ANNALES

DE

L'INSTITUT FOURIER

Xiaojun HUANG & Shanyu JI

Cartan-Chern-Moser theory on algebraic hypersurfaces and an application to the study of automorphism groups of algebraic domains

Tome 52, nº 6 (2002), p. 1793-1831.

<http://aif.cedram.org/item?id=AIF_2002__52_6_1793_0>

Ann. Inst. Fourier, Grenoble
52, 6 (2002), 1793–1831

CARTAN-CHERN-MOSER THEORY ON ALGEBRAIC HYPERSURFACES AND AN APPLICATION TO THE STUDY OF AUTOMORPHISM GROUPS OF ALGEBRAIC DOMAINS

by **X. HUANG** (*) and **S. JI**

0. Introduction.

It is known that for a projective compact Riemann surface S, the number of elements in its automorphism group $\mathrm{Aut}(S)$ is finite (when $g(S) > 1$), which is moreover bounded by a certain constant depending only on the degree of the equations defining S. It would be interesting to find an analogue of this fact for a bounded strongly pseudoconvex domain $D \subset \mathbb{C}^{n+1}$ defined by a real polynomial. Motivated by this problem, we shall prove in this paper, that for a strongly pseudoconvex domain D defined by a real polynomial of degree k_0, the Lie group $\mathrm{Aut}(D)$ can be identified with a constructible Nash algebraic smooth variety in the CR structure bundle Y of ∂D, and the sum of its Betti numbers is bounded by a certain constant C_{n,k_0} depending only on n and k_0. In case D is simply connected, we further give an explicit but quite rough bound in terms of the dimension and the degree of the defining polynomial.

Our approach is to adapt the Cartan-Chern-Moser theory to algebraic hypersurfaces. Since the domain under consideration is strongly pseudoconvex, the study of its automorphism group can be pushed to that of the CR

(*) Supported in part by NSF-0200689 and a grant from the Rutgers Research Council.
Keywords: Real algebraic hypersurface – Automorphism group – Algebraic domains – Cartan-Chern-Moser theory – Strongly pseudoconvex domain – Betti numbers.
Math. classification: 32V40 – 14P15 – 32E99 – 32H02 – 32T15.

automorphism group of its boundary. Applying the CR equivalence theory, this can be reduced to investigating a certain variety in its structure bundle defined by certain curvature equations. The precise estimate of the total Betti number of the just mentioned variety is then obtained by making use of a result of Milnor [M] and the specific construction of its defining equations.

Now, we give our main result, whose statement requires some terminology to be explained in §1–§3:

THEOREM 0.1. — *Let $D, \widetilde{D} \subset \mathbb{C}^{n+1}$, $n \geq 1$, be bounded strongly pseudoconvex domains defined by real polynomials of degree $\leq k_0$. Then there is a smooth submanifold $\mathcal{V} \subset \widetilde{Y}$ such that the following holds:*

(i) *There exists a bijective map Λ between $\mathcal{V}$ and the collection of proper holomorphic maps from D to $\widetilde{D}$ denoted by* $\mathrm{Prop}(D, \widetilde{D})$.

(ii) *If we equip* $\mathrm{Prop}(D, \widetilde{D})$ *with its classical topology, then Λ is a homeomorphism.*

(iii) *The sum of the Betti numbers of $\mathcal{V}$ is bounded by a constant $C_{n,k_0} > 0$, which depends only on n and k_0.*

(iv) *When D is further assumed to be simply connected, the above C_{n,k_0} can be taken as*

$$\left\{4 \cdot 3^{3+2l_0^3}(16n + 33 + 2l_0^3)k_0\right\}^{2(n+2)^2},$$

where $l_0 = [(n+2)^2 - 1]\left[54 \cdot 3^{(n+2)^2}(n^2 + 20n + 37)k_0^3\right]^{2+3\cdot 5^{(n+2)^2-1}}$.

From the proof of Theorem 0.1, we will see that $\mathcal{V}$ carries a "Nash real algebraic structure": It admits a finite open covering $\{\mathcal{V}_j^i\}$, where each $\mathcal{V}_j^i$ is diffeomorphic to a smooth piece of a Nash algebraic variety and the associated transition functions are smooth Nash algebraic functions.

