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Karl-Hermann NEEB

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# CENTRAL EXTENSIONS OF INFINITE-DIMENSIONAL LIE GROUPS

by Karl-Hermann NEEB

## Introduction.

The purpose of this paper is to describe the structure of the abelian group of central extensions of an infinite-dimensional Lie group in the sense of Milnor ([Mi83], resp. [Gl01a]). These Lie groups are manifolds modeled on locally convex spaces. A serious difficulty one has to face in this context is that even Banach manifolds are in general not smoothly paracompact, which means that not every open cover has a subordinated smooth partition of unity. Therefore de Rham's Theorem is not available for these manifolds. Typical examples of Banach–Lie groups which are not smoothly paracompact are the additive groups of the Banach spaces $C([0,1],\mathbb{R})$ and $l^1(\mathbb{N},\mathbb{R})$.

In the Lie theoretic context, the central extensions $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ of interest are those which are principal bundles. For $G$ and $Z$ fixed, the equivalence classes of such extensions can be described by an abelian group $\mathrm{Ext}_{\mathrm{Lie}}(G,Z)$, so that the problem is to describe this group as explicitly as possible. This means in particular to relate it to the Lie algebra cohomology

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group $H^2_c(\mathfrak{g},\mathfrak{z})$ which classifies the central extensions $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ of the topological Lie algebra $\mathfrak{g}$ by the abelian Lie algebra $\mathfrak{z}$ (assumed to be a sequentially complete locally convex space) for which there exists a continuous linear section $\mathfrak{g} \to \widehat{\mathfrak{g}}$. Our central result is the following exact sequence for a connected Lie group $G$, its universal covering group $\widetilde{G}$, the discrete central subgroup $\pi_1(G) \subseteq \widetilde{G}$, and an abelian Lie group $Z$ which can be written as $Z = \mathfrak{z}/\Gamma$, where $\Gamma \subseteq \mathfrak{z}$ is a discrete subgroup (Theorem 7.12):

$$\begin{aligned}&\mathrm{Hom}(G,Z) \hookrightarrow \mathrm{Hom}(\widetilde{G},Z) \to \mathrm{Hom}\,\big(\pi_1(G),Z\big) \xrightarrow{\ C\ } \mathrm{Ext}_{\mathrm{Lie}}(G,Z)\\ (1)\qquad &\xrightarrow{\ D\ } H^2_c(\mathfrak{g},\mathfrak{z}) \xrightarrow{\ P\ } \mathrm{Hom}\,\big(\pi_2(G),Z\big) \times \mathrm{Hom}\,\big(\pi_1(G),\mathrm{Lin}(\mathfrak{g},\mathfrak{z})\big),\end{aligned}$$

where $\mathrm{Lin}(\mathfrak{g},\mathfrak{z})$ denotes the space of continuous linear maps $\mathfrak{g} \to \mathfrak{z}$. Here $C$ assigns to a group homomorphism $\gamma\colon \pi_1(G) \to Z$ the quotient of $\widetilde{G} \times Z$ modulo the graph of $\gamma^{-1}$ (here inversion is meant pointwise in $Z$) and $D$ assigns to a group extension the corresponding Lie algebra extension. The definition of $P$ is more subtle. Let $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ be a continuous Lie algebra cocycle and $\Omega$ be the corresponding left invariant closed $\mathfrak{z}$-valued 2-form on $G$. To obtain the first component $P_1([\omega])$ of $P([\omega])$, we first define a period homomorphism $\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{z}$ by integrating $\Omega$ over sufficiently smooth representatives of homotopy classes. Then $P_1([\omega]) := q_Z \circ \mathrm{per}_\omega$, where $q_Z\colon \mathfrak{z} \to Z$ is the quotient map. The second component $P_2([\omega])$ is defined as follows. For each $X \in \mathfrak{g}$ we write $X_r$ for the corresponding right invariant vector field on $G$. Then $i(X_r).\Omega$ is a closed $\mathfrak{z}$-valued 1-form to which we associate a homomorphism $\pi_1(G) \to \mathfrak{z}$ via an embedding $H^1_{\mathrm{dR}}(G,\mathfrak{z}) \hookrightarrow \mathrm{Hom}(\pi_1(G),\mathfrak{z})$. This embedding is established directly, so that we have it also if $G$ is not smoothly paracompact (Theorem 3.6). In terms of symplectic geometry, the condition $P_2([\omega]) = 0$ means that the action of $G$ on $(G,\Omega)$ has a moment map, but we won't emphasize this point of view.

If the space $H^2_c(\mathfrak{g},\mathfrak{z})$ is trivial or if at least $D = 0$, then (1) leads to

$$\mathrm{Ext}_{\mathrm{Lie}}(G,Z) \cong \mathrm{Hom}\,\big(\pi_1(G),Z\big)/\,\mathrm{Hom}\,\big(\widetilde{G},Z\big)\,|_{\pi_1(G)},$$

a formula which has first been obtained for connected compact Lie groups by A. Shapiro ([Sh49]). Likewise many particular results which essentially are consequences of (1) have been obtained in the setting of finite-dimensional Lie groups by G. Hochschild ([Ho51]).

For a simply connected Lie group $G$ the sequence (1) reduces to

$$(2)\qquad \mathrm{Ext}_{\mathrm{Lie}}(G,Z) \hookrightarrow H^2_c(\mathfrak{g},\mathfrak{z}) \to \mathrm{Hom}\,\big(\pi_2(G),Z\big),$$

showing that in this case the group $\mathrm{Ext}_{\mathrm{Lie}}(G, Z)$ can be identified with the subgroup of $H^2_c(\mathfrak{g}, \mathfrak{z})$ consisting of those classes $[\omega]$ for which the image $\Pi_\omega$ of $\mathrm{per}_\omega$, the so-called period group, is contained in $\Gamma$.

Similar conditions are well-known in the theory of geometric quantization of smoothly paracompact Dirac manifolds $(M, \Omega)$, i.e., $\Omega$ is a closed 2-form on $M$. Here the integrality of the cohomology class $[\Omega]$ of the 2-form $\Omega$ is equivalent to the existence of a so-called pre-quantum bundle, i.e., a $\mathbb{T}$-principal bundle $\mathbb{T} \hookrightarrow \widehat{M} \twoheadrightarrow M$ whose curvature 2-form is $\Omega$ (cf. [TW87] and [Bry93] for the infinite-dimensional case). Based on these observations, Tuynman and Wiegerinck gave a proof of the exactness of (1) in $H^2_c(\mathfrak{g}, \mathbb{R})$ for finite-dimensional Lie algebras $\mathfrak{g}$ ([TW87, Th. 5.4]). As was observed in [Ne96], for finite-dimensional groups $G$ the map $P$ is simpler because the vanishing of $\pi_2(G)$ makes the first component of $P$ superfluous. That the vanishing of $\pi_2(G)$, resp., $H^2_{\mathrm{dR}}(G, \mathbb{R})$ for finite-dimensional Lie groups $G$ permits to construct arbitrary central extensions for simply connected groups is a quite old observation of E. Cartan ([Ca52a]). He used it to prove Lie's Third Theorem by constructing a Lie group associated to a Lie algebra $\mathfrak{g}$ as a central extension of the simply connected covering group of the group $\mathrm{Inn}(\mathfrak{g}) = \langle e^{\mathrm{ad}\,\mathfrak{g}} \rangle$ of inner automorphisms (see also [Est88] for an elaboration of Cartan's method). This method has been extended to Banach–Lie groups by van Est and Korthagen who characterize the existence of a Banach–Lie group with Lie algebra $\mathfrak{g}$ by the discreteness of a topological period group corresponding to the Lie algebra extension $\mathfrak{z}(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathrm{ad}\,\mathfrak{g}$ and the simply connected covering of the group $\mathrm{Inn}(\mathfrak{g})$ endowed with its intrinsic Banach–Lie group structure ([EK64]). It is remarkable that their approach does not require the existence of smooth local sections, which do not always exist for Banach–Lie groups. The reason for this is that they can use van Est's theory of local group extensions and their enlargeability to global extensions because for Banach–Lie groups the existence of local groups corresponding to central extensions of Lie algebras can be obtained from the Baker–Campbell–Hausdorff series, but for more general Lie algebras, this series need not converge on a 0-neighborhood in $\mathfrak{g}$ (see [Gl01b] for a discussion of a large class of groups where the Baker–Campbell–Hausdorff formalism still works well). In a previous version of this paper we have used a result of van Est and Korthagen to show that for a simply connected Lie group $G$ the vanishing of $P([\omega])$ implies the extendability of a local group cocycle $f$ to a global one, and hence the existence of a corresponding global group extension (this is needed for the exactness of (1) in $H^2_c(\mathfrak{g}, \mathfrak{z})$). In the present version we give a much more direct argument which was inspired by the construction of group cocycles in [Est54] by using the symplectic area of geodesic triangles.

For smooth loop groups central extensions are discussed in [PS86], but in this case many difficulties are absent because these groups are modeled on nuclear Fréchet spaces which are smoothly regular ([KM97, Th. 16.10]), hence smoothly paracompact because this holds for every smoothly Hausdorff second countable manifold modeled over a smoothly regular space ([KM97, 27.4]). In [TL99, Prop. 5.3.1] Toledano Laredo discusses central extensions of Lie groups obtained from projective representations with a smooth vector by constructing a corresponding locally smooth 2-cocycle. In Section 5 of his paper he applies results of Pressley and Segal to general groups, which restricts the scope of his theory to smoothly paracompact groups. In Omori's book one also finds some remarks on central $\mathbb{T}$-extensions including in particular Cartan's construction for simply connected regular Fréchet–Lie groups ([Omo97, pp. 252/254]). If the singular cohomology class associated to $\omega$ does not vanish but is integral, then Omori uses simple open covers (the Poincaré Lemma applies to all finite intersections) to construct the $\mathbb{T}$-bundle from the corresponding integral Čech cocycle. Unfortunately it is not known to the author whether all infinite-dimensional Lie groups have such open covers.

It would be very interesting to extend the results and the methods of the present paper to general smooth Lie group extensions. In this context the work of Hochschild ([Ho51]) and Eilenberg-MacLane ([EML47]) contains results which have good potential to extend to infinite-dimensional Lie groups. Another interesting project is to establish the corresponding results for prequantization of manifolds $M$ endowed with a closed 2-form $\Omega$. Here the question is under which conditions there exists a prequantization, i.e., a principal $\mathbb{T}$-bundle $\mathbb{T} \hookrightarrow \widehat{M} \xrightarrow{q} M$ with a connection 1-form $\alpha$ such that $d\alpha = q^*\Omega$, i.e., $\Omega$ is the curvature form of the bundle. For smoothly paracompact manifolds this condition is the discreteness of the group of periods of $\Omega$ ([TW87], [Bry93]). Is this still true for infinite-dimensional manifolds? The results of the present paper cover the case of a Lie group $G$ with a closed left invariant 2-form, where we do not have to assume that $G$ is smoothly paracompact. Unfortunately our methods rely on the group structure of the underlying manifold, hence do not apply to a non-homogeneous setting.

We approach the problem to describe $\mathrm{Ext}_{\mathrm{Lie}}(G, Z)$ by first discussing for abstract groups the exact sequence in Eilenberg–MacLane cohomology induced by a central extension $A \hookrightarrow B \twoheadrightarrow C$ (Theorem I.5, [MacL63]):

$$\begin{aligned}\mathrm{Hom}(C,Z) \hookrightarrow \mathrm{Hom}(B,Z) \to \mathrm{Hom}(A,Z) \to \mathrm{Ext}(C,Z)& \\ \to \mathrm{Ext}_A(B,Z) \to \mathrm{Ext}_{\mathrm{ab}}(A,Z),& \qquad (3)\end{aligned}$$

where $\mathrm{Ext}_A(B,Z)$ denotes the equivalence classes of central extensions $q\colon \widehat{B} \to B$ for which the subgroup $\widehat{A} := q^{-1}(A)$ is central, and $\mathrm{Ext}_{\mathrm{ab}}(A,Z)$ denotes the equivalence classes of abelian group extensions of $A$ by $Z$. This long exact sequence remains valid for central extensions of topological groups and Lie groups as well, if we interpret the Hom- and Ext-groups in the appropriate sense.

In Section 2 we collect the necessary results on central extensions of topological groups and in Section 3 we provide some results on infinite-dimensional manifolds and Lie groups which are well-known in the finite-dimensional case. In particular we show that for general manifolds we have a natural embedding $H^1_{\mathrm{dR}}(M,\mathfrak{z}) \hookrightarrow \mathrm{Hom}(\pi_1(M),\mathfrak{z})$ for the first $\mathfrak{z}$-valued de Rham cohomology group. In Section 4 we explain how the setting for abstract, resp., topological groups has to be modified to deal with central extensions of Lie groups with smooth local sections. Section 5 is dedicated to the construction of the period homomorphism $\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{z}$. If $G$ is smoothly paracompact, it can be obtained quite directly from the de Rham Theorem, but in general one has to construct it directly. This is done by considering piecewise smooth maps from triangulated manifolds with boundary with values in $G$ and by showing that the prescription $\mathrm{per}_\omega([\sigma]) = \int_\sigma \Omega$ works if $\sigma\colon \mathbb{S}^2 \to G$ is piecewise smooth with respect to a triangulation. We also show that whenever we have a central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$, the connecting homomorphism $\delta\colon \pi_2(G) \to \pi_1(Z)$ from the long exact homotopy sequence of the $Z$-bundle $\widehat{G} \to G$ coincides with $-\mathrm{per}_\omega$.

Section 6, which is the heart of the paper, contains the construction of a global group cocycle $f\colon G \times G \to Z$ for simply connected groups $G$ and any Lie algebra cocycle $\omega$, where $Z$ can be defined as $\mathfrak{z}/\Pi_\omega$. The so defined group $Z$ is a Lie group if and only if $\Pi_\omega$ is discrete, so that we obtain a Lie group extension if and only if $\Pi_\omega$ is discrete. In Section 7 we eventually put all pieces together to prove the exactness of (1). An interesting byproduct is that the vanishing of $P_2([\omega])\colon \pi_1(G) \to \mathrm{Lin}(\mathfrak{g},\mathfrak{z})$ precisely describes the condition under which the adjoint action of $\mathfrak{g}$ on the central extension $\widehat{\mathfrak{g}}$ integrates to a smooth representation of the group $G$. In this sense the adjoint and coadjoint action of $G$ on $\widehat{\mathfrak{g}}$ may exist even if the group $\widehat{G}$ does not. This happens in particular if $G$ is simply connected and $\Pi_\omega$ is not discrete.

It is a well-known fact in finite-dimensional Lie theory that extensions of simply connected Lie groups are topologically trivial in the sense that they have a global smooth section, hence can be defined by a global cocycle. For central extensions of simply connected Lie groups the existence of a global smooth section is equivalent to the exactness of the corresponding left invariant closed 2-form $\Omega$ (Proposition 8.4). If $G$ is smoothly paracompact, then each central $\mathfrak{z}$-extension of $G$ has a smooth global section, and one can give more accessible criteria for the existence of a smooth global section. The central result of Section 8 is Theorem 8.8 which gives a version of the exact sequence (1) for central Lie group extensions with smooth global sections.

Section 9 is a collection of examples displaying various aspects in the description of the group $\mathrm{Ext}_{\mathrm{Lie}}(G, Z)$ by the exact sequence (1).

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## 1. The abstract setting for central extensions of groups.

In this section we discuss several aspects of central extensions of groups on the level where no topology or manifold structure is involved. The focus of this section is on the Hom-Ext exact sequence for central extensions of groups (Theorem 1.5; see also [MacL63]). This result can also be obtained by more elaborate spectral sequence arguments which basically are also suited for non-central extensions, but for central extensions it can be obtained quite directly. Moreover, we shall later need explicit information on the maps in this exact sequence to generalize it to central extensions of topological and Lie groups, which will be done by verifying that the crucial steps generalize to the topological and the Lie group context.

Throughout this section $G$ denotes a group and $Z$ an abelian group.

Definition 1.1. — *We define the group*

$$\begin{aligned} Z^2(G,Z) := \{f\colon G\times G\to Z\colon (\forall x,y,z\in G)\\ f(\mathbf{1},x)=f(x,\mathbf{1})=\mathbf{1},\ f(x,y)f(xy,z)=f(x,yz)f(y,z)\} \end{aligned}$$

*of $Z$-valued 2-cocycles and the subgroup*

$$B^2(G,Z) := \{f: G\times G \to Z: (\exists h: G\to Z) h(\mathbf{1}) = \mathbf{1}, \\ (\forall x,y\in G)\, f(x,y) = h(xy)h(x)^{-1}h(y)^{-1}\}$$

*of $Z$-valued 2-coboundaries. In both cases the group structure is given by pointwise multiplication. Since both groups are abelian, it makes sense to define the group*

$$\mathrm{Ext}(G,Z) := H^2(G,Z) := Z^2(G,Z)/B^2(G,Z). \qquad \square$$

*Remark 1.2.* — To each $f \in Z^2(G,Z)$ we associate a central extension of $G$ by $Z$ via

$$\widehat{G} := G \times_f Z, \quad (g,z)(g',z') := \big(gg', zz'f(g,g')\big). \tag{1.1}$$

This multiplication turns $\widehat{G}$ into a group with neutral element $(\mathbf{1},\mathbf{1})$ and inversion given by

$$(g,z)^{-1} = \big(g^{-1}, z^{-1}f(g,g^{-1})^{-1}\big). \tag{1.2}$$

The projection $q: \widehat{G} \to G, (g,z) \mapsto g$ is a homomorphism whose kernel is the central subgroup $Z$, hence defines a central extension of $G$ by $Z$. The conjugation in this group is given by

$$(g,z)(h,w)(g,z)^{-1} = \big(ghg^{-1}, wf(g,h)f(ghg^{-1},g)^{-1}\big). \tag{1.3}$$

One can show that every central $Z$-extension of $G$ can be realized this way, and that the group $H^2(G,Z)$ parametrizes the equivalence classes of central extensions of $G$ by $Z$, justifying the notation $\mathrm{Ext}(G,Z)$ (cf. [MacL63, Th. IV.4.1]). $\square$

*Remark 1.3* (The connecting homomorphism). — Let $E: A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be a central extension of $C$ by $A$. We write $[f_E]$ for the corresponding element of $\mathrm{Ext}(C,A)$, where $f_E \in Z^2(C,A)$ is a representing cocycle. Let $Z$ be an abelian group. We define a homomorphism

$$E^*: \mathrm{Hom}(A,Z) \to \mathrm{Ext}(C,Z), \quad E^*(\gamma) := \gamma_*.[f_E] := [\gamma\circ f_E].$$

It is clear that $E^*$ is a well-defined group homomorphism. The central extension of $C$ by $Z$ corresponding to $[\gamma\circ f_E]$ is given by

$$\widehat{C} := (B\times Z)/D \quad \text{with} \quad D := \{(\alpha(a), \gamma(a)^{-1}) \in B\times Z: a\in A\}.$$

Writing its elements as $[b, z] := (b, z)D$, the homomorphism $q: \widehat{C} \to C$ is given by $q([b, z]) = \beta(b)$. This is the standard pushout construction. $\square$

*Remark 1.4.* — (a) If one is only interested in those central extensions of abelian groups $G$ which are abelian, then one requires the cocycle $f$ to satisfy $f(a, b) = f(b, a)$ which leads to the groups $Z^2_{\mathrm{ab}}(G, Z)$ for abelian groups $G, Z$. We have $B^2_{\mathrm{ab}}(G, Z) = B^2(G, Z)$ because $G$ is abelian, so that we get an inclusion

$$\begin{aligned}\mathrm{Ext}_{\mathrm{ab}}(G, Z) &:= H^2_{\mathrm{ab}}(G, Z) \\ &:= Z^2_{\mathrm{ab}}(G, Z)/B^2_{\mathrm{ab}}(G, Z) \hookrightarrow Z^2(G, Z)/B^2(G, Z) = H^2(G, Z).\end{aligned}$$

(b) Although $\mathrm{Ext}_{\mathrm{ab}}(G, \mathbb{R}) = \mathbf{0}$ holds for each abelian group $G$ because $\mathbb{R}$ is divisible, we might have $\mathrm{Ext}(G, \mathbb{R}) \neq \mathbf{0}$ for certain abelian groups $G$. A typical example is the central extension $\widehat{G}$ of $G := \mathbb{R}^2$ given by $\widehat{G} = \mathbb{R}^3$ with the multiplication

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy')$$

in the three-dimensional Heisenberg group. $\square$

The exact sequence discussed below provides crucial information on how the group $\mathrm{Ext}(C, Z)$ of a quotient $C \cong B/A$ is related to the Ext-groups of $A$ and $B$. Later we will see that it generalizes in an appropriate sense to topological groups and Lie groups. It is instructive to compare Theorems 1.5 and 1.6 below with the corresponding results for abelian groups ([Fu70, Th. 51.3]) which are sharper in the sense that the last map in the sequence is surjective.

THEOREM 1.5. — *Let $E: A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be a central extension of $C$ by $A$, and $Z$ an abelian group. Then*

$$\begin{aligned}&\mathrm{Hom}(C, Z) \hookrightarrow \mathrm{Hom}(B, Z) \longrightarrow \mathrm{Hom}(A, Z) \xrightarrow{E^*} \\ &\qquad \mathrm{Ext}(C, Z) \xrightarrow{\beta^*} \mathrm{Ext}_{\alpha(A)}(B, Z) \xrightarrow{\alpha^*} \mathrm{Ext}_{\mathrm{ab}}(A, Z)\end{aligned}$$

*is exact. Here $\beta^*.[f] := [f \circ (\beta \times \beta)]$ is the inflation map and $\alpha^*.[f] := [f \circ (\alpha \times \alpha)]$ is the restriction map.*

*Proof.* — This is a slight refinement of the well-known exact sequence which stops at $\mathrm{Ext}(B, Z)$ (cf. [MacL63, p. 354] or [We95, 6.8.3]). Essentially the proof for the corresponding exact sequence for abelian groups still works for central extensions (see [Fu70, Th. 51.3] for the corresponding result for abelian groups). Therefore we only discuss the exactness at $\mathrm{Ext}_{\alpha(A)}(B, Z)$.

In view of $\alpha^*\beta^* = (\beta\circ\alpha)^* = 1$, it remains to see that $\ker\alpha^* \subseteq \operatorname{im}\beta^*$. Let $f \in Z^2_{\alpha(A)}(B,Z)$ and $q_B\colon \widehat{B} := B\times_f Z \to B$ be the corresponding central extension. We assume that $[f\circ(\alpha\times\alpha)] = 1$ and have to show that $[f] \in \operatorname{im}\beta^*$. First we observe that there exists a homomorphism $\sigma\colon A \to \widehat{B}$ with $q_B\circ\sigma = \alpha$. The assumption $f \in Z^2_{\alpha(A)}(B,Z)$ implies that $\sigma(A) \subseteq q_B^{-1}(\alpha(A))$ is central in $\widehat{B}$, so that we may form the quotient group $\widehat{C} := \widehat{B}/\sigma(A)$ which is a central extension of $\widehat{C}/\widehat{A} \cong C/A \cong B$ by $\widehat{A}/\sigma(A) \cong Z$. Let $q_C\colon \widehat{C} \to C$ be the corresponding quotient map. Now it suffices to show that

$$\widehat{B} \cong \beta^*\widehat{C} := \{(b,\widehat{c}) \in B\times\widehat{C}\colon \beta(b) = q_C(\widehat{c})\}.$$

We define a homomorphism

$$\gamma\colon \widehat{B} \to \beta^*\widehat{C}, \quad \gamma := (q_B, \widehat{\beta}),$$

where $\widehat{\beta}\colon \widehat{B} \to \widehat{C}$ is the quotient map. That $\operatorname{im}\gamma \subseteq \beta^*\widehat{C}$ follows from $\beta\circ q_B = q_C\circ\widehat{\beta}$. We claim that $\gamma$ is bijective. The injectivity follows from

$$\ker\gamma = \ker q_B \cap \ker\widehat{\beta} = \ker q_B \cap \sigma(A) = \mathbf{1}.$$

To see that $\gamma$ is surjective, let $(b,\widehat{c}) \in \beta^*\widehat{C}$ and pick $\widehat{b} \in \widehat{B}$ with $b = q_B(\widehat{b})$. Then $q_C\widehat{\beta}(\widehat{b}) = \beta q_B(\widehat{b}) = \beta(b) = q_C(\widehat{c})$ implies that there exists a $z \in Z$ with $\widehat{\beta}(\widehat{b})z = \widehat{c}$. Now $\gamma(\widehat{b}z) = (b,\widehat{c})$. □

THEOREM 1.6. — *Let $E\colon A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be an extension of abelian groups and $G$ be a group. Then*

$$\operatorname{Hom}(G,A) \hookrightarrow \operatorname{Hom}(G,B) \longrightarrow \operatorname{Hom}(G,C) \xrightarrow{E_*}$$
$$\operatorname{Ext}(G,A) \xrightarrow{\alpha_*} \operatorname{Ext}(G,B) \xrightarrow{\beta_*} \operatorname{Ext}(G,C)$$

*is exact. Here $\alpha_*.[f] = [\alpha\circ f]$, $\beta_*.[f] = [\beta\circ f]$, and $E_*.\gamma = \gamma^*E$ is the pullback of $E$ to a central extension of $G$.*

*Proof.* — Again, the proof of the corresponding result for abelian groups [Fu70, Th. 51.3] works. □

## 2. Central extensions of topological groups.

For a topological group $G$ and an abelian topological group $Z$ we consider only those central $Z$-extensions $q\colon \widehat{G} \to G$ which are $Z$-principal bundles, i.e., for which there exist an open $\mathbf{1}$-neighborhood $U \subseteq G$ and a continuous map $\sigma: U \to \widehat{G}$ with $q \circ \sigma = \mathrm{id}_U$ (cf. also the approach in [He73]). As we will see below, these are precisely those central extensions that can be represented by a cocycle $f: G \times G \to Z$ which is continuous in a neighborhood of $\mathbf{1} \times \mathbf{1}$, and this leads to a generalization of Theorems 1.5 and 1.6 to central extensions of topological groups. Before we can derive these facts, we collect some generalities on topological groups. Throughout this paper, all topological groups are assumed to be Hausdorff.

LEMMA 2.1.— *Let $G$ be a connected simply connected topological group and $T$ a group. Let $U$ be an open symmetric connected identity neighborhood in $G$ and $f: U \to T$ a function with*

$$f(xy) = f(x)f(y) \quad \text{for} \quad x, y, xy \in U.$$

*Then there exists a unique group homomorphism extending $f$. If, in addition, $T$ is a topological group and $f$ is continuous, then its extension is also continuous.*

*Proof* [HoMo98, Cor. A.2.26]. □

PROPOSITION 2.2.— *Let $G$ and $Z$ be topological groups, where $G$ is connected, and $Z \hookrightarrow \widehat{G} \to G$ a central extension of $G$ by $Z$. Then $\widehat{G}$ carries the structure of a topological group such that $\widehat{G} \to G$ is a $Z$-principal bundle if and only if the central extension can be described by a cocycle $f: G \times G \to Z$ which is continuous in a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$.*

*Proof.*— First we assume that $\widehat{G}$ is a $Z$-principal bundle over $G$. Then there exist a $\mathbf{1}$-neighborhood $U \subseteq G$ and a continuous section $\sigma: U \to \widehat{G}$ of the map $q: \widehat{G} \to G$. We extend $\sigma$ to a global section $G \to \widehat{G}$. Then $f(x, y) := \sigma(x)\sigma(y)\sigma(xy)^{-1}$ defines a 2-cocycle $G \times G \to Z$ which is continuous in a neighborhood of $(\mathbf{1}, \mathbf{1})$.

Conversely, we assume that $\widehat{G} \cong G \times_f Z$ holds for a 2-cocycle $f: G \times G \to Z$ which is continuous in a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$. Let $U \subseteq G$ be an open symmetric $\mathbf{1}$-neighborhood such that $f$ is continuous on $U \times U$, and consider the subset

$$K := U \times Z = q^{-1}(U) \subseteq \widehat{G} = G \times_f Z.$$

Then $K = K^{-1}$. We endow $K$ with the product topology of $U \times Z$. Since the multiplication $m_G|_{U \times U}: U \times U \to G$ is continuous, the set

$$V := \{((x, z), (x', z')) \in K \times K: xx' \in U\}$$

is an open subset of $K \times K$ such that the multiplication map

$$V \to K, \quad ((x,z),(x',z')) \to (xx', zz'f(x,x'))$$

is continuous. In addition, the inversion

$$K \to K, (x,z) \mapsto \left(x^{-1}, z^{-1}f(x,x^{-1})^{-1}\right)$$

is continuous. Since $G$ is connected, it is generated by $U$, and therefore $\widehat{G}$ is generated by $K = q^{-1}(U)$. Therefore [Ti83, p. 62] applies and shows that $\widehat{G}$ carries a unique group topology for which the inclusion map $K = U \times Z \hookrightarrow \widehat{G}$ is an open embedding. It is clear that with respect to this topology, the map $q\colon \widehat{G} \to G$ is a $Z$-principal bundle. □

*Remark 2.3.* — To derive a generalization of Proposition 2.2 to groups which are not necessarily connected, one has to make the additional assumption that for each $g \in G$ the corresponding conjugation map $I_g\colon \widehat{G} \to \widehat{G}$ is continuous in the identity. In view of Proposition 2.2, this condition is automatically satisfied for all elements in the open subgroup generated by $U$, hence redundant if $G$ is connected. □

DEFINITION 2.4. — *Let $G$ and $Z$ be topological groups, where $G$ is connected. We have seen in Proposition 2.2 that the central extensions of $G$ by $Z$ which are principal $Z$-bundles can be represented by 2-cocycles $f\colon G \times G \to Z$ which are continuous in a neighborhood of $(\mathbf{1},\mathbf{1})$ in $G \times G$. We write $Z^2_c(G,Z)$ for the group of these cocycles. Likewise we have a group $B^2_c(G,Z)$ of 2-coboundaries $f(x,y) = h(xy)h(x)^{-1}h(y)^{-1}$, where $h\colon G \to Z$ is continuous in a $\mathbf{1}$-neighborhood. Then the group*

$$\mathrm{Ext}_c(G,Z) := H^2_c(G,Z) := Z^2_c(G,Z)/B^2_c(G,Z)$$

*classifies the central extensions of $G$ by $Z$ which are principal bundles.* □

A typical example of a central extension of a compact group which has no continuous local section is the sequence $\{1,-1\}^{\mathbb{N}} \hookrightarrow \mathbb{T}^{\mathbb{N}} \xrightarrow{q} \mathbb{T}^{\mathbb{N}}$, where $q(x) = x^2$ is the squaring map on the infinite-dimensional torus $\mathbb{T}^{\mathbb{N}}$.

*Remark 2.5.* — (a) We consider the setting of Remark 1.3, where $B$ is a principal $A$-bundle. This means that there exists a local section $\sigma\colon U_C \to B$ which can be used to obtain a local section of $\widehat{C} \to C$, so that $E^*$ maps continuous homomorphisms to central extensions with continuous local sections.

Therefore the maps in Theorem I.5 are compatible with the topological situation, and we thus obtain for connected groups $A$, $B$ and $C$ the sequence of maps

$$\mathrm{Hom}(C,Z) \hookrightarrow \mathrm{Hom}(B,Z) \longrightarrow \mathrm{Hom}(A,Z) \xrightarrow{E^*} \mathrm{Ext}_c(C,Z) \xrightarrow{\beta^*} \mathrm{Ext}_{c,\alpha(A)}(B,Z) \xrightarrow{\alpha^*} \mathrm{Ext}_{c,\mathrm{ab}}(A,Z),$$

where Hom denotes continuous homomorphisms.

