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CONVERGENCE OF RIEMANNIAN MANIFOLDS AND LAPLACE OPERATORS. I

by Atsushi KASUE *

Dedicated to Professor Hung-Hsi Wu on his 60th birthday

Introduction.

Riemannian manifolds are considered as metric spaces equipped with Riemannian distances. From this point of view, a set of compact, connected Riemannain manifolds has uniform structure defined by the Gromov-Hausdorff distance, and there are intensive activities around the convergence theory of Riemannian manifolds, which include some works from the viewpoint of spectral geometry and also diffusion processes (cf. e.g., [3], [4], [10], [16], [24]). In [18] and [19], Kumura and the present author indroduced a spectral distance on a set of compact, (weighted) Riemannian manifolds, using heat kernels instead of Riemannian distances, and proved some results on the spectral convergence of Riemannian manifolds. Kumura, Ogura and the present author [20] also investigated another metric topology on a set of pairs of Riemanian metrics and weights on a manifold and discussed the convergence of energy forms. In this paper, we shall continue the study

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for further developments and prove some basic results concerning the Riemannian distances and the energy forms under the spectral convergence of Riemannian manifolds.

0.1. To begin with, we recall the classical notion of Hausdorff distance on the set of closed subsets of a compact metric space. Let $K = (K, d)$ be a compact metric space. The Hausdorff distance of two closed subsets A and B of K is by definition the greatest lower bound of positive numbers ε such that the ε-neighborhoods A_ε and B_ε of A and B respectively include B and A. Then the set of closed subsets of K turns out to be a compact metric space with the Hausdorff distance. We observe that if the Hausdorff distance of A and B is less than $\varepsilon > 0$ and if we write d_A and d_B, respectively, for the restriction of the distance of K to the subspaces A and B, then by sending points $a \in A$ and $b \in B$, respectively, to points $f(a) \in B$ and $h(b) \in A$ in such a way that $d(a, f(a)) < \varepsilon$ and $d(b, h(b)) < \varepsilon$, we can define a pair of maps $f : A \to B$ and $h : B \to A$ satisfying the following properties:

$$|d_A(a,a') - d_B(f(a), f(a'))| < 2\varepsilon \quad (a, a' \in A),$$
$$|d_B(b,b') - d_A(h(b), h(b'))| < 2\varepsilon \quad (b, b' \in B),$$
$$d_A(a, h(f(a))) < 2\varepsilon \quad (a \in A); \ d_B(b, f(h(b))) < 2\varepsilon \quad (b \in B).$$

In general, given two compact metric spaces (A, d_A) and (B, d_B), we call two maps $f : A \to B$ and $h : B \to A$ a pair of ε-Hausdorff approximating maps of A and B if these satisfy the above properties, and we define the Gromov-Hausdorff distance, denoted by $HD(A, B)$, of A and B by the greatest lower bound of positive numbers ε such that there exits a pair of ε-Hausdorff approximating maps of A and B. (Although HD does not exactly satisfy the triangle inequality, it defines the same uniform topology as the original distance due to Gromov [12], and we shall call this the Gromov-Hausdorff distance on the set of isometry classes of compact metric spaces.)

In this paper, we shall consider a compact, connected Riemannian manifold (M, g) as a metric space equipped with the Riemannian distances d_M, unless otherwise stated.

0.2. Let us briefly recall some definitions on regular Dirichlet spaces, refering to the monograph [11], Chap. 1. Let X be a locally compact, separable, Hausdorff space and μ a nonnegative Radon measure on X. A Dirichlet form $\mathcal{E}$ is by definition a nonnegative definite symmetric bilinear form defined on a dense subspace $D[\mathcal{E}]$ in $L^2(X,\mu)$, which is closed, that is $D[\mathcal{E}]$ is complete with respect to the $\mathcal{E}_1$-norm: $\|u\|_{\mathcal{E}_1} = (\mathcal{E}(u,u)+\|u\|_{L^2}^2)^{1/2}$, and further satisfies the (Markovian) property:

$$u \in \mathcal{E},\ v = \min\{\max\{u,o\},1\} \Longrightarrow v \in D[\mathcal{E}],\ \mathcal{E}(v,v) \leqslant \mathcal{E}(u,u).$$

The generator $\mathcal{L}$ of the Dirichlet form $\mathcal{E}$ is the (uniquely determined) positive self-adjoint operator with $(\sqrt{\mathcal{L}}u, \sqrt{\mathcal{L}}v)_{L^2} = \mathcal{E}(u,v)$ and $D[\sqrt{\mathcal{L}}] = D[\mathcal{E}]$. In terms of the generator, we define the strongly continuous semigroup $P_t = e^{-t\mathcal{L}}$ on $L^2(X,\mu)$. The Dirichlet form $\mathcal{E}$ is said to be local if $\mathcal{E}(u,v)=0$ for u and v with disjoint supports. We denote by $C_0(X)$ the space of continuous functions with compact supports, and we call the form regular if $D[\mathcal{E}] \cap C_0(X)$ is dense in $D[\mathcal{E}]$ with respect to the $\mathcal{E}_1$-norm and dense in $C_0(X)$ with respect to the uniform norm.

Note that for our convenience, the measure is not assumed here to be fully supported in the state space X.

The Dirichlet form $\mathcal{E}$ can be written as

$$\mathcal{E}(u,v) = \int_X d\mu_{\langle u,v\rangle},$$

where $\mu_{\langle *,*\rangle}$ is a positive semi-definite, symmetric bilinear form on $D[\mathcal{E}]$ with values in the signed Radon measures on X (the so called energy measure). It can be defined by the formula

$$\int_X \phi\ d\mu_{\langle u,u\rangle} = \mathcal{E}(\phi u, u) - \frac{1}{2}\mathcal{E}(u^2,\phi)$$

for any $u \in D[\mathcal{E}] \cap L^\infty$ and every $\phi \in D[\mathcal{E}] \cap C(X)$ (cf. [11], Chap. 3).

Riemannian manifolds may be viewed as regular Dirichlet spaces with the Riemannian measures and the energy forms. From this point of view, we would like to study convergence of compact, connected Riemannian manifolds.

For a compact, connected Riemannian manifold $M = (M,g)$, we consider the Riemannian measuer μ_M normalized by $\mu_M(M) = 1$, that is,

$$\int_M u(x) d\mu_M(x) = \frac{1}{\mathrm{Vol}(M)} \int_M u(x)\ dv_g(x) \quad (u \in C(M)).$$

A natural energy form on the Hilbert space $L^2(M, \mu_M)$ of square integrable functions is defined by

$$\mathcal{E}_M(u,u) = \int_M |du|_g{}^2(x)\ d\mu_M(x)$$
$$(u \in D[\mathcal{E}_M](= H^1(M,g)) \subset L^2(M,\mu_M)).$$

Let Δ_M, $P_{M;t} = e^{-t\Delta_M}$ and $p_M(t,x,y)$ respectively denote the Laplace operator, the heat semigroup, and the heat kernel of M. We note that

$$\begin{aligned}\mathcal{E}_M(u,u) &= \lim_{t\to 0}\frac{1}{t}(u - P_{M;t}u, u)_{L^2}\\ &= \lim_{t\to 0}\frac{1}{2t}\int\int_{M\times M}(u(x)-u(y))^2 p_M(t,x,y)d\mu_M(x)d\mu_M(y).\end{aligned}$$

0.3. From the point of spectral geometry, Bérard, Besson and Gallot [3], [4] introduced a spectral distance on a set of compact Riemannian manifolds and showed a precompactness theorem as interpretation of several estimates on the heat kernels and the spectra in the presence of a uniform lower bound of the Ricci curvatures and a uniform upper bound of the diameters. Relevantly, Kumura and the author [18], [19] defined another spectral distance on a set of compact Riemannian manifolds and investigated some properties of the distance, which will be explained below in order to illustrate the contents of the present paper.

First we introduce a distance on the set of isometry classes of compact, connected Riemannian manifolds. Let M and N be compact connected Riemannian manifolds. A Borel measurable map $f : M \to N$ is called an ε-spectral approximating map if it satisfies

$$e^{-(t+1/t)}|p_M(t,x,x') - p_N(t,f(x),f(x'))| < \varepsilon, \quad t > 0, x, x' \in M.$$

The spectral distance $SD(M,N)$ of M and N is by definition the greatest lower bound for positive numbers ε such that there exist ε-spectral approximating maps $f : M \to N$ and $h : N \to M$. The spectral distance SD gives a uniform topology on the set of isometry classes of compact connected Riemannian manifolds.

To study convergence of Riemannian manifolds with respect to the spectral distance, we embed given manifolds into a Banach space, using complete orthonormal systems of eigenfunctions of the L^2 spaces. To be precise, let us denote by $C_0([0,\infty],\ell^2)$ the set of continuous curves $\gamma(t)$ ($t \in [0,\infty]$) with values in ℓ^2 such that $\gamma(0) = \gamma(\infty) = 0$. Here ℓ^2 stands for the Hilbert space consisting of square summable sequences. The space $C_0([0,\infty],\ell^2)$ is considered as a metric space with a distance

$$\Theta(\gamma,\sigma) = \sup\{\|\gamma(t)-\sigma(t)\|_{\ell^2} \mid t\in[0,\infty]\}.$$

Let M be a compact, connected Riemannian manifold and $\Phi = \{\phi_i\}$ a complete orthonormal system of eigenfunctions of M. The eigenfunction ϕ_i has the i-th eigenvalue $\lambda_i(M)$ of M. For such a pair $(M, \Phi=\{\phi_i\})$, we define a map of M into $C_0([0,\infty],\ell^2)$ by ·

$$I_\Phi[x](t) = (e^{-(t+1/t)/2}e^{-\lambda_i(M)t/2}\phi_i(x))_{i=0,1,2,\dots} \quad (x\in M).$$

Then I_Φ turns out to be a continuous embedding of M into $C_0([0,\infty],\ell^2)$ and furthermore it follows from its definition that

$$\Theta(I_\Phi[x], I_\Phi[x'])^2 = \sup_{t>0} e^{-(t+1/t)}(p_M(t,x,x)+p_M(t,x',x')-2p_M(t,x,x')).$$

In other words, if we define a distance d_M^{spec} on M by

$$d_M^{\mathrm{spec}}(x,x')^2 = \sup_{t>0} e^{-(t+1/t)}(p_M(t,x,x)+p_M(t,x',x')-2p_M(t,x,x')),$$

then I_Φ is a distance-preserving embedding of the metric space (M, d_M^{spec}) into $C_0([0,\infty],\ell^2)$. Therefore for a family $\mathcal{F}=\{M\}$ of compact, connected Riemannian manifolds, if there exists a compact set K in $C_0([0,\infty],\ell^2)$ such that each $M\in\mathcal{F}$ can be embedded into K, then we see that the family $\mathcal{F}=\{(M, d_M^{\mathrm{spec}})\}$ is precompact as the set of compact subsets of K with respect to the Hausdorff distance. This would suggest the existence of limits of such a family $\mathcal{F}$ with respect to the spectral distance SD.

In this paper, we consider a family $\mathcal{F}=\{M\}$ of compact, connected Riemannian manifolds M, and assume that there exist positive constants ν and C_U such that for any $M\in\mathcal{F}$,

$$[\mathrm{H}_0] \qquad p_M(t,x,x) \leqslant \frac{C_U}{t^{\nu/2}}, \quad t\in(0,1],\ x\in M.$$

It is well known that $[\mathrm{H}_0]$ is equivalent to the condition that the Sobolev inequality holds with some constant $C_S>0$ independent of $M\in\mathcal{F}$, that is for any $M\in\mathcal{F}$,

$$[\mathrm{H}_0]' \qquad \|u\|_{L^{2\nu/\nu-2}} \leqslant C_S(\mathcal{E}_M(u,u)^{1/2}+\|u\|_{L^2}), \quad u\in C^\infty(M)$$

when $\nu>2$; and also the condition that the Nash inequality holds with some constant C_N independent of $M\in\mathcal{F}$, that is for any $M\in\mathcal{F}$,

$$[\mathrm{H}_0]'' \quad \|u\|_{L^2}{}^{1+2/\nu} \leqslant C_N(\mathcal{E}_M(u,u)^{1/2}+\|u\|_{L^2})\|u\|_{L^1}{}^{2/\nu}, \quad u\in C^\infty(M)$$

(see, e.g., [13] and the references therein for these inequalities and related ones). As a consequence of the above inequalities, we get a lower bound for the measure of the geodesic ball $B(x,r)$ around a point x of M with radius r as follows:

$$\mu_M(B(x,r)) \geqslant C_1 r^\nu,$$

where C_1 is a constant depending only on C_U and ν (cf. [1], [7], [19]). This implies that the family $\mathcal{F}$ is precompact with respect to the Gromov-Hausdorff distance HD (cf. [12]). Moreover by deriving certain uniform estimates on the eigenfunctions and eigenvalues of M from the above conditions (cf. Lemma 3.1), we can show that all $M \in \mathcal{F}$ can be embedded into a compact subset K in $C_0([0,\infty],\ell^2)$ by the maps described as above. Based on these observations, Kumura and the present author [19] proved the following

THEOREM 0.1 ([19]).— *Let $\mathcal{F} = \{M\}$ be a family of compact, connected Riemannian manifolds satisfying condition $[H_0]$ with constants $C_U > 0$ and $\nu > 0$. Then the following assertions hold:*

(i) *The family $\mathcal{F}$ is precompact with respect to both the spectral distance SD and the Gromov-Hausdorff distance HD. In the latter case, each $M \in \mathcal{F}$ is considered as a metric space with its Riemannian distance d_M.*

(ii) *Let $\{M_n\}$ be an SD-Cauchy sequence in $\mathcal{F}$. Then there exists a compact, connected Hausdorff space X, a nonnegative Radon measure μ_X on X and a regular Dirichlet form $(\mathcal{E}_X, D[\mathcal{E}_X])$ defined on $L^2(X,\mu_X)$ such that the strongly continuous semigroup $P_{X;t}$ on $L^2(X,\mu_X)$ associated with the Dirichlet form possesses a continuous kernel function $p_X(t,x,x')$ $(t>0, x,x' \in X)$; further there exists a pair of ε_n-spectral approximating maps $f_n : (M_n, p_{M_n}) \to (X, p_X)$ and $h_n : (X,p_X) \to (M_n, p_{M_n})$ with $\lim_{n\to\infty} \varepsilon_n = 0$, and in addition, these maps f_n and h_n are also a pair of ε_n-Hausdorff approximating maps with respect to the distances $d^{\rm spec}_{M_n}$ on M_n and $d^{\rm spec}_X$ on X, where $d^{\rm spec}_X$ is defined by*

$$d^{\rm spec}_X(x,x')^2 = \sup_{t>0} e^{-(t+1/t)}(p_X(t,x,x) + p_X(t,x',x') - 2p_X(t,x,x')).$$

Moreover a sequence of the (image) measures $f_{n*}\mu_{M_n}$ converges to μ_X with respect to the weak topology as $n \to \infty$.*

(iii) *For each $i = 0,1,2,\ldots$, the i-th eigenvalue $\lambda_i(M_n)$ converges to the i-th eigenvalue $\lambda_i(X)$ of the generator $\mathcal{L}_X$ of the Dirichlet form $\mathcal{E}_X$ and* *if u is an eigenfunction of M_n with eigenvalue $\lambda_i(M_n)$ and unit L^2-norm, $\|u\|_{L^2} = 1$, there exists an eigenfunction v of $\mathcal{L}_X$ with eigenvalue $\lambda_i(X)$ and unit L^2-norm, $\|v\|_{L^2} = 1$, such that*

$$\sup_{a\in M_n} |u(a) - v(f_n(a))| < \varepsilon_n(i); \ \sup_{x\in X} |u(h_n(x)) - v(x)| < \varepsilon_n(i),$$

where $\varepsilon_n(i)$ tends to zero as $n \to \infty$. The eigenfunctions of $\mathcal{L}_X$ are all continuous on X.