Theorem 0.1 is also in the spirit of the study of the parameterization problem for the CR automorphisms of real analytic hypersurfaces. Along these lines, there have recently appeared many papers (see [BER3] for a survey and the references therein). Here, we only mention the work done by Chern-Moser [CM], Vitushkin [V] and Baouendi-Ebenfelt-Rothschild [BER1], to name a few.

The main idea of the proof of Theorem 0.1 can be explained as follows (for detailed accounts of the notation and definitions, see §1–§3): First we consider local maps from $M = \partial D$ to $\widetilde{M} = \partial\widetilde{D}$. Since

$M = \{r(z,\overline{z}) = 0\}$ is a real analytic hypersurface, we can define its complexification $\mathcal{M} = \{r(z,\zeta) = 0\}$, which is called the associated Segre family of M. Let $(\mathcal{Y}, \pi, \mathcal{M})$ be the $\mathcal{G}$-structure bundle associated with $\mathcal{M}$. The following is the key fact for our argument: The existence of a Segre-isomorphism Φ between $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ with $\Phi(p) = \widetilde{p}$ if and only if the invariant function spaces at certain points $P \in \pi^{-1}(p)$ and $\widetilde{P} \in \pi^{-1}(\widetilde{p})$ coincide:

$$\Gamma_k(\Omega, \mathcal{Y})(P) = \Gamma_k(\widetilde{\Omega}, \widetilde{\mathcal{Y}})(\widetilde{P}) \tag{0.1}$$

as a lexicographically ordered set for any k (see Lemma 3.3 (ii) and Lemma 4.2 for the notation and explanation). Here all invariant functions are defined on the projective structure bundle $\mathcal{Y}$ ($\widetilde{\mathcal{Y}}$) associated with $\mathcal{M}$ ($\widetilde{\mathcal{M}}$, respectively). Notice that the above is only an equation for the value of the invariant functions at a point, instead of the typical version (see Theorem 3.2) where the relations are for functions on a certain open subset. Fix a point $P \in Y \cap \mathcal{Y}$. Then the complex analytic variety $\mathcal{V}^* = \{Q \in \widetilde{\mathcal{Y}} \mid \Gamma_k(Q) = \Gamma_k(P), \forall k\}$ is bijective to the set of all local Segre-isomorphisms Φ from $\mathcal{M}$ to $\widetilde{\mathcal{M}}$. This relates the set of maps to a complex analytic variety. To study the set of all local Segre-isomorphisms induced by CR-isomorphisms from M to $\widetilde{M}$, we need to consider the CR structure bundle $\widetilde{Y}$ naturally embedded in $\widetilde{\mathcal{Y}}$. We have a real analytic variety $\mathcal{V} = \mathcal{V}^* \cap \widetilde{Y}$. In case M is defined by a real polynomial, the variety $\mathcal{V}$ turns out to be a real algebraic variety. Next we consider global maps from ∂D to $\partial \widetilde{D}$. When ∂D is simply connected, a local CR isomorphism can be holomorphically extended to a neighborhood of ∂D. Hence, it extends to a proper holomorphic map from D to $\widetilde{D}$. Thus, the real algebraic variety $\mathcal{V}$ is the desired one in Theorem 0.1. In general, an arbitrary local CR isomorphism is not necessarily a global CR isomorphism. We will then find a subset $\mathcal{V}_0$ of $\mathcal{V}$ by taking certain intersections such that it represents all global CR isomorphisms.

Once $\mathcal{V}_0$ is constructed, in light of the Milnor Theorem [M], the estimate of its total Betti number can be done by studying the defining functions of $\mathcal{V}_0$.

The paper is organized as follows. In Section 1, we collect some basic results related to the Milnor Theorem and Segre families. In Section 2, we prepare some needed properties for algebraic functions. We review and outline Cartan's method on the equivalence problem in differential geometry in Section 3. We then apply Cartan's method to CR geometry in Section 4. In Section 5, we shall construct the variety $\mathcal{V}$ in Theorem 0.1 and prove (i)–(iii) of Theorem 0.1. We shall prove the last part (iv) of Theorem

0.1 in Section 6 by an explicit computation, following up a procedure in Chern's paper [Ch]. The computation in [Ch] was carried out over the projectivized cotangent space $\mathbb{C}^{n+1} \times \mathbb{P}^n$, instead of over the Segre family $\mathcal{M}$. Computing over $\mathbb{C}^{n+1} \times \mathbb{P}^n$ simplifies the computation. For our purpose here, we will carry out the computation over $\mathcal{M}$ that seems to provide a better degree estimate.