It is easy to verify that the proof of Theorem 1.5 remains valid in this topological context (cf. [Se70, Prop. 4.1]). One has to use the following easy facts:

(1) Pull-backs and pushout constructions preserve the existence of continuous local sections.

(2) For central extensions $\beta: B \to C$ with continuous local sections a continuous homomorphism $f: B \to Z$ factors through a continuous homomorphism $C \to Z$ if and only if $\ker\beta \subseteq \ker f$.

(3) A group homomorphism between topological groups is continuous if and only if it is continuous in the identity, resp., on a neighborhood of the identity.

(b) Similar arguments show that each extension $E: A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ of abelian topological groups which is a principal $A$-bundle leads for each connected topological group $G$ to an exact sequence

$$\mathrm{Hom}(G,A) \hookrightarrow \mathrm{Hom}(G,B) \longrightarrow \mathrm{Hom}(G,C) \xrightarrow{E_*} \mathrm{Ext}_c(G,A) \xrightarrow{\alpha_*} \mathrm{Ext}_c(G,B) \xrightarrow{\beta_*} \mathrm{Ext}_c(G,C). \qquad \square$$

It is instructive to describe the image of $E^*$ corresponding to a universal covering map $q_G: \widetilde{G} \to G$ for a topological group $G$.

Proposition 2.6. — *Let $G$ be a connected, locally arcwise connected and semilocally simply connected topological group and $q_G: \widetilde{G} \to G$ a universal covering homomorphism. We identify $\pi_1(G)$ with $\ker q_G$. For a central extension of topological groups $Z \hookrightarrow \widehat{G} \xrightarrow{q} G$ the following conditions are equivalent:*

(1) *There exists a continuous local section $\sigma_U: U \to \widehat{G}$ with $\sigma_U(xy) = \sigma_U(x)\sigma_U(y)$ for $x,\ y,\ xy \in U$.*

(2) *$\widehat{G} \cong G \times_f Z$, where $f \in Z^2(G,Z)$ takes the value $\mathbf{1}$ on a neighborhood of $(\mathbf{1},\mathbf{1})$ in $G \times G$.*

(3) *There exists a homomorphism* $\gamma\colon \pi_1(G) \to Z$ *and an isomorphism* $\Phi\colon (\widetilde{G} \times Z)/\Gamma(\gamma^{-1}) \to \widehat{G}$ *with* $q\Phi([x, \mathbf{1}]) = q_G(x)$, $x \in \widetilde{G}$, *where* $\Gamma(\gamma^{-1})$ *is the graph of* $d \mapsto \gamma(d)^{-1}$.

*Proof.* — (1) $\Leftrightarrow$ (2) follows directly from the definitions.

(1) $\Rightarrow$ (3): We may w.l.o.g. assume that $U$ is connected, $U = U^{-1}$, and that there exists a continuous section $\widetilde{\sigma}\colon U \to \widetilde{G}$ of the universal covering map $q_G$. Then

$$\sigma_U \circ q_G \,|_{\widetilde{\sigma}(U)}\colon \widetilde{\sigma}(U) \to \widehat{G}$$

extends uniquely to a continuous homomorphism $f\colon \widetilde{G} \to \widehat{G}$ with $f \circ \widetilde{\sigma} = \sigma_U$ and $q \circ f = q_G$ (Lemma II.1). We define $\psi\colon \widetilde{G} \times Z \to \widehat{G}$, $(g, z) \mapsto f(g)z$. Then $\psi$ is a continuous group homomorphism which is a local homeomorphism because

$$\psi(\widetilde{\sigma}(x), z) = f(\widetilde{\sigma}(x))z = \sigma_U(x)z \quad \text{for} \quad x \in U, z \in Z.$$

We conclude that $\psi$ is a covering homomorphism. Moreover, $\psi$ is surjective because its range is a subgroup of $\widehat{G}$ containing $Z$ and mapped surjectively by $q$ onto $G$. This proves that

$$\widehat{G} \cong (\widetilde{G} \times Z)/\ker\psi, \quad \ker\psi = \{(g, f(g)^{-1})\colon g \in f^{-1}(Z)\}.$$

On the other hand, $f^{-1}(Z) = \ker(q \circ f) = \ker q_G = \pi_1(G)$, so that

$$\ker\psi = \{(d, \gamma(d)^{-1})\colon d \in \pi_1(G)\} = \Gamma(\gamma^{-1}), \quad \gamma := f\,|_{\pi_1(G)}.$$

(3) $\Rightarrow$ (1) follows directly from the fact that the map $\widetilde{G} \times Z \to \widehat{G}$ is a covering morphism. □

## 3. Topology of infinite-dimensional manifolds.

So far we have only dealt with abstract groups or topological groups. In this section we turn to manifolds and specifically to infinite-dimensional ones modeled on locally convex spaces. Sometimes we will have to require the model space to be sequentially complete to ensures the existence of Riemann integrals. For more details on this setting we refer to [Mi83], [Gl01a] and [Ne01a]. As we will explain in some more detail below, the approach of Kriegl and Michor ([KM97]) is slightly different, but coincides with the other one for Fréchet manifolds, i.e., manifolds modeled on Fréchet spaces. An unpleasant obstacle one has to face when dealing with infinite-dimensional manifolds is that they need not be smoothly paracompact, i.e., not every open cover has a subordinate smooth partition of unity (cf. [KM97]). Hence there is no a priori reason for de Rham isomorphisms $H^n_{\rm dR}(M,\mathbb{R}) \cong H^n_{\rm sing}(M,\mathbb{R})$ to hold because the sheaf theoretic proofs break down. This is a problem that already arises in the classical setting of Banach manifolds because there are Banach spaces $M$ for which there exists no smooth function supported by the unit ball, so that $M$ is in particular not smoothly paracompact. Simple examples are the spaces $C([0,1])$ and $l^1(\mathbb{N})$ (cf. [KM97, 14.11]). On the topological side, paracompactness is a natural assumption on manifolds. In view of Theorem 1 in [Pa66], a manifold is metrizable if and only if it is first countable and paracompact which for sequentially complete model spaces implies in particular that the model space is Fréchet (cf. [KM97, Lemma 27.8]). Fréchet–Lie groups are always paracompact because they are first countable topological groups, hence metrizable.

It is a central idea in this paper that all those parts of the de Rham isomorphism that are essential to study central extensions of Lie groups still remain true to a sufficient extent. Here a key point is that the Poincaré Lemma is still valid. In particular we will see that we have an injection

$$H^1_{\rm dR}(M,\mathbb{R}) \hookrightarrow H^1_{\rm sing}(M,\mathbb{R}) \cong \mathrm{Hom}(\pi_1(M),\mathbb{R}),$$

where the isomorphism $H^1_{\rm sing}(M,\mathbb{R}) \cong \mathrm{Hom}(\pi_1(M),\mathbb{R})$ is a direct consequence of the Hurewicz Theorem (Remark A.1.4).

Definition 3.1. — (a) *Let $X$ and $Y$ be topological vector spaces, $U \subseteq X$ open and $f\colon U \to Y$ a continuous map. Then the* derivative of $f$ at $x$ in the direction of $h$ *is defined as*

$$df(x)(h) := \lim_{t\to 0} \frac{1}{t}\big(f(x+th) - f(x)\big)$$

*whenever it exists. The function $f$ is called* differentiable in $x$ *if $df(x)(h)$ exists for all $h \in X$. It is called* continuously differentiable or $C^1$ *if it is differentiable in all points of $U$ and*

$$df\colon U \times X \to Y, \quad (x,h) \mapsto df(x)(h)$$

*is a continuous map. It is called a $C^n$-map if $df$ is a $C^{n-1}$-map, and $C^\infty$ if it is $C^n$ for all $n \in \mathbb{N}$. This is the notion of differentiability used in [Mi83], [Ha82], [Gl01a] and [Ne01a].*

(b) *We briefly recall the basic definitions underlying the convenient calculus in [KM97]. Let $E$ be a locally convex space. The* $c^\infty$-topology *on $E$ is the final topology with respect to the set $C^\infty(\mathbb{R}, E)$. We call $E$* convenient *if for each smooth curve $c_1: \mathbb{R} \to E$ there exists a smooth curve $c_2: \mathbb{R} \to E$ with $c_2' = c_1$ (cf. [KM97, p. 20]).*

*Let $U \subseteq E$ be an open subset and $f: U \to F$ a function, where $F$ is a locally convex space. Then we call $f$* conveniently smooth *if*

$$f \circ C^\infty(\mathbb{R}, U) \subseteq C^\infty(\mathbb{R}, F).$$

*This concept quite directly implies nice cartesian closedness properties for smooth maps (cf. [KM97, p. 30]).* □

*Remark 3.2.* — If $E$ is a sequentially complete locally convex (s.c.l.c.) space, then it is convenient because the sequential completeness implies the existence of Riemann integrals ([KM97, Th. 2.14]). If $E$ is a Fréchet space, then the $c^\infty$-topology coincides with the original topology ([KM97, Th. 4.11]).

Moreover, for an open subset $U$ of a Fréchet space, a map $f: U \to F$ is conveniently smooth if and only if it is smooth in the sense of [Mi83]. This can be shown as follows. Since $C^\infty(\mathbb{R}, E)$ is the same space for both concepts of differentiability, the chain rule shows that smoothness in the sense of [Mi83] implies smoothness in the sense of convenient calculus. Now we assume that $f: U \to F$ is conveniently smooth. Then the derivative $df: U \times E \to F$ exists and defines a conveniently smooth map $df: U \to L(E, F) \subseteq C^\infty(E, F)$ ([KM97, Th. 3.18]). Hence $df: U \times E \to F$ is also conveniently smooth, hence continuous with respect to the $c^\infty$-topology. As $E \times E$ is a Fréchet space, it follows that $df$ is continuous. Therefore $f$ is $C^1$ in the sense of [Mi83], and now one can iterate the argument. □

If $M$ is a differentiable manifold and $\mathfrak{z}$ a locally convex space, then a $\mathfrak{z}$-valued $k$-form $\omega$ on $M$ is a function $\omega$ which associates to each $p \in M$ a $k$-linear alternating map $T_p(M)^k \to \mathfrak{z}$ such that in local coordinates the map $(p, v_1, \ldots, v_k) \mapsto \omega(p)(v_1, \ldots, v_k)$ is smooth. We write $\Omega^k(M, \mathfrak{z})$ for the space of smooth $k$-forms on $M$ with values in $\mathfrak{z}$.

LEMMA 3.3 (Poincaré Lemma). — *Let $E$ be locally convex, $\mathfrak{z}$ an s.c.l.c. space and $U \subseteq E$ an open subset which is star-shaped with respect to* 0. *Let $\omega \in \Omega^{k+1}(U, \mathfrak{z})$ be a $\mathfrak{z}$-valued closed $k+1$-form. Then $\omega$ is exact. Moreover,*

$\omega = d\varphi$ *for some* $\varphi \in \Omega^k(U, \mathfrak{z})$ *with* $\varphi(0) = 0$ *given by*

$$\varphi(x)(v_1, \ldots, v_k) = \int_0^1 t^k \omega(tx)(x, v_1, \ldots, v_k)\, dt.$$

*Proof.* — For the case of Fréchet spaces Remark 3.2 implies that the assertion follows from [KM97, Lemma 33.20]. On the other hand, one can prove it directly in the context of locally convex spaces by using the fact that one may differentiate under the integral for a function of the type $\int_0^1 H(t,x)\,dt$, where $H$ is a smooth function $]-\varepsilon, 1+\varepsilon[\times U \to \mathfrak{z}$ (cf. [KM97, p. 32]). The existence of the integrals follows from the sequential completeness of $\mathfrak{z}$. For the calculations needed for the proof we refer to [La99, Th. V.4.1]. □

PROPOSITION 3.4. — *Let $M$ be a connected manifold, $\mathfrak{z}$ an s.c.l.c. space and $\alpha \in \Omega^1(M, \mathfrak{z})$ a closed 1-form. Then there exists a connected covering $q: \widehat{M} \to M$ and a smooth function $f: \widehat{M} \to \mathfrak{z}$ with $df = q^*\alpha$.*

*Proof* (cf. Sect. XIV.2 in [God71] for the finite-dimensional case). — On $M$ we consider the pre-sheaf $\mathcal{F}$ given for an open subset $U \subseteq M$ by

$$\mathcal{F}(U) := \{f \in C^\infty(U, \mathfrak{z}) : df = \alpha|_U\}.$$

It is easy to verify that $\mathcal{F}$ is a sheaf on $M$ (cf. [We80, Sect. 2.1]).

To determine the stalks $\mathcal{F}_x$, $x \in M$, of the sheaf $\mathcal{F}$, we use the Poincaré Lemma. Let $x \in M$. Since $M$ is a manifold, there exists a neighborhood $U$ of $x$ which is diffeomorphic to a convex subset of a locally convex space. Then the Poincaré Lemma implies for each $y \in \mathfrak{z}$ the existence of a smooth function $f_U$ on $U$ with $df_U = \alpha|_U$ and $f_U(x) = y$. Since $U$ is connected, the function $f_U$ is uniquely determined by its value in $x$. Now let $V$ be another open set containing $x$, and $f_V \in \mathcal{F}(V)$ with $[f_U]_x = [f_V]_x$. Choosing an open neighborhood $W \subseteq U \cap V$ of $x$ which is diffeomorphic to a convex domain, we conclude from $f_U(x) = f_V(x) = y$ that $f_V|_W = f_U|_W$. Therefore the map $\mathcal{F}_x \to \mathfrak{z}$, $[f]_x \mapsto f(x)$ is a linear bijection.

Now let $p: \widetilde{\mathcal{F}} = \bigcup_{x \in X} \mathcal{F}_x \to M$ denote the étale space over $M$ associated to the sheaf $\mathcal{F}$. We claim that $p$ is a covering map. Let $x \in X$ and $U$ as above. Then $\mathcal{F}(U) \cong \mathfrak{z}$, as we have seen above. Therefore $\Gamma(U, \widetilde{\mathcal{F}}) \cong \mathcal{F}(U) \cong \mathcal{F}_x$ (cf. [We80, Th. 2.2.2]). For each $z \in \mathfrak{z}$ we write $s_z: U \to \widetilde{\mathcal{F}}$ for the continuous section given by $s_z(y) = [f_z]_y$, where $f_z \in \mathcal{F}(U)$ satisfies $f_z(x) = z$. Then the sets $s_z(U)$ are open subsets of $\widetilde{\mathcal{F}}$ by the definition of the topology on $\widetilde{\mathcal{F}}$ ([We80, p. 42]). Moreover, these sets are disjoint because $[f_z]_y = [f_w]_u$ first implies $u = y$ and further $f_z(y) = f_w(u)$, so that $f_z = f_w$ and therefore $z = w$. This proves that $p^{-1}(U) = \dot\bigcup_{z\in\mathfrak{z}} s_z(U)$ is a disjoint union of open sets, where $s_z : U \to s_z(U)$ is a homeomorphism for each $z$ by the construction of $\widetilde{\mathcal{F}}$. Thus $p$ is a covering map.

Pick $x_0 \in M$ and an inverse image $y_0 \in \widetilde{\mathcal{F}}$. Then the connected component $\widehat{M}$ of $\widetilde{\mathcal{F}}$ containing $y_0$ is a manifold with a covering map $q: \widehat{M} \to M$. Moreover, the function $f: \widehat{M} \to \mathfrak{z}, [s]_y \mapsto s(y)$ is smooth. It remains to show that $q^*\alpha = df$. So let $s: U \to \widetilde{\mathcal{F}}$ be a smooth section of $\widetilde{\mathcal{F}}$. Then $f \circ s \in C^\infty(U, \mathfrak{z})$ is a smooth function with $df(s(x))ds(x) = d(f \circ s)(x) = \alpha(x)$ for all $x \in U$. Since $ds(x) = (dq(s(x)))^{-1}$, it follows that $df(s(x)) = (q^*\alpha)(s(x))$, and therefore that $df = q^*\alpha$. □

COROLLARY 3.5. — *If $M$ is a simply connected manifold and $\mathfrak{z}$ an s.c.l.c. space, then $H^1_{\mathrm{dR}}(M, \mathfrak{z})$ vanishes.*

*Proof.* — Let $\alpha$ be a closed $\mathfrak{z}$-valued 1-form on $M$. Using Proposition 3.4, we find a covering $q: \widehat{M} \to M$ and a smooth function $f: \widehat{M} \to \mathfrak{z}$ with $df = q^*\alpha$. Since $M$ is simply connected, the covering $q$ is trivial, hence a diffeomorphism. Therefore $\alpha$ is exact. □

THEOREM 3.6. — *Let $M$ be a connected manifold, $\mathfrak{z}$ an s.c.l.c. space, $x_0 \in M$, and $\pi_1(M) := \pi_1(M, x_0)$. Then we have an inclusion*

$$\zeta: H^1_{\mathrm{dR}}(M, \mathfrak{z}) \hookrightarrow \mathrm{Hom}(\pi_1(M), \mathfrak{z})$$

*which is given on a piecewise differentiable loop $\gamma: [0,1] \to M$ in $x_0$ for $\alpha \in Z^1_{\mathrm{dR}}(M, \mathfrak{z})$ by*

$$\zeta(\alpha)(\gamma) := \zeta([\alpha])([\gamma]) = \int_\gamma \alpha := \int_0^1 \gamma^*\alpha.$$

*The homomorphism $\zeta([\alpha])$ can also be calculated as follows: Let $f_\alpha \in C^\infty(\widetilde{M}, \mathfrak{z})$ with $df_\alpha = q^*\alpha$, where $q: \widetilde{M} \to M$ is the universal covering map, and write $\widetilde{M} \times \pi_1(M) \to \widetilde{M}, (x, g) \mapsto \mu_g(x)$ for the right action of $\pi_1(M)$ on $\widetilde{M}$. Then the function $f_\alpha \circ \mu_g - f_\alpha$ is constant equal to $\zeta([\alpha])(g)$.*

*Proof* (cf. Theorem 14.1.7 in [God71]). — Let $q: \widetilde{M} \to M$ be a simply connected covering manifold and $y_0 \in q^{-1}(x_0)$. In view of Corollary 3.5, for each closed 1-form $\alpha$ on $M$, the closed 1-form $q^*\alpha$ on $\widetilde{M}$ is exact. Let $f_\alpha \in C^\infty(\widetilde{M}, \mathfrak{z})$ with $\widetilde{f}_\alpha(y_0) = 0$ and $d\widetilde{f}_\alpha = q^*\alpha$.

Let $\widetilde{M} \times \pi_1(M) \to \widetilde{M}, (y,g) \mapsto \mu_g(y) := y.g$ denote the action of $\pi_1(M)$ on $\widetilde{M}$ by deck transformations. We put

$$\zeta(\alpha)(g) := f_\alpha(y_0.g).$$

Then $\zeta(\alpha)(\mathbf{1}) = 0$ and

$$\begin{aligned}\zeta(\alpha)(g_1g_2) = f_\alpha(y_0.g_1g_2) &= f_\alpha(y_0.g_1g_2) - f_\alpha(y_0.g_1) + f_\alpha(y_0.g_1)\\ &= f_\alpha(y_0.g_1g_2) - f_\alpha(y_0.g_1) + \zeta(\alpha)(g_1).\end{aligned}$$

For each $g \in \pi_1(M)$ the function $h := \mu_g^* f_\alpha - f_\alpha$ satisfies $h(y_0) = \zeta(\alpha)(g) = f_\alpha(y_0.g)$ and

$$dh = \mu_g^* df_\alpha - df_\alpha = \mu_g^* q^*\alpha - q^*\alpha = (q \circ \mu_g)^*\alpha - q^*\alpha = q^*\alpha - q^*\alpha = 0.$$

Therefore $h$ is constantly $\zeta(\alpha)(g)$, and we obtain $\zeta(\alpha)(g_1g_2) = \zeta(\alpha)(g_2) + \zeta(\alpha)(g_1)$. This proves that $\zeta(\alpha) \in \mathrm{Hom}(\pi_1(M), \mathfrak{z})$.

Suppose that $\zeta(\alpha) = 0$. Then $\mu_g^* f_\alpha - f_\alpha = 0$ holds for each $g \in \pi_1(M)$, showing that the function $f_\alpha$ factors through a smooth function $f: M \to \mathfrak{z}$ with $f \circ q = f_\alpha$. Now $q^*df = df_\alpha = q^*\alpha$ implies $df = \alpha$, so that $\alpha$ is exact. If, conversely, $\alpha$ is exact, then the function $f_\alpha$ is invariant under $\pi_1(M)$, and we see that $\zeta(\alpha) = 0$. Therefore $\zeta: Z^1_{\mathrm{dR}}(M, \mathfrak{z}) \to \mathrm{Hom}(\pi_1(M), \mathfrak{z})$ factors through an inclusion $H^1_{\mathrm{dR}}(M, \mathfrak{z}) \hookrightarrow \mathrm{Hom}(\pi_1(M), \mathfrak{z})$.

Finally, let $[\gamma] \in \pi_1(M)$, where $\gamma: [0,1] \to M$ is piecewise smooth. Let $\widetilde{\gamma}: [0,1] \to \widetilde{M}$ be a lift of $\gamma$ with $\widetilde{\gamma}(0) = y_0$. Then

$$\begin{aligned}&\zeta([\alpha])([\gamma]) = f_\alpha([\gamma]) = f_\alpha(\widetilde{\gamma}(1)) = f_\alpha(\widetilde{\gamma}(0)) + \int_0^1 df_\alpha(\widetilde{\gamma}(t))(\widetilde{\gamma}'(t))\, dt\\ &= f_\alpha(y_0) + \int_0^1 (q^*\alpha)(\widetilde{\gamma}(t))(\widetilde{\gamma}'(t))\, dt = \int_0^1 \alpha(\gamma(t))(\gamma'(t))\, dt = \int_0^1 \gamma^*\alpha = \int_\gamma \alpha.\end{aligned}$$

□

The following lemma shows that exactness of a vector-valued 1-form can be tested by looking at the associated scalar-valued 1-forms.

Lemma 3.7. — *Let $\alpha \in \Omega^1(M, \mathfrak{z})$ be a closed 1-form. If for each continuous linear functional $\lambda$ on $\mathfrak{z}$ the 1-form $\lambda \circ \alpha$ is exact, then $\alpha$ is exact.*

*Proof.* — If $\lambda \circ \alpha$ is exact, then the group homomorphism $\zeta(\alpha) : \pi_1(M) \to \mathfrak{z}$ satisfies $\lambda \circ \zeta(\alpha) = 0$ (Theorem 3.6). If this holds for each

$\lambda \in \mathfrak{z}' = \mathrm{Lin}(\mathfrak{z}, \mathbb{R})$, then the fact that the continuous linear functionals on the locally convex space $\mathfrak{z}$ separate points implies that $\zeta(\alpha) = 0$ and hence that $\alpha$ is exact. □

To see that the map $\zeta$ is surjective, one needs smooth paracompactness which is not always available, not even for Banach manifolds. For an infinite-dimensional version of de Rham's Theorem for smoothly paracompact manifolds we refer to [KM97, Thm. 34.7]. The following proposition is a particular consequence:

PROPOSITION 3.8. — *If $M$ is a connected smoothly paracompact manifold, then the inclusion map $\zeta: H^1_{\mathrm{dR}}(M, \mathfrak{z}) \to \mathrm{Hom}(\pi_1(M), \mathfrak{z})$ is bijective.*

□

PROPOSITION 3.9. — *Let $M$ be a connected manifold, $\mathfrak{z}$ an s.c.l.c. space and $\Gamma \subseteq \mathfrak{z}$ a discrete subgroup. Then $\mathfrak{z}/\Gamma$ carries a natural manifold structure such that the tangent space in every element of $\mathfrak{z}/\Gamma$ can be canonically identified with $\mathfrak{z}$. For a smooth function $f: M \to \mathfrak{z}/\Gamma$ we can thus identify the differential $df$ with a $\mathfrak{z}$-valued 1-form on $M$. For a closed $\mathfrak{z}$-valued 1-form $\alpha$ on $M$ the following conditions are equivalent:*

(1) *There exists a smooth function $f: M \to \mathfrak{z}/\Gamma$ with $df = \alpha$.*

(2) $\zeta(\alpha)\big(\pi_1(M)\big) \subseteq \Gamma$.

*Proof.* — Let $q: \widetilde{M} \to M$ denote the universal covering map and fix a point $x_0 \in \widetilde{M}$. Then the closed 1-form $q^*\alpha$ on $\widetilde{M}$ is exact (Theorem 3.6), so that there exists a unique smooth function $\widetilde{f}: \widetilde{M} \to \mathfrak{z}$ with $d\widetilde{f} = q^*\alpha$ and $\widetilde{f}(x_0) = 0$. In Theorem 3.6 we have seen that for each $g \in \pi_1(M)$ we have

$$\mu_g^* \widetilde{f} - \widetilde{f} = \zeta(\alpha)(g). \tag{3.1}$$

(1) $\Rightarrow$ (2): Let $p: \mathfrak{z} \to \mathfrak{z}/\Gamma$ denote the quotient map. We may w.l.o.g. assume that $f\big(q(x_0)\big) = p(0)$. The function $p \circ \widetilde{f}: \widetilde{M} \to \mathfrak{z}/\Gamma$ satisfies $d(p \circ \widetilde{f}) = q^*\alpha$, and the same is true for $f \circ q: \widetilde{M} \to \mathfrak{z}/\Gamma$. Since both have the same value in $x_0$, we see that $p \circ \widetilde{f} = f \circ q$. This proves that $p \circ \widetilde{f}$ is invariant under $\pi_1(M)$, and therefore (3.1) shows that $\zeta(\alpha)\big(\pi_1(M)\big) \subseteq \Gamma$.

(2) $\Rightarrow$ (1): If (2) is satisfied, then (3.1) implies that the function $p \circ \widetilde{f}: \widetilde{M} \to \mathfrak{z}/\Gamma$ is $\pi_1(M)$-invariant, hence factors through a function $f: M \to \mathfrak{z}/\Gamma$ with $f \circ q = p \circ \widetilde{f}$. Then $f$ is smooth and satisfies $q^* df = d\widetilde{f} = q^*\alpha$, which implies that $df = \alpha$. □

**Applications to Lie groups.**

Next we apply the results of this section to homomorphisms of Lie groups. A Lie group $G$ is a group and a manifold modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. We write $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for the left, resp., right multiplication on $G$. Then each $X \in T_{\mathbf{1}}(G)$ corresponds to a unique left invariant vector field $X_l$ with $X_l(g) := d\lambda_g(\mathbf{1}).X, g \in G$. The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak{g} := T_{\mathbf{1}}(G)$ a continuous Lie bracket which is uniquely determined by $[X,Y]_l = [X_l, Y_l]$. Similarly we obtain right invariant vector fields $X_r(g) = d\rho_g(\mathbf{1}).X$, and they satisfy $[X_r, Y_r] = -[X,Y]_r$ (cf. [Mi83], [Gl01a], [Ne01a], [KM97]).

Lemma 3.10. — *Let $G$ be a Lie group, $\mathfrak{z}$ an s.c.l.c. space and $C^n_c(\mathfrak{g},\mathfrak{z})$ the space of alternating continuous $n$-linear maps $\mathfrak{g}^n \to \mathfrak{z}$. Then the isomorphisms of vector spaces*

$$L: C^n_c(\mathfrak{g},\mathfrak{z}) \to \Omega^n(G,\mathfrak{z}),$$
$$L(\alpha)(g)(v_1,\ldots,v_n) := \alpha\big(d\lambda_{g^{-1}}(g).v_1,\ldots,d\lambda_{g^{-1}}(g).v_n\big)$$

*assigning to $\alpha \in C_c(\mathfrak{g},\mathfrak{z})$ the corresponding left invariant $n$-form $L(\alpha) \in \Omega^n(G,\mathfrak{z})$ intertwine the differentials on $C^*_c(\mathfrak{g},\mathfrak{z})$ and $\Omega^*(G,\mathfrak{z})$. In particular, $L(Z^n_c(\mathfrak{g},\mathfrak{z}))$ consists of closed forms and $L(B^n_c(\mathfrak{g},\mathfrak{z}))$ of exact forms.*

*Proof.* — It suffices to evaluate $L(\alpha)$ on left invariant vector fields $X_i$. Then the formula

$$d(L(\alpha))(X_1,\ldots,X_n) = L(d\alpha)(X_1,\ldots,X_n)$$

follows directly from the definition of the differentials on both sides. □

Lemma 3.11. — *Let $G$ be a Lie group, $\mathfrak{z}$ an s.c.l.c. space, $\Omega \in \Omega^2(G,\mathfrak{z})$ a left invariant closed 2-form, and $X \in \mathfrak{g}$. Then the $\mathfrak{z}$-valued 1-form $i(X_r).\Omega = \Omega(X_r,\cdot)$ on $G$ is closed. If $\Omega = d\alpha$ for a left invariant 1-form $\alpha$, then all the forms $i(X_r).\Omega$ are exact.*

*Proof.* — Since the right invariant vector fields on $G$ correspond to the left multiplication action of $G$ on itself, Lemma A.2.5 implies that for each $X \in \mathfrak{g}$

$$d\big(i(X_r).\Omega\big) = \mathcal{L}_{X_r}.\Omega - i(X_r).d\Omega = \mathcal{L}_{X_r}.\Omega = 0.$$

If $\alpha$ is a left invariant 1-form with $\Omega = d\alpha$, then for each $X \in \mathfrak{g}$ Lemma A.2.5 leads to

$$0 = \mathcal{L}_{X_r}.\alpha = i(X_r).d\alpha + d(i(X_r).\alpha),$$

which implies that $i(X_r).d\alpha = i(X_r).\Omega$ is exact. □

DEFINITION 3.12. — *A Lie group $G$ is called* regular *if for each closed interval $I \subseteq \mathbb{R}$, $0 \in I$, and $X \in C^\infty(I, \mathfrak{g})$ the ordinary differential equation*

$$\gamma(0) = \mathbf{1}, \quad \gamma'(t) = d\rho_{\gamma(t)}(\mathbf{1}).X(t)$$

*has a solution $\gamma_X \in C^\infty(I, G)$ and the evolution map*

$$\mathrm{evol}_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to G, \quad X \mapsto \gamma_X(1)$$

*is smooth.* □

*Remark 3.13.* — If $\mathfrak{z}$ is an s.c.l.c. vector space, then $\mathfrak{z}$ is a regular Lie group because the Fundamental Theorem of Calculus holds for curves in $\mathfrak{z}$. The smoothness of the evolution map is trivial in this case because it is a continuous linear map. Regularity is trivially inherited by all Lie groups $Z = \mathfrak{z}/\Gamma$, where $\Gamma \subseteq \mathfrak{z}$ is a discrete subgroup.

If, conversely, $Z$ is a regular Fréchet–Lie group, then the exponential function $\exp : \mathfrak{z} \to Z_0$ is a universal covering homomorphism, so that $Z_0 \cong \mathfrak{z}/\Gamma$ holds for the identity component $Z_0$ of $Z$, where $\Gamma := \ker \exp \cong \pi_1(Z)$ ([MT99]). □

For an example of a Lie group without exponential function we refer to [Gl01c, Sect. 7]. In this example the group is not modeled over a s.c.l.c. space and so far, no example of a non-regular s.c.l.c. Lie group is known.

LEMMA 3.14 ([Mi83, Lemma 7.1]). — *Let $G$ and $H$ be connected Lie groups and $\varphi_{1/2} : G \to H$ two Lie group homomorphisms for which the corresponding Lie algebra homomorphisms $d\varphi_1(\mathbf{1})$ and $d\varphi_2(\mathbf{1})$ coincide. Then $\varphi_1 = \varphi_2$.* □

COROLLARY 3.15. — *If $G$ is a connected Lie group, then* $\ker \mathrm{Ad} = Z(G)$.

