(f^i(p), f^{i+1}(p)\big) \geq r \quad \text{and} \quad d\big(g^i(q), g^{i+1}(q)\big) \geq r$$

for all $i \in \mathbb{N}$. Then $d\big(\gamma_f^p, \gamma_g^q\big) \geq r$.

Proof. — Suppose not. Then there are two points

$$a \in \big[g^i(q), g^{i+1}(q)\big] \text{ and } b \in \big[f^j(p), f^{j+1}(p))\big] \quad \text{such that } d(a, b) < r.$$

Therefore

$$d\big(g^i(q), f^j(p)\big) \leq d\big(g^i(q), g^{i+1}(q)\big) + r + d\big(f^j(p), f^{j+1}(p)\big)$$
$$\leq 3 \max \big(d\big(g^i(q), g^{i+1}(q)\big), d\big(f^j(p), f^{j+1}(p)\big)\big).$$

Now the same argument used in the proof of Lemma 4.3 applies to conclude the lemma.   □

5. The main lemma.

For each integer $k \geq 1$ let $\mathcal{V}_k$ be the following open neighborhood of the identity map of S^2 in the C^1 topology:

$$\mathcal{V}_k = \left\{ f \in \mathrm{Diff}^1(S^2) \,;\, \|f - \mathrm{Id}\,\|_1 \leq \frac{1}{5^{\frac{1}{2}(k-1)k} \cdot 60} \right\}.$$

It is an elementary exercise to verify that

PROPOSITION 5.1. — *If* $f, g \in \mathcal{V}_1$ *then*

$$\big\|[f, g] - \mathrm{Id}\big\|_1 \leq 5 \max \big\{ \|f - \mathrm{Id}\,\|_1, \|g - \mathrm{Id}\,\|_1 \big\}.$$

Furthermore, if $f_1, \ldots, f_{k+1} \in \mathcal{V}_{k+1}$ *and* $k \geq 1$ *then*

$$[f_1, [f_2, \ldots, [f_i, f_{i+1}] \ldots]] \in \mathcal{V}_k \quad \text{for all} \quad 1 \leq i \leq k.$$

For the sake of simplicity we use f_0 to denote the identity map.

MAIN LEMMA. — *Let* $G \subset \mathrm{Diff}^1(S^2)$ *be a k-nilpotent subgroup finitely generated by elements in* $\mathcal{V}_k$ *and let* $f_1, \ldots, f_n \in \mathcal{V}_k \cap G$. *Let*

$$p \in \mathrm{Fix}(G_{(1)}, f_0, \ldots, f_{n-1}) - \mathrm{Fix}(f_n)$$

be an ω-recurrent point for f_n and $\gamma \subset \gamma_{f_n}^p$ be the character curve of f_n at p. If D *is a disk enclosed by γ then* $G_{(1)}, f_1, \ldots, f_n$ *have a common fixed point in the interior of* D.

Proof. — The proof will be by induction on the nilpotency length of the group G. When $k = 1$, the group G is abelian and the lemma is just Proposition 4.2. Assume now that for some $k \geq 1$ the lemma is true whenever the nilpotency length of G is ℓ, for all $1 \leq \ell \leq k$ and for all $n \in \mathbb{Z}^+$. Suppose that G is a $(k + 1)$-nilpotent group finitely generated by elements in $\mathcal{V}_{k+1}$ as in the lemma. We now proceed by induction on the number n of diffeomorphisms $f_1, \ldots, f_n$.

Let $f_1 \in \mathcal{V}_{k+1} \cap G$ and let $p \in \mathrm{Fix}(G_{(1)}) - \mathrm{Fix}(f_1)$ be an ω-recurrent point for f_1. Let $\gamma \subset \gamma_{f_1}^p$ be the character curve of f_1 and let D be a disk enclosed by γ. Joining Propositions 2.3 and 5.1 we obtain a finite set $\{g_1, \ldots, g_m\} \subset \mathcal{V}_k$ which generates $G_{(1)}$. Let H be the subgroup of G

generated by $G_{(1)}$ and f_1. Note that $H_{(1)} \subset G_{(2)}$. Therefore H is a nilpotent group with nilpotency length at most k and

$$p \in \mathrm{Fix}(H_{(1)}, g_1, \ldots, g_m) - \mathrm{Fix}(f_1).$$

By the induction assumption there exists a point q in the interior of the disk D such that

$$q \in \mathrm{Fix}(H_{(1)}, g_1, \ldots, g_m, f_1) = \mathrm{Fix}(G_{(1)}, f_1).$$

This proves the case $n = 1$.