In this paper, all small Greek indices have the range $1 \le \alpha, \beta \cdots \le n$, small Latin indices have the range $1 \le i, j, k, \ldots \le n+1$, and repeated indices imply summation.

1. Preliminaries.

• **Algebraic functions.** — Let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$. Let $U \subset \mathbb{K}^n$ be an open subset. We recall a smooth function g defined on U is said to be a *Nash algebraic function* (or simply an *algebraic* function) on U if there is a non-constant irreducible polynomial $P(x,y)$ in $(x,y) \in \mathbb{K}^n \times \mathbb{K}$ of degree k such that $P(x,g(x)) = 0$, $\forall x \in U$. Here, k is called the degree of g.

• **Betti numbers of sub-algebraic sets.** — By the qth *Betti number* of a topological space X, we mean the rank of the Čech cohomology group $H^q(X)$, using coefficients in $\mathbb{K}$. In our proof, we need the following result of Milnor:

THEOREM 1.1 (Milnor, cf. [M]). — *Let $p_1, .., p_m \in \mathbb{R}[\mathbb{R}^n]$ (resp. $\mathbb{C}[\mathbb{C}^n]$) be real polynomials (resp. complex polynomials) with $\deg(p_j) \le k$. Then the sum of the Betti numbers of the zero locus $Z(p_1, \ldots, p_m) \subset \mathbb{R}^n$ is $\le k(2k-1)^{n-1}$ (resp. $\subset \mathbb{C}^n$ is $\le k(2k-1)^{2n-1}$).*

PROPOSITION 1.2. — *Let $p_1, \ldots, p_m, g_1, \ldots, g_s \in \mathbb{R}[\mathbb{R}^n]$ be real polynomials with $\deg(p_j) \le k$ and $d := \deg(g_1) + \cdots + \deg(g_s)$. Let*

$$X := \{x \in \mathbb{R}^n \mid p_1(x) = \cdots = p_m(x) = 0,\ g_1(x) \ge 0, \cdots, g_s(x) \ge 0\}.$$

Then the sum of the Betti numbers of X is bounded by $\frac{1}{2}k_0(k_0-1)^{n-1}$, where $k_0 = \max\{2k, d+2\}$.

Proof of Proposition 1.2. — When $m = 0$, the proof was given in [M]. The proof in the general case can be similarly done as follows:

We first let $X_r = X \cap B^n(r)$, where $B^n(r)$ is the ball in $\mathbb{R}^n$ centered at the origin and with radius r. Let $g_0 = r^2 - |x|^2$.

For any constant ϵ, δ with $\epsilon > \epsilon^{s+2} > \delta > 0$, let $L(\epsilon, \delta)$ denote the set of points

$$g_0 + \epsilon \geq 0, \dots, \; g_s + \epsilon \geq 0, \; \epsilon(g_0 + \epsilon) \cdot \dots \cdot (g_s + \epsilon) \geq \delta + p_1^2 + \dots + p_m^2.$$

Then $L(\epsilon, \delta)$ is compact, and its boundary is obtained by setting only the last expression equal to zero. Notice that the polynomial $p_1^2 + \cdots p_m^2 - \epsilon(g_0 + \epsilon) \cdot \dots \cdot (g_s + \epsilon)$ has degree bounded by k_0.

Given a small ϵ, we will choose δ so that the boundary $\partial L(\epsilon, \delta)$ is non-singular. Then the argument in [M, Theorem 1] shows that

$$\text{rank } \; H^*(\partial L(\epsilon, \delta)) \leq k_0 \cdot (k_0 - 1)^{n-1}.$$

Hence by the Alexander duality Theorem (cf. [ES]), we have

$$\text{rank } \; H^*(L(\epsilon, \delta)) \leq \frac{k_0}{2}(k_0 - 1)^{n-1}.$$

We choose ϵ_j converging monotonically to *zero* and suitably choose δ_j so that $L(\epsilon_1, \delta_1) \supset L(\epsilon_2, \delta_2) \supset \cdots$ with intersection $X \cap B^n(r)$. Then by the argument in [M, Theorem 2], it follows that

$$\text{rank } \; H^*(X_r) \leq \limsup(\text{rank } \; H^*(L(\epsilon_j, \delta_j)) \leq \frac{k_0}{2}(k_0 - 1)^{n-1}.$$

Since X can be triangulated, it thus follows that rank $H^*(X) \leq \frac{k_0}{2}(k_0 - 1)^{n-1}$.