*Proof.* — Let $I_g(x) = gxg^{-1}$. In view of Lemma 3.14, for $g \in G$ the conditions $I_g = \mathrm{id}_G$ and $dI_g(\mathbf{1}) = \mathrm{Ad}(g) = \mathrm{id}_{\mathfrak{g}}$ are equivalent. This implies the assertion. □

Theorem 3.16. — *If $H$ is a regular Lie group, $G$ is a simply connected Lie group, and $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is a continuous homomorphism of Lie algebras, then there exists a unique Lie group homomorphism $\alpha\colon G \to H$ with $d\alpha(\mathbf{1}) = \varphi$.*

*Proof.* — This is Theorem 8.1 in [Mi83] (see also [KM97, Th. 40.3]). The uniqueness assertion does not require the regularity of $H$, it follows from Lemma 3.14. □

Corollary 3.17. — *Let $G$ be a simply connected Lie group, $\mathfrak{z}$ an s.c.l.c. space, and $\alpha\colon \mathfrak{g} \to \mathfrak{z}$ a continuous Lie algebra homomorphism. Then there exists a unique smooth group homomorphism $f\colon G \to \mathfrak{z}$ with $df(\mathbf{1}) = \alpha$.*

*Proof.* — Since every s.c.l.c. vector space $\mathfrak{z}$ is a regular Lie group (Remark 3.13), the assertion follows from Theorem 3.16. □

Corollary 3.18. — *Let $G$ be a connected Lie group, $\mathfrak{z}$ an s.c.l.c. space, $\Gamma \subseteq \mathfrak{z}$ a discrete subgroup, and $\lambda\colon \mathfrak{g} \to \mathfrak{z}$ a continuous Lie algebra homomorphism. Then there exists a smooth group homomorphism $f\colon G \to Z := \mathfrak{z}/\Gamma$ with $df(\mathbf{1}) = \lambda$ if and only if $\zeta(\alpha)\big(\pi_1(G)\big) \subseteq \Gamma$ holds for the left invariant closed $\mathbf{1}$-form $\alpha$ on $G$ with $\alpha_{\mathbf{1}} = \lambda$.*

*Proof.* — Let $q\colon \widetilde{G} \to G$ denote the universal covering morphism and $\widetilde{f}\colon \widetilde{G} \to \mathfrak{z}$ the unique Lie group homomorphism with $d\widetilde{f}(\mathbf{1}) = \lambda$ (Corollary 3.17). Let $q_Z\colon \mathfrak{z} \to Z$ denote the quotient map. Then $f_Z := q_Z \circ \widetilde{f}\colon \widetilde{G} \to Z$ is a Lie group homomorphism with $df_Z = \alpha$. Whenever a homomorphism $f$ as required exists, its differential $df$ is a left invariant 1-form, hence coincides with $\alpha$. Therefore $f \circ q = f_Z$.

This proves that $f$ exists if and only if $\ker q \subseteq \ker f_Z$ which in turn means that $\widetilde{f}(\ker q) \subseteq \Gamma$. On the other hand $\widetilde{f}(\ker q) = \zeta(\alpha)(\pi_1(G))$, and this concludes the proof. □

## 4. Cocycles for central extensions of Lie groups.

In the setting of Lie groups, we consider only those central extensions $\widehat{G} \to G$ which are smooth principal bundles, i.e., have a smooth local section. We simply call them *smooth central extensions* (cf. [KM97, Sect. 38.6]). A typical example of an extension which does not have this property is

$$c_0(\mathbb{N}) \hookrightarrow l^\infty(\mathbb{N}) \twoheadrightarrow l^\infty(\mathbb{N})/c_0(\mathbb{N})$$

which does not have any smooth local section because the closed subspace $c_0(\mathbb{N})$ of $l^\infty(\mathbb{N})$ is not complemented (cf. [We95, Satz 4.6.5]). Nevertheless, according to Michael's Theorem ([Mi59]), the quotient map has a continuous section.

In this section we collect general material on central extensions of Lie groups. In particular we discuss the representability of Lie group extensions by locally smooth cocycles and explain how group and Lie algebra cocycles are related.

### Central extensions and cocycles.

LEMMA 4.1. — *Let $G$ be a connected topological group and $K = K^{-1}$ be an open $\mathbf{1}$-neighborhood in $G$. We further assume that $K$ is a smooth manifold such that the inversion is smooth on $K$ and there exists an open $\mathbf{1}$-neighborhood $V \subseteq K$ with $V^2 \subseteq K$ such that the group multiplication $m: V \times V \to K$ is smooth. Then there exists a unique structure of a Lie group on $G$ for which the inclusion map $K \hookrightarrow G$ induces a diffeomorphism on open neighborhoods of $\mathbf{1}$.*

*If $G$ is not connected, then we have to assume in addition that for each $g \in G$ there exists an open $\mathbf{1}$-neighborhood $K_g \subseteq K$ such that $I_g(x) := gxg^{-1}$ maps $K_g$ into $K$ and $I_g|_{K_g}: K_g \to K$ is smooth.*

*Proof.* — This is proved exactly as in the finite-dimensional case (cf. [Ch46, §14, Prop. 2] or [Ti83, p. 14]). □

PROPOSITION 4.2. — *Let $G$ and $Z$ be Lie groups, where $G$ is connected, and $Z \hookrightarrow \widehat{G} \to G$ a central extension of $G$ by $Z$. Then $\widehat{G}$ carries the structure of a Lie group such that $\widehat{G} \to G$ is a smooth central extension if and only if the central extension can be described by a cocycle $f: G \times G \to Z$ which is smooth in a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$.*

*Proof* (cf. [TW87, Prop. 3.11] for the finite-dimensional case). — First we assume that $\widehat{G} \to G$ is a smooth central extension of $G$. Then there exists a $\mathbf{1}$-neighborhood $U \subseteq G$ and a smooth section $\sigma: U \to \widehat{G}$ of the map $q: \widehat{G} \to G$. We extend $\sigma$ to a global section $G \to \widehat{G}$. Then

$$f(x, y) := \sigma(x)\sigma(y)\sigma(xy)^{-1}$$

defines a 2-cocycle $G \times G \to Z$ which is smooth in a neighborhood of $(\mathbf{1}, \mathbf{1})$.

Conversely, we assume that $\widehat{G} \cong G \times_f Z$ holds for a 2-cocycle $f: G \times G \to Z$ which is smooth in a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$. We endow $\widehat{G}$ with the unique group topology such that $\widehat{G} \to G$ is a topological principal bundle (Proposition 2.2). Then Lemma 4.1 implies the existence of a unique Lie group structure on $\widehat{G}$ compatible with the topology and such that there exists a $\mathbf{1}$-neighborhood of the product type $U_G \cdot U_Z$, where $U_G$ is a $\mathbf{1}$-neighborhood in $G$, $U_Z$ is a $\mathbf{1}$-neighborhood in $Z$, and the product map $U_G \times U_Z \to U_G U_Z$ is a diffeomorphism. Hence there exists a smooth local section $\sigma: U_G \to \widehat{G}$, showing that $\widehat{G} \to G$ is a smooth central extension. □

In [Va85, Th. 7.21] one finds a version of Proposition 4.2 for finite-dimensional Lie groups, where Lie groups are considered as special locally compact groups. The existence of Borel cross sections for locally compact groups implies that their central extensions can be described by measurable cocycles which, for Lie groups, can be replaced by equivalent cocycles which are smooth near to the identity (cf. also [Ca51] and [Ma57]).

*Remark 4.3.* — If the group $G$ is not connected, then one has to make the additional assumption that for each $g \in G$ the corresponding conjugation map $I_g: \widehat{G} \to \widehat{G}$ is smooth in an identity neighborhood, but this is only relevant for the elements not contained in the open subgroup generated by $U$ (cf. Remark 2.3 for the continuous case).

For Banach–Lie groups and in particular for finite-dimensional Lie groups every automorphism of the topological structure is automatically smooth, which can be deduced from the fact that the exponential function is a local diffeomorphism around $\mathbf{1}$. Therefore Proposition 4.2 requires for Banach–Lie groups which are not connected no additional requirements, once we have a group topology on $\widehat{G}$ with the required properties. □

Definition 4.4. — (a) *Let $G$ and $Z$ be Lie groups, where $G$ is connected. We have seen in Proposition 4.2 that the central extensions of $G$ by $Z$ which are smooth principal $Z$-bundles can be represented by 2-cocycles $f: G \times G \to Z$ which are smooth in a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$. We write $Z^2_s(G, Z)$ for the group of these cocycles. Likewise we have a group $B^2_s(G, Z)$ of 2-coboundaries $f(x, y) = h(xy)h(x)^{-1}h(y)^{-1}$, where $h: G \to Z$ is smooth in a $\mathbf{1}$-neighborhood. Then the group*

$$\mathrm{Ext}_{\mathrm{Lie}}(G, Z) := H^2_s(G, Z) := Z^2_s(G, Z)/B^2_s(G, Z)$$

*classifies the smooth central extensions of $G$ by $Z$.*

(b) *Let $\mathfrak{z}$ be a topological vector space and $\mathfrak{g}$ a topological Lie algebra. A* continuous $\mathfrak{z}$-valued 2-cocycle *is a continuous skew-symmetric function*

*$\omega\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ with*

$$\omega([x,y],z) + \omega([y,z],x) + \omega([z,x],y) = 0.$$

*It is called* a coboundary *if there exists a continuous linear map $\alpha \in \mathrm{Lin}(\mathfrak{g},\mathfrak{z})$ with $\omega(x,y) = \alpha([x,y])$ for all $x,y \in \mathfrak{g}$. We write $Z_c^2(\mathfrak{g},\mathfrak{z})$ for the space of continuous $\mathfrak{z}$-valued 2-cocycles and $B_c^2(\mathfrak{g},\mathfrak{z})$ for the subspace of coboundaries defined by continuous linear maps. We also define the* second continuous Lie algebra cohomology space

$$H_c^2(\mathfrak{g},\mathfrak{z}) := Z_c^2(\mathfrak{g},\mathfrak{z})/B_c^2(\mathfrak{g},\mathfrak{z}).$$

(c) *If $\omega$ is a continuous $\mathfrak{z}$-valued cocycle on $\mathfrak{g}$, then we write $\mathfrak{g} \oplus_\omega \mathfrak{z}$ for the topological Lie algebra whose underlying topological vector space is the product space $\mathfrak{g} \times \mathfrak{z}$, and the bracket is defined by*

$$[(x,z),(x',z')] = \big([x,x'],\omega(x,x')\big).$$

*Then $q\colon \mathfrak{g}\oplus_\omega\mathfrak{z} \to \mathfrak{g}$, $(x,z) \mapsto x$ is a central extension and $\sigma\colon \mathfrak{g} \to \mathfrak{g}\oplus_\omega\mathfrak{z}$, $x \mapsto (x,0)$ is a continuous linear section of $q$.* □

*Remark 4.5.* — We consider the setting of Remark 2.5, where $A$, $B$, $C$, $G$ and $Z$ are Lie groups such that $B \to C$ is a smooth central, resp., abelian extension. In this context everything in Remark 2.5 carries over to the smooth context. In particular we obtain an exact sequence of maps

$$\begin{aligned}&\mathbf{1} \to \mathrm{Hom}(C,Z) \longrightarrow \mathrm{Hom}(B,Z) \longrightarrow \mathrm{Hom}(A,Z)\\ &\xrightarrow{E^*} \mathrm{Ext}_{\mathrm{Lie}}(C,Z) \xrightarrow{\beta^*} \mathrm{Ext}_{\mathrm{Lie},\alpha(A)}(B,Z) \xrightarrow{\alpha^*} \mathrm{Ext}_{\mathrm{Lie,ab}}(A,Z),\end{aligned}$$

where Hom denotes smooth homomorphisms and the groups $A$, $B$ and $C$ are connected. Likewise we obtain for a connected Lie group $G$ an exact sequence

$$\begin{aligned}&\mathbf{1} \to \mathrm{Hom}(G,A) \longrightarrow \mathrm{Hom}(G,B) \longrightarrow \mathrm{Hom}(G,C)\\ &\xrightarrow{E_*} \mathrm{Ext}_{\mathrm{Lie}}(G,A) \xrightarrow{\alpha_*} \mathrm{Ext}_{\mathrm{Lie}}(G,B) \xrightarrow{\beta_*} \mathrm{Ext}_{\mathrm{Lie}}(G,C).\end{aligned}$$

□

### From group cocycles to Lie algebra cocycles.

For the following lemma we define for a smooth map $f\colon M \times N \to \mathfrak{z}$ and $(p,q) \in M \times N$ the bilinear map

(4.1)

$$d^2f(p,q)\colon T_p(M)\times T_q(N) \to \mathfrak{z}, \quad d^2f(p,q)(v,w) := \frac{\partial^2}{\partial s\partial t}\,\Big|_{t,s=0} f(\gamma(t),\eta(s)),$$

where $\gamma: ]-\varepsilon, \varepsilon[\to M$, resp., $\eta: ]-\varepsilon, \varepsilon[\to N$ are curves with $\gamma(0) = p$, $\gamma'(0) = v$, resp., $\eta(0) = q$, $\eta'(0) = w$. It is easy to see that the right hand side does not depend on the choice of curves $\gamma$ and $\eta$.

LEMMA 4.6.— *Let $Z \cong \mathfrak{z}/\Gamma$, $f \in Z^2_s(G, Z)$ and $\widehat{G} \cong G \times_f Z$ the corresponding central extension of $G$ by $Z$. We use a smooth local section of $q_Z : \mathfrak{z} \to Z$ to write $f = q_Z \circ f_{\mathfrak{z}}$ on an open neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$. Then the Lie algebra cocycle*

$$(Df)(x, y) := d^2 f_{\mathfrak{z}}(\mathbf{1}, \mathbf{1})(x, y) - d^2 f_{\mathfrak{z}}(\mathbf{1}, \mathbf{1})(y, x)$$

*satisfies $\widehat{\mathfrak{g}} \cong \mathfrak{g} \times_{D(f)} \mathfrak{z}$.*

*Proof.*— We identify $\widehat{\mathfrak{g}}$ with $\mathfrak{g} \times \mathfrak{z}$ via the differential $d\sigma(\mathbf{1}): \mathfrak{g} \to \widehat{\mathfrak{g}}$ in $\mathbf{1}$ of the section $\sigma: G \to \widehat{G}, g \mapsto (g, \mathbf{1})$ which is smooth in a neighborhood of the identity. The relation

$$(g, z)(h, w)(g, z)^{-1} = \big(ghg^{-1}, wf(g, h)f(ghg^{-1}, g)^{-1}\big)$$

(Remark 1.2) leads to the following formula for the adjoint action of $\widehat{G}$ on $\widehat{\mathfrak{g}}$:

$$\mathrm{Ad}_{\widehat{G}}(g, z)(y, s) = (\mathrm{Ad}_G(g).y, s + d_2 f_{\mathfrak{z}}(g, \mathbf{1}).y - d_1 f_{\mathfrak{z}}(\mathbf{1}, g)\, \mathrm{Ad}_G(g).y).$$

Here $d_1 f$, resp., $d_2 f$ denotes the derivative of $f$ in the first, resp., the second argument. Now we recall $d\big(\mathrm{Ad}(\cdot).(X, z)\big)(\mathbf{1}) = \mathrm{ad}(\cdot).(X, z)$ ([Mi83, p. 1036]) and observe that $f_{\mathfrak{z}}(g, \mathbf{1}) \equiv 0$ implies $d_1 f_{\mathfrak{z}}(\mathbf{1}, \mathbf{1}) = 0$ Taking derivatives with respect to $(g, z)$ in $\mathbf{1} \in \widehat{G}$, we therefore obtain the formula

$$\begin{aligned}
[(x, t), (y, s)] &= \big([x, y], d^2 f_{\mathfrak{z}}(\mathbf{1}, \mathbf{1})(x, y) - d^2 f_{\mathfrak{z}}(\mathbf{1}, \mathbf{1})(\mathrm{Ad}_G(\mathbf{1}).y, x) \\
&\quad - d_1 f_{\mathfrak{z}}(\mathbf{1}, \mathbf{1})([x, y])\big) \\
&= \big([x, y], d^2 f_{\mathfrak{z}}(\mathbf{1}, \mathbf{1})(x, y) - d^2 f_{\mathfrak{z}}(\mathbf{1}, \mathbf{1})(y, x)\big) \\
&= \big([x, y], (Df)(x, y)\big).
\end{aligned}$$

□

The preceding lemma implies in particular that for each $f \in Z^2_s(G, Z)$ we obtain an element $Df \in Z^2_c(\mathfrak{g}, \mathfrak{z})$, and further that $D$ induces a group homomorphism

$$D: H^2_s(G, Z) \to H^2_c(\mathfrak{g}, \mathfrak{z}), \quad [f] \mapsto [D(f)]$$

because equivalent Lie group extensions lead to equivalent Lie algebra extensions. This can also be verified more directly by observing that for

$f(g,h) = \ell(g)\ell(h)\ell(gh)^{-1}$ with a locally smooth function $\ell: G \to Z$ with $\ell(\mathbf{1}) = \mathbf{1}$ we obtain

$$(Df)(x,y) = -d\ell(\mathbf{1})([x,y]).$$

In fact, we can write $\ell = q_Z \circ \ell_{\mathfrak{z}}$ with a locally smooth function $\ell_{\mathfrak{z}}: G \to \mathfrak{z}$ and accordingly

$$f_{\mathfrak{z}}(g,h) = \ell_{\mathfrak{z}}(g) + \ell_{\mathfrak{z}}(h) - \ell_{\mathfrak{z}}(gh).$$

We then obtain

$$\begin{aligned}(Df)(\mathbf{1},\mathbf{1})(x,y) &= -x_r y_l \ell_{\mathfrak{z}}(\mathbf{1}) + y_r x_l \ell_{\mathfrak{z}}(\mathbf{1}) \\ &= -(x_r - x_l) y_l \ell_{\mathfrak{z}}(\mathbf{1}) - x_l y_l \ell_{\mathfrak{z}}(\mathbf{1}) + (y_r - y_l) x_l \ell_{\mathfrak{z}}(\mathbf{1}) \\ &\qquad + y_l x_l \ell_{\mathfrak{z}}(\mathbf{1}) \\ &= -x_l y_l \ell_{\mathfrak{z}}(\mathbf{1}) + y_l x_l \ell_{\mathfrak{z}}(\mathbf{1}) \\ &= d\ell_{\mathfrak{z}}(\mathbf{1})([y,x]).\end{aligned}$$

Here we use that the vector field $x_r - x_l$ vanishes in $\mathbf{1}$, which implies f.i. $(x_r - x_l) y_l \ell_{\mathfrak{z}}(\mathbf{1}) = d(y_l \ell_{\mathfrak{z}})(\mathbf{1}).(x_r - x_l)(\mathbf{1}) = 0$.

## 5. The period homomorphism of a Lie algebra cocycle.

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{z}$ an s.c.l.c. space. In this section we associate to each continuous Lie algebra cocycle $\omega \in Z_c^2(\mathfrak{g},\mathfrak{z})$ a period homomorphism

$$\mathrm{per}_\omega: \pi_2(G) \to \mathfrak{z}.$$

The main difficulty of the construction lies in the fact that we do not assume that $G$ is smoothly paracompact. If this is the case, then the construction is straight-forward: We consider the left invariant $\mathfrak{z}$-valued 2-form $\Omega$ on $G$ with $\Omega_{\mathbf{1}} = \omega$. Its cohomology class $[\Omega] \in H^2_{\mathrm{dR}}(G,\mathfrak{z}) \cong H^2_{\mathrm{sing}}(G,\mathfrak{z})$ can be interpreted as a homomorphism $H_2(G) \to \mathfrak{z}$ which in particular induces a homomorphism $\pi_2(G) \to \mathfrak{z}$ if composed with the Hurewicz homomorphism $\pi_2(G) \to H_2(G)$. For this argument we use de Rham's Theorem which is not available for non smoothly paracompact manifolds. Nevertheless, we will see that a direct construction works.

In the next section it will become apparent that the period homomorphism provides the main obstruction to the integrability of the central Lie algebra extension $\mathfrak{g} \oplus_\omega \mathfrak{z}$ to a group extension in the sense that such an extension exists if and only if the period group $\Pi_\omega := \mathrm{im}(\mathrm{per}_\omega)$ is discrete in $\mathfrak{z}$.

**Piecewise smooth maps on simplicial complexes.**

Definition 5.1. — *In the following $\Delta^p = \{(x_1,\dots,x_p): x_i \geqslant 0, \sum_j x_j \leqslant 1\}$ denotes the $p$-dimensional standard simplex in $\mathbb{R}^p$. We also write $\langle v_0,\dots,v_p\rangle$ for the affine simplex in a vector space spanned by the affinely independent points $v_0,\dots,v_p$. In this sense $\Delta^p = \langle 0, e_1,\dots,e_p\rangle$, where $e_i$ denotes the $i$th basis vector in $\mathbb{R}^p$.*

*Let $Y$ be a smooth manifold. A continuous map $f\colon \Delta^p \to Y$ is called a $C^1$-map if it is differentiable in the interior $\mathrm{int}(\Delta^p)$ and in each local chart of $Y$ all directional derivatives $x \mapsto df(x)(v)$ of $f$ extend continuously to the boundary $\partial\Delta^p$ of $\Delta^p$. We call $f$ a $C^k$-map if all maps $x \mapsto df(x)(v)$ are $C^{k-1}$, and we say that $f$ is smooth if $f$ is $C^k$ for every $k \in \mathbb{N}$. We write $C^\infty(\Delta^p, Y)$ for the set of smooth maps $\Delta^p \to Y$.*

*This point of view can also be used to define smooth maps on convex subsets $C$ of finite-dimensional vector spaces with $E := C - C$. In this context we call a continuous map $f\colon C \to Y$ a $C^1$-map if it is $C^1$ on the relative interior $\mathrm{int}_{\mathrm{aff}(C)}(C)$ of $C$ with respect to the affine subspace $\mathrm{aff}(C)$ it generates, and for which the differential $df\colon \mathrm{int}_{\mathrm{aff}(C)}(C) \times E \to Y$ extends to a continuous map $C \times E \to Y$, where $\mathrm{aff}(C)$ denotes the affine span of $C$. As for simplices, iteration leads to a smoothness concept for maps $C \to Y$.*

*If $\Sigma$ is a simplicial complex, then we call a map $f\colon \Sigma \to Y$ piecewise smooth if it is continuous and its restrictions to all simplices in $\Sigma$ are smooth. We write $C^\infty_{pw}(\Sigma, Y)$ for the set of piecewise smooth maps $\Sigma \to Y$. There is a natural topology on this space inherited from the natural embedding of $C^\infty_{pw}(\Sigma, Y)$ into the space $\prod_{S\subseteq\Sigma} C^\infty_{pw}(S, Y)$, where $S$ runs through all simplices of $\Sigma$ and the topology on $C^\infty_{pw}(S,Y)$ is defined as in Definition A.3.5 as the topology of uniform convergence of all directional derivatives of arbitrarily high order.* □

Lemma 5.2. — *If $f\colon \Delta^p \to Y$ is a smooth map, then each restriction of $f$ to a face of $\Delta^p$ is a smooth map.*

*Proof.* — Locally we have

$$f(x) = f(x_0) + \int_0^1 \langle df(x_0 + t(x - x_0)), x - x_0\rangle\, dt.$$

As the integrand on the right hand side extends continuously to the boundary, the restriction of $f$ to a face $F$ has directional derivatives which coincide with the continuous extensions of the directional derivatives of $f$

on $\mathrm{int}(\Delta^p)$. From that we derive that $f \mid_F$ is $C^1$ and by induction the assertion follows. $\square$

*Remark 5.3.* — With the differentiability concept of Definition 5.1 we obtain in particular a natural concept of smooth singular chains, cycles etc. We write $C_p(Y)$ for the group of smooth singular chains in the manifold $Y$, $Z_p(Y)$ for the group of smooth singular cycles, and

$$\partial \colon C_p(Y) \to C_{p-1}(Y), \quad n > 0$$

for the boundary map which is given on a smooth singular simplex $f \colon \Delta^p \to Y$ by

$$\partial f := \sum_{j=0}^{p} (-1)^j f \circ \sigma_j,$$

where

$$\sigma_j(x_1, \ldots, x_{p-1}) = \begin{cases} (x_1, \ldots, x_j, 0, x_{j+1}, \ldots, x_{p-1}), & \text{for } j \leqslant p-1 \\ (1 - \sum_i x_i, x_1, \ldots, x_{p-1}), & \text{for } j = p. \end{cases}$$

The maps $f_j := f \circ \sigma_j$, $j = 0, \ldots, p$ are called the *faces* of the smooth singular simplex $f$ and the sets $\Delta_j^p := \sigma_j(\Delta^{p-1})$ are called the faces of $\Delta^p$.

If $\omega$ is a smooth $\mathfrak{z}$-valued $p$-form on $Y$, then for each smooth singular simplex $\sigma \colon \Delta^p \to Y$ we define the integral

$$\int_\sigma \omega := \int_{\Delta^p} \sigma^* \omega,$$

where $\sigma^*\omega$ is a smooth $p$-form on $\mathrm{int}(\Delta^p)$ which extends continuously to the boundary. By linear extension we define the integral of $\omega$ over any smooth singular $p$-chain, and it is easy to verify that in this context Stoke's Theorem

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

([Wa83, p. 144]) holds. This formula extends directly to oriented simplicial complexes $\Sigma$ which are triangulations of compact manifolds with boundary and piecewise smooth maps $\sigma \colon \Sigma \to Y$. $\square$

LEMMA 5.4. — *If $G$ is a Lie group and $\Sigma$ a finite simplicial complex, then the group $C^\infty_{pw}(\Sigma, G)$ is a Lie group with respect to pointwise multiplication and Lie algebra $C^\infty_{pw}(\Sigma, \mathfrak{g})$.*

*Proof.* — This follows with the same argument as for the groups $C^\infty(M, G)$, where $M$ is a compact manifold (cf. [Gl01b] and Definition A.3.5). $\square$

For a simplicial complex $\Sigma$ we write $\Sigma_{(j)}$ for the $j$-th *barycentric subdivision of* $\Sigma$. Note that for a manifold $Y$ we have a natural inclusion

$$C^\infty_{pw}(\Sigma, Y) \to C^\infty_{pw}(\Sigma_{(1)}, Y)$$

which in general is not surjective because the requirement of piecewise smoothness for a function $\Sigma_{(1)} \to Y$ is weaker than for a function $\Sigma \to Y$. If $\Sigma = \Delta^1 = [0,1]$ is the unit interval, then a smooth function on $\Sigma_{(1)} = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$ need not be smooth around $\frac{1}{2}$.

In the following lemma we have to consider the barycentric subdivision of $\Delta^p$ because we otherwise run into problems if we want to extend a piecewise smooth map on the boundary to the interior.

LEMMA 5.5. — *Let $Y$ be a locally convex space. Then each piecewise smooth function $f\colon \partial\Delta^p_{(1)} \to Y$ extends to a piecewise smooth function $f\colon \Delta^p_{(1)} \to Y$.*

*Proof.* — The key idea of the proof is the following. If $f\colon \Delta^p_p = \langle e_1, \ldots, e_p\rangle \to Y$ is a smooth map, then we obtain for every $y_0 \in Y$ a natural extension of $f$ to a smooth map $\Delta^p \to Y$ by

$$E(f)(x_1, \ldots, x_p) := (1 - x_1 - \ldots - x_p)y_0 + \sum_{j=1}^p x_j f\Big(x_1, \ldots, x_{j-1}, 1 - \sum_{i\neq j} x_i, x_{j+1}, \ldots, x_p\Big).$$

Moreover, for each face $\Delta^p_j, 0 \leqslant j \leqslant p-1$, given by $x_{j+1} = 0$, the extension of the restriction of $f$ to $\Delta^p_p \cap \Delta^p_j$ coincides with the restriction of $E(f)$ to the face $\Delta^p_j$ of $\Delta^p$. This implies that if we have a piecewise smooth function on $\partial\Delta^p$ and we apply the preceding extension procedure to each simplex of the first barycentric subdivision $\Delta^p_{(1)}$ of $\Delta^p$, then we obtain a collection of smooth functions on all simplices in $\Delta^p_{(1)}$ matching to a piecewise smooth function on the simplicial complex $\Delta^p_{(1)}$. Here we need only that the function $f$ on $\partial\Delta^p$ is piecewise smooth on $(\partial\Delta^p)_{(1)}$. □

PROPOSITION 5.6. — *If $G$ is a simply connected Lie group, then the restriction map*

$$R\colon C^\infty_{pw}(\Delta^2_{(1)}, G) \to C^\infty_{pw}(\partial\Delta^2_{(1)}, G)$$

*is surjective.*

*Proof.* — Lemma 5.5 implies that if $f \in H := C^\infty_{pw}(\partial\Delta^2_{(1)}, G)$ is sufficiently close to the constant function $\mathbf{1}$ with respect to the compact open topology, then $f \in \operatorname{im}(R)$. Therefore the image of the group homomorphism $R$ is a subgroup of $H$ which is open with respect to the compact open topology.

We write $|\Sigma|$ for the topological space underlying a simplicial complex $\Sigma$. Topologically we then have $|\partial\Delta^2_{(1)}| = |\partial\Delta^2| \cong \mathbb{S}^1$. Therefore the density of $C^\infty(\mathbb{S}^1, G)$ in $C(\mathbb{S}^1, G)$ (Lemma A.3.6) implies in particular that $C^\infty(\mathbb{S}^1, G)$ is dense in $H$ with respect to the compact open topology. Now the connectedness of $H$ follows from the connectedness of $C^\infty(\mathbb{S}^1, G)$, which in turn follows from $\pi_0(C^\infty(\mathbb{S}^1, G)) \cong \pi_0(C(\mathbb{S}^1, G)) \cong \pi_1(G) = \mathbf{1}$ (Theorem A.3.7). □

Note that in the preceding proof we cannot argue directly by the density of the connected subgroup $C^\infty(\mathbb{S}^1, G)$ in $H$ because the former carries the subgroup topology inherited from $H$ and is complete, hence closed.

### Construction of the period homomorphism.

Now we return to the context where $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{z}$ is an s.c.l.c. space and $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$. In this subsection we construct the corresponding period homomorphism $\operatorname{per}_\omega \colon \pi_2(G) \to \mathfrak{z}$.

If $f \colon \Sigma \to G$ is a function from a topological space $\Sigma$ into a group $G$, then the closure of the set $f^{-1}(G \setminus \{\mathbf{1}\})$ is called the *support of* $f$.

LEMMA 5.7. — *If $\Sigma$ is an oriented simplicial complex which is a triangulation of $|\Sigma| = \mathbb{S}^2$, and $\Omega \in \Omega^2(G, \mathfrak{z})$ a closed $\mathfrak{z}$-valued 2-form, then the map*

$$\widetilde{\operatorname{per}}_\Omega \colon C^\infty_{pw}(\Sigma, G) \to \mathfrak{z}, \quad \sigma \mapsto \int_\sigma \Omega$$

*is locally constant and a group homomorphism.*

*Proof.* — As the group $C^\infty(\mathbb{S}^2, G)$ is dense in $C(\mathbb{S}^2, G)$ with respect to the compact open topology (Lemma A.3.6), it is in particular dense in $C^\infty_{pw}(\Sigma, G)$ with respect to the compact open topology.