The limit Dirichlet space $(X, \mu_X, \mathcal{E}_X)$ in this theorem can be obtained as follows: For a complete orthonormal system $\Phi = \{\phi_i\}$ of eigenfunctions on $M \in \mathcal{F}$, the image $I_\Phi[M]$ lies in a compact subset K of $C_0([0,\infty], \ell^2)$. Then given any sequence $\{M_n\} \subset \mathcal{F}$, taking such a system $\Phi_n = \{\phi_i^{(n)}\}$ of M_n for each n and choosing a subsequence of $\{M_n\}$, denoted by the same letter, we may assume that as $n \to \infty$, $I_{\Phi_n}[M_n]$ converges to a compact subset X in K via a pair of ε_n-Hausdorff approximating maps $\bar{f}_n : I_{\Phi_n}[M_n] \to X$ and $\bar{h}_n : X \to I_{\Phi_n}[M_n]$ with $\lim_{n\to\infty} \varepsilon_n = 0$. We put $f_n = I_{\Phi_n} \circ \bar{f}_n$ and $h_n = {I_{\Phi_n}}^{-1} \circ \bar{h}_n$. Then we may further assume that the image measure $f_{n*}\mu_{M_n}$ weakly converges to a nonnegative Radon measure μ_X on X. Each element $x \in X$ can be expressed as $x(t) = (e^{-(t+1/t)/2} e^{-\lambda_i t/2} \phi_i(x))_{i=0,1,2,\ldots} \in \ell^2$ $(0 \leqslant t \leqslant +\infty)$ for some sequence of nonnegative numbers λ_i and some sequence of continuous functions ϕ_i on X. Define a continuous function $p_X(t,x,y)$ on $(0,\infty) \times X \times X$ by $p_X(t,x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x)\phi_i(y)$. Then p_X turns out to be the kernel function of a strongly continuous semigroup $P_{X;t}$ on $L^2(X, \mu_X)$, which is associated with a regular Dirichlet form $\mathcal{E}_X$ on $L^2(X, \mu_X)$. The set of functions $\{\phi_i\}$ is a complete orthonormal system of eigenfunctions of the generator $\mathcal{L}_X$ of $\mathcal{E}_X$ with eigenvalues $\{\lambda_i\}$. In this way, we obtain a regular Dirichlet form $(X, \mu_X, \mathcal{E}_X)$, to which M_n converges as $n \to \infty$ with respect to the spectral distance via a pair of the approximating maps f_n and h_n. Note that we may obtain another space $(X', \mu_{X'}, \mathcal{E}_{X'})$ by taking a different choice of a complete orthonormal system Φ'_n of eigenfunctions on M_n, but $(X', \mu_{X'}, \mathcal{E}_{X'})$ can be identified with $(X, \mu_X, \mathcal{E}_X)$ in the sense that there exists a homeomorphism $\eta : X \to X'$ which preserves the measures and the kernel functions.

In Theorem 0.1, the support X_0 of the measure μ_X may be disconnected in general, X_0 may not coincide with the whole space X (although X_0 is connected), and $\mathcal{E}_X$ may be nonlocal (see Example 2.1 in Section 2). Although the complement $X \setminus X_0$ does not play any role in the Dirichlet space $(X, \mu_X, \mathcal{E}_X)$, this part is nevertheless valid in the topology of the spectral distance.

0.4. Let us now mention the main results of the present paper. Let $\{M_n\}$ be as in Theorem 0.1. Then according to the first assertion, $\{M_n\}$ contains HD-Cauchy sequences which converge to compact length spaces. In Section 1, we shall prove the following

THEOREM 0.2. — *Let $f_n : M_n \to X$, $h_n : X \to M_n$, $\mathcal{E}_X$, and $p_X(t, x, x')$ be as in Theorem 0.1. Then the following assertions hold:*

(i) *There exist a subsequence $\{M_m\}$, a sequence of positive numbers $\{\varepsilon_m\}$ tending to zero as $m \to \infty$, and a continuous pseudo-distance δ on X such that the maps $f_m : M_m \to X$ and $h_m : X \to M_m$ are a pair of ε_m-Hausdorff approximating maps of (M_m, d_{M_m}) and (X, δ).*

(ii) *The kernel p_X has the following off-diagonal upper bound:*

$$p_X(t, x, x') \leqslant \frac{C_U(\alpha)}{t^{\nu/2}} \exp\left(-\frac{\delta(x, x')^2}{(4+\alpha)t}\right), \quad t \in (0, 1], \ x, x' \in X,$$

where α is any positive constant and $C_U(\alpha)$ is a positive constant depending only on C_U and α.

(iii) *Let δ be one of the continuous pseudo-distances on X obtained in the first assertion and let $C^{0,1}(X, \delta)$ be the space of functions on X which are Lipschitz continuous with respect to δ. Then $C^{0,1}(X, \delta) \subset D[\mathcal{E}_X] \cap C(X)$ and for $u \in C^{0,1}(X, \delta)$ and $v \in D[\mathcal{E}_X]$, $\mathcal{E}_X(u, v) = 0$ if the support of u does not intersect that of v. Moreover the energy measure of $u \in C^{0,1}(X, \delta)$ is absolutely continuous with respect to the measure μ_X and the Radon-Nikodym derivative $\Gamma(u, u) = d\mu_{\langle u,u \rangle}/d\mu_X$ is bounded from above by the square of the local dilatation of u,*

$$\Gamma(u, u)(x) \leqslant \mathrm{dil}_\delta u(x)^2, \quad a.a.\ x \in X. \tag{1}$$

Let us recall the definition of the local dilatation of a Lipschitz function in this theorem. Given a Lipschitz function u on a subspace A of (X, δ), the dilatation of u on A, that is the infimal number λ satisfying $|u(x) - u(y)| \leqslant \lambda \delta(x, y)$ for all $x, y \in A$ is denoted by $\mathbf{dil}_\delta(u)$, and for a Lipschitz function u on X, the local dilatation of u at a point x is the number

$$\mathrm{dil}_\delta u(x) = \lim_{r \to 0} \mathbf{dil}_\delta(u_{|B_\delta(x,r)}),$$

where $B_\delta(x,r)$ stands for the metric ball around x with radius r with respect to the pseudo-distance δ, $B_\delta(x,r) = \{y \in X \mid \delta(x,y) < r\}$.

Let us now denote by $\mathcal{A}[\mathcal{E}_X]$ (resp., $\Gamma(u,u)$) the subspace of $D[\mathcal{E}_X]$ which consists of functions whose energy measures are absolutely continuous with respect to μ_X (resp., the density $d\mu_{\langle u,u\rangle}/d\mu_X$ of the energy measure of a funcion $u \in \mathcal{A}[\mathcal{E}_X]$). The energy measure defines in an intrinsic way a pseudo-metric $\rho_{\mathcal{E}_X} : X \times X \to [0,+\infty]$, so called Carathéodory metric (cf. e.g., [6], [27], [28], [29]), by

$$\rho_{\mathcal{E}_X}(x,y) = \sup\{u(x) - u(y) \mid u \in \mathcal{A}[\mathcal{E}_X] \cap C(X),\ \Gamma(u,u) \leqslant 1\}.$$

Then as a result of (1), we have

$$(0 \leqslant)\ \delta(x,y) \leqslant \rho_{\mathcal{E}_X}(x,y)\ (\leqslant +\infty), \quad x,y \in X. \tag{2}$$

This holds for all pseudo-distances δ obtained in the first assertion of Theorem 0.2. In general, we can expect neither the equality in (1) nor (2) even in case δ becomes a distance on X and induces the topology of X (cf. Examples 2.4 and 2.6).

0.5. We shall now consider, instead of condition [H_0], the following stronger conditions on a given family $\mathcal{F} = \{M\}$ of compact, connected Riemannian manifolds: There exist positive constants C_D, C_P and C_B such that

$$[H_1] \qquad \mu_M(B(x,2r)) \leqslant C_D \mu_M(B(x,r));$$

$$[H_2] \qquad \int_{B(x,r)} |u - u_{x,r}|^2 d\mu_M \leqslant C_P r^2 \int_{B(x,2r)} |du|^2 d\mu_M,$$

where $u_{x,r} = \mu_M(B(x,r))^{-1} \int_{B(x,r)} u d\mu_M$;

$$[H_3] \qquad \mu_M(B(x,1)) \geqslant C_B$$

for all $r \in (0,1]$, $x \in M$, $u \in C^\infty(M)$ and $M \in \mathcal{F}$. According to Saloff-Coste [26], these conditions ensure that the family $\mathcal{F}$ satisfies condition [H_0] with constants $\nu = \max\{\log_2 C_D, 3\}$ and C_U depending only on C_D, C_P and C_B, and further that *a priori* estimates on Hölder continuity of the eigenfunctions and the heat kernels hold, which implies

$$e^{-(t+1/t)}|p_M(t,x,x) - p_M(t,x,y)| \leqslant C_2 d_M(x,y)^\alpha, \qquad t > 0,\ x,y \in M,\ M \in \mathcal{F},$$

where the exponent α (resp., the constant C_2) depends only on C_D and C_P (resp., C_D, C_P and C_B). Hence we have

$$d_M^{\text{spec}}(x,y) \leqslant (2C_2)^{1/2} d_M(x,y)^{\alpha/2}, \quad x,y \in M,\ M \in \mathcal{F}.$$

This estimate continues to hold on an SD-limit space X of $\mathcal{F}$ and the pseudo-distances δ on X, and thus we have

$$d_X^{\text{spec}}(x,y) \leqslant (2C_2)^{1/2} \delta(x,y)^{\alpha/2}, \quad x,y \in X.$$

This shows that any pseudo-distance δ obtained in Theorem 0.2 (i) becomes a distance on X and induces the topology of X. We also note that each eigenfunction of X belongs to a class of Hölder continuous functions of the exponent α with respect to the distance δ. The distances δ indeed belong to the same Lipschitz equivalence class as the intrinsic metric $\rho_{\mathcal{E}_X}$ on X, as is shown in the following

THEOREM 0.3. — *Let $\{M_n\}$ be an SD-Cauchy sequence of compact, connected Riemannian manifolds satisfying conditions* [H$_1$], [H$_2$] *and* [H$_3$] *with positive constants C_D, C_P and C_B, respectively. Let X and δ be respectively the Dirichlet space and a distance on X as in Theorem 0.2.*

(i) *The limit space X also satisfies the same conditions as above, namely,*

[H$_1$] $$\mu_X(B_\delta(x,2r)) \leqslant C_D \mu_X(B_\delta(x,r));$$

[H$_2$]$'$ $$\int_{B_\delta(x,r)} |u - u_{x,r}|^2 d\mu_X \leqslant C_P r^2 \int_{B_\delta(x,2r)} d\mu_{\langle u,u\rangle},$$

where $u_{x,r} = \mu_X(B_\delta(x,r))^{-1} \int_{B_\delta(x,r)} u \, d\mu_X$;

[H$_3$] $$\mu_X(B_\delta(x,1)) \geqslant C_B$$

for all $r \in (0,1]$, $x \in X$ *and* $u \in D[\mathcal{E}_X]$.

(ii) *There exists a constant $\Lambda \geqslant 1$, depending only on C_D and C_P, such that*

$$\rho_{\mathcal{E}_X}(x,y) \leqslant \Lambda \delta(x,y), \quad x,y \in X$$

and for any function $u \in C^{0,1}(X,\delta)$,

$$\operatorname{dil}_\delta u(x)^2 \leqslant \Lambda \Gamma(u,u)(x), \quad a.a.\ x \in X.$$

In view of (1) and [H$_2$]$'$ for Lipschitz functions, we can apply a result by Cheeger [8] to the limit metric measure space (X, μ_X, δ), and conclude that a finite dimensional L^∞ vector bundle T^*X on X can be constructed, Lipschitz functions u define L^∞ sections du of this bundle, and then the energy densities $\Gamma(u,u)$ yield an L^∞ Riemannian structure on T^*X as follows:

$$\langle du, dv\rangle_\Gamma = \frac{1}{4}\{\Gamma(u+v,u+v) - \Gamma(u-v,u-v)\}.$$

This is in general not true for the squares of the local dilatations of Lipschitz functions.

In this paper, we confine ourselves to a family of compact, connected Riemannian manifolds. However all of the results stated so far can be extended to the case of a family of regular Dirichlet spaces satisfying certain properties (see Remark 3.6), which were studied in a series of papers by K.T. Sturm [27], [28], [29]. Such families include, for instance, compact connected manifolds endowed with Riemannian metrics and smooth probability measures.