□

Similarly, we have the following:

PROPOSITION 1.3. — *Let $p_1, \dots, p_m, g_1, \dots, g_s$ be as in Proposition 1.2. Let*

$$X := \{x \in \mathbb{R}^n \mid p_1(x) = \cdots = p_m(x) = 0, \; g_1(x) > 0, \; g_2 \neq 0, \dots, g_s(x) \neq 0\}. \tag{1.1}$$

Then the sum of the Betti numbers of X is $\leq k_0(k_0 - 1)^{n-1}$, where $k_0 = \max\{2k, d + 2\}$.

Proof of Proposition 1.3. — Let $g^{\pm} = \pm g_2 \cdots g_s$. For any constant $r > 1, \epsilon, \delta$ with $\epsilon > \epsilon^4 > \delta > 0$. Denote by $L^{\pm}(r, \epsilon, \delta)$ the set of points

$$r^2 - |x|^2 + \epsilon \geq 0, g_1 + \epsilon - 1/r \geq 0,$$

$$g^{\pm} + \epsilon - 1/r \geq 0, \; \epsilon(r^2 - |x|^2 + \epsilon)(g_1 + \epsilon - 1/r)(g^{\pm} + \epsilon - 1/r) \geq \delta + p_1^2 + \dots + p_m^2.$$

Then a similar argument as before shows that the study of $H^*(L^{\pm}(r, \epsilon, \delta))$ gives the estimate

$$\text{rank } \; H^*(X_r^{\pm}) \leq 1/2k_0 \cdot (k_0 - 1)^{n-1},$$

where $X_r^{\pm} = X \cap B^n(r) \cap \{x : g^{\pm} \geq 1/r, g_1 \geq 1/r\}$. Write $X_r = X_r^+ \cup X_r^-$. Then, rank $H^*(X_r) \leq k_0 \cdot (k_0 - 1)^{n-1}$. Let $1 < r_1 < r_2 < \cdots$ with $r_j \to \infty$. Since $X_{r_1} \subset\subset X_{r_2} \subset\subset X_{r_3} \subset\subset \cdots$ and their union is precisely X, we see the proof of Proposition 3.1 as in [M, p. 278]. □

• **Segre families.** — Let $M_r = \{z \in U \mid r(z, \overline{z}) = 0\} \subset \mathbb{C}^{n+1}$ be a real analytic strongly pseudoconvex hypersurface, where U is a neighborhood of 0 in $\mathbb{C}^{n+1}$ and r is a real analytic function on U with $r(0,0) = 0$ and $dr \neq 0$. Replacing $\overline{z}$ by new variables ζ, we obtain a holomorphic function $r(z,\zeta)$ on $U \times \mathrm{Conj}(U) \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$, where $\mathrm{Conj}(U) := \{z : \overline{z} \in U\}$. We then have a complex analytic variety $\mathcal{M}_r := \{(z,\zeta) \in U \times \mathrm{Conj}(U) \mid r(z,\zeta) = 0\}$, which is called the *Segre family* associated with M_r.

Suppose

$$(1.2)\quad r_{n+1}(0,0) := \frac{\partial r}{\partial z^{n+1}}(0,0) \neq 0 \ \text{ and } \ r^{n+1}(0,0) := \frac{\partial r}{\partial \zeta_{n+1}}(0,0) \neq 0.$$

Let M_r and $\mathcal{M}_r$ be as above. We define a holomorphic map $\mathcal{S} : \mathcal{M}_r \to PT\mathbb{C}^{n+1}$ given by

$$(1.3)\qquad \mathcal{S}(z^\alpha, z^{n+1}, \zeta_\alpha) := \Big(z^\alpha, z^{n+1}, \Big[r_1 : \cdots : r_{n+1}\Big](z^\alpha, z^{n+1}, \zeta_\alpha)\Big) \longrightarrow \Big(z^\alpha, z^{n+1}, -p_\alpha\Big)$$

where $r_\alpha = \frac{\partial r}{\partial z^\alpha}$ and $p_\alpha := -\frac{r_\alpha}{r_{n+1}}$. Clearly $\mathcal{S}$ is independent of the choice of the local defining function r.