Let $U \subseteq \mathfrak{g}$ be a convex open 0-neighborhood for which there exists a diffeomorphism $\varphi \colon U \to G$ with $\varphi(0) = \mathbf{1}$. Suppose that $\eta, \sigma \in C^\infty_{pw}(\Sigma, G)$ satisfy $\operatorname{im}(\eta^{-1} \cdot \sigma) \subseteq \varphi(U)$. Then there exists a piecewise smooth map

$$F \colon \Sigma \times [0,1] \to G \quad \text{with} \quad F(x,0) = \sigma(x), \quad F(x,1) = \eta(x), \quad x \in \Sigma.$$

Here the piecewise smoothness of a map on $\Sigma\times[0,1]$ refers to any simplicial decomposition of $\Sigma\times[0,1]$ refining the decomposition into the sets $S\times[0,1]$, where $S$ is a simplex in $\Sigma$. To verify this assertion, we first define a map

$$F'\colon \Sigma\times[0,1]\to U,\quad F'(x,t)=\varphi\Big((1-t)\varphi^{-1}\big(\eta(x)^{-1}\sigma(x)\big)\Big)$$

and then $F(x,t):=\eta(x)F'(x,t)$. Now Stoke's Theorem implies that

$$\int_\sigma\Omega-\int_\eta\Omega=\int_{\partial F}\Omega=\int_F d\Omega=0.$$

It follows in particular that $\widetilde{\mathrm{per}}_\Omega$ is locally constant, even in the compact open topology.

Next we show that $\widetilde{\mathrm{per}}_\Omega$ is a group homomorphism. First we observe that each $\sigma\in C^\infty_{pw}(\mathbb{S}^2,G)$ lies in the same connected component of a smooth map supported by the (opposite) hemisphere of a base point $x_0$ of $\mathbb{S}^2$. Let $\sigma_1,\sigma_2\in C^\infty_{pw}(\Sigma,G)$. Then the connected components of $\sigma_j$ contain elements $\zeta_j$ whose supports are contained in disjoint hemispheres of $\mathbb{S}^2$. We therefore obtain

$$\widetilde{\mathrm{per}}_\Omega(\sigma_1\sigma_2)=\widetilde{\mathrm{per}}_\Omega(\zeta_1\zeta_2)=\int_{\mathrm{supp}(\zeta_1)}(\zeta_1\zeta_2)^*\Omega+\int_{\mathrm{supp}(\zeta_2)}(\zeta_1\zeta_2)^*\Omega$$
$$=\int_{\mathrm{supp}(\zeta_1)}\zeta_1^*\Omega+\int_{\mathrm{supp}(\zeta_2)}\zeta_2^*\Omega=\widetilde{\mathrm{per}}_\Omega(\zeta_1)+\widetilde{\mathrm{per}}_\Omega(\zeta_2)=\widetilde{\mathrm{per}}_\Omega(\sigma_1)+\widetilde{\mathrm{per}}_\Omega(\sigma_2).$$

□

Definition 5.8. — *Now we define for a closed $\mathfrak{z}$-valued 2-form $\Omega$ on $G$ the* period homomorphism

$$\mathrm{per}_\Omega\colon \pi_2(G)\cong\pi_0(C^\infty_*(\mathbb{S}^2,G))\cong\pi_0(C^\infty(\mathbb{S}^2,G))\to\mathfrak{z}$$

*by factorization of $\widetilde{\mathrm{per}}_\Omega$ through the quotient map $C^\infty_*(\mathbb{S}^2,G)\to\pi_2(G)\cong\pi_0(C^\infty_*(\mathbb{S}^2,G))$ (cf. Theorem A.3.7).*

*The group $\Pi_\Omega:=\mathrm{im}(\mathrm{per}_\Omega)\subseteq\mathfrak{z}$ is called the* period group *of $\Omega$.*

*For the special case where $\Omega$ is a left invariant $\mathfrak{z}$-valued 2-form on $G$ with $\Omega_{\mathbf{1}}=\omega$, corresponding to $\omega\in Z^2_c(\mathfrak{g},\mathfrak{z})$, we also write $\mathrm{per}_\omega:=\mathrm{per}_\Omega$ and $\Pi_\omega:=\Pi_\Omega$.* □

If $G$ is a simply connected Lie group, then the singular homology group $H_2(G)$ is isomorphic to $\pi_2(G)$ (Remark A.1.4), so that the period homomorphism defines in particular a singular cohomology class.

*Remark* 5.9.— (a) The preceding construction is not restricted to degree 2. It provides for any closed $p$-form $\Omega \in \Omega^p(G, \mathfrak{z})$ a period homomorphism $\pi_k(G) \to \mathfrak{z}$. The extension to a singular cohomology class in $\operatorname{Hom}(H_p(G), \mathfrak{z}) \cong H^p_{\text{sing}}(G, \mathfrak{z})$ would require refined approximation arguments connecting singular and smooth singular homology of $G$.

(b) The period homomorphism $\operatorname{per}_\omega : \pi_2(G) \to \mathfrak{z}$ depends linearly on $\omega$. If $\omega$ is a coboundary, i.e., $\omega = d\alpha$ for some $\alpha \in \operatorname{Lin}(\mathfrak{g}, \mathfrak{z})$, then the left invariant 1-form $A \in \Omega^1(G, \mathfrak{z})$ with $A_{\mathbf{1}} = \alpha$ satisfies $dA = \Omega$ by Lemma 3.10. Hence Stoke's Theorem implies that $\operatorname{per}_\omega = 0$, and hence that $\operatorname{per}_\omega$ only depends on the Lie algebra cohomology class of $\omega$. □

### The connecting homomorphism and the period map.

In this subsection we will relate the period homomorphism of a Lie algebra cocycle to the connecting homomorphism $\delta : \pi_2(G) \to \pi_1(Z)$ from the long exact homotopy sequence of the bundle $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ provided that such a central Lie group extension exists.

DEFINITION 5.10.— *We recall the definition of* relative homotopy groups. *Let $I^n := [0,1]^n$ denote the $n$-dimensional cube. Then the boundary $\partial I^n$ of $I^n$ can be written as $I^{n-1} \cup J^{n-1}$, where $I^{n-1}$ is called the* initial face *and $J^{n-1}$ is the union of all other faces.*

*Let $X$ be a topological space, $A \subseteq X$ a subspace, and $x_0 \in A$. A* map

$$f : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$$

*is a continuous map $f : I^n \to X$ satisfying $f(I^{n-1}) \subseteq A$ and $f(J^{n-1}) = \{x_0\}$. We write $F^n(X, A, x_0)$ for the set of all such maps and $\pi_n(X, A, x_0)$ for the homotopy classes of such maps, i.e., the arc-components of the topological space $F^n(X, A, x_0)$ endowed with the compact open topology (cf. [Ste51]). Likewise we define the space $F^n(X, x_0) := F^n(X, \{x_0\}, x_0)$ and $\pi_n(X, x_0) := \pi_n(X, \{x_0\}, x_0)$. We have a canonical map*

$$\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0), \quad [f] \mapsto [f|_{I^{n-1}}]. \qquad \square$$

Let $q : \widehat{G} \to G$ be an extension of Lie groups with kernel $Z \cong \mathfrak{z}/\pi_1(Z)$, where $\mathfrak{z}$ is a s.c.l.c. space and $q$ has a smooth local section. For the following result we do not have to assume that $Z$ is central in $\widehat{G}$. Then $q$ defines in particular the structure of a $Z$-principal bundle on $\widehat{G}$, so that we have a natural homomorphism

$$\delta := \partial \circ (q_*)^{-1} \colon \pi_2(G) \to \pi_1(Z),$$

where the map

$$q_* \colon \pi_2(\widehat{G}, Z) := \pi_2(\widehat{G}, Z, \mathbf{1}) \to \pi_2(G), \quad [f] \mapsto [q \circ f]$$

is an isomorphism ([Ste51, Cor. 17.2]).

Let $\theta \in \Omega^1(\widehat{G}, \mathfrak{z})$ a 1-form with the property that for each $g \in \widehat{G}$ the orbit map $\eta_g \colon Z \hookrightarrow \widehat{G}, z \mapsto gz$ satisfies $\eta_g^*\theta = \theta_Z$, where $\theta_Z \in \Omega^1(Z, \mathfrak{z})$ is the canonical invariant 1-form on $Z$ with $\theta_Z(\mathbf{1}) = \mathrm{id}_{\mathfrak{z}}$. In the language of principal bundles, this means that $\theta$ is a connection 1-form for the principal $Z$-bundle $\widehat{G} \to G$. Further let $\Omega \in \Omega^2(G, \mathfrak{z})$ be the corresponding curvature form, i.e., $q^*\Omega = -d\theta$.

For the proof of Proposition 5.11 below we also observe that for $\gamma \in C^\infty_*(\mathbb{S}^1, Z)$ we have a natural identification

$$\int_\gamma \theta_Z = [\gamma] \in \pi_1(Z) \subseteq \mathfrak{z}. \tag{5.1}$$

PROPOSITION 5.11. — $\delta = -\operatorname{per}_\Omega$.

*Proof.* — Let $[\sigma] \in \pi_2(G)$ (Definition 5.10) and $\gamma \in C^\infty_{pw}(\partial\Delta^2, Z)$ a piecewise smooth representative of $\delta([\sigma])$ in $\pi_1(Z)$. The long exact homotopy sequence of the bundle $\widehat{G} \to G$ implies that the image of $[\gamma]$ in $\pi_1(\widehat{G})$ is trivial, so that Proposition V.6 implies the existence of $f \in C^\infty_{pw}(\Delta^2_{(1)}, \widehat{G})$ with $f|_{\partial\Delta^2} = \gamma$. Now $(q \circ f)(\partial\Delta^2) = \{\mathbf{1}\}$ shows that $q \circ f$ represents an element of $\pi_2(G)$ corresponding to the class of $f$ in $\pi_2(\widehat{G}, Z)$ via $q_*$.

According to the long exact homotopy sequence of the principal $Z$-bundle $\widehat{G} \to G$, the difference between $[\sigma]$ and $[q \circ f]$ in $\pi_2(G)$ lies in the image of $\pi_2(\widehat{G})$. Hence there exists a smooth base point preserving map $h \colon \mathbb{S}^2 \to \widehat{G}$ with $[\sigma] = [q \circ f] + [q \circ h]$ (Theorem A.3.7). We may assume that $h$ is constant in a neighborhood of the base point, so that we can view it also as a smooth map $h \colon \Delta^2 \to \widehat{G}$ with $h(\partial\Delta^2) = \{\mathbf{1}\}$ which is constant on a neighborhood of the boundary. Now $[\sigma] = [q \circ (h \cdot f)]$, so that, after replacing $f$ by $h^{-1} \cdot f$, we may assume that $[\sigma] = [q \circ f]$.

We now have

$$\begin{aligned}\mathrm{per}_\Omega([\sigma]) = \mathrm{per}_\Omega([q\circ f]) &= \int_{q\circ f}\Omega = \int_{\Delta^2} f^*q^*\Omega = -\int_{\Delta^2} f^*d\theta = -\int_{\Delta^2} df^*\theta \\ &= -\int_{\partial\Delta^2} f^*\theta = -\int_\gamma \theta = -\int_\gamma \theta_Z = -[\gamma] = -\delta([\sigma]),\end{aligned}$$

where we use (5.1) for the next to last equality. This completes the proof. □

*Remark 5.12.* — (a) We consider the special case of Proposition 5.11, where $q\colon \widehat{G}\to G$ is a central $Z$-extension corresponding to the Lie algebra cocycle $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ in the sense that $\widehat{\mathfrak{g}} \cong \mathfrak{g}\oplus_\omega \mathfrak{z}$. Let $\Omega$ be the left invariant 2-form on $G$ with $\Omega_{\mathbf{1}} = \omega$ and $p_{\mathfrak{z}}\colon \widehat{\mathfrak{g}} \cong \mathfrak{g}\oplus_\omega\mathfrak{z} \to \mathfrak{z}$ the projection onto $\mathfrak{z}$. Further let $\theta$ denote the left invariant $\mathfrak{z}$-valued 1-form on $\widehat{G}$ with $\theta_{\mathbf{1}} = p_{\mathfrak{z}}$. Then the left invariant 2-form $q^*\Omega$ is exact with $q^*\Omega = -d\theta$ because $-dp_{\mathfrak{z}}((x,z),(x',z')) = p_{\mathfrak{z}}([(x,z),(x',z')]) = \omega(x,x')$. Therefore Proposition 5.11 implies that $\mathrm{per}_\omega = -\delta$.

(b) Let $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ be an extension of connected Lie groups and assume that $Z$ is connected and abelian as above. In view of $\pi_2(Z)\cong \pi_2(\mathfrak{z}) = \mathbf{1}$, the long exact homotopy sequence of this bundle leads to an exact sequence

$$\mathbf{1} = \pi_2(Z) \to \pi_2(\widehat{G}) \to \pi_2(G) \to \pi_1(Z) \to \pi_1(\widehat{G}) \to \pi_1(G) \to \pi_0(Z) = \mathbf{1},$$

and therefore to

$$\pi_2(\widehat{G}) \hookrightarrow \pi_2(G) \xrightarrow{\mathrm{per}_\omega} \pi_1(Z) \to \pi_1(\widehat{G}) \twoheadrightarrow \pi_1(G).$$

This implies that

$$\pi_2(\widehat{G}) \cong \ker \mathrm{per}_\omega \subseteq \pi_2(G) \quad\text{ and }\quad \pi_1(G) \cong \pi_1(\widehat{G})/\operatorname{coker}\mathrm{per}_\omega .$$

These relations show how the period homomorphism controls how the first two homotopy groups of $G$ and $\widehat{G}$ are related. In particular we see that $\pi_2(\widehat{G})$ is smaller than $\pi_2(G)$. □

*Remark 5.13.* — We have just seen that every central extension of $G$ by $\mathbb{T}$ defines a homomorphism $\pi_2(G)\to\pi_1(\mathbb{T})\cong\mathbb{Z}$. Let $B\mathbb{T}$ be the classifying space of $\mathbb{T}$. For topological spaces $X$ and $Y$ we write $[X,Y]$ for the set of homotopy classes of continuous maps $f\colon X\to Y$. Since $\mathbb{T}$ is an Eilenberg–MacLane space of type $K(\mathbb{Z},1)$, we have for each paracompact locally contractible topological group $G$ natural isomorphisms

$$[G, B\mathbb{T}] = [G, BK(\mathbb{Z},1)] \cong [G, K(\mathbb{Z},2)] \cong H^2_{\mathrm{sing}}(G,\mathbb{Z})$$

because for these spaces Čech and singular cohomology are isomorphic (cf. [Hub61], [Br97, p. 184]). If $G$ is simply connected, we thus obtain an isomorphism

$$[G, B\mathbb{T}] \to H^2_{\mathrm{sing}}(G, \mathbb{Z}) \cong \mathrm{Hom}(\pi_2(G), \mathbb{Z}),$$

showing that each homomorphism $\delta\colon \pi_2(G) \to \mathbb{Z} \cong \pi_1(\mathbb{T})$ is the connecting homomorphism of a principal $\mathbb{T}$-bundle $\mathbb{T} \hookrightarrow \widehat{G} \twoheadrightarrow G$ (Section 4.4 in [tD91]), but it corresponds to a central extension of Lie groups if and only if it is the period homomorphism of a left invariant closed 2-form on $G$. □

## 6. From Lie algebra cocycles to group cocycles.

In this section we assume that $G$ is a connected simply connected Lie group and $\mathfrak{z}$ an s.c.l.c. space. Let $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$, $\Pi_\omega \subseteq \mathfrak{z}$ the corresponding period group (Definition 5.8), and consider the quotient group $Z := \mathfrak{z}/\Pi_\omega$ with the quotient map $q_Z\colon \mathfrak{z} \to Z$. If $\Pi_\omega$ is discrete, then $Z$ carries a natural Lie group structure. The main result of this section is the existence of a locally smooth group cocycle $f\colon G \times G \to Z$ corresponding to the Lie algebra cocycle $\omega$. The construction in this section was inspired by the beautiful idea in [Est54] to obtain group cocycles by integrating 2-forms over suitable triangles.

For $g \in G$ we choose a smooth path $\alpha_{\mathbf{1},g}\colon [0,1] \to G$ from $\mathbf{1}$ to $g$. We thus obtain a left invariant system of smooth arcs $\alpha_{g,h} := \lambda_g \circ \alpha_{\mathbf{1},g^{-1}h}$ from $g$ to $h$, where $\lambda_g(x) := gx$ denotes left translation. For $g, h, u \in G$ we then obtain a singular smooth cycle

$$\alpha_{g,h,u} := \alpha_{g,h} + \alpha_{h,u} - \alpha_{g,u},$$

which corresponds to the piecewise smooth map $\alpha_{g,h,u} \in C^\infty_{pw}(\partial\Delta^2, G)$ with

$$\alpha_{g,h,u}(s,t) = \begin{cases} \alpha_{g,h}(s), & \text{for } t = 0 \\ \alpha_{h,u}(1-s), & \text{for } s + t = 1 \\ \alpha_{g,u}(t), & \text{for } s = 0. \end{cases}$$

According to Proposition 5.6, each map $\alpha_{g,h,u}$ can be obtained as the restriction of a piecewise smooth map $\sigma\colon \Delta^2_{(1)} \to G$. Let $\sigma'\colon \Delta^2_{(1)} \to G$ be another piecewise smooth map with the same boundary values as $\sigma$. We claim that $\int_\sigma \Omega - \int_{\sigma'} \Omega \in \Pi_\omega$. In fact, we consider the sphere $\mathbb{S}^2$ as an oriented simplicial complex $\Sigma$ obtained by glueing two copies $D$ and $D'$ of

$\Delta^2$ along their boundary, where the inclusion of $D$ is orientation preserving and the inclusion on $D'$ reverses orientation. Then $\sigma$ and $\sigma'$ combine to a piecewise smooth map $\gamma\colon \Sigma \to G$ with $\gamma|_D = \sigma$ and $\gamma|_{D'} = \sigma'$, so that we get with Lemma 5.7,

$$\int_\sigma \Omega - \int_{\sigma'} \Omega = \int_\gamma \Omega \in \Pi_\omega.$$

We thus obtain a well-defined map

$$F\colon G^3 \to Z, \quad (g,h,u) \mapsto q_Z\Big(\int_\sigma \Omega\Big),$$

where $\sigma \in C^\infty(\Delta^2_{(1)}, G)$ is a piecewise smooth map whose boundary values coincide with $\alpha_{g,h,u}$.

LEMMA 6.1. — *The function*

$$f\colon G^2 \to Z, \quad (g,h) \mapsto F(\mathbf{1}, g, gh)$$

*is a group cocycle.*

*Proof.* — First we show that for $g, h \in G$ we have

$$f(g,\mathbf{1}) = F(\mathbf{1}, g, g) = 0 \quad \text{and} \quad f(\mathbf{1}, h) = F(\mathbf{1}, \mathbf{1}, h) = 0.$$

If $g = h$ or $h = u$, then we can choose the map $\sigma\colon \Delta^2 \to G$ extending $\alpha_{g,h,u}$ in such a way that $\mathrm{rk}(d\sigma) \leqslant 1$ in every point, so that $\sigma^*\Omega = 0$. In particular we obtain $F(g,h,u) = 0$ in these cases.

From $\alpha_{g,h,u} = \lambda_g \circ \alpha_{\mathbf{1}, g^{-1}h, g^{-1}u}$ we immediately obtain

$$F(g,h,u) = F(\mathbf{1}, g^{-1}h, g^{-1}u), \tag{6.1}$$

i.e., that $F$ is a left invariant function on $G^3$.

Let $\Delta^3 \subseteq \mathbb{R}^3$ be the standard 3-simplex. Then we define a piecewise smooth map $\gamma$ of its 1-skeleton to $G$ by

$$\gamma(t,0,0) = \alpha_{\mathbf{1},g}(t), \quad \gamma(0,t,0) = \alpha_{\mathbf{1},gh}(t), \quad \gamma(0,0,t) = \alpha_{\mathbf{1},ghu}(t)$$

and

$$\gamma(1-t,t,0) = \alpha_{g,gh}(t), \;\; \gamma(0,1-t,t) = \alpha_{gh,ghu}(t), \;\; \gamma(1-t,0,t) = \alpha_{g,ghu}(t).$$

As $G$ is simply connected, we obtain with Proposition V.6 for each face $\Delta^3_j$, $j = 0, \ldots, 3$, of $\Delta^3$ a piecewise smooth map $\gamma_j$ of the first barycentric subdivision to $G$ extending the given map on the 1-skeleton. These maps combine to a piecewise smooth map $\gamma\colon (\partial\Delta^3)_{(1)} \to G$. Modulo the period group $\Pi_\omega$ we now have

$$\begin{aligned}\int_\gamma \Omega &= \int_{\partial\Delta^3} \gamma^*\Omega = \sum_{i=0}^{3} \int_{\gamma_i} \Omega \\ &= F(g, gh, ghu) - F(\mathbf{1}, gh, ghu) + F(\mathbf{1}, g, ghu) - F(\mathbf{1}, g, gh) \\ &= f(h,u) - f(gh,u) + f(g,hu) - f(g,h).\end{aligned}$$

Since $\int_\gamma \Omega \in \Pi_\omega$, this proves that $f$ is a group cocycle. □

In the next lemma we will see that for an appropriate choice of paths from $\mathbf{1}$ to group elements close to $\mathbf{1}$ the cocycle $f$ will be smooth in an identity neighborhood.

Lemma 6.2. — *Let $U \subseteq \mathfrak{g}$ be an open convex 0-neighborhood and $\varphi\colon U \to G$ a chart of $G$ with $\varphi(0) = \mathbf{1}$ and $d\varphi(0) = \mathrm{id}_{\mathfrak{g}}$. We then define the arcs $[\mathbf{1}, \varphi(x)]$ by $\alpha_{\varphi(x)}(t) := \varphi(tx)$. Let $V \subseteq U$ be an open convex 0-neighborhood with $\varphi(V)\varphi(V) \subseteq \varphi(U)$ and define $x * y := \varphi^{-1}(\varphi(x)\varphi(y))$ for $x, y \in V$. If we define $\sigma_{x,y} := \varphi \circ \gamma_{x,y}$ with*

$$\gamma_{x,y}\colon \Delta^2 \to U, \quad (t,s) \mapsto t(x * sy) + s(x * (1-t)y),$$

*then the function*

$$f_V\colon V \times V \to \mathfrak{z}, \quad (x,y) \mapsto \int_{\sigma_{x,y}} \Omega$$

*is smooth with $d^2 f_V(0,0)(x,y) = \frac{1}{2}\omega(x,y)$.*

*Proof.* — First we note that the function $V \times V \to U, (x,y) \mapsto x * y$ is smooth. We consider the cycle

$$\alpha_{\mathbf{1},\varphi(x),\varphi(x)\varphi(y)} = \alpha_{\mathbf{1},\varphi(x),\varphi(x*y)} = \alpha_{\mathbf{1},\varphi(x)} + \alpha_{\varphi(x),\varphi(x*y)} - \alpha_{\mathbf{1},\varphi(x*y)}.$$

The arc connecting $x$ to $x * y$ is given by $s \mapsto x * sy$, so that we may define $\sigma_{x,y} := \varphi \circ \gamma_{x,y}$ with $\gamma_{x,y}$ as above. Then

$$f_V\colon V \times V \to \mathfrak{z}, \quad (x,y) \mapsto \int_{\varphi\circ\gamma_{x,y}} \Omega = \int_{\Delta^2} \gamma_{x,y}^* \varphi^*\Omega,$$

and

$$(6.2) \quad f_V(x,y) = \int_{\Delta^2} (\varphi^*\Omega)\big(\varphi(\gamma_{x,y}(t,s))\big)\Big(\frac{\partial}{\partial t}\gamma_{x,y}(t,s), \frac{\partial}{\partial s}\gamma_{x,y}(t,s)\Big)\, dt\, ds$$

implies that $f_V$ is a smooth function in $V \times V$.

The map $\gamma\colon (x,y) \mapsto \gamma_{x,y}$ satisfies

(1) $\gamma_{0,y}(t,s) = sy$ and $\gamma_{x,0}(t,s) = (t+s)x$.

(2) $\frac{\partial}{\partial t}\gamma_{x,y} \wedge \frac{\partial}{\partial s}\gamma_{x,y} = 0$ for $x=0$ or $y=0$.

In particular we obtain $f_V(x,0) = f_V(0,y) = 0$. Therefore the second order Taylor polynomial

$$T_2(f_V)(x,y) = f_V(0,0) + df_V(0,0)(x,0) + df_V(0,0)(0,y) + \frac{1}{2}d^2 f_V(0,0)\big((x,y),(x,y)\big)$$

of $f_V$ in $(0,0)$ is given by

$$T_2(f_V)(x,y) = \frac{1}{2}d^2 f_V(0,0)\big((x,0),(0,y)\big) + \frac{1}{2}d^2 f_V(0,0)\big((0,y),(x,0)\big) = d^2 f_V(0,0)(x,y)$$

with the convention for $d^2 f_V$ from (4.1).

Next we observe that (1) implies that $\frac{\partial}{\partial t}\gamma_{x,y}$ and $\frac{\partial}{\partial s}\gamma_{x,y}$ vanish in $(0,0)$. Therefore the chain rule for Taylor expansions and (1) imply that for each pair $(t,s)$ the second order term of

$$(\varphi^*\Omega)(\gamma_{x,y}(t,s))\Big(\frac{\partial}{\partial t}\gamma_{x,y}(t,s), \frac{\partial}{\partial s}\gamma_{x,y}(t,s)\Big)$$

is given by

$$(\varphi^*\Omega)(\gamma_{0,0}(t,s))(x,y) = (d\varphi(0)^*\Omega_{\mathbf{1}})(x,y) = (d\varphi(0)^*\omega)(x,y) = \omega(x,y),$$

and eventually

$$d^2 f_V(0,0)(x,y) = T_2(f_V)(x,y) = \int_{\Delta^2} dt\, ds \cdot \omega(x,y) = \frac{1}{2}\omega(x,y). \qquad \square$$

Corollary 6.3. — *Suppose that $\Pi_\omega$ is discrete and construct for $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ the group cocycle $f \in Z^2(G,Z)$ as above. If the paths $\alpha_{\mathbf{1},g}$ for $g \in \varphi(U)$ are chosen as in* Lemma 6.2, *then $f \in Z^2_s(G,Z)$ with $D(f) = \omega$.*

*Proof.* — In the notation of Lemma 6.2 we have for $x,y \in V$ the relation

$$f(\varphi(x),\varphi(y)) = q_Z(f_V(x,y)),$$

so that $f$ is smooth on $\varphi(V) \times \varphi(V)$, and further

$$Df(x,y) = d^2 f_V(\mathbf{1},\mathbf{1})(x,y) - d^2 f_V(\mathbf{1},\mathbf{1})(y,x) = \omega(x,y). \qquad \square$$

*Remark 6.4* (Changing cocycles). — Let $\lambda \in \mathrm{Lin}(\mathfrak{g},\mathfrak{z})$ and $\omega'(x,y) := \omega(x,y) - \lambda([x,y])$. Then the corresponding left invariant 2-form $\Omega'$ satisfies $\Omega' = \Omega + dL$, where $L$ is the left invariant 1-form with $L_{\mathbf{1}} = \lambda$. We therefore obtain $\mathrm{per}_{\omega'} = \mathrm{per}_\omega$ from Remark 5.9(b).

For the corresponding cocycle we have

$$f_\Omega(g,h) = \int_\sigma \Omega \quad \text{for} \quad \sigma \in C^\infty_{pw}(\Delta^2_{(1)}, G) \quad \text{with} \quad \partial\sigma = \alpha_{\mathbf{1},g,gh}.$$

This implies that modulo $\Pi_\omega$ we have

$$\begin{aligned} f_{\Omega'}(g,h) - f_\Omega(g,h) &= \int_\sigma dL = \int_{\partial\sigma} L = \int_{\alpha_{\mathbf{1},g,gh}} L \\ &= \int_{\alpha_{\mathbf{1},g}} L + \int_{\alpha_{g,gh}} L - \int_{\alpha_{\mathbf{1},gh}} L = \int_{\alpha_{\mathbf{1},g}} L + \int_{\alpha_{\mathbf{1},h}} L - \int_{\alpha_{\mathbf{1},gh}} L \\ &= \psi(g) + \psi(h) - \psi(gh). \end{aligned}$$

Hence $f_\Omega - f_{\Omega'}$ is a coboundary defined by the function

$$\psi\colon G \to \mathfrak{z}, \quad g \mapsto \int_{\alpha_{\mathbf{1},g}} L.$$

In local coordinates we have with the paths chosen as in Lemma 6.2 the formula

$$\psi(\varphi(x)) = \int_0^1 \langle (\varphi^* L)(tx), x\rangle \, dt,$$

which shows that $\psi$ is smooth in an identity neighborhood and satisfies $\psi(\mathbf{1}) = 0$. $\square$

## 7. The exact sequence for smooth central extensions.

In this section we eventually turn to the exact sequence for central extensions of Lie groups. Throughout this section $G$ will denote a connected Lie group and $Z$ satisfies $Z_0 \cong \mathfrak{z}/\Gamma$, where $\Gamma \subseteq \mathfrak{z}$ is a discrete subgroup in the s.c.l.c. space $\mathfrak{z}$. We write $q_Z\colon \mathfrak{z} \to Z_0$ for the quotient map. The main result of this section is the exact sequence described in the introduction.

In particular we will see that a Lie algebra cocycle $\omega$ integrates to a smooth central extension of a simply connected Lie group if and only if the corresponding period group is discrete (Theorem 7.9).

We start with the definition of the maps showing up in the exact sequence.

DEFINITION 7.1. — (a) *Let* $\gamma \in \operatorname{Hom}(\pi_1(G), Z)$. *We identify* $\pi_1(G)$ *with* $\ker q_G \subseteq \widetilde{G}$, *where* $q_G\colon \widetilde{G} \to G$ *is the universal covering homomorphism. Then*

$$\Gamma(\gamma^{-1}) := \{(d, \gamma(d)^{-1}) \in \widetilde{G} \times Z : d \in \pi_1(G)\}$$

*is a discrete central subgroup of* $\widetilde{G} \times Z$, *so that* $\widehat{G} := (\widetilde{G} \times Z)/\Gamma(\gamma^{-1})$ *carries a natural Lie group structure which is a* $Z$*-principal bundle over* $G$: *the quotient map* $\pi\colon \widehat{G} \to G$ *is given by* $\pi([g,t]) := q_G(g)$, *and its kernel coincides with* $(\pi_1(G) \times Z)/\Gamma(\gamma^{-1}) \cong Z$. *We write*

$$C\colon \operatorname{Hom}(\pi_1(G), Z) \to \operatorname{Ext}_{\mathrm{Lie}}(G, Z)$$

*for the group homomorphism defined this way. If* $E$ *stands for the central extension* $\pi_1(G) \hookrightarrow \widetilde{G} \twoheadrightarrow G$, *this is the homomorphism* $E^*$ *from Remarks 4.5 and 1.3.*

(b) *We recall the map*

$$D\colon \operatorname{Ext}_{\mathrm{Lie}}(G, Z) = H^2_s(G, Z) \to H^2_c(\mathfrak{g}, \mathfrak{z})$$

*from Section 4, which is given on the level of cocycles by*

$$(Df)(x, y) = d^2 f_{\mathfrak{z}}(\mathbf{1}, \mathbf{1})(x, y) - d^2 f_{\mathfrak{z}}(\mathbf{1}, \mathbf{1})(y, x),$$

*where* $f \in Z^2_s(G, Z)$ *is written on a sufficiently small identity neighborhood* $U$ *of* $G$ *as* $q_Z \circ f_{\mathfrak{z}}$ *with a smooth function* $f_{\mathfrak{z}}\colon U \times U \to \mathfrak{z}$ *(Lemma 4.6).*

*The image of* $D$ *are those cohomology classes* $[\omega] \in H^2_c(\mathfrak{g}, \mathfrak{z})$ *for which there exists a Lie group* $\widehat{G}$ *which is a* $Z$*-extension of* $G$. *If* $G$ *is simply connected, then we call the elements* $[\omega] \in \operatorname{im} D$ *and the corresponding Lie algebras* $\widehat{\mathfrak{g}}$ *integrable.*

(c) *Let* $[\omega] \in H^2_c(\mathfrak{g}, \mathfrak{z})$ *and write* $\Omega$ *for the* $\mathfrak{z}$*-valued left invariant closed 2-form on* $G$ *with* $\Omega_{\mathbf{1}} = \omega$. *Further let* $\operatorname{per}_\omega\colon \pi_2(G) \to \mathfrak{z}$ *be the period homomorphism (Definition 5.8). We define*

$$P_1([\omega]) := q_Z \circ \operatorname{per}_\omega\colon \pi_2(G) \to Z.$$

Now let $X \in \mathfrak{g}$ and consider the corresponding right invariant vector field $X_r$ on $G$. Then $i(X_r).\Omega$ is a closed $\mathfrak{z}$-valued 1-form whose cohomology class only depends on the cohomology class of $\omega$ in $H^2_c(\mathfrak{g}, \mathfrak{z})$ (Lemma 3.11). For each piecewise differentiable loop $\gamma\colon [0,1] \to G$ with $\gamma(0) = \mathbf{1}$ we now put

$$P_2([\omega])([\gamma])(X) := \int_\gamma i(X_r).\Omega = \zeta([i(X_r).\Omega])([\gamma])$$

(Theorem 3.6). It is clear that $P_2([\omega])$ can be viewed as a homomorphism $\pi_1(G) \to \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})$. We claim that its range consists of continuous linear maps. In fact, for each piecewise differentiable loop $\gamma\colon [0,1] \to G$ we have

$$P_2([\omega])([\gamma])(X) = \int_0^1 \Omega(X_r(\gamma(t)), \gamma'(t))\,dt = \int_0^1 \omega\big(\mathrm{Ad}(\gamma(t))^{-1}.X, \gamma_l'(t)\big)\,dt,$$

where $\gamma_l'(t) := d\lambda_{\gamma(t)}^{-1}(\gamma(t)).\gamma'(t) \in \mathfrak{g} \cong T_{\mathbf{1}}(G)$ denotes the left logarithmic derivative of $\gamma$ in $t$. Since the integrand is a continuous map $[0,1] \times \mathfrak{g} \to \mathfrak{z}$, the integral is a continuous map $\mathfrak{g} \to \mathfrak{z}$. We combine these two maps to a group homomorphism

$$P\colon H^2_c(\mathfrak{g}, \mathfrak{z}) \to \mathrm{Hom}\,\big(\pi_2(G), Z\big) \times \mathrm{Hom}\,\big(\pi_1(G), \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})\big),$$
$$[\omega] \mapsto (P_1([\omega]), P_2([\omega])). \qquad \square$$

First we take a closer look at the homomorphism $C$.