In fact, such weighted Riemannian manifolds may be taken as the spectral limits of Riemannian manifolds. To be precise, let g and w be respectively a Riemannian metric and a smooth positive function on a compact, connected manifold M, and assume that $\int_M w\,d\mu_g = 1$. Consider the warped product metrics g_ε ($\varepsilon > 0$) on the product space $M \times S^1$ of M and a unit circle $S^1 = \{e^{\sqrt{-1}x} \mid x \in \mathbb{R}\}$, defined by $g_\varepsilon = g + \varepsilon w^2 dx^2$. Then as $\varepsilon \to 0$, $(M \times S^1, g_\varepsilon)$ converges to the weighted Riemannian manifold $(M, \mu_w = w\,\mu_g, \mathcal{E}_{g,w})$ with respect to the spectral distance, where the energy form $\mathcal{E}_{g,w}$ is given by $\mathcal{E}_{g,w}(u,u) = \int_M |du|_g^2\, d\mu_w \quad (u \in C^\infty(M))$. A pair of a sub-Riemannian metric and a smooth probability measure also sits on the boundary of Riemannian manifolds with respect to the spectral distance (cf. Example 2.5).

So far as Theorem 0.3 is concerned, the results can be generalized to a family of complete, noncompact, pointed Dirichlet spaces having certain properties: The convergence of such a family will be the subject of the second part of the present paper [22], in which the convergence of harmonic functions and harmonic maps into nonpositively curved manifolds will be also discussed.

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1. Spectral convergence of Riemannian manifolds and Riemannian distances.

In this section, we shall prove the first two assertions of Theorem 0.2 and a proposition as a result of them.

To begin with, we shall recall a basic estimate $[\mathrm{H}_{0,\alpha}]$ below, which is due to Davies [9]: Given a family $\mathcal{F}$ of compact, connected Riemannian manifolds satisfying condition $[\mathrm{H}_0]$ with constants ν and C_U, the heat kernel p_M of a compact Riemannian manifold $M \in \mathcal{F}$ satisfies

$$[\mathrm{H}_{0,\alpha}] \;\; p_M(t,x,x') \leqslant \frac{C_U(\alpha)}{t^{\nu/2}} \exp\left(-\frac{d_M(x,x')^2}{(4+\alpha)t}\right), \quad t \in (0,1],\ x,x' \in M,$$

where α is any positive constant and $C_U(\alpha)$ is a positive constant dependeng only on C_U and α (see also [19], Theorem 2.2 (2.6) and the references therein).

LEMMA 1.1. — *There exists a continuous, increasing function θ on $[0,\infty)$, satisfying $\theta(0) = 0$ and depending only on ν and C_U, such that given $M \in \mathcal{F}$, the Riemannian distance d_M and the distance d_M^{spec} satisfy*

$$d_M(x,y) \leqslant \theta(d_M^{\mathrm{spec}}(x,y)),$$

$$|d_M(x,y) - d_M(x',y')| \leqslant \theta(d_M^{\mathrm{spec}}(x,x')) + \theta(d_M^{\mathrm{spec}}(y,y'))$$

for all $x,x',y,y' \in M$.

Proof. — The first inequality is an easy consequence of $[\mathrm{H}_{0,\alpha}]$. Indeed, noting that $p_M(t,x,x) \geqslant 1$ for all $t > 0$ and $x \in M$, we have

$$\begin{aligned} 2 &\leqslant p_M(t,x,x) + p_M(t,x',x') \\ &\leqslant 2p_M(t,x,x') + e^{(t+1/t)} d_M^{\mathrm{spec}}(x,x')^2 \\ &\leqslant 2\frac{C_U(1)}{t^{\nu/2}} \exp\left(-\frac{d_M(x,x')^2}{5t}\right) + e^{(t+1/t)} d_M^{\mathrm{spec}}(x,x')^2 \end{aligned}$$

and hence if $e^{(t+1/t)} d_M^{\mathrm{spec}}(x,x')^2 \leqslant 1$ for some $t \in (0,1]$, then

$$d_M(x,x')^2 \leqslant 5t\left(\log 2C_U(1) - \frac{\nu}{2}\log t\right).$$

This shows the first inequality of the lemma. The second one follows from the triangle inequality. Indeed, we have

$$\begin{aligned}|d_M(x,y) - d_M(x',y')| &\leqslant |d_M(x,y) - d_M(y,x')| + |d_M(y,x') - d_M(x',y')| \\ &\leqslant d_M(x,x') + d_M(y,y') \\ &\leqslant \theta(d_M^{\mathrm{spec}}(x,x')) + \theta(d_M^{\mathrm{spec}}(y,y')).\end{aligned}$$

This completes the proof of Lemma 1.1.

Proof of Theorem 0.2 (i) and (ii). — Let $\{M_n\}$ be an SD-Cauchy sequence in $\mathcal{F}$ which converges to $X \in \overline{\mathcal{F}}$. Let $f_n : M_n \to X$ and $h_n : X \to M_n$ be ε_n-spectral approximating maps between M_n and X with $\lim_{n\to\infty} \varepsilon_n = 0$, which are also a pair of ε_n-Hausdorff approximating maps between $(M_n, d_M^{\mathrm{spec}})$ and (X, d_X^{spec}). We define a sequence $\{\delta_n\}$ of Borel measurable functions on $X \times X$ by $\delta_n(x,x') = d_{M_n}(h_n(x), h_n(x'))$. Note that

$$\delta_n(x,x') \leqslant \theta(d_X^{\mathrm{spec}}(x,x') + \varepsilon_n) \tag{3}$$

$$|\delta_n(x,y) - \delta_n(x',y')| \leqslant \theta(d_X^{\mathrm{spec}}(x,x') + \varepsilon_n) + \theta(d_X^{\mathrm{spec}}(y,y') + \varepsilon_n) \tag{4}$$

for all $x, x', y, y' \in X$. By choosing an increasing family $\{A_k\}$ of finite subsets of X whose union $A_\infty = \cup A_k$ is dense in X, and then passing to a subsequence, we may assume that δ_n converges pointwise to a function δ on a dense subset $A_\infty \times A_\infty$ of $X \times X$. Obviously δ is nonnegative and satisfies the triangle inequality. Moreover letting n tend to infinity in (3) and (4), we have

$$\delta(x,x') \leqslant \theta(d_X^{\mathrm{spec}}(x,x')) \tag{5}$$

$$|\delta(x,y) - \delta(x',y')| \leqslant \theta(d_X^{\mathrm{spec}}(x,x')) + \theta(d_X^{\mathrm{spec}}(y,y')) \tag{6}$$

for all $x, x', y, y' \in A_\infty$. The latter shows that δ is uniformly continuous on the subspace $A_\infty \times A_\infty$. Hence δ extends uniquely to a continuous function on $X \times X$, which is also denoted by the same letter δ, and inequalities (5) and (6) hold everywhere on X.

Now we claim that δ_n uniformly converges to δ on $X \times X$. Indeed, given $\varepsilon > 0$, we choose $r > 0$ and N so that if $d_X^{\mathrm{spec}}(x,x') \leqslant r$, then $\delta(x,x') < \varepsilon$ and in addition if $n \geqslant N$, then $\delta_n(x,x') < \varepsilon$. For any $x, y \in X$, we take $x', y' \in A_\infty$ in such a way that $d_X^{\mathrm{spec}}(x,x') + d_X^{\mathrm{spec}}(y,y') \leqslant r$, and assume $n \geqslant N$. Then

$$\begin{aligned}
|\delta_n(x,y) - \delta(x,y)| & \\
&\leqslant |\delta_n(x,y) - \delta_n(x',y')| + |\delta_n(x',y') - \delta(x',y')| + |\delta(x',y') - \delta(x,y)| \\
&\leqslant \delta_n(x,x') + \delta_n(y,y') + |\delta_n(x',y') - \delta(x',y')| + \delta(x,x') + \delta(y,y') \\
&\leqslant 4\varepsilon + |\delta_n(x',y') - \delta(x',y')|.
\end{aligned}$$

Hence choosing N' so large that $|\delta_n(x',y') - \delta(x',y')| < \varepsilon$ for any $n \geqslant N'$, we have

$$|\delta_n(x,y) - \delta(x,y)| < 5\varepsilon$$

for any $n \geqslant N'$. This shows that as $n \to \infty$, δ_n uniformly converges to δ on $X \times X$, that is, for some ε'_n going to zero as $n \to \infty$,

$$|d_{M_n}(h_n(x), h_n(x')) - \delta(x,x')| \leqslant \varepsilon'_n, \quad x,\ x' \in X.$$

In addition, we have

$$\begin{aligned}
d_{M_n}(a, h_n(f_n(a))) \leqslant \theta(d^{\mathrm{spec}}_{M_n}(a, h_n(f_n(a))) \leqslant \theta(\varepsilon_n) \ (a \in M_n), \\
\delta(x, f_n(h_n(x))) \leqslant \theta(d^{\mathrm{spec}}_X(x, f_n(h_n(x))) \leqslant \theta(\varepsilon_n) \ (x \in X)
\end{aligned}$$

and further, for some ε''_n going to zero as $n \to \infty$,

$$|d_{M_n}(a,b) - \delta(f_n(a), f_n(b))| \leqslant \varepsilon''_n, \quad a, b \in M_n,$$

because

$$\begin{aligned}
|d_{M_n}(a,b) - \delta(f_n(a), f_n(b))| & \\
&\leqslant |d_{M_n}(a,b) - d_{M_n}(h_n(f_n(a)), h_n(f_n(b)))| \\
&\quad + |d_{M_n}(h_n(f_n(a)), h_n(f_n(b))) - \delta(f_n(a), f_n(b))| \\
&\leqslant d_{M_n}(a, h_n(f_n(a))) + d_{M_n}(b, h_n(f_n(b))) + \varepsilon'_n \\
&\leqslant 2\theta(\varepsilon_n) + \varepsilon'_n.
\end{aligned}$$

Hence f_n and h_n are a pair of $\widehat{\varepsilon}_n$-Hausdorff approximating maps with $\lim_{n\to\infty} \widehat{\varepsilon}_n = 0$ between (M_n, d_{M_n}) and (X, δ). Thus we have shown the first assertion of Theorem 0.2, which together with $[\mathrm{H}_{0,\alpha}]$, obviously implies the second one.

Let X be an SD-limit of a family $\mathcal{F}$ of compact, connected Riemannian manifolds satisfying $[\mathrm{H}_0]$. Let $\{M_n\}$, μ_X, $\mathcal{E}_X$, p_X and δ be as in Theorem 0.2. We denote the support of μ_X by X_0 and note that $\delta(x, X_0) = 0$ for all $x \in X$. Indeed, let $B_\delta(x,r)$ be a pseudo-ball around a point $x \in X$

with radius r with respect to δ, $B_\delta(x,r) := \{y \in X \mid \delta(y,x) < r\}$. Then it is easy to see that

$$\limsup_{n\to\infty} \mu_{M_n}(B_{M_n}(h_n(x), r-\varepsilon)) \leqslant \mu_X(B_\delta(x,r)) \leqslant \liminf_{n\to\infty} \mu_{M_n}(B_{M_n}(h_n(x), r+\varepsilon))$$

for any $\varepsilon > 0$. Therefore we have

$$\mu_X(B_\delta(x,r)) \geqslant C_3 r^\nu, \quad x \in X,\ r \leqslant 1,$$

where C_3 is a constant depending only on ν and C_U. This implies in particular that $B_\delta(x,r) \cap X_0 \neq \varnothing$ for all $x \in X$ and $r > 0$, and hence $\delta(x, X_0) = 0$ for all $x \in X$.

We also note that it may occur that δ is trivial, $\delta = 0$, namely, the quotient metric space X_δ obtained by the equivalence relation $\sim_\delta$ on X, $x \sim_\delta y \Leftrightarrow \delta(x,y) = 0$, reduces to a single point, (although X_0 is isomorphic to a smooth Riemannian manifold as a Dirichlet space). See Example 2.2.

The kernel function $p_X(t,x,y)$ in Theorem 0.2 is continuous, so that it defines a semigroup on the Banach space $C(X)$, which is denoted by the same letter $P_{X;t}$ as the semigroup on $L^2(X, \mu_X)$. Since, for any $u \in C(X)$ and $\varepsilon > 0$, we have

$$\begin{aligned} |u(x) - P_{X;t}(x)| &= \left| \int_X p_X(t,x,y)(u(x) - u(y))\, d\mu_X(y) \right| \\ &\leqslant \int_{\delta(x,y)\geqslant\varepsilon} |u(x) - u(y)| \frac{C_U(1)}{t^{\nu/2}} \exp\left(-\frac{\varepsilon^2}{5t}\right) d\mu_X \\ &\qquad + \int_{\delta(x,y)\leqslant\varepsilon} |u(x) - u(y)| p_X(t,x,y)\, d\mu_X, \end{aligned}$$

it is easy to verify the following

Proposition 1.2. — *Let X, $P_{X;t}$ and δ be as in Theorem 0.2. For a function $u \in C^{0,1}(X,\delta)$, $P_{X;t}u$ uniformly converges to u as $t \to 0$. Moreover $P_{X;t}$ defines a strongly continuous semigroup on $C(X)$, that is $P_{X;t}u$ uniformly converges to u as $t \to 0$, for any $u \in C(X)$, provided that δ is a distance on X.*

2. Examples.

In this section, we shall exhibit some elementary examples of spectral convergent sequences of compact (weighted) Riemannian manifolds.

Example 2.1 (cf. [20], Section 7). — Let $\mathcal{F}$ be a family of Riemannian metrics on the product of unit circles $S^1 \times S^1 = \{(e^{\sqrt{-1}x}, e^{\sqrt{-1}y}) |\ x, y \in \mathbb{R}\}$ such that

$$g_F = dx^2 + F(x)^2 dy^2, \quad 0 < F \in C^\infty(S^1).$$

We take a finite number of points $\{p_i = e^{\sqrt{-1}x_i} | 0 < x_1 < x_2 < \cdots < x_k < 2\pi\}$ on S^1 and $2k$ intervals $I_i^- = [x_i - a_i, x_i]$, $I_i^+ = [x_i, x_i + a_i]$ with $0 < a_i < (1/2)\min\{x_{i+1} - x_i |\ i = 0, 1, \ldots, k\}$, where we set $x_0 = 0$ and $x_{k+1} = 2\pi$. We now assume that all $g_F \in \mathcal{F}$ satisfy the following conditions:

$$F'(x) \leqslant 0 \quad \text{for } x \in I_i^-, \ F'(x) \geqslant 0 \quad \text{for } x \in I_i^+,$$
$$b_i \left| \int_{x_i}^{x} F(t)dt \right|^{1/c_i} \leqslant F(x) \quad \text{for } x \in I_i^- \cup I_i^+,$$
$$d_i^- \leqslant F(x) \leqslant d_i^+ \quad \text{for } x \in S^1 \setminus \cup_{i=1}^k I_i^+ \cup I_i^-,$$

where b_i, c_i, d_i^+ and d_i^- are positive constants with $c_i > 1$ $(i = 1, 2, \ldots, k)$. Then the family $\mathcal{F}$ satisfies condition $[\mathrm{H}_0]$ with constants ν and C_U depending only on the given a_i, b_i, c_i, d_i^+ and d_i^- $(i = 1, \ldots, k)$. (Indeed, we see that $\nu = \max\{2, c_1/(c_1 - 1), \ldots, c_k/(c_k - 1)\}$.)