Assume now that the lemma holds for some $n \geq 1$. In order to prove it for $n + 1$ let $f_1, \ldots, f_{n+1} \in \mathcal{V}_{k+1} \cap G$, and let

$$p \in \mathrm{Fix}(G_{(1)}, f_1, \ldots, f_n) - \mathrm{Fix}(f_{n+1})$$

be an ω-recurrent point for f_{n+1}. Let $\gamma \subset \gamma^p_{f_{n+1}}$ be the character curve of $\gamma^p_{f_{n+1}}$ and let us fix a disk D enclosed by γ.

Suppose by contradiction that $G_{(1)}, f_1, \ldots, f_n, f_{n+1}$ do not have common fixed points in the interior of D. The induction assumption on n asserts that $G_{(1)}, f_0, \ldots, f_{n-1}, f_{n+1}$ have a common fixed point $\tilde{y}_0$ in $\mathrm{Int}(D)$. In view of our contradiction assumption we have that

$$\tilde{y}_0 \in \mathrm{Fix}(G_{(1)}, f_0, \ldots, f_{n-1}, f_{n+1}) - \mathrm{Fix}(f_n).$$

Applying Corollary 4.4 to the diffeomorphisms f_n and f_{n+1} we conclude that $\mathcal{O}^+_{\tilde{y}_0}(f_n) \subset \mathrm{Int}(D)$. According to Proposition 3.1 the map f_n has an ω-recurrent point $y_0 \in \overline{\mathcal{O}^+_{\tilde{y}_0}(f_n)} \subset \mathrm{Fix}(G_{(1)}, f_0, \ldots, f_{n-1}, f_{n+1})$. Observe that $y_0 \in \mathrm{Int}(D)$ is a nonfixed ω-recurrent point for f_n, that is

$$y_0 \in \mathrm{Fix}(G_{(1)}, f_0, \ldots, f_{n-1}, f_{n+1}) - \mathrm{Fix}(f_n).$$

From Proposition 4.1 and Corollary 4.4 we obtain a simple closed curve $\gamma_0 \subset \gamma^{y_0}_{f_n}$ and a closed disk D_0 enclosed by γ_0 so that $D_0 \subset \mathrm{Int}(D)$.
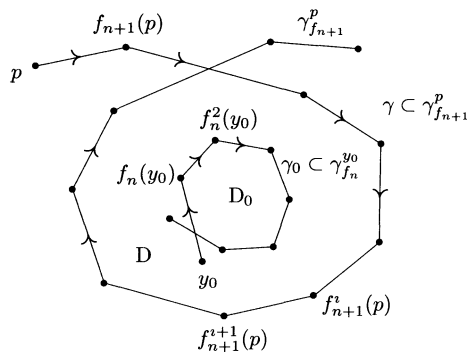
Once again the induction assumption on n asserts that there exists a point $\tilde{x}_1 \in \mathrm{Int}(\mathrm{D}_0)$ such that

$$\tilde{x}_1 \in \mathrm{Fix}(G_{(1)}, f_0, \ldots, f_{n-1}, f_n) - \mathrm{Fix}(f_{n+1}).$$

Now we shall repeat the above argument. Corollary 4.4 applied to the diffeomorphisms f_n and f_{n+1} gives $\overline{\mathcal{O}^+_{\tilde{x}_1}(f_{n+1})} \subset \mathrm{Int}(\mathrm{D}_0)$ and Proposition 3.1 asserts that f_{n+1} has an ω-recurrent point

$$x_1 \in \overline{\mathcal{O}^+_{\tilde{x}_1}(f_{n+1})} \subset \mathrm{Fix}(G_{(1)}, f_0, \ldots, f_{n-1}, f_n).$$

Thereby $x_1 \in \mathrm{Int}(\mathrm{D}_0)$ is an ω-recurrent point for f_{n+1} and

$$x_1 \in \mathrm{Fix}(G_{(1)}, f_0, \ldots, f_{n-1}, f_n) - \mathrm{Fix}(f_{n+1}).$$

Proposition 4.1 and Corollary 4.4 give a simple closed curve $\gamma_1 \subset \gamma^{x_1}_{f_{n+1}}$ and an enclosed disk $D_1 \subset \mathrm{Int}(\mathrm{D}_0)$.

Remember we are assuming that $G_{(1)}, f_1, \ldots, f_n, f_{n+1}$ do not have common fixed points in the interior of D. Consequently, they do not have common fixed points in D since by Proposition 4.1 the diffeomorphism f_{n+1} does not have fixed points along γ.