Lemma 7.2. — *Let $G$ and $\widehat{G}$ be connected Lie groups, $q\colon \widehat{G} \to G$ a covering homomorphism with kernel $Q$ and $Z_0 \cong \mathfrak{z}/\Gamma$. Then $Q$ is a discrete central subgroup of $\widehat{G}$ and $q$ induces an exact sequence*

$$\mathbf{0} \to \mathrm{Hom}(G, Z) \to \mathrm{Hom}(\widehat{G}, Z) \to \mathrm{Hom}(Q, Z) \xrightarrow{C} \mathrm{Ext}_{\mathrm{Lie}}(G, Z) \xrightarrow{q^*} \mathrm{Ext}_{\mathrm{Lie},Q}(\widehat{G}, Z).$$

*If, in addition, the group $Z$ is connected, then $q^*$ is surjective.*

*Proof.* — The kernel $Q$ of $q$ is a discrete normal subgroup of the connected group $\widehat{G}$, hence central. In view of Remark 4.5, the central extension $q\colon \widehat{G} \to G$ leads to the exact sequence

$$\mathrm{Hom}(G, Z) \hookrightarrow \mathrm{Hom}(\widehat{G}, Z) \xrightarrow{res} \mathrm{Hom}(Q, Z) \xrightarrow{C} \mathrm{Ext}_{\mathrm{Lie}}(G, Z) \xrightarrow{q^*} \mathrm{Ext}_{\mathrm{Lie},Q}(\widehat{G}, Z) \to \mathrm{Ext}_{\mathrm{Lie,ab}}(Q, Z)$$

because $C$ coincides with the map $E^*$ in Theorem 1.5. This means in particular that $C$ is a group homomorphism and that the range of $E^*$ consists entirely of Lie group extensions.

If the group $Z$ is connected, then $Z \cong \mathfrak{z}/\Gamma$ is divisible, so that $\mathrm{Ext}_{\mathrm{ab}}(Q, Z) = \mathbf{0}$. Therefore $q^*$ is surjective. $\square$

*Remark 7.3.* — If $\widehat{\mathfrak{g}}$ is topologically perfect, i.e., its commutator algebra is dense, then each Lie algebra homomorphism to an abelian Lie algebra is trivial. Since $\widehat{G}$ is connected, it follows from Lemma 3.14 that $\mathrm{Hom}(\widehat{G}, Z) = \mathbf{0}$. In the setting of Lemma 7.2, we therefore obtain for abelian groups $Z$ the short exact sequence

$$\mathbf{0} \to \mathrm{Hom}(Q, Z) \hookrightarrow \mathrm{Ext}_{\mathrm{Lie}}(G, Z) \twoheadrightarrow \mathrm{Ext}_{\mathrm{Lie},Q}(\widehat{G}, Z) \to \mathbf{0}. \qquad \square$$

PROPOSITION 7.4. — *For every connected Lie group $G$ and $Z_0 \cong \mathfrak{z}/\Gamma$ we have* $\ker D = \operatorname{im} C$.

*Proof.* — "$\supseteq$": Let $f\colon \pi_1(G) \to Z$ and consider the corresponding central extension

$$\widehat{G} := \widetilde{G} \times_f Z \cong (\widetilde{G} \times Z)/\Gamma(f^{-1}) \to G, \quad [g, t] \mapsto q(g).$$

The map $\widetilde{G} \times Z \to \widehat{G}$ is a covering with kernel $\Gamma(f^{-1})$ isomorphic to $\pi_1(G)$. Hence $\widehat{\mathfrak{g}}$, the Lie algebra of $\widehat{G}$, is isomorphic to $\mathfrak{g} \times \mathfrak{z}$, showing that the corresponding Lie algebra extension $\widehat{\mathfrak{g}} \to \mathfrak{g}$ is trivial. Thus $\operatorname{im} C \subseteq \ker D$.

"$\subseteq$": Suppose that $D(E) = \mathbf{0}$ holds for the central extension $E\colon Z \hookrightarrow \widehat{G} \xrightarrow{q} G$. Then the Lie algebra extension $\widehat{\mathfrak{g}} \to \mathfrak{g}$ splits, so that we have a continuous Lie algebra homomorphism $\lambda\colon \widehat{\mathfrak{g}} \to \mathfrak{z}$ extending the identity on $\mathfrak{z} \subseteq \widehat{\mathfrak{g}}$. Let $q_{\widehat{G}}\colon G^\sharp \to \widehat{G}$ denote a universal covering of $\widehat{G}$. In view of Theorem 3.16, there exists a unique Lie group homomorphism $\varphi\colon G^\sharp \to \mathfrak{z}$ with $d\varphi(\mathbf{1}) = \lambda$. On the other hand the embedding $\eta_Z|_{Z_0}\colon Z_0 \to \widehat{G}$ lifts to a homomorphism $\eta_{\mathfrak{z}}\colon \mathfrak{z} \to G^\sharp$ of the universal covering groups with $\varphi \circ \eta_{\mathfrak{z}} = \mathrm{id}_{\mathfrak{z}}$. We fix a smooth local section $\sigma\colon U \to \widehat{G}$, where $U \subseteq G$ is an open symmetric $\mathbf{1}$-neighborhood. In addition, we assume that there exists a smooth local section $\widehat{\sigma}\colon \widehat{U} \to G^\sharp$, where $\widehat{U} \subseteq \widehat{G}$ is an open $\mathbf{1}$-neighborhood containing $\sigma(U)$. Then $\widetilde{\sigma} := \widehat{\sigma} \circ \sigma\colon U \to G^\sharp$ is a smooth map with

$$q \circ q_{\widehat{G}} \circ \widetilde{\sigma} = q \circ \sigma = \mathrm{id}_U\,.$$

Let $\sigma_1(x) := \widetilde{\sigma}(x)\eta_{\mathfrak{z}}\big(\varphi(\widetilde{\sigma}(x))\big)^{-1}$. Then $\sigma_1\colon U \to G^\sharp$ also is a smooth section of $q \circ q_{\widehat{G}}$, and, in addition, $\operatorname{im}(\sigma_1) \subseteq \ker \varphi$. Since $q^{-1}(U) = \sigma(U)Z \cong U \times Z$, the group $G^\sharp$ contains a $\mathbf{1}$-neighborhood of the form

$$\widetilde{U} := \sigma_1(U)\eta_{\mathfrak{z}}(U_{\mathfrak{z}}),$$

where $U_{\mathfrak{z}} \subseteq \mathfrak{z}$ is an open 0-neighborhood. Then $\varphi(\sigma_1(x)\eta_{\mathfrak{z}}(z)) = z$ implies that $\ker\varphi \cap \widetilde{U} = \sigma_1(U)$. Let $x, y \in U$ with $xy \in U$ and $\sigma_1(x)\sigma_1(y) \in \widetilde{U}$. Then $\sigma_1(x)\sigma_1(y) \in \ker\varphi \cap \widetilde{U} = \sigma_1(U)$ and $q \circ q_{\widehat{G}}(\sigma_1(x)\sigma_1(y)) = xy$ leads to $\sigma_1(xy) = \sigma_1(x)\sigma_1(y)$. Now Proposition 2.6 implies that $\widehat{G} \cong (\widetilde{G} \times Z)/\Gamma(\gamma^{-1})$ for some $\gamma \in \mathrm{Hom}(\pi_1(G), Z)$. ◻

*Remark 7.5.* — In Proposition 7.4 we have determined the kernel of $D$ as the image of $C$. On the other hand we have the exact sequence

$$\mathrm{Hom}(\widetilde{G}, Z) \to \mathrm{Hom}(\pi_1(G), Z) \to \mathrm{Ext}_{\mathrm{Lie}}(G, Z) \xrightarrow{q_G^*} \mathrm{Ext}_{\mathrm{Lie}}(\widetilde{G}, Z)$$

(Lemma 7.2). Since $G$ and $\widetilde{G}$ have the same Lie algebra, we also have a homomorphism

$$\widetilde{D} \colon \mathrm{Ext}_{\mathrm{Lie}}(\widetilde{G}, Z) \to H^2_c(\mathfrak{g}, \mathfrak{z})$$

which is injective because $\pi_1(\widetilde{G})$ is trivial (Proposition 7.4). It is easy to see that $\widetilde{D} \circ q_G^* = D$, showing that $\operatorname{im} C = \ker D = \ker q_G^*$. ◻

It is interesting that, in spite of the fact that not every central extension $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$ corresponds to a central group extension of a simply connected group $G$, Proposition 7.6 below implies that we can always construct the adjoint action of $G$.

It is fairly easy to see that to each continuous Lie algebra module $\mathfrak{g} \times V \to V$, $V$ a locally convex space, there exists at most one smooth representation of $G$ on $V$ for which the Lie algebra action is the derived representation. In the finite-dimensional case the simple connectedness of $G$ suffices to ensure the existence, but for infinite-dimensional spaces $V$ the group $\mathrm{GL}(V)$ is not a Lie group, so that one cannot expect that Lie algebra representations integrate to representations of the corresponding simply connected groups. A simple example where no group representation exists is given by the action of the Lie algebra $\mathfrak{g} = \mathbb{R}X$ of $G = \mathbb{R}$ on the space $V := C^\infty(]0,1[, \mathbb{R})$ by $X.f = f'$. This example shows that the problems come from the bad structure of the group $\mathrm{GL}(V)$ and not from the group $G$.

Proposition 7.6. — *Let $G$ be a connected Lie group, $\mathfrak{z}$ an s.c.l.c. space, and $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$. Then the adjoint action of $\mathfrak{g}$ on $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus_\omega \mathfrak{z}$ integrates to a smooth action $\mathrm{Ad}_{\widehat{\mathfrak{g}}}$ of $G$ if and only if $P_2([\omega]) = 0$. In this case the corresponding cocycle can be obtained as follows. For each $X \in \mathfrak{g}$ let $f_X \in C^\infty(G, \mathfrak{z})$ be the unique function with $df_X = i(X_r).\Omega$ and*

$f_X(\mathbf{1}) = 0$. *Then*

$$\theta: G \times \mathfrak{g} \to \mathfrak{z}, \quad (g, X) \mapsto f_X(g^{-1})$$

*defines a smooth* 1*-cocycle with*

$$\mathrm{Ad}_{\widehat{\mathfrak{g}}}(g).(X, z) := (\mathrm{Ad}(g).X, \theta(g, X) + z).$$

*Proof.* — First we assume that $P_2([\omega]) = 0$, which means that for each $X \in \mathfrak{g}$ the closed 1-form $i(X_r).\Omega$ on $G$ is exact, so that the functions $f_X$, $X \in \mathfrak{g}$, exist. We have to show that for $g_1, g_2 \in G$ and $X \in \mathfrak{g}$ we have

$$\theta(g_1 g_2, X) = \theta(g_2, X) + \theta(g_1, g_2.X), \tag{7.1}$$

which means that

$$f_X(g_2^{-1} g_1^{-1}) = f_X(g_2^{-1}) + f_{g_2.X}(g_1^{-1})$$

for all $g_1, g_2 \in G$, and this is equivalent to $f_X(g_2 g_1) = f_X(g_2) + f_{g_2^{-1}.X}(g_1)$ for all $g_1, g_2 \in G$, which in turn means that $f_X \circ \lambda_{g_2} = f_X(g_2) + f_{g_2^{-1}.X}$. In $\mathbf{1}$ both functions have the same value $f_X(g_2)$. Hence it suffices to show that both have the same differential. This follows from

$$d(f_X \circ \lambda_{g_2}) = \lambda_{g_2}^* df_X = \lambda_{g_2}^*\big(i(X_r).\Omega\big) = i((g_2^{-1}.X)_r).\Omega,$$

where the last equality is a consequence of

$$\begin{aligned}\big(\lambda_{g_2}^*(i(X_r).\Omega)\big)_g(v) &= \big(i(X_r).\Omega\big)_{g_2 g}(d\lambda_{g_2}(g).v) = \Omega_{g_2 g}\big(d\rho_{g_2 g}(\mathbf{1})X, d\lambda_{g_2}(g).v\big)\\ &= \Omega_g\big(d\lambda_{g_2^{-1}}(g_2 g) d\rho_{g_2 g}(\mathbf{1})X, v\big) = \Omega_g\big((g_2^{-1}.X)_r(g), v\big).\end{aligned}$$

We further have

$$d(f_X(g_2) + f_{g_2^{-1}.X}) = df_{g_2^{-1}.X} = i((g_2^{-1}.X)_r).\Omega.$$

This proves that $\theta$ is a 1-cocycle.

Now we show that $\theta$ is smooth. Since $\theta$ is linear in the second argument and a cocycle, it suffices to verify this in a neighborhood of $(\mathbf{1}, 0) \in G \times \mathfrak{g}$. Let $U \subseteq G$ be an open $\mathbf{1}$-neighborhood for which there exists a chart $\varphi: V \to U$ with $\varphi(0) = \mathbf{1}$, where $V \subseteq \mathfrak{g}$ is a an open star-shaped neighborhood of 0. Then for each $x \in V$ and $X \in \mathfrak{g}$ we have

$$\begin{aligned}f_X(\varphi(x)) &= \int_{\varphi([0,1]x)} i(X_r).\Omega\\ &= \int_0^1 \omega\big(\mathrm{Ad}(\varphi(tx))^{-1}.X, d\lambda_{\varphi(tx)^{-1}}(\varphi(tx)) d\varphi(tx).x\big)\, dt,\end{aligned}$$

and this formula shows that the function $V \times \mathfrak{g} \to \mathfrak{g}, (x, X) \mapsto f_X(\varphi(x))$ is smooth. We conclude that $\theta$ is a smooth cocycle for the adjoint action, and therefore

$$\mathrm{Ad}_{\widehat{\mathfrak{g}}}(g).(X, z) := (\mathrm{Ad}(g).X, \theta(g, X) + z)$$

defines a smooth action of $G$ on $\widehat{\mathfrak{g}}$. The corresponding derived action $\mathrm{ad}_{\widehat{\mathfrak{g}}}$ of $\mathfrak{g}$ is given by

$$\begin{aligned}\mathrm{ad}_{\widehat{\mathfrak{g}}}(Y).(X, z) &= ([Y, X], -df_X(\mathbf{1}).Y) = ([Y, X], -(i(X_r).\Omega)_{\mathbf{1}}(Y)) \\ &= ([Y, X], \omega(Y, X)) = [(Y, 0), (X, z)].\end{aligned}$$

Suppose, conversely, that there exists a smooth action $\mathrm{Ad}_{\widehat{\mathfrak{g}}}$ of $G$ on $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$ such that the derived action is $\mathrm{ad}_{\widehat{\mathfrak{g}}}$. Since $\mathrm{ad}_{\widehat{\mathfrak{g}}}$ is an action of $\mathfrak{g}$ by derivations on $\widehat{\mathfrak{g}}$, it follows by an easy differentiation argument and the connectedness of $G$ that $G$ acts by automorphisms on $\widehat{\mathfrak{g}}$. Let $X \in \mathfrak{g} \subseteq \widehat{\mathfrak{g}}$ and consider the function $f_X : G \to \mathfrak{z}$ given by $f_X(g) := \theta(g^{-1}, X) = p_\mathfrak{z}(g^{-1}.X)$, where $p_\mathfrak{z} : \widehat{\mathfrak{g}} \to \mathfrak{z}$ is the projection onto $\mathfrak{z}$ along the subspace $\mathfrak{g} \times \{0\}$. By assumption we have

$$df_X(\mathbf{1})(Y) = p_\mathfrak{z}(-[Y, X]) = \omega(X, Y).$$

Therefore

$$\begin{aligned}df_X(g)Y_r(g) &= df_X(g)d\rho_g(\mathbf{1}).Y = p_\mathfrak{z}\big(\mathrm{Ad}_{\widehat{\mathfrak{g}}}(g^{-1})\,\mathrm{ad}_{\widehat{\mathfrak{g}}}(-Y).X\big) \\ &= p_\mathfrak{z}\big(\mathrm{Ad}_{\widehat{\mathfrak{g}}}(g^{-1}).[X, Y]_{\widehat{\mathfrak{g}}}\big) = p_\mathfrak{z}\big([\mathrm{Ad}_{\widehat{\mathfrak{g}}}(g^{-1}).X, \mathrm{Ad}_{\widehat{\mathfrak{g}}}(g^{-1}).Y]_{\widehat{\mathfrak{g}}}\big) \\ &= \omega\big(\mathrm{Ad}_{\widehat{\mathfrak{g}}}(g^{-1}).X, \mathrm{Ad}_{\widehat{\mathfrak{g}}}(g^{-1}).Y\big) = \Omega(X_r, Y_r)(g),\end{aligned}$$

and therefore $df_X = i(X_r).\Omega$. Hence the 1-forms $i(X_r).\Omega$ are exact, and this means that $P_2([\omega]) = 0$. □

Corollary 7.7. — *If $G$ is a connected simply connected Lie group and $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$, then the adjoint action of $\mathfrak{g}$ on $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus_\omega \mathfrak{z}$ integrates to a smooth action $\mathrm{Ad}_{\widehat{\mathfrak{g}}}$ of $G$.* □

The following lemma implies $\mathrm{im}\, D \subseteq \ker P$.

Lemma 7.8. — *If there exists a Lie group extension $Z \hookrightarrow \widehat{G} \to G$ corresponding to $[\omega] \in H^2_c(\mathfrak{g}, \mathfrak{z})$, then $P([\omega]) = 0$.*

*Proof.* — First we consider the homomorphism $P_1([\omega]) = q_Z \circ \mathrm{per}_\omega : \pi_2(G) \to Z$. We have seen in Remark 5.12(a) that $-\mathrm{per}_\omega$ is the connecting homomorphism $\delta\colon \pi_2(G) \to \pi_1(Z) \subseteq \mathfrak{z}$ in the long exact homotopy sequence of the principal $Z$-bundle $\widehat{G} \to G$. It follows in particular that $\operatorname{im}(\mathrm{per}_\omega) \subseteq \pi_1(Z)$ and hence that $P_1([\omega]) = 0$.

Now we turn to $P_2\colon \pi_1(G) \to \operatorname{Lin}(\mathfrak{g}, \mathfrak{z})$. We write the Lie algebra of $\widehat{G}$ as $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$ with the bracket

$$[(X,z),(X',z')] = ([X,X'], \omega(X,X')).$$

Since $Z \subseteq \widehat{G}$ is central and $\widehat{G} \to G$ is a locally trivial bundle, the adjoint action of $\widehat{G}$ on $\widehat{\mathfrak{g}}$ factors to a smooth action $\mathrm{Ad}_{\widehat{\mathfrak{g}}}$ of $G$ on $\widehat{\mathfrak{g}}$ whose derived action is given by

$$\mathrm{ad}_{\widehat{\mathfrak{g}}}(X)(Y,z) = [(X,0),(Y,z)] = ([X,Y], \omega(X,Y)).$$

In view of Lemma 7.6, the existence of $\mathrm{Ad}_{\widehat{\mathfrak{g}}}$ means that $P_2([\omega]) = 0$. $\square$

The following theorem describes the bridge from the infinitesimal central extension corresponding to a Lie algebra cocycle to a global central extension of a Lie group.

THEOREM 7.9 (Integrability Criterion). — *Let $\mathfrak{g}$ be the Lie algebra of the simply connected Lie group $G$ and $[\omega] \in H^2_c(\mathfrak{g}, \mathfrak{z})$. Then there exists a corresponding smooth central extension of $G$ by some group $Z = \mathfrak{z}/\Gamma$ if and only if* $\operatorname{im}(\mathrm{per}_\omega)$ *is a discrete subgroup of $\mathfrak{z}$. If $Z$, resp., $\Gamma$ is given, then the central extension exists if and only if* $\operatorname{im}(\mathrm{per}_\omega) \subseteq \Gamma$.

*Proof.* — First we assume that the image of $\mathrm{per}_\omega$ is discrete and contained in the discrete subgroup $\Gamma$. Using Corollary 6.3, we obtain for $Z := \mathfrak{z}/\operatorname{im}(\mathrm{per}_\omega)$ a cocycle $f \in Z^2_s(G,Z)$ with $D(f) = \omega$. In view of Proposition 4.2, the corresponding group $\widehat{G} := G \times_f Z$ carries a natural Lie group structure such that $Z \hookrightarrow \widehat{G} \to G$ is a smooth central extension.

If, conversely, a smooth central extension of $G$ by $Z = \mathfrak{z}/\Gamma$ exists, then Lemma 7.8 implies that $\operatorname{im}(\mathrm{per}_\omega) \subseteq \Gamma$. $\square$

LEMMA 7.10. — *If $Z$ is an abelian Lie group with $Z_0 \cong \mathfrak{z}/\Gamma$, then the exact sequence*

$$Z_0 \hookrightarrow Z \twoheadrightarrow \pi_0(Z)$$

*splits and $Z \cong Z_0 \times \pi_0(Z)$, where the group $\pi_0(Z)$ is discrete.*

*Proof.* — Since the abelian group $Z_0$ is divisible, there exists a group homomorphism $\sigma\colon \pi_0(Z) \to Z$ with $\sigma(zZ_0) \in zZ_0$. As $\pi_0(Z) \cong Z/Z_0$ is a discrete group, $\sigma$ is continuous, and therefore $Z \cong Z_0 \times \sigma(\pi_0(Z)) \cong Z_0 \times \pi_0(Z)$. $\square$

LEMMA 7.11. — *If $P([\omega]) = 0$, then there exists a Lie group extension $Z \cong \mathfrak{z}/\Gamma \hookrightarrow \widehat{G} \to G$ with Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$.*

*Proof.* — Let $q_G: \widetilde{G} \to G$ be the universal covering group. Since the canonical map $\pi_2(\widetilde{G}) \to \pi_2(G)$ is an isomorphism (Remark A.1.4), $P_1([\omega]) = 0$ and Theorem 7.9 imply the existence of a central extension $Z \hookrightarrow H \xrightarrow{q_H} \widetilde{G}$ such that the Lie algebra of $H$ is $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$. It is clear that the central subgroup $Z \subseteq H$ acts trivially on $\widehat{\mathfrak{g}}$ by the adjoint action, so that we obtain an action $\mathrm{Ad}_{\widehat{\mathfrak{g}}}$ of $\widetilde{G}$ on $\widehat{\mathfrak{g}}$ with

$$g.(X, z) = (\mathrm{Ad}(g).X, z + \theta(g, X)),$$

where $\theta: \widetilde{G} \times \mathfrak{g} \to \mathfrak{z}$ is a smooth function. In view of $P_2([\omega]) = 0$, this action factors through the universal covering map $q_G: \widetilde{G} \to G$ (Proposition 7.6). Therefore the subgroup $\pi_1(G) \subseteq \widetilde{G}$ acts trivially on $\widehat{\mathfrak{g}}$, and hence the group $D_Z := q_H^{-1}(\pi_1(G)) \subseteq H$ is central because $H$ is connected (Corollary 3.15). Now $Z \cong (D_Z)_0$ is divisible, so that Lemma 7.10 yields $D_Z \cong Z \times E$ with a discrete group $E \cong \pi_1(G)$. Therefore $\widehat{G} := H/E$ carries a natural Lie group structure. The homomorphism $q_G \circ q_H: H \to G$ has the kernel $D_Z$, hence factors through a homomorphism $q: \widehat{G} \to G$ which is a principal bundle with structure group $D_Z/E \cong Z$. $\square$

At this point we have all the pieces to obtain the exact sequence, the main result of this section and the heart of the paper.

THEOREM 7.12 (Exact sequence for central Lie group extensions). *Let $G$ be a connected Lie group, $\mathfrak{z}$ an s.c.l.c. space, $\Gamma \subseteq \mathfrak{z}$ a discrete subgroup, and $Z$ a Lie group with $Z_0 \cong \mathfrak{z}/\Gamma$. Then the sequence*

$$\mathrm{Hom}(G, Z) \hookrightarrow \mathrm{Hom}(\widetilde{G}, Z) \to \mathrm{Hom}\big(\pi_1(G), Z\big) \xrightarrow{C} \mathrm{Ext}_{\mathrm{Lie}}(G, Z)$$
$$\xrightarrow{D} H^2_c(\mathfrak{g}, \mathfrak{z}) \xrightarrow{P} \mathrm{Hom}\big(\pi_2(G), Z\big) \times \mathrm{Hom}\big(\pi_1(G), \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})\big)$$

*is exact.*

*Proof.* — The exactness in $\mathrm{Hom}(G, Z)$ and $\mathrm{Hom}(\widetilde{G}, Z)$ is trivial. The exactness in the group $\mathrm{Hom}(\pi_1(G), Z)$ follows from Lemma 7.2, and Proposition 7.4 yields the exactness in $\mathrm{Ext}_{\mathrm{Lie}}(G, Z)$, so that it remains to verify the exactness in $H^2_c(\mathfrak{g}, \mathfrak{z})$.

For the case where $Z$ is connected, it follows from Lemmas 7.8 and 7.11. If $Z$ is not connected, then Lemma 7.10 implies that $Z \cong Z_0 \times \pi_0(Z)$.

If $q: \widehat{G} \to G$ is a central $Z$-extension, then $\widehat{G}/\pi_0(Z)$ is a central $Z_0$-extension with the same Lie algebra because $\widehat{G} \to \widehat{G}/\pi_0(Z)$ is a covering map. Therefore $P \circ D = 0$ follows directly from the case of connected groups because $\operatorname{im}(D) \subseteq H^2_c(\mathfrak{g}, \mathfrak{z})$ is the same for $Z$ and $Z_0$. To verify $\ker P \subseteq \operatorname{im} D$, we first obtain from $P([\omega]) = 0$ a central $Z_0$-extension $\widehat{G} \to G$ (Lemma 7.11), and then $\widehat{G} \times \pi_0(Z) \to G$ is a central $Z$-extension with the same Lie algebra. □

The following proposition clarifies how central extensions by non-connected groups can be reduced to central extensions by discrete and connected groups. For finite-dimensional groups $G$ Proposition 7.13 can be found as Theorem 3.4 in [Ho51, II].

PROPOSITION 7.13. — *If $\Gamma \subseteq \mathfrak{z}$ is a discrete subgroup and $Z$ an abelian Lie group with $Z_0 \cong \mathfrak{z}/\Gamma$, then for each connected Lie group $G$ we have*

$$\begin{aligned}\operatorname{Ext}_{\mathrm{Lie}}(G, Z) &\cong \operatorname{Ext}_{\mathrm{Lie}}(G, Z_0) \times \operatorname{Ext}_{\mathrm{Lie}}(G, \pi_0(Z)) \\ &\cong \operatorname{Ext}_{\mathrm{Lie}}(G, Z_0) \times \operatorname{Hom}(\pi_1(G), Z/Z_0).\end{aligned}$$

*Proof.* — First we obtain from Lemma 7.10 that $Z \cong Z_0 \times \pi_0(Z)$. Using this product structure, one easily verifies that

$$\operatorname{Ext}_{\mathrm{Lie}}(G, Z) \cong \operatorname{Ext}_{\mathrm{Lie}}(G, Z_0) \times \operatorname{Ext}_{\mathrm{Lie}}(G, Z/Z_0)$$

holds for every Lie group $G$. For the discrete group $Z/Z_0$ Theorem 7.12 shows that

$$\operatorname{Ext}_{\mathrm{Lie}}(G, Z/Z_0) \cong \operatorname{Hom}(\pi_1(G), \pi_0(Z))$$

because $\operatorname{Hom}(\widetilde{G}, Z/Z_0)$ is trivial. □

*Remark 7.14.* — If $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ is a central extension of $G$ by the connected group $Z \cong \mathfrak{z}/\Gamma$ and $Z \hookrightarrow H \twoheadrightarrow \widetilde{G}$ is the pull-back to the universal covering group $\widetilde{G}$ of $G$, then $H \to G$ is still a central extension of $G$ because its kernel acts trivially on the Lie algebra $\widehat{\mathfrak{g}}$. The kernel of this action is isomorphic to $Z \times \pi_1(G)$ (Lemma 7.10). In terms of Proposition 7.13, this corresponds to replacing the extension $E \in \operatorname{Ext}(G, Z)$ by the element

$$(E, \mathrm{id}_{\pi_1(G)}) \in \operatorname{Ext}(G, Z) \times \operatorname{Hom}(\pi_1(G), \pi_1(G)) \cong \operatorname{Ext}(G, Z \times \pi_1(G)). \quad □$$

COROLLARY 7.15. — *Let $G$ be a connected Lie group and $Z_0 \cong \mathfrak{z}/\Gamma$ for a discrete subgroup $\Gamma \subseteq \mathfrak{z}$. Then the following assertions hold:*

(i) *If $G$ is simply connected, then the sequence*

$$\mathbf{0} \to \mathrm{Ext}_{\mathrm{Lie}}(G, Z) \xrightarrow{D} H^2_c(\mathfrak{g}, \mathfrak{z}) \xrightarrow{P_1} \mathrm{Hom}\,\big(\pi_2(G), Z\big)$$

*is exact.*

(ii) *If $Z$ is connected, then the sequence*

$$\begin{aligned}\mathbf{0} &\to \mathrm{Hom}(G, \mathfrak{z}) \longrightarrow \mathrm{Hom}(G, Z) \xrightarrow{E_*} \mathrm{Ext}_{\mathrm{Lie}}(G, \Gamma)\\ &\longrightarrow \mathrm{Ext}_{\mathrm{Lie}}(G, \mathfrak{z}) \xrightarrow{(q_Z)_*} \mathrm{Ext}_{\mathrm{Lie}}(G, Z) \xrightarrow{\zeta} \mathrm{Hom}(\pi_2(G), \Gamma)\end{aligned}$$

*is exact, where $\zeta$ assigns to a central $Z$-extension of $G$ the homomorphism $\mathrm{per}_\omega \colon \pi_2(G) \to \mathfrak{z}$ and $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ is a corresponding Lie algebra cocycle.*

*Proof.* — (i) follows directly from Theorem 7.12.