Let $g_n = g_{F_n}$ be an SD-Cauchy sequence in $\mathcal{F}$ which converges to a regular Dirichlet space $(X, \mu_X, \mathcal{E}_X)$, and suppose that as $n \to \infty$, F_n uniformly converges to a continuous function F on S^1 satisfying: $F(x) > 0$ for $x \neq x_i$, $F(x_i) = 0$ and $F(x) \leqslant e_i F_n(x)$ for $x \in I_i^- \cup I_i^+$ $(i = 1, \ldots, k)$, where e_i are some positive constants.

Under this situation, we shall describe the Dirichlet space $(X_0, \mathcal{E}_X)$ in three cases, where $X_0 = \operatorname{supp} \mu_X$.

(i) In the case where

$$\int_{x_i - a_i}^{x_i} \frac{1}{F(x)} dx = \int_{x_i}^{x_i + a_i} \frac{1}{F(x)} dx = +\infty, \quad i = 1, \ldots, k,$$

X_0 consists of k connected components $X_{0;i}$ $(i = 1, \ldots, k)$. Each $(X_{0;i}, \mathcal{E}_{X|X_{0;i}})$ can be identified with the singular Riemannian manifolds

$([x_i, x_{i+1}] \times S^1, g_F)$ equipped with the energy forms $\mathcal{E}_{g_F}$, where two circles $\{x_i\} \times S^1$ and $\{x_{i+1}\} \times S^1$ respectively reduce to points z_i^- and z_i^+, and the form $\mathcal{E}_{g_F}$ is the smallest closed extension of the energy form on $C_0^\infty((x_i, x_{i+1}) \times S^1)$ with respect to the metric g_F, so that it may be viewed as a 2-sphere with metric singular at z_i^- and z_i^+. The complement $X \setminus X_0$ of X_0 consists of k connected open subsets Σ_i $(i = 1, \dots, k)$ and each Σ_i joins $X_{0;i-1}$ to $X_{0;i}$ in such a way that $\overline{\Sigma_i} \cap X_0 = \{z_{i-1}^+, z_i^-\}$.

(ii) In the case where

$$\int_{x_i - a_i}^{x_i} \frac{1}{F(x)} dx < +\infty; \int_{x_i}^{x_i + a_i} \frac{1}{F(x)} dx < +\infty, \quad i = 1, \dots, k,$$

X_0 is connected and $(X_0, \mathcal{E}_X)$ can be identified with the singular Riemannian manifold $(S^1 \times S^1, g_F)$ equipped with the energy form $\mathcal{E}_{g_F}$. The domain $D[\mathcal{E}_{g_F}]$ consists of functions in $H^1((S^1 \setminus \{p_1, \dots, p_k\}) \times S^1, g_F)$ whose traces on each circle $\{p_i\} \times S^1$ from the both sides conincide and $C^\infty(S^1 \times S^1)$ is dense in the domain. The pseudo-distance δ obviously degenerates along the k circles.

(iii) In the case where

$$\int_{x_i - a_i}^{x_i} \frac{1}{F(x)} dx = +\infty; \int_{x_i}^{x_i + a_i} \frac{1}{F(x)} dx < +\infty, \quad i = 1, \dots, k,$$

X_0 consists of k connected components $X_{0;i}$ $(i = 1, \dots, k)$, and each $(X_{0;i}, \mathcal{E}_{X|X_{0;i}})$ can be identified with a singular Riemannian manifold $M_i = ([x_i, x_{i+1}] \times S^1, g_F)$ with boundary $S_i^- = \{x_{i+1}\} \times S^1$ equipped with a nonlocal energy form $\mathcal{E}_i$. The circle $\{x_i\} \times S^1$ reduces to a single point z_i^- and the energy form $\mathcal{E}_i$ is given by

$$\mathcal{E}_i(u, u) = \mathcal{E}_{g_F}(u, u) + \frac{1}{\pi \mathrm{Vol}(M_i, g_F)} \mathcal{C}(u_{|S_i^-}, u_{|S_i^-})$$
$$(u \in D[\mathcal{E}_i] = H^1(M_i, g_F)),$$

where

$$\mathcal{C}(\phi, \phi) = \frac{1}{16\pi} \int_0^{2\pi} \int_0^{2\pi} (\phi(y) - \phi(y'))^2 \sin^{-2}\left(\frac{y - y'}{2}\right) dy dy',$$
$$D[\mathcal{C}] = \{\phi \in S_i^-(= S^1) | \ \mathcal{C}(\phi, \phi) < +\infty\}.$$

The complement $X \setminus X_0$ of X_0 consists of k connected open subsets Σ_i' $(i = 1, \dots, k)$ and each Σ_i' joins $X_{0;i-1}$ to $X_{0;i}$ in such a way that $\overline{\Sigma_i} \cap X_0 = S_{i-1}^- \cup \{z_i^-\}$.

Example 2.2. — Let $M = (M, g_M)$ be a compact, connected Riemannian manifold of dimension 2. We consider a family $\mathcal{F}$ of Riemannian metrics on the product space $M \times S^1$ of M and a unit circle $S^1 = \{e^{\sqrt{-1}x} |\ x \in \mathbb{R}\}$ such that

$$g_\omega = g_M + (dx + \omega)^2,$$

where ω is a one-form on M. We assume that M satisfies condition $[\mathrm{H}_0]$ with constants ν and C_U. Then $\mathcal{F}$ satisfies condition $[\mathrm{H}_0]$ with constants $\nu+1$ and C'_U depending only on C_U. We take a finite number of points $\{p_1, \dots, p_k\}$ and coordinates neighborhoods $(U_i, (x_i, y_i))$ around p_i $(i = 1, \dots, k)$ which are mutually disjoint. Let $\{(M \times S^1, g_n = g_{\omega_n})\}$ be an SD-Cauchy sequence in $\mathcal{F}$ which converges to a regular Dirichlet space $(X, \mu_X, \mathcal{E}_X)$ such that ω_n converges to a continuous one-form ω uniformly on compact sets in M, and the 2-forms $\Omega_n = d\omega_n$ satisfy

$$\liminf_{\varepsilon \to 0} \liminf_{n \to \infty} \left| \int_{B_M(p_i, \varepsilon)} \Omega_n \right| \geqslant a, \quad i = 1, \dots, k$$

for some positive constant a. Then $(X_0, \mu_X, \mathcal{E}_X)$ can be identified with $(M \times S^1, g_\omega)$, because the heat kernel of g_n converges to that of g_ω uniformly on compact sets in M. On the other hand, the Riemannian distance d_n of g_n tends to zero along each circle $\{p_i\} \times S^1$ $(i = 1, \dots, k)$. Indeed, for each i, we take a closed curve $\gamma_{i;\varepsilon}(t)$ $(0 \leqslant t \leqslant \ell)$ of unit speed, which joins p_i to a point $q_{i;\varepsilon}$ on the geodesic circle $\partial B_M(p_i, \varepsilon)$ by the geodesic segment, moves along the circle to a point $r_{i;\varepsilon}$, and then goes back to p_i along the geodesic segment. Let $\overline{\gamma}^{(n)}_{i;\varepsilon}(t)$ $(0 \leqslant t \leqslant \ell)$ be the horizontal lift of $\gamma_{i;\varepsilon}$ starting at $(p_i, 1) \in M \times S^1$, namely the curve $\overline{\gamma}^{(n)}_{i;\varepsilon}(t) = (\gamma_{i;\varepsilon}(t), \theta^{(n)}_{i;\varepsilon}(t))$ on $M \times S^1$ given by

$$\theta^{(n)}_{i;\varepsilon}(t) = -\int_0^t \omega_n\left(\frac{d}{dt}\gamma_{i;\varepsilon}(s)\right)ds.$$

Since the length ℓ of $\gamma_{i;\varepsilon}$ is less than $4\pi\varepsilon$ for ε small, we get

$$d_n(\overline{\gamma}^{(n)}_{i;\varepsilon}(0), \overline{\gamma}^{(n)}_{i;\varepsilon}(\ell)) \leqslant 4\pi\varepsilon.$$

On the other hand, if we denote by $A_{i;\varepsilon}$ the region enclosed by $\gamma_{i;\varepsilon}$, then we have

$$\theta^{(n)}_{i,\varepsilon}(\ell) = -\int_{A_{i;\varepsilon}} \Omega_n,$$

and hence if $\left|\int_{B_M(p_i,\varepsilon)} \Omega_n\right| \geqslant a$, then the interval $[0,a]$ is covered by the range of $\theta^{(n)}_{i;\varepsilon}(\ell)$, as $q_{i;\varepsilon}$ and $r_{i;\varepsilon}$ are varied. This implies that the circle $\{p_i\} \times S^1$ is contained in a geodesic ball around $(p_i, 1)$ with radius less than $b\varepsilon$, where b is a positive constant depending only on a. This shows that the Riemannian distance d_n degenerates along each circle $\{p_i\} \times S^1$ $(i = 1, \ldots, k)$ as $n \to \infty$.

Now we consider a family of Riemannian metrics defined by

$$g_{\tau,n} = \tau g_M + (dx + \omega_n)^2, \quad \tau > 0.$$

By choosing a sequence $\tau(n)$ with $\lim_{n\to\infty} \tau(n) = 0$ appropriately, we obatin an SD-Cauchy sequence $\{(M \times S^1, g_n')\}$ which converges to X with $(X_0, \mathcal{E}_X) = S^1$, but collapses to a single point with respect to the Gromov-Hausdorff distance.

Now we make an observation before proceeding to the next example. Let M be a compact, connected manifold and fix a Riemannian metric g_0 as a reference one. We consider pairs (g, w) of Riemannian metrics g and positive smooth functions w such that $\int_M w d\mu_{g_0} = 1$. Such pairs (g, w) define the Dirichlet forms $\mathcal{E}_{g,w}$ on M. In view of condition $[\mathrm{H}_0]'$ or $[\mathrm{H}_0]''$, a family of the Dirichlet forms $\mathcal{E}_{g,w}$ on $L^2(M, \mu_w)$ satisfies condition $[\mathrm{H}_0]$ (resp., $[\mathrm{H}_1]$, $[\mathrm{H}_2]$ and $[\mathrm{H}_3]$), if there exist positive constants α_i $(i = 1, 2)$ (resp. β_i $(i = 1, 2, 3, 4)$) such that $g \leqslant \alpha_1 w g_0$ and $w \leqslant \alpha_2$ (resp., $\beta_1 g_0 \leqslant g \leqslant \beta_2 g_0$ and $\beta_3 \leqslant w \leqslant \beta_4$).

Example 2.3 (cf. [14]; also [11], 3.1(2°)). — Let (N, h) be a compact connected Riemannian manifold. We consider a sequence of Riemannian metrics g_n on the product space $M = S^1 \times N$ of a unit circle $S^1 = \{e^{\sqrt{-1}x} \mid x \in \mathbb{R}\}$ and N such that

$$g_n = dx^2 + \frac{1}{1 + f_n(x)} h$$

and assume that f_n is a nonnegative smooth function on S^1 supported in $[-1/n, 1/n]$ and $f_n(x)dx$ weakly converges to a delta function δ_0 at 0 as $n \to \infty$. Let $\mathcal{E}_n$ be the energy form on $L^2(M, \mu)$ defined by

$$\mathcal{E}_n(u, v) = \int_M \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + (1 + f_n(x))\langle du_{|\{x\}\times N}, dv_{|\{x\}\times N}\rangle_h \ \frac{dx}{2\pi} \times d\mu_N.$$

Then as $n \to \infty$, $(M, dx/(2\pi) \times d\mu_N, \mathcal{E}_n)$ converges to the regular Dirichlet space $(M, dx/(2\pi) \times d\mu_N, \mathcal{E}_\infty)$ on $L^2(M, dx/(2\pi) \times \mu_N)$, where

$$\mathcal{E}_\infty(u,v) = \int_M \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + (1+\delta_0(x))\langle du_{|\{x\}\times N}, dv_{|\{x\}\times N}\rangle_h \, \frac{dx}{2\pi} \times d\mu_N.$$

The energy measure of the limit form $\mathcal{E}_\infty$ is singular along the hypersurface $\{0\} \times N$, and the limit of the Riemannian distance d_{g_n} is a pseudo-distance which is degenerate along the hypersurface.

Example 2.4 (cf. e.g., [20], Section 4).— We consider a sequence of metrics g_n on $S^1 \times S^1$ such that

$$g_n = dx^2 + F(nx)^2 dy^2,$$

where F is a positive smooth function on S^1. Then as $n \to \infty$, $(S^1 \times S^1, g_n)$ converges to $(S^1 \times S^1, g_\infty)$ with respect to the spectral distance via the identity map, where g_∞ is given by

$$g_\infty = ab\,dx^2 + \frac{b}{a}\,dy^2, \quad a = \int_0^{2\pi} \frac{1}{F(x)}\,dx, \quad b = \int_0^{2\pi} F(x)dx.$$

Therefore if we set $u(x,y) = \min\{|x|, |2\pi - x|\}$, then $\mathcal{E}_{g_n}(u,u) = 1$ and $\mathcal{E}_{g_\infty}(u,u) = 1/ab \leqslant 1$. The equality holds if and only if F is a constant.

Example 2.5.— In this example, we shall see that sub-Riemannian metrics lie on the boundary of Riemannian metrics with respect to the topology of not only the Gromov-Hausdorff distance but also the spectral distance.