Since M_r is strongly pseudoconvex at p, the map $\mathcal{S}$ is locally biholomorphic at $(p,\overline{p}) \in \mathcal{M}_r$ and $L_r(p,\overline{p}) \neq 0$, where

$$L_r(z,\zeta) := \det \begin{pmatrix} 0 & r^j \\ r_i & r_i^j \end{pmatrix} (z,\zeta)$$

is the *Levi determinant* of r([CJ2]). Here, we use the notation $r^i = \frac{\partial r}{\partial \zeta_i}$, $r_i = \frac{\partial r}{\partial z_i}$, $r_j^i = \frac{\partial^2 r}{\partial \zeta_i \partial z_j}$. For any $1 \leq i, j \leq n+1$, write

$$\mathcal{M}_j^i := \{(z,\zeta) \mid r(z,\zeta) = 0, r_j(z,\zeta) \neq 0, r^i(z,\zeta) \neq 0, L_r(z,\zeta) \neq 0\},$$

and $M_j^i := \mathcal{M}_j^i \cap \{(z,\zeta) \mid \zeta = \overline{z}\}$. Also, when there is no confusion, we identify M_j^i with its projection to the z-coordinates space. Let D be as in Theorem 0.1 with a polynomial defining function r. Then M_i^j described above is strongly pseudoconvex and $\{M_j^i\}_{1 \leq i,j \leq n+1}$ forms an open covering of ∂D. For simplicity, we write, in what follows, $M := M_{n+1}^{n+1}$, and $\mathcal{M} := \mathcal{M}_{n+1}^{n+1}$.

Making use of the implicit function Theorem and (1.2), we can replace the local defining function $r(z,\zeta)$ of $\mathcal{M}$ near $(0,0)$ by $z^{n+1}-p(z^\alpha,\zeta_\alpha,\zeta_{n+1})$. Then the functions p_α in (1.3) become $\frac{\partial p}{\partial z^\alpha}$. Restricting $dp_\alpha = p_{\alpha\beta}dz^\beta + p_\alpha^\beta d\zeta_\beta + p_\alpha^{n+1}d\zeta_{n+1}$ to $\mathcal{M}$, we get

$$(1.4)\quad dp_\alpha|_{\mathcal{M}} = \Big(p_{\alpha\beta} - \frac{p_\alpha^{n+1}p_\beta}{p^{n+1}}\Big)dz^\beta + \frac{p_\alpha^{n+1}}{p^{n+1}}dz^{n+1} + \Big(p_\alpha^\beta - \frac{p_\alpha^{n+1}p^\beta}{p^{n+1}}\Big)d\zeta_\beta.$$

As above, we write $p_\alpha = \frac{\partial p}{\partial z^\alpha}, p_\alpha^\beta = \frac{\partial^2 p}{\partial z^\alpha \partial \zeta_\beta}$, *etc.* Then we have the following coframe (cf. [CJ1, p.587]) on $\mathcal{M}$ near $(0,0)$:

$$(1.5)\quad \begin{cases} \theta = dz^{n+1} + \dfrac{r_\alpha}{r_{n+1}}dz^\alpha = dz^{n+1} - p_\alpha dz^\alpha, \\ \theta^\alpha = dz^\alpha, \\ \theta_\alpha = \dfrac{p_\alpha^{n+1}}{p^{n+1}}\theta + \Big(p_\alpha^\beta - \dfrac{p_\alpha^{n+1}p^\beta}{p^{n+1}}\Big)d\zeta_\beta\ . \end{cases}$$

Since $dp_\alpha - p_{\alpha\beta}dz^\beta = \theta_\alpha$ on $\mathcal{M}$ by (1.4), we see $d\theta = \theta^\alpha \wedge \theta_\alpha$. From the chain rule, it follows that