(ii) Since $G$ is connected, we have $\mathrm{Hom}(G, \Gamma) = \mathbf{0}$, so that, in view of the second part of Remark 4.5, it only remains to verify the exactness at $\mathrm{Ext}_{\mathrm{Lie}}(G, Z)$.

Let $\mathfrak{z} \hookrightarrow \widehat{G} \to G$ be a central $\mathfrak{z}$-extension of $G$ and $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ a corresponding Lie algebra cocycle. Then $\mathrm{per}_\omega = 0$ (Theorem 7.9), and this shows that $\zeta \circ (q_Z)_* = 0$. If, conversely, $E \colon Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ is a central extension with $\zeta(E) = \mathrm{per}_\omega = 0$, then Theorem 7.12 implies that $E = (q_Z)_* \widetilde{E}$ holds for a central $\mathfrak{z}$-extension $\widetilde{E}$ of $G$ because $P_2([\omega]) = 0$ follows from the existence of the central extension $E$. □

LEMMA 7.16. — *For each $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ we have*

$$\mathrm{tor}\,\big(\pi_2(G)\big) \subseteq \ker P_1([\omega]) \quad \textit{and} \quad \mathrm{tor}\,\big(\pi_1(G)\big) \subseteq \ker P_2([\omega]).$$

*In particular $P_1([\omega])$ and $P_2([\omega])$ factor through homomorphisms of the rational homotopy groups*

$$\pi_2(G) \otimes \mathbb{Q} \to Z \quad \textit{and} \quad \pi_1(G) \otimes \mathbb{Q} \to \mathrm{Lin}(\mathfrak{g}, \mathfrak{z}).$$

*Proof.* — The first assertion follows from the fact that the range of the homomorphism $P_2([\omega])$ is a vector space. Similarly we see that $\mathrm{tor}(\pi_2(G)) \subseteq \ker \mathrm{per}_\omega$, and this implies that $\mathrm{tor}(\pi_2(G)) \subseteq \ker P_1([\omega])$. The second assertion follows from the fact that for an abelian group $A$ the kernel $A$ of the natural map $A \to A \otimes \mathbb{Q}, a \mapsto a \otimes 1$ coincides with $\mathrm{tor}(A)$. □

*Example 7.17.* — Suppose that $\dim G < \infty$. Then $\pi_2(G)$ is trivial (cf. [God71]), so that we obtain from Theorem 7.12 a simpler exact sequence

$$\mathrm{Hom}\,\big(\pi_1(G), Z\big) \xrightarrow{C} \mathrm{Ext}_{\mathrm{Lie}}(G, Z) \xrightarrow{D} H^2_c(\mathfrak{g}, \mathfrak{z}) \xrightarrow{P} \mathrm{Hom}\,\big(\pi_1(G), \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})\big)$$

(cf. [Ne96]). If, in addition, $G$ is simply connected, then we obtain an isomorphism

$$\mathrm{Ext}_{\mathrm{Lie}}(G, Z) \cong H^2_c(\mathfrak{g}, \mathfrak{z}) \tag{7.2}$$

(cf. [TW87, Cor. 5.7]). □

## 8. Central extensions with global smooth sections.

In this subsection we discuss the existence of a smooth cross section for a central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ which is equivalent to the existence of a smooth global cocycle $f: G \times G \to Z$ with $\widehat{G} \cong G \times_f Z$. Moreover, we will show that for simply connected groups, it is equivalent to the exactness of the left invariant 2-form $\Omega$ on $G$.

We shall see below that both conditions in the following proposition are also necessary for the existence of a smooth global cocycle. The vanishing of $P_2([\omega])$ is already necessary for the existence of the central extension, and (1) corresponds to the existence of a smooth global section.

Proposition 8.1 (Cartan's construction). — *Let $G$ be a connected Lie group, $\mathfrak{z}$ an s.c.l.c. space, $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ a continuous 2-cocycle, and $\Omega \in \Omega^2(G, \mathfrak{z})$ the corresponding left invariant 2-form on $G$ with $\Omega_{\mathbf{1}} = \omega$. We assume that*

*(1) $\Omega = d\theta$ for some $\theta \in \Omega^1(G, \mathfrak{z})$ and*

*(2) $P_2([\omega]) = 0$.*

*Then the product manifold $\widehat{G} := G \times \mathfrak{z}$ carries a Lie group structure which is given by a smooth 2-cocycle $f \in Z^2(G, \mathfrak{z})$ via*

$$(g, z)(g', z') := (gg', z + z' + f(g, g')).$$

*The Lie algebra of this group is isomorphic to $\mathfrak{g} \oplus_\omega \mathfrak{z}$.*

*Proof.* — First we note that (2) means that for each $X \in \mathfrak{g}$ the 1-form $i(X_r).\Omega$ is exact (Theorem 3.6). For the natural left action of $G$ on $G$ given by $\sigma(g, x) = gx$, we have $\dot{\sigma}(X) = -X_r$ and therefore

$$\mathcal{L}_{X_r}.\theta = d(i(X_r).\theta) + i(X_r).d\theta = d(i(X_r).\theta) + i(X_r).\Omega,$$

implies the exactness of $\mathcal{L}_{X_r}.\theta$ for each $X \in \mathfrak{g}$. Using Lemma A.2.6, we now see that the 1-forms $\lambda_g^*\theta - \theta$ are exact. Hence there exists for each $g \in G$ a unique $f_g \in C^\infty(G, \mathfrak{z})$ with $f_g(\mathbf{1}) = 0$ and $df_g = \lambda_g^*\theta - \theta$.

Now we show that $f(g,h) := f_g(h)$ is a $\mathfrak{z}$-valued 2-cocycle. Our construction shows that

$$f(\mathbf{1},x) = f(x,\mathbf{1}) = 0 \quad \text{for} \quad x \in G.$$

For $g, h \in G$ the functions $f_g \circ \lambda_h$ and $f_{gh}$ satisfy

$$d(f_g \circ \lambda_h + f_h) = \lambda_h^*(\lambda_g^*\theta - \theta) + \lambda_h^*\theta - \theta = \lambda_{gh}^*\theta - \theta = df_{gh}.$$

Therefore the connectedness of $G$ implies $f_g \circ \lambda_h + f_h = f_{gh} + f_g(h)$ because both sides have the same differential and the same value in $\mathbf{1}$. This leads to

$$f(g,hu) + f(h,u) = f(gh,u) + f(g,h) \quad \text{for} \quad g,h,u \in G.$$

Moreover, the concrete local formula for $f_g$ in the Poincaré Lemma and the smooth dependence of the integral on $g$ imply that $f$ is smooth on a neighborhood of $(\mathbf{1},\mathbf{1})$. We write the cocycle condition as

$$f(gh,u) = f(h,u) - f(g,h) + f(g,hu).$$

For $g$ fixed, this function is smooth as a function of the pair $(h,u)$ in a neighborhood of $(\mathbf{1},\mathbf{1})$. This implies that $f$ is smooth on a neighborhood of the points $(g,\mathbf{1})$, $g \in G$. Fixing $g$ and $u$ shows that there exists a $\mathbf{1}$-neighborhood $V \subseteq G$ such that the functions $f(\cdot,u)$, $u \in V$, are smooth in a neighborhood of $g$. Since $g \in G$ was arbitrary, we conclude that the functions $f(\cdot,u)$, $u \in V$, are smooth. Now

$$f(\cdot,hu) = f(\cdot h,u) - f(h,u) + f(\cdot,h)$$

shows that the same holds for the functions $f(\cdot,x)$, $x \in V^2$, and iterating this process, using $G = \bigcup_{n\in\mathbb{N}} V^n$, we derive that all functions $f(\cdot,x)$, $x \in G$, are smooth. Finally we conclude that the function

$$(g,h) \mapsto f(g,hu) = f(gh,u) - f(h,u) + f(g,h)$$

is smooth in a neighborhood of each point $(g_0,\mathbf{1})$, hence that $f$ is smooth in each point $(g_0,u_0)$, and this proves that $f$ is smooth on $G \times G$. We therefore obtain on the space $\widehat{G} := G \times \mathfrak{z}$ a Lie group structure with the multiplication given by

$$(g,z)(g',z') := (gg', z + z' + f(g,g')),$$

and Lemma 4.6 implies that the corresponding Lie bracket is given by

$$[(X',z'),(X,z)] = \big([X',X], d^2f(\mathbf{1},\mathbf{1})(X',X) - d^2f(\mathbf{1},\mathbf{1})(X,X')\big).$$

Now we relate this formula to the Lie algebra cocycle $\omega$. The relation $df_g = \lambda_g^*\theta - \theta$ leads to

$$d_2f(g,\mathbf{1})(Y) = (\lambda_g^*\theta - \theta)_{\mathbf{1}}(Y) = \langle\theta, Y_l\rangle(g) - \theta_{\mathbf{1}}(Y),$$

where $Y_l$ denotes the left invariant vector field with $Y_l(\mathbf{1}) = Y$. Taking second derivatives, we further obtain for $X \in \mathfrak{g}$:

$$\begin{aligned} d^2f(\mathbf{1},\mathbf{1})(X,Y) &= X_l(\langle\theta, Y_l\rangle)(\mathbf{1}) \\ &= d\theta(X_l, Y_l)(\mathbf{1}) + Y_l(\langle\theta, X_l\rangle)(\mathbf{1}) + \theta([X_l, Y_l])(\mathbf{1}) \\ &= \omega(X,Y) + Y_l(\langle\theta, X_l\rangle)(\mathbf{1}) + \theta_{\mathbf{1}}([X,Y]), \end{aligned}$$

so that

$$d^2f(\mathbf{1},\mathbf{1})(X,Y) - d^2f(\mathbf{1},\mathbf{1})(Y,X) = \omega(X,Y) + \theta_{\mathbf{1}}([X,Y]).$$

Since this cocycle is equivalent to $\omega$, the assertion follows. □

COROLLARY 8.2. — *If $G$ is simply connected and $\Omega$ is exact, then there exists a smooth cocycle $f\colon G \times G \to \mathfrak{z}$, so that $\widehat{G} := G \times_f \mathfrak{z}$ is a Lie group with Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$.*

*Proof.* — Since $\pi_1(G)$ is trivial, the condition $P_2([\omega]) = 0$ is automatically satisfied. □

*Remark 8.3.* — For finite-dimensional groups, the construction described in Proposition 8.2 is due to E. Cartan, who used it to construct a central extension of a simply connected finite-dimensional Lie group $G$ by the group $\mathfrak{z}$. Since in this case

$$H^2_{\mathrm{dR}}(G,\mathfrak{z}) \cong \mathrm{Hom}(\pi_2(G),\mathfrak{z}) = \mathbf{0} \quad \text{and} \quad H^1_{\mathrm{dR}}(G,\mathfrak{z}) \cong \mathrm{Hom}(\pi_1(G),\mathfrak{z}) = \mathbf{0},$$

(cf. [God71]), the requirements of the construction are satisfied for every Lie algebra cocycle $\omega \in Z^2(\mathfrak{g},\mathfrak{z})$.

The construction can in particular be found in the survey article of Tuynman and Wiegerinck [TW87] (see also [Tu95], [Go86] and [Ca52b]). Actually E. Cartan gave three proofs for Lie's Third Theorem ([Ca52a], [Ca52b] and [Ca52c]), where [Ca52a] and [Ca52c] rely on splitting of a Levi subalgebra and hence reducing the problem to the semisimple and the solvable case, but the second one is geometric (in the spirit of the argument in Example 7.17) and uses $H^2_{\mathrm{dR}}(G,\mathfrak{z}) = \mathbf{0}$ for a simply connected Lie group $G$ (see also [Est88]). □

PROPOSITION 8.4. — *For a connected Lie group $G$ and $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ the following are equivalent:*

(1) *There exists a corresponding smooth central extension* $q\colon \widehat{G} \to G$ *of* $G$ *by* $Z = \mathfrak{z}/\Gamma$ *with a smooth global section.*

(2) *The left invariant 2-form* $\Omega$ *on* $G$ *with* $\Omega_1 = \omega$ *is exact and* $P_2([\omega]) = 0$.

*Proof.* — (2) $\Rightarrow$ (1) is Proposition 8.1.

(1) $\Rightarrow$ (2): First $P_2([\omega]) = 0$ follows from Theorem 7.12. Let $\sigma\colon G \to \widehat{G}$ be a smooth section and $\alpha \in \Omega^1(\widehat{G}, \mathfrak{z})$ be the left invariant $\mathfrak{z}$-valued 1-form with $\alpha_1 = p_{\mathfrak{z}}$ the linear projection $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{z}$. Then $d\alpha = -q^*\Omega$ follows from

$$d\alpha_1((X,z),(X',z')) = -p_{\mathfrak{z}}([(X,z),(X',z')]) = -\omega(X,X') = -(q^*\Omega)_1((X,z),(X',z'))$$

and the left invariance of $\alpha$ and $\Omega$. Then $\sigma^*\alpha$ is a $\mathfrak{z}$-valued 1-form on $G$ with

$$d\sigma^*\alpha = \sigma^* d\alpha = -\sigma^* q^*\Omega = -(q\circ\sigma)^*\Omega = -\Omega,$$

so that $\Omega$ is exact. $\square$

Proposition 8.5. — (a) *If a smooth central extension* $Z = \mathfrak{z}/\Gamma \to \widehat{G} \xrightarrow{q} G$ *corresponding to a Lie algebra cocycle* $\omega$ *has a smooth section, then the following assertions hold:*

(1) *the left-invariant 2-form* $\Omega \in \Omega^2(G,\mathfrak{z})$ *with* $\Omega_1 = \omega$ *is exact.*

(2) $\mathrm{per}_\omega = 0$.

(3) *The natural homomorphism* $\pi_1(q)\colon \pi_1(\widehat{G}) \to \pi_1(G)$ *has a homomorphic section.*

*If, conversely,* (1)–(3) *are satisfied, and* $G$ *is smoothly paracompact, then a smooth section exists.*

*Proof* (see [TW87, Prop. 4.14] for the finite-dimensional case). "$\Rightarrow$" The exactness of $\Omega$ follows from Proposition 8.4. For each piecewise smooth map $\gamma\colon \mathbb{S}^2 \to G$ with respect to a triangulation of $\mathbb{S}^2$ we then have

$$\mathrm{per}_\omega([\gamma]) = \int_\gamma \Omega = \int_\gamma d\theta = \int_{\partial\gamma} \theta = 0.$$

Therefore $\mathrm{per}_\omega$ vanishes.

Let $\sigma\colon G \to \widehat{G}$ be a smooth section, so that $q\circ\sigma = \mathrm{id}_G$. Then the induced homomorphisms of the fundamental groups satisfy $\pi_1(q)\circ\pi_1(\sigma) = \mathrm{id}_{\pi_1(G)}$, and the assertion follows.

"$\Leftarrow$" Now we assume (1)-(3). Since $\Omega$ is exact and $P_2([\omega]) = 0$ by Theorem 7.12, Corollary 8.2 implies the existence of a central extension $G \times_{f_{\mathfrak{z}}^1} \mathfrak{z}$ of $G$ by $\mathfrak{z}$ which can be written as a product and $f_{\mathfrak{z}}^1: G \times G \to \mathfrak{z}$ is a smooth group cocycle. Passing to a different but cohomologous Lie algebra cocycle $\omega$, we may assume that $D(f_{\mathfrak{z}}^1) = \omega$. Let $f_{\mathfrak{z}} := f_{\mathfrak{z}}^1 \circ (q_G \times q_G)$ and $f := q_Z \circ f_{\mathfrak{z}}$ denote the corresponding smooth cocycles with values in $\mathfrak{z}$ and $Z$ on the universal covering group $\widetilde{G}$ of $G$. Then $G^\sharp := \widetilde{G} \times_f Z$ is a central extension of $\widetilde{G}$ by $Z$ corresponding to $\omega$ and with a global smooth section. Since the pull-back of the central extension $\widehat{G} \to G$ to $\widetilde{G}$ corresponds to the same Lie algebra cocycle, both extensions are equivalent (Corollary 7.15(i)). Therefore we have a covering homomorphism $\varphi: \widetilde{G} \times_f Z \to \widehat{G}$ with $q \circ \varphi = q_G$. In particular the universal covering group of $\widehat{G}$ is isomorphic to $\widetilde{G} \times_{f_{\mathfrak{z}}} \mathfrak{z}$. We identify the group $\pi_1(\widehat{G})$ with the kernel of the natural homomorphism $q_{\widehat{G}}: \widetilde{G} \times_{f_{\mathfrak{z}}} \mathfrak{z} \to \widehat{G}$. Then $\pi_1(\widehat{G}) \subseteq \pi_1(G) \times \mathfrak{z}$, and since $f_{\mathfrak{z}}$ is pulled back from a cocycle on $G$, it is trivial on $\pi_1(G)$, so that $\pi_1(G) \times \mathfrak{z}$ is a product subgroup of $\widetilde{G} \times_{f_{\mathfrak{z}}} \mathfrak{z}$. The natural projection $\pi_1(\widehat{G}) \to \pi_1(G)$ is simply the projection onto the first component in $\pi_1(G) \times \mathfrak{z}$ and $\Gamma \cong \{\mathbf{1}\} \times \Gamma$ is contained in $\pi_1(G)$. Therefore the image of $\pi_1(\widehat{G})$ under $\mathrm{id} \times q_Z$ in $\pi_1(G) \times Z$ is isomorphic to $\pi_1(G)$. Hence there exists a group homomorphism $\gamma: \pi_1(G) \to Z$ with

$$(\mathrm{id} \times q_Z)(\pi_1(\widehat{G})) = \Gamma(\gamma) = \{(d, \gamma(d)): d \in \pi_1(G)\}.$$

We conclude that

$$\widehat{G} \cong (\widetilde{G} \times_{f_{\mathfrak{z}}} \mathfrak{z})/\pi_1(\widehat{G}) \cong (\widetilde{G} \times_f Z)/\Gamma(\gamma).$$

Now our assumption (3) implies the existence of a group homomorphism $\gamma_{\mathfrak{z}}: \pi_1(G) \to \mathfrak{z}$ such that

$$\pi_1(\widehat{G}) = \Gamma\{(d, \gamma_{\mathfrak{z}}(d)): d \in \pi_1(G)\} = \{(d, z) \in \pi_1(G) \times \mathfrak{z}: q_Z(z) = q_Z(\gamma_{\mathfrak{z}}(d))\}.$$

This means that $q_Z \circ \gamma_{\mathfrak{z}} = \gamma$, i.e., $\gamma$ lifts to a homomorphism $\gamma_{\mathfrak{z}}: \pi_1(G) \to \mathfrak{z}$. Therefore

$$\widehat{G} \cong (\widetilde{G} \times_{f_{\mathfrak{z}}} \mathfrak{z})/\pi_1(\widehat{G}) \cong (\widetilde{G} \times_{f_{\mathfrak{z}}} \mathfrak{z})/\Gamma(\gamma_{\mathfrak{z}})\Gamma \cong \Big((\widetilde{G} \times_{f_{\mathfrak{z}}} \mathfrak{z})/\Gamma(\gamma_{\mathfrak{z}})\Big)/\Gamma.$$

Since the central $\mathfrak{z}$-extension of $G$ given by $(\widetilde{G} \times_{f_{\mathfrak{z}}} \mathfrak{z})/\Gamma(\gamma_{\mathfrak{z}})$ has convex fibers, the fact that $G$ is smoothly paracompact implies that it has a smooth global section. Therefore $q: \widehat{G} \to G$ also has a smooth global section. □

COROLLARY 8.6. — *Suppose that $\widehat{G}$ is defined by a homomorphism $\gamma: \pi_1(G) \to Z \cong \mathfrak{z}/\Gamma$ and that $G$ is smoothly paracompact. Then $\widehat{G} \to G$*

*has a smooth section if and only if $\gamma$ has a lift to a homomorphism $\pi_1(G) \to \mathfrak{z}$.*

*Proof.* — This is the special case of Proposition 8.5, where $[\omega] = 0$ (Theorem 7.12). As

$$\widehat{G} \cong (\widetilde{G} \times Z)/\Gamma(\gamma^{-1}),$$

we have

$$\pi_1(\widehat{G}) \cong \{(d, z) \in \pi_1(G) \times \mathfrak{z} : q_Z(z)\gamma(d) = \mathbf{1}\},$$

where the natural homomorphism to $\pi_1(G)$ is the projection onto the first factor. This homomorphism splits if and only $\gamma$ lifts to a homomorphism $\pi_1(G) \to \mathfrak{z}$, so that the assertion follows from Proposition 8.5. □

Definition 8.7. — *It is also interesting, and was the traditional way to approach central extensions of finite-dimensional Lie groups by Hochschild ([Ho51]) to consider only smooth Lie group extensions with a global smooth cocycle, or, equivalently, with a smooth section.*

With the results in this section, we can easily pinpoint the difference to our approach. Let $G$, $Z$ be Lie groups, $Z$ abelian, and $Z^2_{gs}(G,Z) \subseteq Z^2_s(G,Z)$ the group of smooth cocycles. If $f \in Z^2_{gs}(G,Z)$ is a coboundary, i.e.,

$$f(x,y) = h(xy)h(x)^{-1}h(y)^{-1}, \quad x, y \in G$$

for a locally smooth function $h: G \to Z$, then fixing $x$ in the above formula, we see that $h$ is smooth in a neighborhood of $x$, and therefore that $h$ is smooth. Therefore $Z^2_{gs}(G,Z) \cap B^2_s(G,Z)$ coincides with the group $B^2_{gs}(G,Z)$ of those coboundaries defined by globally smooth functions $h: G \to Z$. Therefore we have a natural inclusion

(9.2)
$$H^2_{gs}(G,Z) := Z^2_{gs}(G,Z)/B^2_{gs}(G,Z) \hookrightarrow H^2_s(G,Z) = Z^2_s(G,Z)/B^2_s(G,Z).$$

The group $H^2_{gs}(G,Z)$ classifies the smooth central extensions of $G$ by $Z$ with global smooth sections.

Since every smooth cocycle $f \in Z^2_{gs}(G,Z)$ has values in the identity component $Z_0$ of $Z$, it is no loss of generality to consider only connected groups $Z$. □

The following result is the version of the exact sequence in Theorem 7.12 for the setting with globally smooth cocycles.

THEOREM 8.8. — *Let $G$ be a smoothly paracompact connected Lie group, $\mathfrak{z}$ an s.c.l.c. space, $\Gamma \subseteq \mathfrak{z}$ a discrete subgroup, and $Z \cong \mathfrak{z}/\Gamma$. Then the sequence*

$$\text{(9.3)} \qquad \mathrm{Hom}\left(\pi_1(G), \mathfrak{z}\right) \xrightarrow{C_{\mathfrak{g}}} H^2_{gs}(G, Z) \xrightarrow{D} H^2_c(\mathfrak{g}, \mathfrak{z}) \xrightarrow{P_{\mathfrak{g}}} H^2_{\mathrm{dR}}(G, \mathfrak{z}) \times \mathrm{Hom}\left(\pi_1(G), \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})\right)$$

*is exact, where $C_{\mathfrak{g}}(\gamma) = C(q_Z \circ \gamma)$, $(P_{\mathfrak{g}})_1([\omega]) = [\Omega]$ is the cohomology class of the corresponding left invariant 2-form and $(P_{\mathfrak{g}})_2([\omega]) = P_2([\omega])$. Moreover, the sequence*

$$\text{(9.4)} \qquad \mathrm{Hom}(\pi_1(G), \Gamma) \xrightarrow{C_\Gamma} H^2_{gs}(G, \mathfrak{z}) \xrightarrow{\eta} H^2_{gs}(G, Z) \to \mathbf{0}$$

*with $\eta([f]) = [q_Z \circ f]$ and $C_\Gamma = C\,|_{\mathrm{Hom}(\pi_1(G),\Gamma)}$ is exact.*

*Proof.* — The exactness of (9.3) in $H^2_c(\mathfrak{g}, \mathfrak{z})$ is Proposition 8.4. In view of Theorem 7.12, the exactness in $H^2_{gs}(G, Z)$ follows from Corollary 8.6.

To show the exactness of (9.4), let $f \in Z^2_{gs}(G, Z)$. Then Proposition 8.4 implies the existence of a smooth cocycle $f_{\mathfrak{z}} \in Z^2_{gs}(G, \mathfrak{z})$ with $D([f]) = D([f_{\mathfrak{z}}])$ in $H^2_c(\mathfrak{g}, \mathfrak{z})$. From the exactness of (9.3), we derive that $[f] - \eta([f_{\mathfrak{z}}]) \in \mathrm{im}(C_{\mathfrak{g}}) \subseteq \mathrm{im}(\eta)$. Hence $\eta$ is surjective. Now let $f \in Z^2_{gs}(G, \mathfrak{z})$ with $\eta([f]) = 0$. Then, in particular $D[f] = 0$, so that $[f] = C(\gamma)$ with $\gamma \in \mathrm{Hom}(\pi_1(G), \mathfrak{z})$. Then $0 = \eta([f]) = C(q_Z \circ \gamma)$ implies that $q_Z \circ \gamma$ extends to a homomorphism $\widetilde{G} \to Z$. As the group $\widetilde{G}$ is simply connected, this homomorphism lifts to a homomorphism $\beta: \widetilde{G} \to \mathfrak{z}$. Now $C(\gamma) = C(\gamma \cdot \beta^{-1} \mid_{\pi_1(G)}) = [f]$, so that we may w.l.o.g. assume that $\mathrm{im}(\gamma) \subseteq \ker q_Z = \Gamma$, which completes the proof of the exactness of (9.4). □

One can even determine the kernel of the map $C_{\mathfrak{g}}$ in Theorem 8.8 as follows. It is clear that $\mathrm{Hom}(\pi_1(G), \Gamma) \subseteq \ker C_{\mathfrak{g}}$. Let $\gamma \in \mathrm{Hom}(\pi_1(G), \mathfrak{z})$ and write $C_{\mathfrak{g}}(\gamma)$ as $\eta(C(\gamma))$ with the natural map $C: \mathrm{Hom}(\pi_1(G), \mathfrak{z}) \to H^2_{gs}(G, \mathfrak{z})$. Then $C_{\mathfrak{g}}(\gamma) = 0$ and the exactness of (9.4) imply $C(\gamma) = C(\delta)$ for $\delta \in \mathrm{Hom}(\pi_1(G), \Gamma)$. Hence $\gamma\delta^{-1} \in \ker C \subseteq \ker C_{\mathfrak{g}}$, and therefore

$$\ker C_{\mathfrak{g}} = \mathrm{Hom}(\pi_1(G), \Gamma) + \mathrm{Hom}(\widetilde{G}, \mathfrak{z})\,|_{\pi_1(G)}.$$

## 9. Examples.

In this section we discuss several important classes of examples which will demonstrate the effectiveness of the long exact sequence for the determination of the central extensions of an infinite-dimensional Lie group $G$.

*Example 9.1.* — Let $H$ be an infinite-dimensional complex Hilbert space, $B_2(H)$ the space of Hilbert-Schmidt operators on $H$, $G := \mathrm{GL}_2(H) := \mathrm{GL}(H) \cap (\mathbf{1} + B_2(H))$, and $\mathfrak{g} = B_2(H)$ its Lie algebra. Then

$$\pi_1(G) \cong \pi_1\big(\operatorname{indlim}_{n\to\infty} \mathrm{GL}(n,\mathbb{C})\big) \cong \mathbb{Z} \quad \text{and} \quad \pi_2(G) \cong \pi_2\big(\operatorname{indlim}_{n\to\infty} \mathrm{GL}(n,\mathbb{C})\big) \cong \mathbf{1}$$

(cf. [Pa65] for the separable case and Lemma 3.5 in [Ne98] for the extension to the general case). Moreover, for each $\omega \in Z^2_c(\mathfrak{g},\mathbb{C})$ there exists an operator $C \in B(H)$ with

$$\omega(X,Y) = \mathrm{tr}([X,Y]C), \quad X,Y \in \mathfrak{g}$$

which leads to

$$H^2_c(\mathfrak{g},\mathbb{C}) \cong B(H)/(B_2(H) + \mathbb{C}\mathbf{1})$$

(cf. [dlH72, p.141]).

We claim that $P([\omega])$ vanishes. Since $\pi_2(G)$ is trivial, this will follow from the exactness of the 1-forms $i(X_r).\Omega$ for every $\omega \in Z^2(\mathfrak{g},\mathbb{C})$ (cf. Lemma 3.11). So let $\omega \in Z^2_c(\mathfrak{g},\mathbb{C})$ and $C \in B(H)$ with $\omega(X,Y) = \mathrm{tr}([X,Y]C)$ for $X,Y \in \mathfrak{g}$. We consider the function

$$f_X : G \to \mathbb{C}, \quad f_X(g) := \mathrm{tr}\big((gCg^{-1} - C)X\big),$$

and observe that

$$gCg^{-1} - C = (g - \mathbf{1})Cg^{-1} + C(g^{-1} - \mathbf{1}) \in B_2(H),$$

so that $f_X$ is a well-defined smooth function. We have for all $Y \in \mathfrak{g}$:

$$\begin{aligned} df_X(g)d\lambda_g(\mathbf{1}).Y &= \mathrm{tr}(gYCg^{-1}X) - \mathrm{tr}(gCYg^{-1}X) = \mathrm{tr}([g^{-1}Xg, Y]C) \\ &= \omega(\mathrm{Ad}(g)^{-1}.X, Y) = (i(X_r).\Omega)(g).(d\lambda_g(\mathbf{1}).Y). \end{aligned}$$

Hence $df_X = i(X_r).\Omega$, showing that the 1-forms $i(X_r).\Omega$ are all exact, and therefore that $P([\omega])$ vanishes.

Since the space $[\mathfrak{g},\mathfrak{g}] = B_1(H)$ of trace class operators is dense in $\mathfrak{g}$, we have $\mathrm{Hom}(\widetilde{G}, Z) = \mathbf{0}$ for each abelian Lie group $Z$, so that the exact sequence (Theorem 7.12) leads to the short exact sequence

$$\mathrm{Hom}(\pi_1(G), Z) \cong \mathrm{Hom}(\mathbb{Z}, Z) \cong Z \hookrightarrow \mathrm{Ext}(G,Z) \twoheadrightarrow H^2_c(\mathfrak{g},\mathfrak{z}).$$

For the simply connected covering group $\widetilde{G}$ we obtain with $\pi_2(\widetilde{G}) \cong \pi_2(G) = \mathbf{1}$ that

$$\operatorname{Ext}(\widetilde{G}, \mathbb{C}^\times) \cong H^2_c(\mathfrak{g}, \mathbb{C}) \cong B(H)/(\mathbb{C}\mathbf{1} + B_2(H)). \qquad \square$$

*Remark 9.2.* — We explain below how central extensions with non-trivial period homomorphisms can be used to construct non-integrable central extensions. Similar constructions can be found in [EK64] and in [DL66, p.147], where the central extension $\mathbb{T} \hookrightarrow \mathrm{U}(H) \twoheadrightarrow \mathrm{PU}(H)$ is discussed for an infinite-dimensional Hilbert space.