Let us consider a subbundle H of the tangent bundle TM of a compact connected manifold M endowed with a smooth probability measure μ. Let h be a metric on H, and set $h(v,v) = +\infty$ if v is outside H. Then for any absolutely continuous path $\gamma(t)$ $(a \leqslant t \leqslant b)$ in M, we define the length of γ by $\int_a^b h(\gamma'(t), \gamma'(t))^{1/2} dt$, and for points $x, y \in M$, we denote by $\rho_h(x,y)$ the infimum of the length of such paths joining x and y. On the other hand, the metric h is transformed into a degenerate metric h^* on the cotangent bundle T^*M of M by

$$h^*(\xi,\xi) = 2 \sup_{v \in T_xM} \left\{ \langle \xi, v\rangle - \frac{1}{2} h(v,v) \right\}, \quad \xi \in T_x^*M, \ x \in M,$$

which yields the energy form $\mathcal{E}_h$ on $L^2(M,\mu)$ defined by

$$\mathcal{E}_h(u,u) = \int_M h^*(du,du)\ d\mu.$$

Now we assume that ρ_h becomes a distance on M and induces the topology of M, and the volume doubling property [H_1] with a constant $C_D(h)$ and the weak Poincaré inequality [H_2] with a constant $C_P(h)$ hold (cf. [23], [17]). Then we take the orthogonal complement V of the subbundle H in TM with respect to a Riemannian metric g_0 and consider a sequence $\{g_n\}$ of Riemannian metrics such that H and V are orthogonal with respect to every g_n, $g_n \geqslant {\varepsilon_n}^{-1} g_0$ on V and $|g_n - h| \leqslant \varepsilon_n$ on H, where ε_n tends to 0 as $n \to \infty$. For such a sequence $\{g_n\}$ of Riemannian metrics, a sequence of the energy forms

$$\mathcal{E}_n(u,u) = \int_M \langle du, du\rangle_{g_n}\, d\mu$$

converges to $\mathcal{E}_h$ as $n \to \infty$, in the sense of the spectral distance via the identity map (cf. Theorem 3.2 in Section 3; also [20]).

Example 2.6 (cf. [16]). — We consider a sequence of Riemannian metrics g_n on $S^1 \times S^1 = \{(e^{\sqrt{-1}x}, e^{\sqrt{-1}y}) \mid x, y \in \mathbb{R}\}$, such that

$$g_n = E(x,y)^2 dx^2 + \varepsilon_n F(x,y)^2 dy^2,$$

where E and F are positive smooth functions on $S^1 \times S^1$, and $\{\varepsilon_n\}$ is a sequence of positive numbers which tends to 0 as $n \to \infty$. We note that the normalized Riemannian measure of g_n is independent of n and given by

$$\overline{\mu} = \frac{EF(x,y)dxdy}{\int\int_{S^1\times S^1} EF(x,y)dxdy}.$$

Let $\pi_1 : S^1 \times S^1 \to S^1$ denote the projection onto the first S^1 and define a measure μ on this $S^1 = \{e^{\sqrt{-1}x} \mid x \in \mathbb{R}\}$ by

$$\mu = \pi_{1*}\overline{\mu} = \frac{\int_{S^1} EF(x,y)dy}{\int\int_{S^1\times S^1} EF(x,y)dxdy}dx.$$

Moreover we have two metrics on S^1 given by

$$h = \frac{\int_{S^1} EF(x,y)dy}{\int_{S^1} F/E(x,y)dy}dx^2; \ h^* = E^*(x)^2 dx^2,$$

where $E_*(x) = \min\{E(x,y) \mid y \in \mathbb{R}\}$. Then $(S^1 \times S^1, g_n)$ converges to $(S^1, \mu, \mathcal{E}_h)$ (resp., (S^1, h^*)) with respect to the spectral distance (resp., the Gromov-Hausdorff distance). The distance δ on S^1 with respect to h^* is given by $\delta(x_1,x_2) = \left|\int_{x_1}^{x_2} E_*(x)dx\right|$, and hence if we fix a point x_1 and set $\rho(x) = \delta(x_1, x)$, then we have

$$\mathcal{E}_h(\rho,\rho) = \frac{\int_{S^1} \left(\int_{S^1} F/E(x,y)dy\right) E_*(x)dx}{\int\int_{S^1\times S^1} EF(x,y)dxdy} \leqslant 1,$$

where the equality holds if and only if $E(x,y) = E^*(x)$, namely, $E(x,y)$ is independent of the second variable y.

We note that Examples 2.4 and 2.6 satisfy conditions $[\mathrm{H}_1]$, $[\mathrm{H}_2]$ and $[\mathrm{H}_3]$.

Finally we shall mention some geometric classes which satisfy condition $[\mathrm{H}_0]$. Let (M,g) be a compact, connected Riemannian manifold of dimension d.

(i) Let $\mathcal{Y}(M,[g])$ be the Yamabe constant of the conformal class of g and Scal_g denote the scalar curvature of g. If $\mathrm{Vol}(M,g) \leqslant \alpha$, $\mathcal{Y}(M,[g]) \geqslant \beta$, $\int_M (\mathrm{Scal}_g)_+^\gamma dv_g \leqslant \eta$ for some positive constants α, β, $\gamma > d/2 (> 3/2)$, then M satisfies condition $[\mathrm{H}_0]$ with constants $\nu = d$ and $C_U = C_U(d,\alpha,\beta,\gamma,\eta)$.

(ii) Suppose that M is isometrically immersed into a complete manifold whose sectional curvature is bounded from above by a positive constant κ and whose injectivity radius is bounded from below by a positive constant ι. Then if $\mathrm{Vol}(M) \leqslant \alpha$ and the mean curvature H_M satisfies: $\int_M |H_M|^\beta dv_g \leqslant \gamma$ for some constants α, $\beta > d$ and γ, then M satisfies condition $[\mathrm{H}_0]$ with constants $\nu = d$ and $C_U = C_U(d,\kappa,\iota,\alpha,\beta,\gamma)$.

(iii) Suppose that M is the total space of a Riemannian submersion onto a compact, connected Riemannian manifold B such that all fibers are connected and totally geodesic. In this case, all fibers are isometric to a compact, connected Riemannian manifold F, and moreover if B (resp., F) satisfies condition $[\mathrm{H}_0]$ with constants ν' and C'_U (resp., ν'' and C''_U), then so does the total space M with constants $\nu = \nu' + \nu''$ and $C_U = C'_U + C''_U$.

See, e.g., [2], [5], [19], [20], [21], [30] for details and related topics.

3. Convergence of energy forms.

We shall study the convergence of energy forms under the same situation as in the preceding section, and prove Theorem 0.2 (iii) and Theorem 0.3.

To begin with, we recall some consequences from condition $[\mathrm{H}_0]$ in the following

LEMMA 3.1 (cf. [19], Lemmas 2.4 and 2.5). — *Let M be a compact, connected Riemannian manifold satisfying* $[\mathrm{H}_0]$ *with constants ν and C_U.*

(i) *The i-th eigenvalue $\lambda_i(M)$ of M satisfies*

$$\lambda_i(M) \geqslant C_4(i+1)^{2/\nu} \quad \textit{if} \quad \lambda_i(M) \geqslant 1; i+1 \leqslant C_4 \quad \textit{if} \quad \lambda_i(M) \leqslant 1,$$

and further

$$\lambda_i(M) \leqslant C_4(\operatorname{diam} M)^{-2-\nu} i^{2+\nu} \quad \textit{for} \quad i \geqslant \frac{\operatorname{diam} M}{4},$$

where C_4 is a constant depending only on ν and C_U.

(ii) *Let $\{\phi_i\}$ be a complete, orthonormal system of eigenfunctions ϕ_i with eigenvalue $\lambda_i(M)$. Then one has*

$$\|\phi_i\|_{L^\infty} \leqslant C_U e \max\{\lambda_i(M)^{\nu/4}, 1\}$$

and given $\ell \geqslant 0$,

$$e^{-(t+1/t)} \sum_{T<\lambda_i(M)} \lambda_i(M)^\ell e^{-t\lambda_i(M)} \phi_i(x)^2 \leqslant 2C_U e \int_T^\infty \lambda^{\ell+\nu/2} e^{-2\sqrt{\lambda}} d\lambda$$

for all $T \geqslant 1$, $t > 0$, and $x \in M$.

Let $f_n : M_n \to X$ be as in Theorem 0.1. Given an integrable function u_n on M_n for each n, we say u_n weakly converges to an integrable function u on X as $n \to \infty$ (via approximating maps $f_n : M_n \to X$), if

$$\lim_{n\to\infty} \int_X v f_{n*}(u_n \mu_{M_n}) = \lim_{n\to\infty} \int_{M_n} v(f_n(a)) u_n(a) \mu_{M_n}(a) = \int_X vu \, d\mu_X$$

for any $v \in C(X)$. Also we say a sequence of bounded functions u_n on M_n uniformly converges to a bounded function u on X as $n \to \infty$, if

$$\lim_{n\to\infty} \|u_n - f_n^* u\|_{L^\infty} = 0.$$

THEOREM 3.2. — *Let $f_n : M_n \to X$, $h_n : X \to M_n$, $\mathcal{E}_X$ and $\mathcal{L}_X$ be as in Theorem 0.1.*

(i) *Suppose that a sequence $\{u_n\}$ of functions $u_n \in L^2(M_n)$ weakly converges to a function $u \in L^1(X, \mu_X)$ and that the L^2-norms $\|u_n\|_{L^2}$ are bounded as $n \to \infty$, then $u \in L^2(X, \mu_X)$ and*

$$\|u\|_{L^2} \leqslant \liminf_{n\to\infty} \|u_n\|_{L^2}.$$

If, in addition, $\sup_n \mathcal{E}_{M_n}(u_n, u_n) < +\infty$, *then* $u \in D[\mathcal{E}_X]$, $\|u\|_{L^2} = \lim_{n\to\infty} \|u_n\|_{L^2}$ *and*

$$\mathcal{E}_X(u,u) \leqslant \liminf_{n\to\infty} \mathcal{E}_{M_n}(u_n,u_n).$$

Moreover if the L^2*-norms of the Laplacians* $\Delta_{M_n} u_n$ *are bounded as* $n \to \infty$, *then* $u \in D[\mathcal{L}_X]$, $\|u\|_{L^2} = \lim_{n\to\infty}\|u_n\|_{L^2}$, $\mathcal{E}_X(u,u) = \lim_{n\to\infty} \mathcal{E}_{M_n}(u_n,u_n)$ *and*

$$\|\mathcal{L}_X u\|_{L^2} \leqslant \liminf_{n\to\infty} \|\Delta_{M_n} u_n\|_{L^2}.$$

(ii) *For any* $u \in D[\mathcal{E}_X] \cap C(X)$, *there exists a sequence* $\{v_n\}$ *of functions* $v_n \in D[\mathcal{E}_{M_n}] \cap C(M_n)$ *such that* v_n *weakly converges to* u, *and further*

$$\lim_{n\to\infty} \|v_n - f_n^* u\|_{L^2} = 0, \quad \lim_{n\to\infty} \|h_n^* v_n - u\|_{L^2} = 0,$$
$$\mathcal{E}_X(u,u) = \lim_{n\to\infty} \mathcal{E}_{M_n}(v_n,v_n);$$

in addition, if $u \in D[\mathcal{L}_X] \cap C(X)$, *then* $v_n \in D[\Delta_{M_n}] \cap C(M_n)$, $\Delta_{M_n} v_n$ *weakly converges to* $\mathcal{L}_X u$, *and*

$$\lim_{n\to\infty} \|\Delta_{M_n} v_n - f_n^*(\mathcal{L}_X u)\|_{L^2} = 0, \quad \lim_{n\to\infty} \|h_n^*(\Delta_{M_n} v_n) - \mathcal{L}_X u\|_{L^2} = 0.$$

Proof. — We choose a complete orthonormal system $\Phi_n = \{\phi_i^{(n)}\}$ of eigenfunctions of M_n and such a system $\Phi = \{\phi_i\}$ of X, and we shall discuss under the assumption that

$$(7) \qquad |\phi_i^{(n)}(a) - \phi_i(f_n(a))| < \varepsilon_n(i), \quad |\phi_i^{(n)}(h_n(x)) - \phi_i(x)| < \varepsilon_n(i)$$

for all $a \in M_n$ and $x \in X$, where $\varepsilon_n(i)$ tends to 0 as $n \to \infty$.

Given an integrable function u_n on M_n for each n, we suppose that u_n weakly converges to a function $u \in L^1(X, \mu_X)$ via the approximating map $f_n : M_n \to X$. Now suppose each u_n is square integrable. Then u_n has the eigenfunction expansion with respect to the basis $\Phi_n = \{\phi_i^{(n)}\}$, which reads

$$u_n \sim \sum_{i=0}^{\infty} c_i^{(n)}(u_n)\phi_i^{(n)} \quad \left(c_i^{(n)}(u_n) = \int_{M_n} u_n \phi_i^{(n)} d\mu_{M_n}\right).$$

The L^2 norms of u_n, $|du_n|$ and $\Delta_{M_n} u_n$ are respectively given by

$$\|u_n\|_{L^2}^2 = \sum_{i=0}^{\infty} c_i^{(n)}(u_n)^2, \quad \mathcal{E}_{M_n}(u_n,u_n) = \sum_{i=1}^{\infty} \lambda_i(M_n) c_i^{(n)}(u_n)^2,$$

$$\|\Delta_{M_n} u_n\|_{L^2}^2 = \sum_{i=0}^{\infty} \lambda_i(M_n)^2 c_i^{(n)}(u_n)^2,$$

if they are finite.

We claim first that $\|u\|_{L^2}^2 \leqslant \liminf_{n\to\infty} \|u_n\|_{L^2}^2$, if the L^2 norm of u_n is bounded as $n \to \infty$. To see this, given any N, we have $\sum_{i=0}^N c_i(u)^2 = \lim_{n\to\infty} \sum_{i=0}^N c_i^{(n)}(u_n)^2$ ($c_i(u) = \int_X u\phi_i \, d\mu_X$), and hence letting $N \to \infty$, we obtain $\|u\|_{L^2}^2 = \sum_{i=0}^\infty c_i(u)^2 \leqslant \liminf_{n\to\infty} \|u_n\|_{L^2}^2$.