$$(1.6)\quad \begin{cases} p_\alpha = \dfrac{\partial p}{\partial z^\alpha} = -\dfrac{r_\alpha}{r_{n+1}}, \\ p^\alpha = \dfrac{\partial p}{\partial \zeta_\alpha} = -\dfrac{r^\alpha}{r_{n+1}}, \\ p^{n+1} = \dfrac{\partial p}{\partial \zeta_{n+1}} = -\dfrac{r^{n+1}}{r_{n+1}}, \\ p_\alpha^{n+1} = \dfrac{\partial^2 p}{\partial z^\alpha \partial z^{n+1}} \\ \quad = \dfrac{r_{\alpha n+1}r^{n+1}r_{n+1} - (r_{n+1})^2 r_\alpha^{n+1} - r_{(n+1)^2}r^{n+1}r_\alpha + r_{n+1}^{n+1}r_\alpha r_{n+1}}{(r_{n+1})^3} \\ p_\alpha^\beta = \dfrac{\partial^2 p}{\partial z^\alpha \partial \zeta_\beta} \\ \quad = \dfrac{r_{\alpha n+1}r^\beta r_{n+1} - (r_{n+1})^2 r_\alpha^\beta - r_{(n+1)^2}r^\beta r_\alpha + r_{n+1}^\beta r_\alpha r_{n+1}}{(r_{n+1})^3}, \\ p_{\alpha\beta} = \dfrac{\partial^2 p}{\partial z^\alpha \partial z^\beta} \\ \quad = \dfrac{r_{\alpha n+1}r_\beta r_{n+1} - (r_{n+1})^2 r_{\alpha\beta} - r_{(n+1)^2}r_\beta r_\alpha + r_{n+1\beta}r_\alpha r_{n+1}}{(r_{n+1})^3}. \end{cases}$$

2. Algebraic functions.

For non zero polynomials $P\,Q$, $f = P/Q$ and $\deg_*(f) := \max\{\deg(P), \deg(Q)+1\}$. Then $\deg(f) \le \deg_*(f)$. When P and Q have no non-constant common factors, it then holds that $\deg(f) = \deg_*(f)$. It is clear that $\deg_*(f+g) \le \max\{\deg_*(f), \deg_*(g)\}$ for rational functions with the same denominator: $f = \frac{P}{Q}$, $g = \frac{\widetilde{P}}{Q}$. More generally, we have the following degree estimates, which will be used for the proof of Theorem 0.1 (iv).

LEMMA 2.1.

(1) *For any non-zero rational functions f and g in $\mathbb{C}^n$,* $\deg(f+g) \le \deg(f) + \deg(g) - 1$, *and* $\deg(fg) \le \deg(f) + \deg(g)$. *If* $f = \frac{P}{Q}$, $g = \frac{\widetilde{P}}{Q}$ *where P, $\widetilde{P}$, and $Q \not\equiv 0$ are polynomials on $\mathbb{C}^n$, then* $\deg_*(f+g) \le \max\{\deg_*(f), \deg_*(g)\}$, $\deg_*(fg) \le \deg_*(f) + \deg_*(g)$.

(2) *If $f(z,t)$ is a rational function on an open subset of $\mathbb{C}^{n'} \times \mathbb{C}^m$ with $n' + m = n$, and $r(z) = (r^{(1)}(z), \ldots, r^{(m)}(z))$ are non zero Nash algebraic functions defined on an open subset $V \subset \mathbb{C}^{n'}$, then $f(z, r(z))$ is a Nash algebraic function satisfying*

$$\deg\Big(f(z, r(z))\Big) \le 2\ \deg(f) \sum_{\mu=1}^{m} \Big(\deg(r^{(\mu)})\Big)^2.$$

(3) *If f is a rational function, then for any index α, $\frac{\partial f}{\partial z^\alpha}$ is a rational function with*

$$\deg\left(\frac{\partial f}{\partial z^\alpha}\right) \le 2\deg(f) - 1.$$

(4) *Let f be a smooth function on $U \subset \mathbb{C}^n$ given by the equation $F(x, f(x)) \equiv 0$ for a certain algebraic function F. Then f is Nash algebraic with* $\deg(f) \le \deg(F)$.

(5) *Let $(dz^1, \ldots, dz^n)$ be the Euclidean coframe in $\mathbb{C}^n$ and $(\omega^1, \ldots, \omega^n)$ be another coframe in $U \subset \mathbb{C}^n$. Suppose $dz^j = h^j_k \omega^k$ where $h^j_k = \frac{E^j_k}{R}$ with E^j_k and R polynomials. For any rational function f over U, write $df = f_j dz^j = f_{|\omega^j}\omega^j$, then*

$$\deg_*(f_{|\omega^k}) \le \max_j\{\deg_*(f_j)\} + \max_{j,k}\{\deg_*(h^j_k)\}.$$

Proof of Lemma 2.1.