Suppose that $G$ is a simply connected Lie group and $\omega \in Z^2_c(\mathfrak{g}, \mathbb{R})$ with $\mathrm{per}_\omega \neq 0$. If $\mathrm{im}(\mathrm{per}\,\omega)$ is not discrete, then we already have an example of a non-integrable central extension. Suppose that $\mathrm{im}(\mathrm{per}_\omega)$ is discrete, so that we may assume that $\mathrm{im}(\mathrm{per}_\omega) = \mathbb{Z}$. Let $q\colon \widehat{G} \to G$ be the corresponding $\mathbb{T}$-extension of $G$. We put $\mathfrak{g}_1 := \mathfrak{g} \oplus \mathfrak{g}$, $G_1 := G \times G$, and

$$\omega_1((X,Y),(X',Y')) := \omega(X,Y) + \sqrt{2}\omega(X',Y').$$

Then $\mathrm{im}(\mathrm{per}_{\omega_1}) = \mathrm{im}(\mathrm{per}_\omega) + \sqrt{2}\,\mathrm{im}(\mathrm{per}_\omega)$ is not discrete, so that there exists no smooth central extension of $G_1$ corresponding to $\omega_1$ (Theorem 7.9).

This can also be proved more directly as follows: The group $\widehat{G}_2 := \widehat{G} \times \widehat{G}$ is a central extension of $G_1$ by the two-dimensional torus $\mathbb{T}^2$ with period group $\mathbb{Z}^2 \subseteq \mathbb{R}^2$. If a central extension $\widehat{G}_1 \to G_1$ corresponding to $\omega_1$ would exist, then we could construct a local homomorphism of some $\mathbf{1}$-neighborhood in $\widehat{G}_2$ to $\widehat{G}_1$, and then use Lemma 2.1 to extend it to a Lie group homomorphism $\widehat{G}_2 \to \widehat{G}_1$ with the appropriate differential. Then the central torus $\mathbb{T}^2$ in $\widehat{G}_2$ would be mapped onto the subgroup $Z_1$ corresponding to $\mathfrak{z}_1 \cong \mathbb{R}$. So this subgroup would be a quotient of $\mathbb{T}^2$ modulo a dense wind, which is absurd. $\square$

*Example 9.3.* — Let $G := \mathrm{Diff}_+(\mathbb{T})$ be the group of orientation preserving diffeomorphisms of the circle $\mathbb{T}$. Then $\widetilde{G}$ can be identified with the group

$$\widetilde{G} := \{f \in \mathrm{Diff}(\mathbb{R})\colon (\forall x \in \mathbb{R})\, f(x + 2\pi) = f(x) + 2\pi\},$$

and the covering homomorphism $q_G\colon \widetilde{G} \to G$ is given by $q(f)([x]) = [f(x)]$, where $[x] = x + \mathbb{Z} \in \mathbb{T} \cong \mathbb{R}/\mathbb{Z}$. Then $\ker q_G$ consists of all translations $\tau_a$, $a \in \mathbb{Z}$. Moreover, the inclusion map

$$\eta\colon \mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{Diff}_+(\mathbb{T})$$

is a homotopy equivalence (cf. [Fu86, p. 302]). Note also that $\widetilde{G}$ is a convex set of maps $\mathbb{R} \to \mathbb{R}$, so that this group is obviously contractible (cf. [TL99, 6.1]). In particular we have

$$\pi_1(G) \cong \mathbb{Z} \quad \text{and} \quad \pi_k(G) = \{\mathbf{1}\}, \quad k > 1.$$

As a consequence, we obtain $\operatorname{Hom}(\pi_1(G), \mathbb{T}) \cong \mathbb{T}$. Moreover,

$$H^2_{\text{sing}}(G, \mathbb{T}) \cong H^2_{\text{sing}}(\mathbb{T}, \mathbb{T}) \cong \operatorname{Hom}(H_2(\mathbb{T}), \mathbb{T}) = \mathbf{0}.$$

Furthermore we have

$$H^2_c(\mathfrak{g}, \mathbb{R}) = \mathbb{R}[\omega]$$

([PS86]). To describe the cocycle $\omega$, we identify $\mathfrak{g}_{\mathbb{C}}$ with $\mathcal{V}(\mathbb{T})_{\mathbb{C}}$. Let $iL_0 \in \mathfrak{g}$ denote the invariant vector field on $\mathbb{T}$. Then there exist suitably normalized eigenvectors $L_n$, $n \in \mathbb{Z}$, for $\operatorname{ad} L_0$ in $\mathfrak{g}_{\mathbb{C}}$ such that the cocycle $\omega$ is given by

$$\omega(L_n, L_{-m}) = n(n-1)(n+1)\delta_{n,m}.$$

Therefore the long exact sequence in Theorem 5.13 leads to an exact sequence

$$\mathbb{T} \hookrightarrow \operatorname{Ext}(G, \mathbb{T}) \to \mathbb{R} \to \operatorname{Hom}(\pi_1(G), \mathfrak{g}').$$

Now one has to show that the standard generator $[\omega]$ of $H^2_c(\mathfrak{g}, \mathbb{R})$ has trivial image in the space $\operatorname{Hom}(\pi_1(G), \mathfrak{g}')$ to get an exact sequence

$$\mathbb{T} \hookrightarrow \operatorname{Ext}(G, \mathbb{T}) \twoheadrightarrow \mathbb{R}, \qquad \text{and hence} \quad \operatorname{Ext}(G, \mathbb{T}) \cong \mathbb{T} \times \mathbb{R}$$

(cf. [Se81, Cor. 7.5]). The formula for $\omega$ implies that it is trivial on on $\operatorname{span}_{\mathbb{C}}\{L_0, L_1, L_{-1}\} \cong \mathfrak{sl}(2, \mathbb{C})$. Therefore $i(X_r).\Omega \mid_{\mathrm{PSL}(2,\mathbb{R})} = 0$ for all $X \in \mathfrak{sl}(2, \mathbb{R})$: In fact, for $g \in \mathrm{PSL}(2, \mathbb{R})$, $X \in \mathfrak{sl}(2, \mathbb{R})$ and $Y \in \mathfrak{g}$ we have

$$\Omega_g(X_r(g), d\lambda_g(\mathbf{1}).Y) = \omega(\operatorname{Ad}(g)^{-1}.X, Y) \in \omega(\mathfrak{sl}(2, \mathbb{R}), \mathfrak{g}) = \{0\}.$$

Hence $P_2([\omega]) = 0$, and since $P_1([\omega]) = 0$ follows from the triviality of $\pi_2(G)$, we have $P([\omega]) = 0$. We conclude that there exists a central $\mathbb{T}$-extension Vir of $G$, which is called the *Virasoro group*.

For the simply connected covering group $\widetilde{G}$ of $G$ we have

$$\operatorname{Ext}(\widetilde{G}, \mathbb{T}) \cong H^2_c(\mathfrak{g}, \mathbb{R}) \cong \mathbb{R}.$$

This implies in particular that $G$ has a universal central extension $Z \hookrightarrow \widehat{G} \to G$ with $Z \cong \mathbb{Z} \times \mathbb{R}$ (cf. [Ne01b]). One can realize the group $\widehat{G}$ as a central extension of $\widetilde{G}$ by $\mathbb{R}$. This is the universal covering group of the Virasoro group Vir.

Since $G$ is modeled on the nuclear Fréchet space $\mathfrak{g} \cong C^\infty(\mathbb{T}, \mathbb{R})$ it is a smoothly paracompact manifold ([KM97, 16.10, 27.4]). In particular the de Rham isomorphism holds for $G$ ([KM97]), and we conclude that

$$H^2_{\mathrm{dR}}(G, \mathbb{R}) \cong H^2_{\mathrm{sing}}(G, \mathbb{R}) \cong H^2_{\mathrm{sing}}(\mathbb{T}, \mathbb{R}) = \mathbf{0}.$$

It follows in particular that $\Omega$ is exact, so that Cartan's construction (Proposition 8.4) implies that $\mathrm{Vir} \cong G \times \mathbb{T}$ as smooth manifolds, and that it can be defined by a smooth cocycle $f\colon G \times G \to \mathbb{T}$. Such cocycles are known explicitly, and one is the famous Bott-Thurston cocycle (cf. [Ro95, p.237]). $\square$

*Example 9.4 (Current groups).* — (a) Let $\mathfrak{k}$ be a finite-dimensional Lie algebra, $M$ a compact manifold and $\mathfrak{g} := C^\infty(M, \mathfrak{k})$ endowed with the pointwise bracket. Then we assign to every symmetric invariant bilinear form $\kappa$ on $\mathfrak{k}$ the continuous Lie algebra cocycle of $\mathfrak{g}$ with values in the Fréchet space $\mathfrak{z} := \Omega^1(M, \mathbb{R})/dC^\infty(M, \mathbb{R})$ given by

$$\omega(f, g) := [\kappa(f, dg)], \quad f, g \in \mathfrak{g}.$$

In [MN01] we calculate the period group for such cocycles and the group $G := C^\infty(M, K)_0$, where $K$ is a connected group with Lie algebra $\mathfrak{k}$. In turns out that $\Pi_\omega$ is always contained in the subspace $H^1_{\mathrm{dR}}(M, \mathbb{R})$. If $\mathfrak{k}$ is simple and $\kappa$ is suitably normalized, then $\Pi_\omega$ coincides with the discrete subgroup of all cohomology classes with integral periods. In [MN01] we give more detailed criteria for the discreteness of $\Pi_\omega$. Moreover, we show that in all cases $P_2([\omega])$ vanishes, so that the results of Section 7 lead to central Lie group extensions of $G$ corresponding to $\omega$ if and only if $\Pi_\omega$ is discrete.

(b) Now let $G := \Omega(\mathrm{SU}(2))$ be the loop group of $\mathrm{SU}(2)$, i.e., the group of continuous base point preserving maps $\mathbb{S}^1 \to \mathrm{SU}(2)$. Then

$$\pi_2(G) \cong \pi_3(\mathrm{SU}(2)) \cong \pi_3(\mathbb{S}^3) \cong \mathbb{Z} \quad \text{and} \quad \pi_1(G) \cong \pi_2(\mathrm{SU}(2)) = \mathbf{1}.$$

On the Lie algebra $\mathfrak{g}^1 := \Omega^1(\mathfrak{su}(2))$ of the group $\Omega^1(\mathrm{SU}(2))$ of $C^1$-loops one has the natural 2-cocycle

$$\omega(\alpha, \beta) := \int_{\mathbb{T}} \kappa(\alpha(t), \beta'(t))\, dt,$$

where $\kappa$ is the Cartan–Killing form of $\mathfrak{su}(2)$. This cocycle has no continuous extension to $\Omega(\mathfrak{su}(2))$. It follows from the results in [Ma01] that

$H^2_c(\Omega(\mathfrak{g}_0), \mathbb{R}) = \mathbf{0}$ for every semisimple compact Lie algebra $\mathfrak{g}_0$. Assuming this, the exact sequence for central extensions leads to

$$\mathrm{Ext}(\Omega(\mathrm{SU}(2)), \mathbb{T}) = \mathbf{1}.$$

In contrast to that, the inclusion $G^1 \hookrightarrow G$ is a homotopy equivalence and

$$H^2_c(\Omega^1(\mathfrak{su}(2)), \mathbb{R}) \cong \mathbb{R},$$

which, in view of [EK64, p.28], leads to $\mathrm{Ext}(\Omega^1(\mathrm{SU}(2)), \mathbb{T}) \cong \mathbb{Z}$. □

*Remark 9.5 (Central extensions of abelian Lie groups).* — (a) Suppose that $G$ is an abelian Lie group with an exponential function $\exp\colon \mathfrak{g} \to G$ which is a universal covering homomorphism (cf. Remark 3.13). Since the covering map $\exp$ induces an isomorphism of the second homotopy groups, $\pi_2(G) \cong \pi_2(\mathfrak{g})$ is trivial. Hence we have the exact sequence

$$\begin{aligned} \mathrm{Lin}(\mathfrak{g}, Z) &\xrightarrow{res} \mathrm{Hom}\big(\pi_1(G), Z\big) \xrightarrow{C} \mathrm{Ext}_{\mathrm{Lie}}(G, Z) \\ &\xrightarrow{D} H^2_c(\mathfrak{g}, \mathfrak{z}) \xrightarrow{P} \mathrm{Hom}\big(\pi_1(G), \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})\big). \end{aligned}$$

For abelian Lie algebras the coboundary operator is trivial, so that $H^2_c(\mathfrak{g}, \mathfrak{z}) = \mathrm{Alt}^2(\mathfrak{g}, \mathfrak{z})$ coincides with the space of continuous alternating bilinear forms $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$. Here the map $P$ is quite simple:

$$P\colon \mathrm{Alt}^2(\mathfrak{g}, \mathfrak{z}) \to \mathrm{Hom}\big(\pi_1(G), \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})\big), \quad P(\omega)(d, X) = \omega(X, d).$$

Therefore the condition for the existence of a Lie group extension $\widehat{G} \to G$ by $Z$ is that

$$\pi_1(G) \subseteq \mathrm{rad}(\omega) := \{X \in \mathfrak{g}\colon \omega(X, \mathfrak{g}) = \mathbf{0}\}.$$

If this condition is satisfied, then $\omega$ factors through $G \times G$ to a smooth 2-cocycle

$$f\colon G \times G \to \mathfrak{z}, \quad (\exp X, \exp Y) \mapsto \omega(X, Y).$$

We thus obtain a group $G \times_f \mathfrak{z}$ which is a covering of the group $G \times_{f_Z} Z$ with $f_Z = q_Z \circ f_{\mathfrak{z}}$.

(b) If $\mathrm{span}\, \pi_1(G)$ is dense in $\mathfrak{g}$, then we call $G$ a *generalized torus.* Then $\ker P = \mathbf{0}$ implies that $D = 0$, and therefore that $C$ is surjective, so that

$$\mathrm{Ext}_{\mathrm{Lie}}(G, Z) \cong \mathrm{Hom}\big(\pi_1(G), Z\big)/\big(\mathrm{Lin}(\mathfrak{g}, Z)\,|_{\pi_1(G)}\big).$$

If $\dim G < \infty$, then $\operatorname{span} \pi_1(G) = \mathfrak{g}$, and $\pi_1(G)$ is a lattice in $\mathfrak{g}$. Therefore $\operatorname{Hom}\big(\pi_1(G), Z\big) = \operatorname{Lin}(\mathfrak{g}, Z)\,|_{\pi_1(G)}$ leads to

$$\operatorname{Ext}(\mathbb{T}^n, Z) = \mathbf{0} \quad \text{for all} \quad n \in \mathbb{N}, Z = \mathfrak{z}/\Gamma.$$

(c) Let $\mathfrak{g}$ be a locally convex space $\mathfrak{g}$ and $D \subseteq \mathfrak{g}$ a discrete subgroup. Then there exists a continuous seminorm $p$ on $\mathfrak{g}$ with $D \cap p^{-1}([0,1]) = \mathbf{0}$, showing that the image in the normed space $\mathfrak{g}_p := \mathfrak{g}/p^{-1}(0)$ is a discrete subgroup isomorphic to $D$. This implies that every discrete subgroup of a locally convex space is isomorphic to a discrete subgroup of a Banach space. As has been shown by Sidney ([Si77, p. 983]), countable discrete subgroups of Banach spaces are free. This implies in particular that discrete subgroups of separable locally convex spaces are free.

Let $E$ be a vector space and $f\colon D \to E$ a homomorphism of additive groups. Since every finitely generated subgroup of $D$ is a discrete subgroup of the vector space it spans, every linear relation $\sum_d \lambda_d d = 0$ implies that $\sum_d \lambda_d f(d) = 0$. Hence $f$ extends to a linear map $f\colon \operatorname{span} D \to E$. Such an extension need not be continuous if $D$ is not finitely generated. Suppose that $D$ is countably infinite and that $\mathfrak{g}$ is a Banach space. Let $(e_n)_{n\in\mathbb{N}}$ be a basis of $D$ as an abelian group. We define $f(e_n) := n\|e_n\|$. Then $f$ extends to a linear map on $\operatorname{span} D$ which obviously is not continuous. We conclude in particular that if $G$ is an infinite-dimensional separable generalized Banach torus, then

$$\operatorname{Ext}_{\mathrm{Lie}}(G, \mathbb{R}) \cong \operatorname{Hom}\big(\pi_1(G), \mathbb{R}\big)/\big(\operatorname{Lin}(\mathfrak{g}, \mathbb{R})\,|_{\pi_1(G)}\big) \neq \mathbf{0}.$$

(d) If $\widehat{G}$ is a connected central extension with abelian Lie algebra, then its universal covering group is the vector space $\widehat{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{z}$, and the fundamental group $\pi_1(\widehat{G})$ is defined by an exact sequence

$$\Gamma = \pi_1(Z) \hookrightarrow \pi_1(\widehat{G}) \xrightarrow{p_{\mathfrak{g}}} \pi_1(G),$$

where $p_{\mathfrak{g}}\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is the projection onto the first factor. In this sense we have a natural map

$$\eta\colon \operatorname{Ext}_{\mathrm{ab}}(G, Z) \to \operatorname{Ext}_{\mathrm{ab}}\big(\pi_1(G), \pi_1(Z)\big).$$

If $\pi_1(G)$ is free, then the group on the right hand side is trivial, so that $\eta$ vanishes, but if $\pi_1(G)$ is not free, there might be non-trivial classes in $\operatorname{Ext}\big(\pi_1(G), \pi_1(Z)\big)$, and therefore $\widehat{G}$ is non-trivial.

The relation $\eta(C(\gamma)) = 0$ for $\gamma \in \operatorname{Hom}(\pi_1(G), Z)$ means that $\gamma$ can be lifted to a homomorphism $\widetilde{\gamma}\colon \pi_1(G) \to \mathfrak{z}$ (cf. Lemma 8.5), so that we have a $\mathfrak{z}$-extension of $G$ covering the $Z$-extension $\widehat{G}$. This extension is trivial if and only if the homomorphism $\pi_1(G) \to \mathfrak{z}$ extends continuously to $\mathfrak{g}$ which might not be possible, as we have seen in (c).

(e) Let $\mathfrak{g}$ be a Banach space, $D \subseteq \mathfrak{g}$ a discrete subgroup with $\operatorname{Ext}(D, \mathbb{Z}) \neq \mathbf{0}$ and $G := \mathfrak{g}/D$. The exactness of the sequence

$$\operatorname{Hom}(D, \mathbb{Z}) \hookrightarrow \operatorname{Hom}(D, \mathbb{R}) \to \operatorname{Hom}(D, \mathbb{T}) \to \operatorname{Ext}_{\mathrm{ab}}(D, \mathbb{Z}) \to \operatorname{Ext}_{\mathrm{ab}}(D, \mathbb{R}) = \mathbf{0}$$

(Theorem A.1.4) shows that there exists a homomorphism $\gamma \colon D \to \mathbb{T}$ which cannot be lifted to a homomorphism $\widetilde{\gamma} \colon D \to \mathbb{R}$. In view of (d), this implies that the corresponding abelian extension

$$\mathbb{T} \hookrightarrow \widehat{G} := (\mathfrak{g} \times \mathbb{T})/\Gamma(\gamma^{-1}) \twoheadrightarrow G \cong \mathfrak{g}/D$$

has no global continuous section.

We do not know of any example of a discrete subgroup of a Banach space which is not free (cf. (c)). $\square$

*Example 9.6.* — We consider the real Banach space $\mathfrak{g} = c_0(\mathbb{N}, \mathbb{R})$ of sequences converging to 0 endowed with the sup-norm. Then $\mathbb{Z}^{(\mathbb{N})} = \mathbb{Z}^{\mathbb{N}} \cap c_0(\mathbb{N}, \mathbb{R})$ is a discrete subgroup spanning a dense subspace, so that $G := \mathfrak{g}/\mathbb{Z}^{(\mathbb{N})}$ is a generalized torus with $\pi_1(G) \cong \mathbb{Z}^{(\mathbb{N})}$. Now Remark 9.5(b) implies that

$$\operatorname{Ext}_{\mathrm{Lie}}(G, \mathbb{R}) \cong \mathbb{R}^{\mathbb{N}}/l^1(\mathbb{N}, \mathbb{R}). \qquad \square$$

*Remark 9.7.* — In [Se81, Prop. 7.4] G. Segal claims that for a connected Lie group $G$ the sequence

$$\operatorname{Hom}\big(\pi_1(G), \mathbb{T}\big) \xrightarrow{C} \operatorname{Ext}(G, \mathbb{T}) \xrightarrow{D} H^2_c(\mathfrak{g}, \mathbb{R}) \xrightarrow{c_{\mathbb{T}}} H^2_{\mathrm{sing}}(G, \mathbb{T}) \tag{9.1}$$

is exact, where $c_{\mathbb{T}}$ assigns to a Lie algebra cohomology class the de Rham cohomology class of the corresponding left invariant 2-form and further the corresponding $\mathbb{T}$-valued singular cohomology class, which can be done with the de Rham Theorem if $G$ is smoothly paracompact.

A simple example of a group where the sequence (9.1) is not exact is $G = \mathbb{T}^2$, the two-dimensional torus. As we have seen in Remark 9.5(b), we have $\operatorname{Ext}(G, \mathbb{T}) = \mathbf{0}$, and Remark 9.5(a) shows that $H^2_c(\mathfrak{g}, \mathbb{R}) \cong \mathbb{R}$. Further $H_2(G) \cong \mathbb{Z}$, where the generator is the fundamental cycle ($G$ is an orientable surface). Hence $H^2_{\mathrm{sing}}(G, \mathbb{T}) \cong \mathbb{T}$. We conclude that the sequence above leads to a concrete sequence

$$\mathbb{T}^2 \xrightarrow{C} \mathbf{0} \xrightarrow{D} \mathbb{R} \xrightarrow{P} \mathbb{T}.$$

On the other hand the definition of $P$ shows that it is continuous, and this contradicts Segal's claim. $\square$

# A. Appendix.

## A.1. Universal coefficients and abelian groups.

THEOREM A.1.1 (Universal Coefficient Theorem). — *Let $K$ be a complex of free abelian groups $K_n$ and $Z$ be any abelian group. Put $H^*(K,Z) := H^*(\mathrm{Hom}(K,Z))$. Then for each dimension $n$ there is an exact sequence*

$$\mathbf{0} \to \mathrm{Ext}_{\mathrm{ab}}\big(H_{n-1}(K), Z\big) \xrightarrow{\ \beta\ } H^n(K,Z) \xrightarrow{\ \alpha\ } \mathrm{Hom}\big(H_n(K), Z\big) \to \mathbf{0}$$

*with homomorphisms $\beta$ and $\alpha$ natural in $Z$ and $K$. This sequence splits by a homomorphism which is natural in $Z$ but not in $K$.*

*The second map $\alpha$ is defined on a cohomology class $[f]$ as follows. Each $n$-cocycle of* $\mathrm{Hom}(K,Z)$ *is a homomorphism $f\colon K_n \to Z$ vanishing on $\partial K_{n+1}$, so induces $f_*\colon H_n(K) \to Z$. If $f = \delta g$ is a coboundary, it vanishes on cycles, so $(\delta g)_* = 0$. Now define $\alpha([f]) := f_*$.*

*Proof* [MacL63, Th. 3.4.1]. $\square$

*Remark A.1.2.* — If the abelian group $Z$ is divisible, then $\mathrm{Ext}_{\mathrm{ab}}(B,Z) = \mathbf{0}$ for each abelian group $B$, so that Theorem A.1.1 leads to an isomorphism $H^n(K,Z) \cong \mathrm{Hom}\big(H_n(K), Z\big)$ of abelian groups. $\square$

*Remark A.1.3.* — For each topological space $X$ we obtain for the singular (co)homology and each abelian group $Z$ a short exact sequence

$$\mathbf{0} \to \mathrm{Ext}_{\mathrm{ab}}\big(H_{n-1}(X), Z\big) \to H^n_{\mathrm{sing}}(X,Z) \to \mathrm{Hom}\big(H_n(X), Z\big) \to \mathbf{0}.$$

If $Z$ is divisible, then we have

$$H^n_{\mathrm{sing}}(X,Z) \cong \mathrm{Hom}\big(H_n(X), Z\big). \qquad \square$$

*Remark A.1.4.* — (a) The Hurewicz-Theorem says that if $n \geqslant 2$ and $X$ is arcwise connected with $\pi_i(X) = \mathbf{0}$ for $1 \leqslant i < n$ ($X$ is $(n-1)$-connected), then

$$\pi_n(X) \cong H_n(X)$$

(cf. [Br93, Cor. 7.10.8]). For $n = 1$ we have the complementary result that for any arcwise connected topological space $X$,

$$\pi_1(X)/(\pi_1(X), \pi_1(X)) \cong H_1(X).$$

In both cases we obtain

$$\mathrm{Hom}\,\big(H_n(X), Z\big) \cong \mathrm{Hom}\,\big(\pi_n(X), Z\big)$$

for every abelian group $Z$.

If $n \geqslant 2$, then we obtain with the Hurewicz Theorem $H_{n-1}(X) = \mathbf{0}$, so that the Universal Coefficient Theorem also shows that

$$H^n_{\mathrm{sing}}(X, Z) \cong \mathrm{Hom}\,\big(H_n(X), Z\big) \cong \mathrm{Hom}\,\big(\pi_n(X), Z\big)$$

for all abelian groups $Z$.

(b) If, in addition, $M$ is a smoothly paracompact manifold (cf. [KM97, Th. 34.7]), then

$$H^n_{\mathrm{dR}}(M, \mathbb{R}) \cong H^n(M, \mathbb{R}) \cong \mathrm{Hom}\,\big(H_n(M), \mathbb{R}\big). \qquad \square$$

### A.2. Differential forms and vector fields.

Let $M$ be a smooth manifold modeled over a locally convex space. For a Lie group $G$ we will use the natural multiplication on the tangent bundle given by $T(m_G)$, where $m_G$ is the group multiplication on $G$. We thus identify $G$ with the subgroup of $TG$ given by the zero section and the Lie algebra $\mathfrak{g}$, as an additive group, with $T_{\mathbf{1}}(G)$. In the following, $\mathfrak{z}$ will always denote a sequentially complete locally convex space.

LEMMA A.2.1. — *Let $G$ be a connected Lie group acting smoothly on $M$ by $\sigma_G: G \times M \to M$. Let*

$$\dot\sigma: \mathfrak{g} \to \mathcal{V}(M), \quad \dot\sigma(X)(p) := -d\sigma(\mathbf{1}, p)(X, 0)$$

*denote the corresponding homomorphism of Lie algebras. Then we have an action of $G$ on $\mathcal{V}(M)$ by $(g.\mathcal{Y})(p) := d\sigma_g(g^{-1}.p)\mathcal{Y}(g^{-1}.p)$, and the derived action is given by $X.\mathcal{Y} := [\dot\sigma(X), \mathcal{Y}]$.*

*Proof.* — It follows from [Ne01a, Prop. 1.18(v)] that $\dot\sigma$ is a homomorphism of Lie algebras. We have

$$(g.\mathcal{Y})(p) = d\sigma_g(g^{-1}.p)\mathcal{Y}(g^{-1}.p) = g.\big(\mathcal{Y}(g^{-1}.p)\big),$$

where the last term refers to the action of $G$ on $TM$. Applying $g.\mathcal{Y}$ to a smooth function $f$, we get

$$\begin{aligned}(g.\mathcal{Y})(f)(p) &= \langle d(f\circ\sigma_g)(g^{-1}.p), \mathcal{Y}(g^{-1}.p)\rangle \\ &= \langle d(f\circ\sigma_g), \mathcal{Y}\rangle(g^{-1}.p) = (\mathcal{Y}.(f\circ\sigma_g))(g^{-1}.p).\end{aligned}$$

Taking derivatives in $g = e$ in the direction of $X \in \mathfrak{g} = T_{\mathbf{1}}(G)$ leads to

$$(X.\mathcal{Y})(f)(p) = -\mathcal{Y}.(\dot\sigma(X).f)(p) + \dot\sigma(X).(\mathcal{Y}.f)(p) = ([\dot\sigma(X), \mathcal{Y}].f)(p).$$

Since locally we have sufficiently many smooth functions to separate vector fields, we conclude that $X.\mathcal{Y} = [\dot\sigma(X), \mathcal{Y}]$. □

*Remark A.2.2.* — If $\alpha\colon ]a,b[ \to G$ is a smooth curve on $G$, then for each $p \in M$ we obtain a smooth curve $\alpha_p\colon ]a,b[ \to M$ given by $\alpha_p(t) = \alpha(t).p$, and the derivative of this curve is given by

$$\alpha_p'(t) = \alpha'(t).p,$$

where we use the action $T\sigma$ of $TG$ on $TM$ (cf. [Ne01a, Prop. 1.18]). This action satisfies

$$T\sigma(Y_{\mathbf{1}}).p = -\dot\sigma(Y)(p) \quad \text{for} \quad Y_{\mathbf{1}} \in T_{\mathbf{1}}(G) \cong \mathfrak{g}$$

and therefore

$$T\sigma(Y_{\mathbf{1}}\cdot g).p = T\sigma(d\rho_g(\mathbf{1}).Y_{\mathbf{1}}).p = T\sigma(Y_{\mathbf{1}})(g.p) = -\dot\sigma(Y)(g.p).$$

We conclude that

$$\alpha_p'(t) = -\dot\sigma(\alpha_r'(t))(\alpha_p(t)),$$

where $\alpha_r'(t) = d\rho_{\alpha(t)^{-1}}(\alpha(t)).\alpha'(t) \in \mathfrak{g}$ is the *right logarithmic derivative of* $\alpha$.

For a smooth vector field $X$ on $M$, a locally convex space $\mathfrak{z}$ and a $\mathfrak{z}$-valued $p$-form $\omega \in \Omega^p(M,\mathfrak{z})$ we define the *Lie derivative* by

$$\mathcal{L}_X.\omega := i(X)d\omega + d(i(X).\omega).$$

Lemma A.2.3. — *For $\omega \in \Omega^p(M,\mathfrak{z})$ and smooth vector fields $X, X_1, \ldots, X_p$ on $M$ we have*

$$\begin{aligned}(\mathcal{L}_X.\omega)(X_1,\ldots,X_p) &= X.\omega(X_1,\ldots,X_p) \\ &\quad - \sum_{j=1}^p \omega(X_1,\ldots,X_{j-1},[X,X_j],X_{j+1},\ldots,X_p).\end{aligned}$$

*Proof.* — By definition, we have $\mathcal{L}_X = d \circ i(X) + i(X) \circ d$, so that

$$\begin{aligned}
&(\mathcal{L}_X.\omega)(X_1,\ldots,X_p) = d(i(X).\omega)(X_1,\ldots,X_p) + d\omega(X,X_1,\ldots,X_p)\\
&= X_1.\omega(X,X_2,\ldots,X_p) - X_2.\omega(X,X_1,X_3,\ldots) \pm \ldots\\
&\quad - \omega(X,[X_1,X_2],X_3,\ldots,X_p) + \ldots + X.\omega(X_1,\ldots,X_p)\\
&\quad - X_1.\omega(X,X_2,\ldots,X_p) \pm \ldots - \omega([X,X_1],X_2,\ldots,X_p)\\
&\quad - \omega([X_1,X_2],X,X_3,\ldots,X_p) \pm \ldots\\
&= X.\omega(X_1,\ldots,X_p) + \sum_{j=1}^{p}(-1)^j\omega([X,X_j],X_1,X_2,\ldots,\widehat{X_j},\ldots,X_p)\\
&= X.\omega(X_1,\ldots,X_p) - \sum_{j=1}^{p}\omega(X_1,\ldots,X_{j-1},[X,X_j],X_{j+1},\ldots,X_p).
\end{aligned}$$

□

LEMMA A.2.4. — *For $\omega \in \Omega^p(M,\mathfrak{z})$, $X \in \mathfrak{g}$ and a smooth curve $\alpha\colon ]-\varepsilon,\varepsilon[ \to G$ with $\alpha(0) = e$ and $\alpha'(0) = X$ we have*

$$\mathcal{L}_{\dot\sigma(X)}.\omega = \frac{d}{dt}\Big|_{t=0} \sigma^*_{\alpha(t)^{-1}}\omega,$$

*where the limit is considered pointwise on $p$-tuples of tangent vectors of $M$.*

*Proof.* — We consider the situation in local coordinates around a point $q \in M$. More precisely, we pick an open neighborhood $U$ of $q \in M$ which is diffeomorphic to an open subset of a locally convex space $V$. Therefore it suffices to test the equality of both sides on constant vector fields $X_j \equiv v_j \in V$, $j = 1,\ldots,p$.