In a similar manner, we can show that if $\mathcal{E}_X(u_n, u_n)$ is bounded as $n \to \infty$, then $u \in D[\mathcal{E}_X]$, $\|u\|_{L^2}^2 = \lim_{n\to\infty} \|u_n\|_{L^2}^2$, and $\mathcal{E}_X(u,u) \leqslant \liminf_{n\to\infty} \mathcal{E}_X(u_n, u_n)$. Indeed, since

$$\begin{aligned}\sum_{i=N+1}^{\infty} c_i^{(n)}(u_n)^2 &\leqslant \frac{1}{\lambda_{N+1}(M_n)^{1/2}} \|u_n\|_{L^2} \mathcal{E}_{M_n}(u_n, u_n)^{1/2} \\ &\leqslant \frac{1}{C_4^{1/2}(N+2)^{1/\nu}} \|u_n\|_{L^2} \mathcal{E}_{M_n}(u_n, u_n)^{1/2},\end{aligned}$$

$\sum_{i=N+1}^\infty c_i^{(n)}(u_n)^2$ tends to 0 as $N \to \infty$, uniformly in n. This implies that $\lim_{n\to\infty} \|u_n\|_{L^2} = \|u\|_{L^2}$. Moreover for each N fixed, $\sum_{i=0}^N \lambda_i(M_n) c_i^{(n)}(u_n)^2$ converges to $\sum_{i=0}^N \lambda_i(X) c_i(u)^2$ as $n \to \infty$, so that $\mathcal{E}_X(u,u) \leqslant \liminf_{n\to\infty} \mathcal{E}_X(u_n, u_n)$.

In addition, if $\|{\Delta_{M_n}}^{\ell} u_n\|_{L^2}$ is bounded as $n \to \infty$ for some $\ell \in \{1, 2, \ldots\}$, then we see that

$$\begin{gathered} u \in D[{\mathcal{L}_X}^{\ell}], \|{\mathcal{L}_X}^{\ell-1} u\|_{L^2}^2 = \lim_{n\to\infty} \|{\Delta_{M_n}}^{\ell-1} u_n\|_{L^2}^2, \\ \|{\mathcal{L}_X}^{\ell} u\|_{L^2}^2 \leqslant \liminf_{n\to\infty} \|{\Delta_{M_n}}^{\ell} u_n\|_{L^2}^2. \end{gathered}$$

In the discussion just above, we have assumed the convergence of complete orthonormal systems of eigenfunctions as in (7), but the results are clearly independent of the choice of such systems, and thus, the first part of Theorem 3.2, the lower semicontinuity of the above norms of functions with respect to the weak convergence, has been shown.

In what follows, we shall prove the second part of the theorem. Given a function $u \in D[\mathcal{E}_X] \cap C(X)$, we set $u_t = P_{X;t} u$ and also we define bounded functions u_n and $u_{n;t}$ ($t > 0$) on M_n by $u_n = f_n^* u$ and $u_{n;t} = P_{M_n;t} u_n$. Then it is easy to see that u_n weakly converges to u; moreover, for each $t > 0$ fixed and any $\ell \in \{0, 1, 2, \ldots\}$, ${\Delta_{M_n}}^{\ell} u_{n;t}$ uniformly converges to

${\mathcal{L}_X}^{\ell} u_t$ as $n \to \infty$, that is

$$\lim_{n\to\infty} \|{\Delta_{M_n}}^{\ell} u_{n;t} - f_n^*({\mathcal{L}_X}^{\ell} u_t)\|_{L^\infty} = 0; \tag{8}$$
$$\lim_{n\to\infty} \|h_n^*({\Delta_{M_n}}^{\ell} u_{n;t}) - {\mathcal{L}_X}^{\ell} u_t\|_{L^\infty} = 0.$$

Indeed, we observe from Lemma 3.1 that for any N , there exists a positive constant $\theta_N = \theta_N(C_U, \nu, \ell, t)$ depending only on C_U, ν, ℓ, t and tending to zero as $N \to \infty$, such that

$$\|\sum_{i>N} \lambda_i(M_n)^{2\ell} e^{-2\lambda_i(M_n)t} (\phi_i^{(n)})^2\|_{L^\infty} \leqslant {\theta_N}^2,$$
$$\|\sum_{i>N} \lambda_i(X)^{2\ell} e^{-2\lambda_i(X)t} {\phi_i}^2\|_{L^\infty} \leqslant {\theta_N}^2.$$

Therefore we have

$$\begin{aligned}
&|{\Delta_{M_n}}^{\ell} u_{n;t} - f_n^*({\mathcal{L}_X}^{\ell} u_t)| \\
&\leqslant |\sum_{i=0}^{N} \lambda_i(M_n)^{\ell} e^{-\lambda_i(M_n)t} c_i^{(n)}(u_n)\phi_i^{(n)} - \lambda_i(X)^{\ell} e^{-\lambda_i(X)t} c_i(u) f_n^*\phi_i| \\
&\quad + |\sum_{i>N} \lambda_i(M_n)^{\ell} e^{-\lambda_i(M_n)t} c_i^{(n)}(u_n)\phi_i^{(n)}| \\
&\quad + |\sum_{i>N} \lambda_i(X)^{\ell} e^{-\lambda_i(X)t} c_i(u) f_n^*\phi_i| \\
&\leqslant |\sum_{i=0}^{N} \lambda_i(M_n)^{\ell} e^{-\lambda_i(M_n)t} c_i^{(n)}(u_n)\phi_i^{(n)} - \lambda_i(X)^{\ell} e^{-\lambda_i(X)t} c_i(u) f_n^*\phi_i| \\
&\quad + (\sum_{i>N} \lambda_i(M_n)^{2\ell} e^{-2\lambda_i(M_n)t} (\phi_i^{(n)})^2)^{1/2} (\sum_{i>N} c_i^{(n)}(u_n)^2)^{1/2} \\
&\quad + (\sum_{i>N} \lambda_i(X)^{2\ell} e^{-2\lambda_i(X)t} (\phi_i)^2)^{1/2} (\sum_{i>N} c_i(u)^2)^{1/2} \\
&\leqslant |\sum_{i=0}^{N} \lambda_i(M_n)^{\ell} e^{-\lambda_i(M_n)t} c_i^{(n)}(u_n)\phi_i^{(n)} - \lambda_i(X)^{\ell} e^{-\lambda_i(X)t} c_i(u) f_n^*\phi_i| \\
&\quad + \theta_N(\|u_n\|_{L^2} + \|u\|_{L^2}).
\end{aligned}$$

Hence in view of (7), we get the first assertion of (8), because $\lim_{n\to\infty} c_i^{(n)}$ $(u_n) = c_i(u)$ by the assumption that u is continuous, and also $\lim_{n\to\infty} \lambda_i$ $(M_n) = \lambda_i(X)$. In exactly the same way, we can show the second one of (8). In particular, we have

$$\lim_{n\to\infty} \mathcal{E}_{M_n}(u_{n;t}, u_{n;t}) = \mathcal{E}_X(u_t, u_t).$$

Since

$$\lim_{t\to 0}\|u_t-u\|_{L^2}+\mathcal{E}_X(u_t-u,u_t-u)=0,$$

for any $k=1,2,\dots$, there exists a positive number $t(k)$ such that

$$\|u_{t(k)}-u\|_{L^2}+\mathcal{E}_X(u_{t(k)}-u,u_{t(k)}-u)<\frac{1}{k}.$$

Then we can take a positive interger $N(k)$ so large that

$$|\mathcal{E}_{M_n}(u_{n;t(k)},u_{n;t(k)})-\mathcal{E}_X(u_{t(k)},u_{t(k)})|<\frac{1}{k}$$

for all $n\geqslant N(k)$. Now for any n, we denote by $k(n)$ the integer k with $N(k)\leqslant n<N(k+1)$, and set $v_n=u_{n;t(k(n))}$. Then it is easy to see that v_n weakly converges to u as $n\to\infty$, and hence

$$\mathcal{E}_X(u,u)\leqslant\liminf_{n\to\infty}\mathcal{E}_{M_n}(v_n,v_n).$$

On the other hand, we have

$$\mathcal{E}_{M_n}(v_n,v_n)\leqslant\mathcal{E}_X(u_{t(k(n))},u_{t(k(n))})+\frac{1}{k(n)}\leqslant\mathcal{E}_X(u,u)+\frac{1}{k(n)}.$$

This implies that

$$\limsup_{n\to\infty}\mathcal{E}_{M_n}(v_n,v_n)\leqslant\mathcal{E}_X(u,u).$$

We thus obtain

$$\lim_{n\to\infty}\mathcal{E}_{M_n}(v_n,v_n)=\mathcal{E}_X(u,u).$$

In the case where $u\in D[{\mathcal{L}_X}^{\ell}]\cap C(X)$, we see in exactly the same way that

$$\lim_{n\to\infty}\|{\Delta_{M_n}}^{\ell}v_n-f_n^*({\mathcal{L}_X}^{\ell}v)\|_{L^2}=0,\quad\lim_{n\to\infty}\|h_n^*({\Delta_{M_n}}^{\ell}v_n)-{\mathcal{L}_X}^{\ell}v\|_{L^2}=0.$$

This completes the proof of Theorem 3.2.

Remark 3.3. — Let u and v_n be as in Theorem 3.2 (ii). Suppose that $P_{X;t}u$ uniformly converges to u as $t\to\infty$. Then it follows from the above proof that v_n uniformly converges to u as $n\to\infty$, that is

$$\lim_{n\to\infty}\|v_n-f_n^*u\|_{L^\infty}=0;\ \lim_{t\to 0}\|h_n^*v_n-u\|_{L^\infty}=0,$$

and in case $u\in D[\mathcal{L}_X]\cap C(X)$ and $\mathcal{L}_Xu\in C(X)$, $\Delta_{M_n}v_n$ uniformly converges to $\mathcal{L}_Xu$, that is

$$\lim_{n\to\infty}\|\Delta_{M_n}v_n-f_n^*(\mathcal{L}_Xu)\|_{L^\infty}=0;\ \lim_{t\to 0}\|h_n^*(\Delta_{M_n}v_n)-\mathcal{L}_Xu\|_{L^\infty}=0.$$

LEMMA 3.4. — *Let M_n, X, $f_n : M_n \to X$ and $h_n : X \to M_n$ be as in Theorem 0.1. Let u_n and v_n be L^2 functions on M_n such that u_n and v_n weakly converge to L^2 functions u and v on X, respectively. Then the following assertions hold:*

(i) *If $\mathcal{E}_{M_n}(u_n, u_n)$ converges to $\mathcal{E}_X(u,u)$ as $n \to \infty$, and if $\mathcal{E}_{M_n}(v_n, v_n)$ is bounded as $n \to \infty$, then*

$$\lim_{n\to\infty} \mathcal{E}_{M_n}(u_n, v_n) = \mathcal{E}_X(u, v).$$

(ii) *If $\|u_n\|_{L^\infty}$, $\|v_n\|_{L^\infty}$, $\mathcal{E}_{M_n}(u_n, u_n)$ and $\mathcal{E}_{M_n}(v_n, v_n)$ are bounded as $n \to \infty$, then the product function $u_n v_n$ weakly converges to uv as $n \to \infty$, and*

$$\begin{aligned}\mathcal{E}_X(uv, uv) &\leqslant \liminf_{n\to\infty} \mathcal{E}_{M_n}(u_n v_n, u_n v_n) \\ &\leqslant 2 \liminf_{n\to\infty} (\|u_n\|^2_{L^\infty} \mathcal{E}_{M_n}(u_n, u_n) + \|v_n\|^2_{L^\infty} \mathcal{E}_{M_n}(v_n, v_n)).\end{aligned}$$

(iii) *If $\mathcal{E}_{M_n}(u_n, u_n)$ and $\mathcal{E}_{M_n}(v_n, v_n)$ respectively converge to $\mathcal{E}_X(u,u)$ and $\mathcal{E}_X(v,v)$, and if $\|u_n\|_{L^\infty}$ and $\|v_n\|_{L^\infty}$ are bounded as $n \to \infty$, then*

$$\lim_{n\to\infty} \mathcal{E}_{M_n}(u_n v_n, u_n) = \mathcal{E}_X(uv, u); \quad \lim_{n\to\infty} \mathcal{E}_{M_n}(u_n^2, v_n) = \mathcal{E}_X(u^2, v).$$

Proof. — Suppose that $\lim_{n\to\infty} \mathcal{E}_{M_n}(u_n, u_n) = \mathcal{E}_X(u,u)$. As in the proof of Theorem 3.2, we take a complete orthonormal system $\Phi_n = \{\phi_i^{(n)}\}$ of eigenfunctions of M_n and such a system $\Phi = \{\phi_i\}$ of X, and we shall discuss under the assumption (7). Then $\sum_{i>N} \lambda_i(M_n) c_i^{(n)}(u_n)^2$ tends to zero as $N \to \infty$, uniformly in n. Hence if $\mathcal{E}_{M_n}(v_n, v_n)$ is bounded as $n \to \infty$, then $\lim_{N\to\infty} \sum_{i>N} \lambda_i(M_n) c_i^{(n)}(u_n) c_i^{(n)}(v_n) = 0$ uniformly in n. This implies that $\lim_{n\to\infty} \mathcal{E}_{M_n}(u_n, v_n) = \mathcal{E}_X(u, v)$. This proves the first assertion of the lemma.