(1) Assume that $f = \frac{P}{Q}$ and $g = \frac{\widetilde{P}}{\widetilde{Q}}$ with $\deg(f) = \max\{\deg(P), \deg(Q)+1\}$ and $\deg(g) = \max\{\deg(\widetilde{P}), \deg(\widetilde{Q})+1\}$. Then $\deg(f+g) \leq \max\{\deg(P\widetilde{Q}+\widetilde{P}Q), \deg(Q\widetilde{Q})+1\} \leq \deg(f)+\deg(g)-1$. When $f = \frac{P}{Q}$ and $g = \frac{\widetilde{P}}{Q}$, we have $\deg_*(f+g) := \max\{\deg(P+\widetilde{P}), \deg Q+1\} \leq \max\{\deg_*(f), \deg_*(g)\}$. Similarly, we have the inequalities for $\deg(fg)$ and $\deg_*(fg)$.

(2) Let $f = \frac{P}{Q}$, where P and Q have no non constant common factors. Take $T(z,t,X) = Q(z,t)X - P(z,t)$. Then $T(z,t,f(z)) = 0$. Since $r^{(v)}(z)$ is algebraic $(1 \leq v \leq m)$, there exists a polynomial $\widetilde{T}^{(v)}(z,X)$ such that $\widetilde{T}^{(v)}(z, r^{(v)}(z)) = 0$ with $\deg(\widetilde{T}^{(v)}) = \deg(r^{(v)})$. Write

$$\begin{aligned}\widetilde{T}^{(v)}(z,X) &= b^{(v)}_{q_v}(z)X^{q_v} + \cdots + b^{(v)}_1(z)X + b^{(v)}_0(z)\\ &= b^{(v)}_{q_v}(z)(X - r^{(v)}_1(z))\cdots(X - r^{(v)}_{q_v}(z))\\ &= b^{(v)}_{q_v}(z)(X^{q_v} - \sigma^{(v)}_1(z)X^{q_v-1} + \cdots + (-1)^{q_v}\sigma^{(v)}_{q_v}(z)),\end{aligned}$$

where $(-1)^j\sigma^{(v)}_j(z)b^{(v)}_{q_v}(z) = b^{(v)}_j(z)$, $r^{(v)}_1(z) = r^{(v)}(z)$, and $r^{(v)}_2(z), \ldots, r^{(v)}_q(z)$ are other branches of $r^{(v)}$.

In the following, we may shrink V if necessary. Define $H(z,X)$ by the expression

$$\begin{aligned}&\prod_v \left(b_{q_v}(z)\right)^{q_v \ \deg(f)} Q(z, r^{(v)}_1(z))\cdots Q(z, r^{(v)}_{q_v}(z))(X - f(z, r^{(v)}_1(z)))\\ &\qquad\qquad\cdots(X - f(z, r^{(v)}_{q_v}(z)))\\ &= \prod_v \left(b^{(v)}_{q_v}(z)\right)^{q_v \ \deg(f)} \left(Q(z, r^{(v)}_1(z))X - P(z, r^{(v)}_1(z))\right)\\ &\qquad\qquad\cdots\left(Q(z, r^{(v)}_{q_v}(z)X - P(z, r^{(v)}_{q_v}(z))\right).\end{aligned}$$

Clearly $H(z, f(z, r(z))) = 0$. We need to show that $H(z,X)$ is a polynomial and estimate $\deg(H)$. In fact, since $Q(z,t)$ and $P(z,t)$ are polynomials, by the properties of Newton symmetric polynomials,

$$\begin{aligned}H(z,X) = \left(b^{(1)}_{q_1}(z)\right)^{q_1 \deg(f)} \cdots \left(b^{(m)}_{q_m}(z)\right)^{q_m \deg(f)} \sum_{j=0}^{q_1+\cdots+q_v}\\ \sum_{|L|+j+\sum_\mu (t^{(\mu)}_1 + 2t^{(\mu)}_2 + \cdots + q_\mu t^{(\mu)}_q) \leq (q_1+\cdots+q_m) \ \deg(f)}\\ \cdot C^{(\mu)}_{Ljt^{(\mu)}_1\cdots t^{(\mu)}_{q_\mu}} z^L \prod_{\mu=1}^m (\sigma^{(\mu)}_1)^{t^{(\mu)}_1} \cdots (\sigma^{(\mu)}_{q_\mu})^{t^{(\mu)}_{q_\mu}} X^j\end{aligned}$$