By the compactness of the sets

$$\mathrm{Fix}(G_{(1)}, f_0, \ldots, f_{n-1}, f_n) \cap \mathrm{D} \quad \text{and} \quad \mathrm{Fix}(G_{(1)}, f_0, \ldots, f_{n-1}, f_{n+1}) \cap \mathrm{D}$$

we get a constant $r > 0$ such that for all

- $x \in \mathrm{Fix}(G_{(1)}, f_0, \ldots, f_{n-1}, f_n) \cap \mathrm{D}$ and
- $y \in \mathrm{Fix}(G_{(1)}, f_0, \ldots, f_{n-1}, f_{n+1}) \cap \mathrm{D}$

the distance d satisfies

(5.1.1) $d\big(x, f_{n+1}(x)\big) \geq r \quad \text{and} \quad d\big(y, f_n(y)\big) \geq r.$

Since $\mathcal{O}^+_{y_0}(f_n)$ and $\mathcal{O}^+_{x_1}(f_{n+1})$ are contained in D we conclude from the inequalities 5.1.1 and Proposition 3.1 that

$$d\big(f^i_{n+1}(x), f^{i+1}_{n+1}(x)\big) \geq r \quad \text{and} \quad d\big(f^i_n(y), f^{i+1}_n(y)\big) \geq r$$

for all $i \in \mathbb{N}$. It follows from Lemma 4.5 that $d(\gamma_1, \gamma_0) \geq r$. Consequently, $\mathrm{Int}(\mathrm{D}_0) - \mathrm{D}_1$ contains a ball of radius $\frac{1}{3}r$.

Iterating this procedure we obtain an infinite decreasing nested sequence of closed disks $(\mathrm{D}_i)_{i \in \mathbb{N}}$ such that $\mathrm{D}_{i+1} \subset \mathrm{Int}(\mathrm{D}_i)$ and $\mathrm{Int}(\mathrm{D}_i) - \mathrm{D}_{i+1}$ contains a ball of radius $\frac{1}{3}r$, contradicting the fact that S^2 has finite diameter. This proves the desired result.   □

6. Proofs of Theorems 1.1 and 1.3.

We are now ready to prove our results addressed in the first section. Let $\{\mathcal{V}_k\}_{k\geq 1}$ be the decreasing nested sequence of C^1 neighborhoods of the identity map of S^2 as defined in Section 5. It is easy to see that Theorem 1.1 follows immediately from the following result.

THEOREM 6.1. — *If $G \subset \mathrm{Diff}^1(S^2)$ is a k-nilpotent subgroup finitely generated by elements in $\mathcal{V}_k$ then*

$$\mathrm{Fix}(G_{(1)}, f_1, \ldots, f_n) \neq \emptyset, \quad \forall f_1, \ldots, f_n \in \mathcal{V}_k.$$

Proof. — We argue by induction on the nilpotency length of the group G and we will follow the same steps as in the proof of the Main Lemma. The case $k = 1$ reduces to Bonatti's Theorem. Assume that the theorem is true for ℓ-nilpotent subgroups with $1 \leq \ell \leq k$. In order to provethe theorem for the nilpotency length of G equal to $k + 1$ we shall use induction on the number n of diffeomorphisms $f_1, \ldots, f_n$.

Suppose G is a $(k + 1)$-nilpotent subgroup finitely generated by elements in $\mathcal{V}_{k+1}$. Let $f_1 \in \mathcal{V}_{k+1} \cap G$. As in the proof of the Main Lemma the subgroup H generated by $G_{(1)}$ and f_1 is nilpotent with length of nilpotency at most k and $G_{(1)}$ has a finite set of generators $\{g_1, \ldots, g_m\} \subset \mathcal{V}_k$. The induction assumption on k implies that

$$\mathrm{Fix}(H_{(1)}, g_1, \ldots, g_m, f_1) = \mathrm{Fix}(G_{(1)}, f_1)$$

is nonempty. This proves the case $n = 1$.

Assume the theorem is true for some $n \geq 1$. Suppose by contradiction that there are $f_1, \ldots, f_n, f_{n+1} \in \mathcal{V}_{k+1} \cap G$ such that

(6.1.1) $$\mathrm{Fix}(G_{(1)}, f_1, \ldots, f_n, f_{n+1}) = \emptyset.$$

From the induction assumption on n it follows that there is a point $\tilde{p} \in S^2$ such that

$$\tilde{p} \in \mathrm{Fix}(G_{(1)}, f_1, \ldots, f_n) - \mathrm{Fix}(f_{n+1}).$$

Proposition 3.1 implies that there is a point

$$p \in \overline{\mathcal{O}_{\tilde{p}}^+(f_{n+1})} \subset \mathrm{Fix}(G_{(1)}, f_1, \ldots, f_n)$$

which is an ω-recurrent point for f_{n+1}. Moreover

$$p \in \mathrm{Fix}(G_{(1)}, f_1, \dots, f_n) - \mathrm{Fix}\,(f_{n+1}).$$

Now fix the character curve $\gamma \subset \gamma^p_{f_{n+1}}$ and a closed disk D enclosed by γ. Applying the Main Lemma we obtain

$$\mathrm{Fix}(G_{(1)}, f_1, \dots, f_n, f_{n+1}) \neq \emptyset.$$

This contradicts the equality (6.1.1). The theorem is proved.   □

Let $p \in S^2$ and $f \in \mathcal{V}_1$ be such that $f(p) \neq p$. Suppose the orbit $\mathcal{O}^+_p(f)$ is finite. In that case, the point p is a nontrivial ω-recurrent point for f and we can consider the character curve $\gamma \subset \gamma^p_f$ given by Proposition 4.1.