We may w.l.o.g. assume that $\alpha(t).q \in U$ for $|t| < \varepsilon$. Then

(A.2.1)
$$(\sigma^*_{\alpha(t)^{-1}}\omega)(X_1,\ldots,X_p)(q) = \omega(\alpha(t)^{-1}.q)(\alpha(t)^{-1}.X_1,\ldots,\alpha(t)^{-1}.X_q),$$

where $\alpha(t)^{-1}.v$ refers to the action of $G$ on $TM$. For each $v \in V$ Lemma A.2.1 further leads to

$$\frac{d}{dt}\Big|_{t=0} \alpha(t).X_j = [\dot\sigma(X),X_j].$$

Taking derivatives in $t = 0$ in (A.3.1), we now get

$$\begin{aligned}
&d_0\omega(q)(X_1,\ldots,X_p)(\dot\sigma(X)(q))\\
&-\sum_{j=1}^{p}\omega(q)(X_1,\ldots,X_{j-1},[\dot\sigma(X),X_j],X_{j+1},\ldots,X_p),
\end{aligned}$$

where $d_0$ stands for the partial derivative of $\omega$ in the first component when considered locally as a function of $p+1$ arguments, $p$-linear in the last $p$ variables. The fact that the vector fields $X_j$ are constant leads to

$$\big(\dot\sigma(X).\omega(X_1,\ldots,X_p)\big)(q) = d_0\omega(q)(X_1,\ldots,X_p)(\dot\sigma(X)).$$

In view of Lemma A.2.3, this proves the assertion. □

LEMMA A.2.5. — *Assume that $G$ is connected. For $\omega \in \Omega^p(M,\mathfrak{z})$ the following are equivalent:*

(1) *$\omega$ is $G$-invariant.*

(2) *$\mathcal{L}_{\dot\sigma(X)}.\omega = 0$ for each $X \in \mathfrak{g}$.*

*Proof.* — "(1) ⇒ (2)" is an immediate consequence of Lemma A.2.4.

"(2) ⇒ (1)": Let $g \in G$. Then there exists a smooth path $\alpha\colon [0,1] \to G$ with $\alpha(0) = e$ and $\alpha(1) = g$. We have to show that

$$\omega = \sigma_g^*\omega = \sigma_{\alpha(1)}^*\omega.$$

In view of $\sigma_{\alpha(0)}^*\omega = \omega$, it suffices to show that $t \mapsto \alpha(t)^*\omega$ is constant. Let $\alpha_r'(t) := \alpha'(t).\alpha(t)^{-1} \in \mathfrak{g}$ denote the right logarithmic derivative of $\alpha$. Then Lemma A.2.4 implies that

$$\frac{d}{dt}\Big|_{t=s} \sigma_{\alpha(t)}^*\omega = \sigma_{\alpha(s)}^*\big(-\mathcal{L}_{\dot\sigma(\alpha_r'(s))}.\omega\big) = \sigma_{\alpha(s)}^*0 = 0. \qquad \square$$

LEMMA A.2.6. — *Assume that $G$ is connected. For $\theta \in \Omega^1(M,\mathfrak{z})$ the following are equivalent:*

(1) *For each $g \in G$ the 1-form $\sigma_g^*\theta - \theta$ is exact.*

(2) *For each $X \in \mathfrak{g}$ the 1-form $\mathcal{L}_{\dot\sigma(X)}.\theta$ is exact.*

*Proof.* — Let $\gamma \in C^\infty(\mathbb{S}^1, G)$. Since we have a natural injection $H^1_{\mathrm{dR}}(M,\mathfrak{z}) \hookrightarrow \mathrm{Hom}(\pi_1(M),\mathfrak{z})$ given by integration over loops (Theorem 3.6), the assertion follows if we show that the condition $\int_\gamma \sigma^*\theta = \int_\gamma \theta$ for all $g \in G$ is equivalent to $\int_\gamma \mathcal{L}_{\dot\sigma(X)}.\theta = 0$ for all $X \in \mathfrak{g}$.

(1) ⇒ (2): Let $\alpha\colon ]-\varepsilon,\varepsilon[ \to G$ be a smooth curve with $\alpha(0) = e$ and $\alpha'(0) = -X$. Then Lemma A.2.4 yields

$$\mathcal{L}_{\dot\sigma(X)}.\theta = \frac{d}{dt}\Big|_{t=0} \sigma_{\alpha(t)}^*\theta$$

and therefore

$$\int_\gamma \mathcal{L}_{\dot\sigma(X)}.\theta = \int_{\mathbb{S}^1} \gamma^* \mathcal{L}_{\dot\sigma(X)}.\theta$$
$$= \lim_{t\to 0} \frac{1}{t} \int_{\mathbb{S}^1} \gamma^*(\sigma^*_{\alpha(t)}\theta - \theta) = \lim_{t\to 0} \frac{1}{t} \int_\gamma (\sigma^*_{\alpha(t)}\theta - \theta) = 0.$$

(2) $\Rightarrow$ (1): Conversely, we use that

$$\sigma^*_{\alpha(1)}\theta - \theta = \int_0^1 \frac{d}{dt}\Big|_{t=s} \sigma^*_{\alpha(t)}\theta \, ds = \int_0^1 \sigma^*_{\alpha(s)}\big( - \mathcal{L}_{\dot\sigma(\alpha_r'(s))}.\theta\big)\, ds.$$

Integrating this relation over $\gamma$ and interchanging the order of integration leads to

$$\int_\gamma (\sigma^*_{\alpha(1)}\theta - \theta) = \int_0^1 \int_{\mathbb{S}^1} \gamma^* \sigma^*_{\alpha(s)}\big( - \mathcal{L}_{\dot\sigma(\alpha_r'(s))}.\theta\big)\, ds = 0$$

because all integrals

$$\int_{\mathbb{S}^1} \gamma^* \sigma^*_{\alpha(s)} \mathcal{L}(\dot\sigma(\alpha_r'(s))).\theta = \int_{\sigma_{\alpha(s)}\circ\gamma} \mathcal{L}_{\dot\sigma(\alpha_r'(s))}.\theta$$

vanish. □

## A.3. An approximation theorem for infinite-dimensional manifolds.

The goal of this section is to explain that continuous functions $f\colon M \to N$, $M$ a finite-dimensional compact manifold and $N$ an infinite-dimensional manifold, can be approximated by smooth functions. This implies in particular that every homotopy class in $[M, N]$ has a smooth representative.

In the following $C(M,N)_c$ denotes the space $C(M,N)$ of continuous maps $M \to N$ endowed with the compact open topology.

THEOREM A.3.1. — *Let $M$ be a finite-dimensional $\sigma$-compact $C^s$-manifold for $s \in \mathbb{N} \cup \{\infty\}$. Then for all locally convex spaces $\mathfrak{z}$ the space $C^\infty(M,\mathfrak{z})$ is dense in $C(M,\mathfrak{z})_c$. If $f \in C(M,\mathfrak{z})$ has compact support and $U$ is an open neighborhood of* $\operatorname{supp}(f)$, *then each neighborhood of $f$ in $C(M,\mathfrak{z})_c$ contains a smooth function whose support is contained in $U$.*

*Proof (based on [Hi76, Th. 2.2]).* — First we observe that $M$ permits $C^s$-partitions of unity. Further the local convexity of $\mathfrak{z}$ is crucial for the "partition of unity arguments" to work.

Let $f \in C(M, \mathfrak{z})$, $(V_\alpha)_{\alpha \in A}$ a locally finite open cover of $M$ and $W_\alpha \subseteq \mathfrak{z}$ convex open 0-neighborhoods. Then there exists a subordinated locally finite open cover $(U_i)_{i \in I}$ of $M$ and constant maps $g_i \colon M \to \mathfrak{z}$ such that for all $y \in U_i \cap V_\alpha$ we have

$$g_i(y) - f(y) \in W_\alpha.$$

Let $(\lambda_i)_{i \in I}$ be a partition of unity subordinated to $(U_i)_{i \in I}$ and define $g := \sum_i \lambda_i g_i \colon M \to \mathfrak{z}$. Then $g \in C^s(M, \mathfrak{z})$ and on $V_\alpha$ we have $g - f \in W_\alpha$ because $W_\alpha$ is convex.

If, in addition, $\mathrm{supp}(f)$ is compact and contained in the open set $U$, then we may assume that each set $U_i$ is either contained in $U$ or satisfies $U_i \cap \mathrm{supp}(f) = \emptyset$. For $U_i \cap \mathrm{supp}(f) = \emptyset$ we then put $g_i = 0$, and the assertion follows. □

COROLLARY A.3.2. — *Let $M$ be a finite-dimensional $\sigma$-compact $C^s$-manifold for $1 \leqslant s \leqslant \infty$. If $V$ is an open subset of the locally convex space $\mathfrak{z}$, then the space $C^\infty(M, V)$ is dense in $C(M, V)_c$.* □

THEOREM A.3.3. — *Let $M$ and $N$ be $C^s$-manifolds with $\dim M < \infty$ and $s \in \mathbb{N} \cup \{\infty\}$. Then $C^s(M, N)$ is dense in $C(M, N)_c$. Let $f \in C(M, N)$ and $K \subseteq M$ such that $f$ is smooth on $M \setminus K$. Then there exists for each neighborhood $\mathcal{N}$ of $f$ in $C(M, N)_c$ and each open neighborhood $U$ of $K$ in $M$ a smooth function $g \in \mathcal{N}$ with $f = g$ on $M \setminus U$.*

*Proof.* — First we need a refinement of Theorem 2.5 in [Hi76]. Let $U \subseteq \mathbb{R}^n$ be open, $\mathfrak{z}$ an s.c.l.c. space $V \subseteq \mathfrak{z}$ open and $f \colon U \to V$ a $C^r$-map. Further let $K \subseteq U$ be closed and $W \subseteq U$ open such that $f$ is $C^s$ on a neighborhood of the closed subset $K \setminus W$. Then the set of all functions $h \in C^r(U, V)$ which are $C^s$ on a neighborhood of $K$ and coincide with $f$ on $U \setminus W$ intersects every neighborhood of $f$ in $C(U, V)_c$. For the proof we may w.l.o.g. assume that $V = \mathfrak{z}$, so that Theorem A.3.1 can be used. The remaining arguments can be copied from [Hi76, Th. 2.5].

To conclude the proof, one uses that $M$ has a countable open cover, and then an inductive argument as in [Hi76, Th. 2.6]. The argument given in [Hi76] shows in particular that if $f$ is smooth outside of a compact subset $K$ of $M$ and $U$ an open neighborhood $U$ of $K$, then we find in each neighborhood of $f$ a smooth function $g$ which coincides with $f$ on $M \setminus U$. □

*Remark A.3.4.* — (a) If $F$ is a locally convex space and $X$ a compact space, then $C(X,F)$ is a locally convex space with respect to the topology of uniform convergence. For each continuous seminorm $p$ on $F$ the prescription

$$p_X(f) := \sup_{x \in X} p(f(x))$$

defines a continuous seminorm on $C(X,F)$, and the set of all these seminorms defines the topology of compact convergence on $C(X,F)$. It is easy to verify that with respect to this topology the space $C(X,F)$ is sequentially complete if $F$ has this property.

(b) If $U \subseteq F$ is an open subset, then $C(X,U)$ is an open subset of $C(X,F)$. Now let $U_j \subseteq F_j$, $j = 1,2$, be open subsets of s.c.l.c. spaces and $\varphi: U_1 \to U_2$ a smooth map. We consider the map

$$\varphi_X: C(X,U_1) \to C(X,U_2), \quad \gamma \mapsto \varphi \circ \gamma.$$

Then $\varphi_X$ is smooth. The continuity follows from [Ne01a, Lemma III.6]. For each $x \in X$ and $\gamma, \eta \in C(X,F_1)$ we have

$$\lim_{t \to 0} \frac{\varphi(\gamma(x) + t\eta(x)) - \varphi(\gamma(x))}{t}$$
$$= \lim_{t \to 0} \int_0^1 d\varphi(\gamma(x) + st\eta(x)).\eta(x)\, ds = d\varphi(\gamma(x)).\eta(x).$$

Since the integrand is continuous in $[0,1]^2 \times X$, the limit exists uniformly in $X$, hence in the space $C(X,F_2)$. Therefore $d\varphi_X(\gamma)(\eta)$ exists. Since $d\varphi: TU_1 \cong U_1 \times F_1 \to F_2$ is a continuous map, the first part of the proof shows that

$$d\varphi_X: C(X,TU_1) \cong C(X,U_1) \times C(X,F_1) \to C(X,F_2)$$

is continuous, so that $\varphi_X$ is $C^1$. Iterating this argument shows that $\varphi_X$ is $C^\infty$. □

Definition A.3.5. — (a) *If $G$ is a Lie group and $X$ is a compact space, then $C(X,G)$, endowed with the topology of uniform convergence is a Lie group with Lie algebra $C(X,\mathfrak{g})$. In view of Remark A.3.4(b), we only have to see that inversion and multiplication in the canonical local charts are smooth. The remaining arguments leading to the Lie group structure on $C(X,G)$ are a routine verification.*

(b) *If $G$ is a Lie group with Lie algebra $\mathfrak{g}$, then the tangent bundle of $G$ is a Lie group isomorphic to $\mathfrak{g} \rtimes G$, where $G$ acts by the adjoint representation on $\mathfrak{g}$ (cf. [Ne01a]). Iterating this procedure, we obtain a Lie group structure on all iterated higher tangent bundles $T^nG$ which are diffeomorphic to $\mathfrak{g}^{2^n-1} \times G$.*

*It follows in particular that for each finite-dimensional manifold $M$ and each $n \in \mathbb{N}_0$ we obtain topological groups $C(T^nM, T^nG)_c$. Therefore the canonical inclusion map*

$$C^\infty(M,G) \hookrightarrow \prod_{n\in\mathbb{N}_0} C(T^nM, T^nG)_c$$

*leads to a natural topology on $C^\infty(M,G)$ turning it into a topological group. For compact manifolds $M$ these groups can even be turned into Lie groups with Lie algebra $C^\infty(M,\mathfrak{g})$ endowed with the topology of compact convergence of all derivatives which coincides with the topology defined above if we consider $\mathfrak{g}$ as an additive Lie group. For details we refer to [Gl01b].* □

Let $G$ be a connected Lie group and $M$ a compact smooth manifold. In $M$ we fix a base point $x_M$ and in any group we consider the unit element $e$ as the base point. We write $C^\infty_*(M,G) \subseteq C^\infty(M,G)$ for the subgroup of base point preserving maps and observe that

$$C^\infty(M,G) \cong C^\infty_*(M,G) \rtimes G$$

as Lie groups, where we identify $G$ with the subgroup of constant maps. This relation already leads to

$$\pi_k(C^\infty(M,G)) \cong \pi_k(C^\infty_*(M,G)) \times \pi_k(G), \quad k \in \mathbb{N}_0. \tag{A.3.1}$$

In particular we have $\pi_0(C^\infty(M,G)) \cong \pi_0(C^\infty_*(M,G))$ because $G$ is connected. On the other hand we have for topological groups $G$ and $k \in \mathbb{N}$ the relation

$$\pi_k(G) \cong \pi_0(C_*(\mathbb{S}^k, G)) \cong \pi_0(C_*(\mathbb{S}^k, G_0)), \tag{A.3.2}$$

where $G_0$ denotes the identity component of $G$.

Lemma A.3.6. — *Let $M$ be a compact manifold and $G$ a Lie group. Then $C^\infty(M,G)$ is dense in $C(M,G)_c$. In particular every connected component of the Lie group $C(M,G)_c$ contains a smooth map. Moreover, we have*

$$C^\infty(M,G) \cap C(M,G)_0 = C^\infty(M,G)_0. \tag{A.3.3}$$

*Proof.* — As $G$ is a topological group, the compact open topology on $C(M,G)$ coincides with the topology of uniform convergence which turns $C(M,G)$ into a Lie group with Lie algebra $C(M,\mathfrak{g})$ (Definition A.3.5). In particular $C(M,G)$ is locally arcwise connected, so that the first assertion follows immediately from Theorem A.3.3.

To verify (A.3.3), we first observe that every smooth map $f\colon M \to G$ which is sufficiently close to the identity is homotopic to the identity in $C^\infty(M,G)$ because its range lies in an open identity neighborhood diffeomorphic to an open convex set. Now homogeneity implies that $C^\infty(M,G)$ is locally connected with respect to the compact open topology, and hence that its connected components are also open in the coarser compact open topology. This implies that the connected components of $C^\infty(M,G)$ are closed in the compact open topology, and therefore that the closure of $C^\infty(M,G)_0$ is open and closed in $C(M,G)$, hence coincides with $C(M,G)_0$ and satisfies (A.3.3). □

THEOREM A.3.7. — *If $M$ is a compact manifold and $G$ a Lie group, then the inclusion $C^\infty(M,G) \to C(M,G)$ induces isomorphisms of all homotopy groups*

$$\pi_k(C^\infty(M,G)) \to \pi_k(C(M,G)), \quad k \in \mathbb{N}_0.$$

*If $x_M \in M$ is a base point, then the same conclusion holds for the inclusion $C^\infty_*(M,G) \to C_*(M,G)$ of the subgroups of base point preserving maps.*

*Proof.* — For $k = 0$ the assertion follows from Lemma A.3.6. Next we observe that for $k \geqslant 1$ the inclusions

$$\begin{aligned} C_*(\mathbb{S}^k, C^\infty(M,G)) &\hookrightarrow C(\mathbb{S}^k, C^\infty(M,G)_0) \hookrightarrow C(\mathbb{S}^k, C(M,G)_0) \\ &\hookrightarrow C(\mathbb{S}^k, C(M,G)) \cong C(\mathbb{S}^k \times M, G) \end{aligned}$$

are continuous homomorphisms of Lie groups, where the inclusion

$$C(\mathbb{S}^k, C(M,G)_0) \hookrightarrow C(\mathbb{S}^k, C(M,G))$$

is an open embedding. On the level of the group of connected components, we obtain with (A.3.2) the homomorphisms

$$\begin{aligned} \pi_k(C^\infty(M,G)) &\cong \pi_0\big(C_*(\mathbb{S}^k, C^\infty(M,G))\big) \cong \pi_0\big(C(\mathbb{S}^k, C^\infty(M,G)_0)\big) \\ &\xrightarrow{\eta} \pi_0\big(C(\mathbb{S}^k, C(M,G)_0)\big) \\ &\hookrightarrow \pi_0\big(C(\mathbb{S}^k, C(M,G))\big) \cong \pi_0\big(C(\mathbb{S}^k \times M, G)\big). \end{aligned}$$

If $f\colon \mathbb{S}^k \times M \to G$ is a continuous map, then Lemma A.3.6 first implies that every neighborhood of $f$ contains a smooth map and that

$$C^\infty(\mathbb{S}^k \times M, G) \cap C(\mathbb{S}^k \times M, G)_0 = C^\infty(\mathbb{S}^k \times M, G)_0.$$

Further every connected component of $C(\mathbb{S}^k \times M, G)$ and therefore every connected component of $C(\mathbb{S}^k, C(M,G)_0)$ contains an element of $C(\mathbb{S}^k, C^\infty(M,G)_0)$. This shows that the natural homomorphism

$$\eta\colon \pi_0\big(C(\mathbb{S}^k, C^\infty(M,G))\big) \to \pi_0\big(C(\mathbb{S}^k, C(M,G))\big)$$

is a surjective, which implies that the homomorphism

$$\pi_k(C^\infty(M,G)) \to \pi_0\big(C(\mathbb{S}^k, C(M,G)_0)\big) \\ \cong \pi_0\big(C_*(\mathbb{S}^k, C(M,G))\big) \cong \pi_k(C(M,G))$$

is surjective for $k \in \mathbb{N}$.

To see that it is also injective, let $U \subseteq G$ be an identity neighborhood for which there exists a chart $\varphi\colon U \to \mathfrak{g}$ whose range is an open convex subset of $\mathfrak{g}$. If two continuous maps $f, g \in C(\mathbb{S}^k, C^\infty(M,G))$, viewed as elements of $C(\mathbb{S}^k \times M, G) \cong C(\mathbb{S}^k, C(M,G))$ are close in the sense that the range of $(x,m) \mapsto f(x,m)g(x,m)^{-1}$ is contained in $U$, then the convexity of $\varphi(U)$ implies the existence of a homotopy from the constant map to $f \cdot g^{-1}$ in $C(\mathbb{S}^k, C^\infty(M,G))$ and hence, after multiplication with $g$ on the right, from $f$ to $g$. This implies that

$$C(\mathbb{S}^k, C(M,G))_0 \cap C(\mathbb{S}^k, C^\infty(M,G)) = C^\infty(\mathbb{S}^k, C^\infty(M,G))_0,$$

which implies that the homomorphism $\pi_k(C^\infty(M,G)) \to \pi_k(C(M,G))$ is also injective. □

*Remark A.3.8.* — If $G$ is a connected Fréchet–Lie group, then $C^\infty(M,G)$ also is a Fréchet–Lie group (cf. [Gl01b] and Definition A.3.5), so that combining [Pa66] with Theorem A.3.7 even implies that the inclusion $C^\infty(M,G) \hookrightarrow C(M,G)$ is a homotopy equivalence. □

## BIBLIOGRAPHY

[Br93] G. E. Bredon, Topology and Geometry, Graduate Texts in Mathematics 139, Springer-Verlag, Berlin, 1993.

[Br97] G. E. Bredon, Sheaf Theory, Graduate Texts in Mathematics 170, Springer-Verlag, Berlin, 1997.

[Bry93] J.-L. BRYLINSKI, Loop Spaces, Characteristic Classes and Geometric Quantization, Progr. in Math. 107, Birkhäuser Verlag, 1993.

[Ca51] E. CALABI, Sur les extensions des groupes topologiques, Brioschi Annali di Mat. Pura et Appl., Ser 4, 32 (1951), 295–370.

[Ca52a] E. CARTAN, Le troisième théorème fondamental de Lie, Oeuvres I, Gauthier–Villars, Paris, 2 (1952), 1143–1148.

[Ca52b] E. CARTAN, La topologie des espaces représentifs de groupes de Lie, Oeuvres I, Gauthier–Villars, Paris, 2 (1952), 1307–1330.

[Ca52c] E. CARTAN, Les représentations linéaires des groupes de Lie, Oeuvres I, Gauthier–Villars, Paris, 2 (1952), 1339–1350.

[Ch46] C. CHEVALLEY, Theory of Lie Groups I, Princeton Univ. Press, 1946.

[tD91] T. tOM DIECK, Topologie, de Gruyter, Berlin – New York, 1991.

[DL66] A. DOUADY and M. LAZARD, Espaces fibrés en algèbres de Lie et en groupes, Invent. Math., 1 (1966), 133–151.

[EML43] S. EILENBERG and S. MACLANE, Relations between homology and homotopy theory, Proc. Nat. Acad. Sci. U.S.A., 29 (1943), 155–158.

[EML47] S. EILENBERG and S. MACLANE, Cohomology theory in abstract groups. II, Annals of Math., 48-2 (1947), 326–341.

[Est54] W. T. VAN EST, A group theoretic interpretation of area in the elementary geometries, Simon Stevin, Wis. en Natuurkundig Tijdschrift, 32 (1954), 29–38.

[Est62] W. T. VAN EST, Local and global groups, Indag. Math., 24 (1962), 391–425; Proc. Kon. Ned. Akad. v. Wet. Series A 65.

[Est88] W. T. VAN EST, Une démonstration de E. Cartan du troisième théorème de Lie, in P. Dazord et al eds., "Séminaire Sud-Rhodanien de Géometrie VIII: Actions Hamiltoniennes de Groupes; Troisième Théorème de Lie," Hermann, Paris, 1988.

[EK64] W. T. VAN EST and Th. J. KORTHAGEN, Non enlargible Lie algebras, Proc. Kon. Ned. Acad. v. Wet. A 67 (1964), 15–31.

[EL88] W. T. VAN EST and M. A. M. VAN DER LEE, Enlargeability of local groups according to Malcev and Cartan-Smith, in P. Dazord et al eds., "Seminaire Sud-Rhodanien de Geometrie VIII: Actions Hamiltoniennes de Groupes; Troisième Théorème de Lie," Hermann, Paris, 1988.

[Fu70] L. FUCHS, Infinite Abelian Groups, I, Acad. Press, New York, 1970.

[Gl01a] H. GLÖCKNER, Infinite-dimensional Lie groups without completeness restriction, in "Geometry and Analysis on Finite- and Infinite-Dimensional Lie groups", A. Strasburger et al eds., Banach Center Publications, pp. 43-5 55, 2002.

[Gl01b] H. GLÖCKNER, Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups, J. Funct. Anal., to appear.

[Gl01c] H. GLÖCKNER, Algebras whose groups of units are Lie groups, Preprint, 2001.

[God71] C. GODBILLON, Eléments de Topologie Algébrique, Hermann, Paris, 1971.

[Go86] V. V. GORBATSEVICH, The construction of a simply connected Lie group with a given Lie algebra, Russian Math. Surveys, 41 (1986), 207–208.

[Ha82] R. HAMILTON, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc., 7 (1982), 65–222.

[dlH72] P. DE LA HARPE, Classical Banach Lie Algebras and Banach-Lie Groups of Operators in Hilbert Space, Lecture Notes in Math. 285, Springer-Verlag, Berlin, 1972.

[He73] A. HELLER, Principal bundles and groups extensions with applications to Hopf algebras, J. Pure and Appl. Algebra, 3 (1973), 219–250.

[Hi76] M. W. HIRSCH, Differential Topology, Graduate Texts in Mathematics 33, Springer-Verlag, 1976.

[Ho51] G. HOCHSCHILD, Group extensions of Lie groups I, II, Annals of Math., 54–1 (1951), 96–109 and 54–3 (1951), 537–551.

[HoMo98] K. H. HOFMANN and S. A. MORRIS, The Structure of Compact Groups, Studies in Math., de Gruyter, Berlin, 1998.

[Hub61] P. J. HUBER, Homotopical Cohomology and Čech Cohomology, Math. Annalen, 144 (1961), 73–76.

[KM97] A. KRIEGL and P. MICHOR, The Convenient Setting of Global Analysis, Math. Surveys and Monographs 53, Amer. Math. Soc., 1997.

[La99] S. LANG, Fundamentals of Differential Geometry, Graduate Texts in Math. 191, Springer-Verlag, 1999.

[Ma57] G. W. MACKEY, Les ensembles boréliens et les extensions des groupes, J. Math., 36 (1957), 171–178.

[MacL63] S. MACLANE, Homological Algebra, Springer-Verlag, 1963.

[Ma01] P. MAIER, Central extensions of topological current algebras, in "Geometry and Analysis on Finite- and Infinite-Dimensional Lie groups", A. Strasburger et al eds., Banach Center Publications, 55 (2002) ; 61–76.

[MN01] P. MAIER and K.-H. NEEB, Central extensions of current groups, Preprint, 2001.

[Mi59] E. MICHAEL, Convex structures and continuous selections, Can. J. Math., 11 (1959), 556–575.

[MT99] P. MICHOR and J. TEICHMANN, Description of infinite dimensional abelian regular Lie groups, J. Lie Theory, 9–2 (1999), 487–489.

[Mi83] J. MILNOR, Remarks on infinite-dimensional Lie groups, Proc. Summer School on Quantum Gravity, B. DeWitt ed., Les Houches, 1983.

[Ne96] K.-H. NEEB, A note on central extensions, J. Lie Theory, 6–2 (1996), 207–213.

[Ne98] K.-H. NEEB, Holomorphic highest weight representations of infinite dimensional complex classical groups, J. reine angew. Math., 497 (1998), 171–222.

[Ne01a] K.-H. NEEB, Representations of infinite dimensional groups, in "Infinite Dimensional Kähler Manifolds", S. Huchleberry et al eds., DMV Seminar 31, (2001), Birkhäuser, to appear.

[Ne01b] K.-H. NEEB, Universal central extensions of Lie groups, Acta Appl. Math., to appear.

[Omo97] H. OMORI, "Infinite-Dimensional Lie Groups," Translations of Math. Monographs 158, Amer. Math. Soc., 1997.

[Pa65] R. S. PALAIS, On the homotopy type of certain groups of operators, Topology, 3 (1965), 271–279.

[Pa66] R. S. PALAIS, Homotopy theory of infinite dimensional manifolds, Topology, 5 (1965), 1–16.

[PS86] A. PRESSLEY and G. SEGAL, Loop Groups, Oxford University Press, Oxford, 1986.

[Ro95] C. ROGER, Extensions centrales d'algèbres et de groupes de Lie de dimension infinie, algèbres de Virasoro et généralisations, Reports on Math. Phys., 35 (1995), 225–266.

[Se70] G. SEGAL, Cohomology of topological groups, Symposia Math., 4 (1970), 377–387.

[Se81] G. SEGAL, Unitary representations of some infinite-dimensional groups, Comm. Math. Phys., 80 (1981), 301–342.

[Sh49] A. SHAPIRO, Group extensions of compact Lie groups, Annals of Math., 50–3 (1949), 581–586.

[Si77] S. J. SIDNEY, Weakly dense subgroups of Banach spaces, Indiana Univ. Math. Journal, 26–6 (1977), 981–986.

[Sp66] E. H. SPANIER, Algebraic Topology, McGraw-Hill Book Company, New York, 1966.

[St78] J. D. STASHEFF, Continuous cohomology of groups and classifying spaces, Bull. of the Amer. Math. Soc., 84–4 (1978), 513–530.

[Ste51] N. STEENROD, The topology of fibre bundles, Princeton University Press, Princeton, New Jersey, 1951.

[Te99] J. TEICHMANN, Infinite-dimensional Lie Theory from the Point of View of Functional Analysis, Ph. D. Thesis, Vienna, 1999.

[Ti83] J. TITS, Liesche Gruppen und Algebren, Springer, New York-Heidelberg, 1983.

[TL99] V. TOLEDANO LAREDO, Integrating unitary representations of infinite-dimensional Lie groups, Journal of Funct. Anal., 161 (1999), 478–508.

[Tu95] G. M. TUYNMAN, An elementary proof of Lie's Third Theorem, unpublished note, 1995.

[TW87] G. M. TUYNMAN and W. A. J. J. WIEGERINCK, Central extensions and physics, J. Geom. Physics, 4–2 (1987), 207–258.

[Va85] V. S. VARADARAJAN, Geometry of Quantum Theory, Second Edition, Springer-Verlag, 1985.

[Wa83] F. W. WARNER, Foundations of Differentiable Manifolds and Lie Groups, Graduate Texts in Mathematics, Springer-Verlag, Berlin, 1983.

[We80] R. O. WELLS, Differential Analysis on Complex Manifolds, Graduate Texts in Mathematics, Springer-Verlag, 1980.

[We95] C. A. WEIBEL, An introduction to homological algebra, Cambridge studies in advanced math. 38, Cambridge Univ. Press, 1995.

[We95] D. WERNER, Funktionalanalysis, Springer-Verlag, Berlin-Heidelberg, 1995.

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Karl-Hermann NEEB,
Technische Universität Darmstadt
Fachbereich Mathematik AG5
Schlossgartenstrasse 7
D-64289 Darmstadt (Germany).
neeb@mathematik.tu-darmstadt.de