Now suppose that $\sup_n \|u_n\|_{L^\infty} < +\infty$, $\sup_n \|v_n\|_{L^\infty} < +\infty$, $\sup_n \mathcal{E}_{M_n}(u_n, u_n) < +\infty$ and $\sup_n \mathcal{E}_{M_n}(v_n, v_n) < +\infty$. Since $\lim_{n\to\infty} \|u_n\|_{L^2} = \|u\|_{L^2}$ and $\lim_{n\to\infty} \|v_n\|_{L^2} = \|v\|_{L^2}$, we see that both $\sum_{i>N} c_i^{(n)}(u_n)^2$ and $\sum_{i>N} c_i^{(n)}(v_n)^2$ tend to zero as $N \to \infty$, uniformly in n. This shows that

$$\lim_{N\to\infty} \int_{M_n} |u_n v_n - (\sum_{i=0}^{N} c_i^{(n)}(u_n)\phi_i^{(n)})(\sum_{i=0}^{N} c_i^{(n)}(v_n)\phi_i^{(n)})| d\mu_{M_n} = 0$$

uniformly in n. Therefore given any $w \in C(X)$, we have

$$\begin{aligned}
&\left|\int_{M_n}(f_n^*w)u_nv_n d\mu_{M_n} - \int_X wuv d\mu_X\right| \\
&\leqslant \left|\int_{M_n}(f_n^*w)\left(u_nv_n - (\sum_{i=0}^N c_i^{(n)}(u_n)\phi_i^{(n)})(\sum_{i=0}^N c_i^{(n)}(v_n)\phi_i^{(n)})\right) d\mu_{M_n}\right| \\
&+ \left|\int_X w\left(uv - (\sum_{i=0}^N c_i(u)\phi_i)(\sum_{i=0}^N c_i(v)\phi_i)\right) d\mu_X\right| \\
&+ \Bigg|\int_{M_n}(f_n^*w)(\sum_{i=0}^N c_i^{(n)}(u_n)\phi_i^{(n)})(\sum_{i=0}^N c_i^{(n)}(v_n)\phi_i^{(n)})d\mu_{M_n} \\
&\qquad\qquad - \int_X w(\sum_{i=0}^N c_i(u)\phi_i)(\sum_{i=0}^N c_i(v)\phi_i)d\mu_X\Bigg| \\
&\leqslant \sup|w|\int_{M_n}|u_nv_n - (\sum_{i=0}^N c_i^{(n)}(u_n)\phi_i^{(n)})(\sum_{i=0}^N c_i^{(n)}(v_n)\phi_i^{(n)})|d\mu_{M_n} \\
&+ \sup|w|\int_X|uv - (\sum_{i=0}^N c_i(u)\phi_i)(\sum_{i=0}^N c_i(v)\phi_i)|d\mu_X \\
&+ \Bigg|\int_{M_n}(f_n^*w)(\sum_{i=0}^N c_i^{(n)}(u_n)\phi_i^{(n)})(\sum_{i=0}^N c_i^{(n)}(v_n)\phi_i^{(n)})d\mu_{M_n} \\
&\qquad\qquad - \int_X w(\sum_{i=0}^N c_i(u)\phi_i)(\sum_{i=0}^N c_i(v)\phi_i)d\mu_X\Bigg|.
\end{aligned}$$

Hence by choosing N sufficiently large and then letting $n \to \infty$, we see that

$$\lim_{n\to\infty}\int_{M_n}(f_n^*w)u_nv_n d\mu_{M_n} = \int_X wuv d\mu_X.$$

Thus the product function u_nv_n weakly converges to uv as $n \to \infty$. This shows the second assertion of the lemma, which, together with the first assertion, implies the third one. This completes the proof of Lemma 3.4.

LEMMA 3.5. — *Let u be a function in $D[\mathcal{E}_X] \cap C(X)$ such that $P_{X;t}u$ uniformly converges to u as $t \to 0$.*

(i) *If a sequence $\{v_n\}$ of functions v_n in $D[\mathcal{E}_{M_n}] \cap C(M_n)$ uniformly converges to u as $n \to \infty$, then*

$$\int_X \phi \; d\mu_{\langle u,u\rangle} \leqslant \liminf_{n\to\infty}\int_{M_n} f_n^*\phi\,|dv_n|^2 \; d\mu_{M_n}$$

for all nonnegative functions $\phi \in D[\mathcal{E}_X] \cap C(X)$ *to which* $P_{X;t}\phi$ *uniformly converges as* $t \to 0$*. Moreover, if the* L^{2p} *norm of* $|dv_n|$, $(\int |dv_n|^{2p}\, d\mu_{M_n})^{1/2p}$, *is bounded as* $n \to \infty$ *for some* p *with* $2 \leqslant p \leqslant \infty$*, then the energy measure* $\mu_{\langle u,u\rangle}$ *of* u *is absolutely continuous with respect to* μ_X*, the density* $\Gamma(u,u) = d\mu_{\langle u,u\rangle}/d\mu_X$ *is* L^p *integrable, and*

$$\int_\Omega \Gamma(u,u)^p\, d\mu_X \leqslant \liminf_{n\to\infty} \int_{f_n^{-1}(\Omega)} |dv_n|^{2p}\, d\mu_{M_n}$$

for any open subset Ω *of* X*.*

(ii) *If a sequence* $\{w_n\}$ *of functions* w_n *in* $D[\mathcal{E}_{M_n}] \cap C(M_n)$ *weakly converges to* u *and further* $\mathcal{E}_{M_n}(w_n,w_n)$ *tends to* $\mathcal{E}_X(u,u)$ *as* $n \to \infty$*, then*

$$\int_X \phi\, d\mu_{\langle u,u\rangle} = \lim_{n\to\infty} \int_{M_n} f_n^*\phi\, |dw_n|^2\, d\mu_{M_n}$$

for any function $\phi \in D[\mathcal{E}_X] \cap C(X)$ *to which* $P_{X;t}\phi$ *uniformly converges as* $t \to 0$*.*

Proof. — In view of Remark 3.3, we take a sequence $\{u_n\}$ of functions $u_n \in D[\mathcal{E}_{M_n}] \cap C(M_n)$ in such a way that as $n \to \infty$, u_n uniformly converges to u and $\mathcal{E}_{M_n}(u_n,u_n)$ tends to $\mathcal{E}_X(u,u)$. For any $\phi \in D[\mathcal{E}_X] \cap C(X)$ such that $P_{X;t}\phi$ uniformly converges to ϕ as $t \to 0$, we can also take a sequence $\{\phi_n\}$ of functions $\phi_n \in D[\mathcal{E}_{M_n}] \cap C(M_n)$ in such a way that ϕ_n uniformly converges to ϕ and $\mathcal{E}_{M_n}(\phi_n,\phi_n)$ tends to $\mathcal{E}_X(\phi,\phi)$ as $n \to \infty$. Then it follows from Lemma 3.4 that

$$\begin{aligned}
\int_X \phi\, d\mu_{\langle u,u\rangle} &= \mathcal{E}_X(u\phi,u) - \frac{1}{2}\mathcal{E}_X(u^2,\phi)\\
&= \lim_{n\to\infty} \mathcal{E}_{M_n}(u_n\phi_n,u_n) - \frac{1}{2}\mathcal{E}_{M_n}(u_n^2,\phi_n)\\
&= \lim_{n\to\infty} \int_{M_n} \phi_n |du_n|^2 d\mu_{M_n}\\
&= \lim_{n\to\infty} \int_{M_n} (\phi_n - f_n^*\phi)|du_n|^2 d\mu_{M_n} + \int_{M_n} f_n^*\phi |du_n|^2 d\mu_{M_n}\\
&= \lim_{n\to\infty} \int_{M_n} f_n^*\phi |du_n|^2 d\mu_{M_n}.
\end{aligned}$$

Now, for a sequence $\{v_n\}$ of functions v_n in $D[\mathcal{E}_{M_n}] \cap C(M_n)$ which uniformly converges to u as $n \to \infty$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_{M_n} \phi_n \langle du_n, dv_n\rangle \, d\mu_{M_n} \\
&\quad = \lim_{n\to\infty} \int_{M_n} \langle du_n, d(\phi_n v_n)\rangle - (v_n - u_n)\langle du_n, d\phi_n\rangle - \frac{1}{2}\langle du_n^2, d\phi_n\rangle \, d\mu_{M_n} \\
&\quad = \mathcal{E}_X(u, \phi u) - \frac{1}{2}\mathcal{E}_X(u^2, \phi) \\
&\quad = \int_X \phi \, d\mu_{\langle u,u\rangle}.
\end{aligned}$$

On the other hand, in the case where $\phi \geqslant 0$ and $\phi_n \geqslant 0$, by

$$\begin{aligned}
&\left| \int_{M_n} \phi_n \langle du_n, dv_n\rangle \, d\mu_{M_n} \right| \\
&\qquad \leqslant \left(\int_{M_n} \phi_n |du_n|^2 \, d\mu_{M_n} \right)^{1/2} \left(\int_{M_n} \phi_n |dv_n|^2 \, d\mu_{M_n} \right)^{1/2},
\end{aligned}$$

we get

$$\begin{aligned}
\int_X \phi \, d\mu_{\langle u,u\rangle} &\leqslant \left(\int_X \phi \, d\mu_{\langle u,u\rangle} \right)^{1/2} \liminf_{n\to\infty} \left(\int_X \phi_n |dv_n|^2 \, d\mu_{M_n} \right)^{1/2} \\
&= \left(\int_X \phi \, d\mu_{\langle u,u\rangle} \right)^{1/2} \liminf_{n\to\infty} \left(\int_{M_n} f_n^* \phi |dv_n|^2 \, d\mu_{M_n} \right)^{1/2},
\end{aligned}$$

which shows

$$\int_X \phi \, d\mu_{\langle u,u\rangle} \leqslant \liminf_{n\to\infty} \int_{M_n} f_n^* \phi |dv_n|^2 \, d\mu_{M_n}.$$

In the case where the L^{2p} norm of $|dv_n|$ is bounded as $n \to \infty$, we have by setting $q = p/(p-1)$,

$$
\begin{aligned}
&\left|\int_X \phi \, d\mu_{\langle u,u\rangle}\right| \\
&\leqslant \liminf_{n\to\infty} \left|\int_{M_n} \phi_n |dv_n|^2 d\mu_{M_n}\right| \\
&= \liminf_{n\to\infty} | \int_{M_n} (\phi_n - f_n^*\phi)|dv_n|^2 \, d\mu_{M_n} + \int f_n^*\phi |dv_n|^2 \, d\mu_{M_n} \\
&\leqslant \liminf_{n\to\infty} (\|\phi_n - f_n^*\phi\|_{L^q} \|dv_n\|^2_{L^{2p}} + \|f_n^*\phi\|_{L^q} \left(\int_{f_n^{-1}(\operatorname{supp}\phi)} |dv_n|^{2p} \, d\mu_{M_n}\right)^{1/p} \\
&\leqslant \|\phi\|_{L^q} \liminf_{n\to\infty} \left(\int_{f_n^{-1}(\operatorname{supp}\phi)} |dv_n|^{2p} \, d\mu_{M_n}\right)^{1/p}
\end{aligned}
$$

for all $\phi \in D[\mathcal{E}_X] \cap C(X)$. Notice that it is assumed here that ϕ_n converges to ϕ not uniformly but L^2 strongly in the sense that $\lim_{n\to\infty} \|\phi_n - f_n^*\phi\|_{L^2} = 0$. We thus conclude that $\mu_{\langle u,u\rangle} = \Gamma(u,u)\mu_X$ for some L^p function $\Gamma(u,u)$, and

$$\int_\Omega \Gamma(u,u)^p \, d\mu_X \leqslant \liminf_{n\to\infty} \int_{f_n^{-1}(\Omega)} |dv_n|^{2p} \, d\mu_{M_n}$$

for any open subset Ω of X.

Proof of the assertion (iii) of Theorem 0.2. — Assuming $C^{0,1}(X,\delta) \subset D[\mathcal{E}_X]$, we first show that for $u \in C^{0,1}(X,\delta)$ and $v \in D[\mathcal{E}_X]$, $\mathcal{E}_X(u,v) = 0$ if $\operatorname{supp} u \cap \operatorname{supp} v = \varnothing$. This is verified as follows. Since there exists a positive constant ε such that $\delta(x,x') > \varepsilon$ for any $x \in \operatorname{supp} u$ and $x' \in \operatorname{supp} v$, we get

$$
\begin{aligned}
\mathcal{E}_X(u,v) &= \lim_{t\to 0} \frac{1}{2t} \iint_{X\times X} p_X(t,x,x')(u(x) - u(x')(v(x) - v(x'))d\mu(x)d\mu(x') \\
&= \lim_{t\to 0} \frac{1}{t} \iint_{\{(x,x')\in X\times X \mid \delta(x,x')>\varepsilon\}} -p_X(t,x,x')u(x)v(x')d\mu(x)d\mu(x') \\
&= 0.
\end{aligned}
$$

Now given $u \in C^{0,1}(X,\delta)$ and an open set Ω of X, we set $L_\Omega = \mathbf{dil}_\delta(u_{|\Omega})$ and define a Lipschitz function u_Ω on X by

$$u_\Omega(x) = \inf\{L_\Omega \delta(x,y) + u(y) \mid y \in \Omega\}, \quad x \in X.$$

Then $u_\Omega = u$ in Ω and $\operatorname{dil}_\delta(u_\Omega) = L_\Omega$.

Let us assume that for a subsequence $\{m\}$, $f_m : (M_m, d_{M_m}) \to (X,\delta)$ and $h_m : (X,\delta) \to (M_m, d_{M_m})$ are a pair of ε_m-Hausdorff approximating maps with ε_m tending to 0 as $m \to \infty$. We would like to construct a sequence of Lipschitz functions v_m on M_m such that $\mathrm{dil}_\delta(v_m) = L_\Omega$ and v_m uniformly converges to u_Ω as $m \to \infty$. For this, we first take an increasing family of finite subsets A_k of Ω such that $\delta(x, A_k) \leqslant \eta_k$ for all $x \in \Omega$ with $\lim_{k\to\infty} \eta_k = 0$. If we define a sequence of Lipschitz functions $\{u_{\Omega,k}\}$ by

$$u_{\Omega,k}(x) = \min\{L_\Omega \delta(x,y) + u_\Omega(y) \mid y \in A_k\}, \quad x \in X,$$

then it is easy to see that

$$-2\eta_k L_\Omega \leqslant u_\Omega(x) - u_{\Omega,k}(x) \leqslant 0 \quad (x \in X).$$

Let $\{v_m\}$ be a sequence of Lipschitz functions v_m on M_m given by

$$v_m(a) = \min\{L_\Omega d_{M_m}(a,b) + u_\Omega(f_m(b)) \mid b \in h_m(A_m)\}, \quad a \in M_m.$$

Then the dilatation of v_m is obviously equal to L_Ω and satisfies

$$\|v_m - f_m^* u_\Omega\|_{L^\infty} \leqslant (6\varepsilon_m + 2\eta_m) L_\Omega$$

so that v_m uniformly converges to u_Ω as $m \to \infty$. Hence it follows from Theorem 3.2 (i) and Lemma 3.5 (i) that u_Ω belongs to $D[\mathcal{E}_X]$ and $\mu_{\langle u_\Omega, u_\Omega\rangle} = \Gamma(u_\Omega, u_\Omega)\mu_X$ with

$$\Gamma(u_\Omega, u_\Omega) \leqslant {L_\Omega}^2 = \mathbf{dil}_\delta(u_{|\Omega})^2.$$

In particular, by considering the case $\Omega = X$, we see that $u \in D[\mathcal{E}_X]$ and $\Gamma(u,u) \leqslant L_X = \mathbf{dil}_\delta(u)$. Moreover since $u = u_\Omega$ on Ω, we see that $\mathcal{E}_X(\phi u, u) = \mathcal{E}_X(\phi u_\Omega, u_\Omega)$ and $\mathcal{E}_X(u^2, \phi) = \mathcal{E}_X({u_\Omega}^2, \phi)$ for any $\phi \in D[\mathcal{E}_X] \cap C(X)$ supported in Ω, and hence it follows that $\Gamma(u,u) = \Gamma(u_\Omega, u_\Omega)$ in Ω. Thus we obtain

$$\Gamma(u,u)(x) \leqslant {L_\Omega}^2, \quad a.a.\ x \in \Omega.$$

This is true for any open set Ω, so that we can conclude that

$$\Gamma(u,u)(x) \leqslant \mathrm{dil}_\delta u(x)^2, \quad a.a.\ x \in X.$$

Proof of Theorem 0.3.— We may assume that $f_n : M_n \to X$ and $h_n : X \to M_n$ are a pair of ε_n-Hausdorff approximating maps with ε_n tending to 0 as $n \to \infty$. Then the inequalities except the second one, the weak Poincaré inequality, in the first assertion of the theorem are obvious, that is we have

$$\mu_X(B_\delta(x, 2r)) \leqslant C_D \mu_X(B_\delta(x,r)) \tag{9}$$

and

$$\mu_X(B_\delta(x,1)) \geqslant C_B.$$

From (9), we can easily deduce that

$$\text{(10)} \quad \mu_X(B_\delta(x,R)) \leqslant \left(\frac{2R}{r}\right)^\kappa \mu_X(B_\delta(x,r)), \quad 0 < r \leqslant R \leqslant 1,\ x \in X,$$

where $\kappa = \log_2 C_D$.