In the proof of the next result we are repeating the arguments used in the proofs of Main Lemma and Theorem 1.1.

To prove Theorem 1.3 it suffices to prove the following result.

THEOREM 6.2. — *Let $G \subset \mathrm{Diff}^1(S^2)$ be a k-nilpotent subgroup finitely generated by elements in $\mathcal{V}_k$. Suppose there exists a point $p \in S^2$ with finite nontrivial G-orbit. If $f \in \mathcal{V}_k \cap G$ is such that $f(p) \neq p$ then the G-action has a fixed point in the interior of each disk enclosed by the character curve $\gamma \subset \gamma^p_f$.*

Proof. — The proof will be by induction on k.

For $k = 1$ the group G is commutative. To prove this case let $p \in S^2$, $f \in \mathcal{V}_1 \cap G$ and $\gamma \subset \gamma^p_f$ be as in the theorem. Let $f_1, \dots, f_n \in \mathcal{V}_1$ be a set of generators of G and fix a point $q \in \mathrm{Fix}(G)$ given by Bonatti's Theorem.

Let $\mathrm{D} \subset S^2 - \{q\}$ be the disk enclosed by γ. From Proposition 4.1 we know that there exists a point $\tilde{p}_1 \in \mathrm{Fix}(f) \cap \mathrm{Int}(\mathrm{D})$. If $\tilde{p}_1$ is not a fixed point for G then there exists an integer $0 \leq \lambda < n$ such that

$$\tilde{p}_1 \in \mathrm{Fix}(f_0, \dots, f_\lambda) \cap \mathrm{Int}(\mathrm{D}) \quad \text{and} \quad f_{\lambda+1}(\tilde{p}_1) \neq \tilde{p}_1$$

where f_0 denotes the identity map.

Now, we use the $f_{\lambda+1}$-invariance of $\mathrm{Fix}(f_0, \dots, f_\lambda)$ and Corollary 4.4 to obtain a point $p_1 \in \mathrm{Fix}(f_0, \dots, f_\lambda) \cap \mathrm{Int}(\mathrm{D})$ which is an ω-recurrent point for $f_{\lambda+1}$. If $f_{\lambda+1}(p_1) \neq p_1$, once again, Proposition 4.2 implies that

there exists a point $\tilde{p}_2 \in \mathrm{Fix}(f_0, \ldots, f_\lambda, f_{\lambda+1}) \cap \mathrm{Int}(\mathrm{D}_1)$ where $\mathrm{D}_1 \subset \mathrm{Int}(\mathrm{D})$ is the disk enclosed by the character curve $\gamma_1 \subset \gamma_{f_{\lambda+1}}^{p_1}$. Repeating these arguments no more than n times we get a point $\tilde{q} \in \mathrm{Fix}(f_1, \ldots, f_n) \cap \mathrm{Int}(\mathrm{D})$ proving the theorem for the case $k = 1$.

Now, suppose for some $k \geq 1$ the theorem is true for all $1 \leq \ell \leq k$. We will prove it for $k + 1$.

For this let $p \in S^2$, $f \in \mathcal{V}_{k+1} \cap G$ and $\gamma \subset \gamma_f^p$ be as in the theorem. Let $f_1, \ldots, f_n \in \mathcal{V}_{k+1}$ such that $G = \langle f_1, \ldots, f_n \rangle$ is a $(k+1)$-nilpotent group and fix $q \in \mathrm{Fix}(G)$ given by Theorem 1.1.

Let $H = \langle G_{(1)}, f \rangle$. We know that H is nilpotent with length of nilpotency at most k. It is finitely generated by elements in $\mathcal{V}_k$ and the H-orbit of p is finite and nontrivial. Thus, from the induction assumption on k there exists a point $\tilde{p}_1 \in \mathrm{Fix}(G_{(1)}, f) \cap \mathrm{Int}(\mathrm{D})$ where D is given as above.

Repeating the arguments as in the case $k = 1$ and using Proposition 3.1 and Corollary 4.4, and the Main Lemma we get, after no more then n steps, that there exists a point $\tilde{q} \in \mathrm{Fix}(f_1, \ldots, f_n) \cap \mathrm{Int}(\mathrm{D})$, and the proof is finished. $\qquad\square$

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