Now we would like to show the weak Poincaré inequality. We are given a function $u \in D[\mathcal{E}_X] \cap C(X)$ and take a sequence of functions $u_n \in D[\mathcal{E}_{M_n}] \cap C(M_n)$ such that as $n \to \infty$, u_n uniformly converges to u and $\lim_{n\to\infty} \mathcal{E}_{M_n}(u_n, u_n) = \mathcal{E}_X(u,u)$. First we notice that for any $x \in X$ and $r > 0$,

$$\lim_{n\to\infty} \int_{B_{M_n}(h_n(x),r)} u_n \, d\mu_{M_n} = \int_{B_\delta(x,r)} u \, d\mu_X.$$

Secondly, we take a positive number ε and a continuous function v_ε such that $0 \leqslant v_\varepsilon \leqslant 1$, $v_\varepsilon = 1$ on $B_\delta(x, 2r+\varepsilon)$ and v_ε vanishes outside of $B_\delta(x, 2r+2\varepsilon)$. Since $B_{M_n}(h_n(x), 2r) \subset f_n^{-1}(B_\delta(x, 2r+\varepsilon))$ for n large, we have

$$\begin{aligned}
\int_X v_\varepsilon \mu_{\langle u,u\rangle} &= \lim_{n\to\infty} \int_{M_n} f_n^* v_\varepsilon |du_n|^2 d\mu_{M_n} \\
&\geqslant \limsup_{n\to\infty} \int_{B_{M_n}(h_n(x),2r)} |du_n|^2 d\mu_{M_n} \\
&\geqslant C_P^{-1} r^{-2} \limsup_{n\to\infty} \int_{B_{M_n}(h_n(x),r)} |u_n - (u_n)_{h_n(x),r}|^2 d\mu_{M_n} \\
&\geqslant C_P^{-1} r^2 \int_{B_\delta(x,r)} |u - u_{x,r}|^2 d\mu_X,
\end{aligned}$$

and hence letting $\varepsilon \to 0$, we obtain

$$\int_{B_\delta(x,2r)} d\mu_{\langle u,u\rangle} \geqslant C_P^{-1} r^{-2} \int_{B_\delta(x,r)} |u - u_{x,r}|^2 d\mu_X.$$

Since $D[\mathcal{E}_X] \cap C(X)$ is dense in $\mathcal{E}_X$ with respect to the $\mathcal{E}_1$-norm, this holds for all $u \in D[\mathcal{E}_X]$.

In order to prove the assertion (ii) and (iii), we shall first recall a basic fact derived from the weak Poincaré inequality: for a function u in

$\mathcal{A}[\mathcal{E}_X]$ and for Lebesgue points x and y of u, it holds that

$$(11)\ |u(x) - u(y)| \leqslant C_5\delta(x,y)\left\{\left(M_{4\delta(x,y)}\Gamma(u,u)(x)\right)^{1/2} + \left(M_{4\delta(x,y)}\Gamma(u,u)(y)\right)^{1/2}\right\},$$

where C_5 is a positive constant depending only on C_D and C_P, and for an integrable function f and a positive number R, $M_R f$ stands for the (restricted) maximal function of f which is defined by

$$M_R f(x) = \sup_{0<r<R} \frac{1}{\mu_X(B_\delta(x,r))} \int_{B_\delta(x,r)} |f|\, d\mu_X, \quad x \in X.$$

The proof of (11) can be found in [15], Lemma 5.14.

As a result of (11), we have

$$|u(x) - u(y)| \leqslant C_5\delta(x,y), \quad x, y \in X$$

if u is a continuous function in $\mathcal{A}[\mathcal{E}_X]$ with $\Gamma(u,u) \leqslant 1$. Therefore it follows that

$$\rho_X(x,y) \leqslant C_5\delta(x,y), \quad x, y \in X.$$

Combining this with (2), we thus conclude that

$$\delta(x,y) \leqslant \rho_{\mathcal{E}_X}(x,y) \leqslant C_5\delta(x,y), \quad x, y \in X.$$

Let us now prove that for a Lipschitz function u with respect to δ,

$$\mathrm{Lip}_\delta u\ (x) \leqslant C_6\Gamma(u,u)^{1/2}(x), \quad a.a.\ x \in X,$$

where $\mathrm{Lip}_\delta u(x)$ is the number defined by

$$\mathrm{Lip}_\delta u(x) = \limsup_{r\to 0} \sup_{\delta(x,y)=r} \frac{|u(y) - u(x)|}{r}$$

and C_6 is a positive constant depending only on C_D and C_P. This estimate can be verified by the same argument as in [8], Proposition 4.26. Indeed, let p be a Lebesgue point of $\Gamma(u,u)$ and let η and ξ be (small) positive numbers which are fixed for a while. Set for simplicity $R(x) = \delta(x, \partial B_\delta(p,\xi))$ for $x \in B_\delta(p,\xi)$, and put

$$\Sigma = \{x \in B_\delta(p,\xi) \mid M_{R(x)}\Gamma(u,u)(x) > \Gamma(u,u)(p) + \eta\}.$$

Then for any $x \in \Sigma$, we can find a number $r(x) \in (0, R(x))$ such that

$$\int_{B_\delta(x,r(x))} (\Gamma(u,u) - \Gamma(u,u)(p))\, d\mu_X > \eta\mu(B_\delta(x,r)).$$

Denote by $\widetilde{\Sigma}$ the union of the balls $B_\delta(x, r(x))$ for all $x \in \Sigma$. Then in view of Vitali covering theorem, we can find a subset $\{x_i\}$ of Σ so that $B_\delta(x_i, r(x_i))$ are mutually disjoint and $\widetilde{\Sigma} \subset \cup_i B_\delta(x_i, 5r(x_i))$. Therefore using (10), we obtain

$$\begin{aligned}
&\int_{B_\delta(p,\xi)} |\Gamma(u,u) - \Gamma(u,u)(p)|\, d\mu_X \\
&\quad \geqslant \sum_i \int_{B_\delta(x_i,r(x_i))} (\Gamma(u,u) - \Gamma(u,u)(p))\, d\mu_X \geqslant \eta \sum_i \mu_X(B_\delta(x_i, r(x_i))) \\
&\quad \geqslant \frac{\eta}{10^\kappa} \sum_i \mu_X(B_\delta(x_i, 5r(x_i))) \geqslant \frac{\eta}{10^\kappa} \mu_X(\widetilde{\Sigma}).
\end{aligned}$$

Let $\varepsilon(\xi)$ be the positive number given by

$$\int_{B_\delta(p,\xi)} |\Gamma(u,u) - \Gamma(u,u)(p)|\, d\mu_X = \eta \left(\frac{\varepsilon(\xi)}{80} \right)^\kappa \mu_X(B_\delta(p,\xi)).$$

Observe that $\varepsilon(\xi)$ tends to zero as $\xi \to 0$, because p is a Lebesgue point of $\Gamma(u,u)$, and moreover that for any $x \in B_\delta(p,\xi)$, there exists a point y_x in $B_\delta(x, \varepsilon(\xi)\xi) \setminus \widetilde{\Sigma}$; otherwise, we would have

$$\begin{aligned}
\int_{B_\delta(p,\xi)} |\Gamma(u,u) - \Gamma(u,u)(p)|\, d\mu_X &\geqslant \frac{\eta}{10^\kappa} \mu_X(\widetilde{\Sigma}) \geqslant \frac{\eta}{10^\kappa} \mu_X(B_\delta(x, \varepsilon(\xi)\xi)) \\
&\geqslant \eta \left(\frac{\varepsilon(\xi)}{40} \right)^\kappa \mu_X(B_\delta(x, 2\xi)) \\
&\geqslant \eta \left(\frac{\varepsilon(\xi)}{40} \right)^\kappa \mu_X(B_\delta(p, \xi)).
\end{aligned}$$

Thus for any $x \in B_\delta(p,\xi)$, we can find a point $y_x \in B_\delta(x, \varepsilon(\xi)\xi)$ such that

$$M_{R(y_x)} \Gamma(u,u)(y_x) < \Gamma(u,u)(p) + \eta.$$

Since $4\delta(y_x, y_p) \leqslant \min\{R(y_x), R(y_p)\}$, for any $x \in B_\delta(p, \xi/5)$, we get

$$\begin{aligned}
M_{4\delta(y_x,y_p)} \Gamma(u,u)(y_x) &< \Gamma(u,u)(p) + \eta, \\
M_{4\delta(y_x,y_p)} \Gamma(u,u)(y_p) &< \Gamma(u,u)(p) + \eta.
\end{aligned}$$

Hence it follows from (11) that

$$\begin{aligned}|u(y_x) - u(y_p)| &\leqslant 2C_5\delta(y_x, y_p)(\Gamma(u,u)(p) + \eta)^{1/2} \\ &\leqslant 2C_5\left(2\varepsilon(\xi) + \frac{1}{5}\right)\xi(\Gamma(u,u)(p) + \eta)^{1/2}.\end{aligned}$$

Noting that $|u(p)-u(y_p)| \leqslant \mathbf{dil}_\delta(u)\,\varepsilon(\xi)\xi$ and $|u(x)-u(y_x)| \leqslant \mathbf{dil}_\delta(u)\,\varepsilon(\xi)\xi$, we obtain

$$|u(p) - u(x)| \leqslant 10\left(\mathbf{dil}_\delta(u)\,\varepsilon(\xi) + C_5\left(2\varepsilon(\xi) + \frac{1}{5}\right)(\Gamma(u,u)(p) + \eta)^{1/2}\right)\frac{\xi}{5},$$

which implies that

$$\begin{aligned}&\sup_{\delta(x,p)=\xi/5} \frac{|u(x) - u(p)|}{\delta(x,p)} \\ &\qquad\leqslant 10\left(\mathbf{dil}_\delta(u)\,\varepsilon(\xi) + C_5\left(2\varepsilon(\xi) + \frac{1}{5}\right)(\Gamma(u,u)(p) + \eta)^{1/2}\right).\end{aligned}$$

Letting ξ tend to 0 and then η go to 0, we obtain

$$\mathrm{Lip}_\delta\, u\,(p) \leqslant 2C_5\Gamma(u,u)(p)^{1/2}$$

for all Lebesgue points p of $\Gamma(u,u)$, and hence

$$\mathrm{dil}_\delta u(p) \leqslant 2C_5\Gamma(u,u)(p)^{1/2}, \quad a.a.\ p \in X,$$

because $\mathrm{Lip}_\delta u = \mathrm{dil}_\delta u$ almost everywhere (cf. the proof of Theorem 6.5 in [8]). This completes the proof of Theorem 0.3.

Remark 3.6. — As is mentioned at the end of the introduction, the results in this paper can be generalized to a family of certain regular Dirichlet spaces. To be precise, a member $(X, \mu, \mathcal{E})$ of the family satisfies the following properties (cf. [27], [28], [29] for details): (i) X is a locally compact, separable, Hausdorff space; (ii) the measuer μ is a Radon measure with support X and unit mass, $\mu(X) = 1$; (iii) the regular Dirichlet form $\mathcal{E}$ is local, the domain $D[\mathcal{E}]$ contains constant 1, and $\mathcal{E}(1,1) = 0$; (iv) the form $\mathcal{E}$ is strongly regular in the sense that the intrinsic metric $\rho_\mathcal{E}$ induces the topology of X and balls are relatively compact; (v) the doubling property $[\mathrm{H}_1]$ with a constant $C_D(X)$, the (weak) Poincaré inequality $[\mathrm{H}_2]'$ with a constant $C_P(X)$ as in Theorem 0.3 hold with respect to the intrinsic metric $\rho_\mathcal{E}$; (vi) $\inf_{x\in X} \mu(B_{\rho_\mathcal{E}}(x,1)) > 0$. Theorems 0.1 and 0.2 are true for a family of such Dirichlet spaces if it satisfies condition $[\mathrm{H}_0]$, and so is Theorem 0.3 provided that the constants in conditions $[\mathrm{H}_1]$, $[\mathrm{H}_2]'$ and $[\mathrm{H}_3]$

can be chosen uniformly in members of the family. Finally we mention the short-time asymptotics of the heat kernel $p(t,x,y)$ of X. K.T. Sturm [28], [29] showed that

$$\lim_{t\to 0} 4t \log p(t,x,y) \leqslant -\rho_{\mathcal{E}}(x,y)^2, \quad x,y\in X,$$

and further under the condition that $\mathcal{A}[\mathcal{E}] = D[\mathcal{E}]$, Ramírez [25] has established that

$$\lim_{t\to 0} 4t \log p(t,x,y) \geqslant -\rho_{\mathcal{E}}(x,y)^2, \quad x,y\in X.$$

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