$$= \sum_{|L|+j+\sum_\mu (t_1^{(\mu)}+2t_2^{(\mu)}+\cdots+q_\mu t_{q_\mu}^{(\mu)})\le(q_1+\cdots+q_m)\deg(f)} C^{(\mu)}_{Ljt_1^{(\mu)}\cdots t_{q_\mu}^{(\mu)}}$$

$$\cdot \Big(b_{q_1}(z)\Big)^{q_1\deg(f)-t_1^{(1)}-\cdots-t_{q_1}^{(1)}} \cdots \Big(b_{q_m}(z)\Big)^{q_m\deg(f)-t_1^{(m)}-\cdots-t_{q_m}^{(m)}}$$

$$\cdot z^L \prod_\mu (\sigma_1^{(\mu)} b_{q_\mu}^{(\mu)})^{t_1^{(\mu)}} \cdots (\sigma_{q_\mu}^{(\mu)} b_{q_\mu})^{t_{q_\mu}^{(\mu)}} X^j$$

$$= \sum_{|L|+j+\sum_\mu (t_1^{(\mu)}+2t_2^{(\mu)}+\cdots+q_\mu t_{q_\mu}^{(\mu)})\le(q_1+\cdots+q_m)\deg(f)} \widetilde{C}^{((\mu))}_{Ljt_1^{(\mu)}\cdots t_{q_\mu}^{(\mu)}}$$

$$\cdot \Big(b_{q_1}(z)\Big)^{q_1\deg(f)-t_1^{(1)}-\cdots-t_{q_1}^{(1)}} \cdots \Big(b_{q_m}(z)\Big)^{q_m\deg(f)-t_1^{(m)}-\cdots-t_{q_m}^{(m)}}$$

$$z^L \prod_\mu (b_1^{(\mu)})^{t_1^{(\mu)}} \cdots (b_\mu^{(\mu)})^{t_{q_\mu}^{(\mu)}} X^j.$$

Since $b_j^{(v)}$ are polynomials, $H(z,X)$ is a polynomial. It remains to estimate $\deg(H)$. For each j, L and $(\mu_1,\dots,\mu_m)$, we have

$$\deg\Big[\Big(b_{q_1}(z)\Big)^{q_1\deg(f)-t_1^{(1)}-\cdots-t_{q_1}^{(1)}} \cdots \Big(b_{q_m}(z)\Big)^{q_m\deg(f)-t_1^{(m)}-\cdots-t_{q_m}^{(m)}}$$

$$\cdot z^L \prod_\mu (b_1^{(\mu)})^{t_1^{(\mu)}} \cdots (b_\mu^{(\mu)})^{t_{q_\mu}^{(\mu)}} X^j\Big]$$

$$\le \sum_\mu \deg(r^{(\mu)})(q_\mu \ \deg(f) - t_1^{(\mu)} - \cdots - t_{q_\mu}^{(\mu)}) + |L|$$

$$+ \sum_\mu \deg(r^{(\mu)})(t_1^{(\mu)} + \cdots + t_{q_\mu}^{(\mu)}) + j$$

$$= \sum_\mu \deg(f)\Big(\deg(r^{(\mu)})\Big)^2 + |L| + j \le 2\deg(f)\sum_\mu \Big(\deg(r^{(\mu)})\Big)^2.$$

Hence $\deg(H) \le 2\deg(f)\sum_v \Big(\deg(r^{(v)})\Big)^2$.

(3) Let $f = \frac{P}{Q}$, where P and Q have no non constant common factors. Then $\deg(f'_\alpha)= \deg\left(\frac{QP'_\alpha - Q'_\alpha P}{Q^2}\right) \le \max\{\deg(QP'_\alpha - Q'_\alpha P), \deg Q^2 + 1\} \le 2\deg(f) - 1$.

(4) Since $F(z,t)$ is an algebraic function with $F(z,f(z)) \equiv 0$, there is an irreducible polynomial $P(z,t,X) = 0$ such that $P(z,t,F(z,t)) \equiv 0$ with $\deg(F) = \deg(P)$. Then $P(z,f(z),0) \equiv 0$ with $\deg(f) \le \deg(P) = \deg(F)$.

(5) The proof follows by applying (1) to $f_{\omega^k} = \sum_j f_j h_k^j$. □

For Nash algebraic functions, we have the following:

LEMMA 2.1′.

(1) *If f, g are two Nash algebraic functions, then* $\deg(f+g) \leq \deg(f)\deg(g)\,[\deg(f)+\deg(g)]$, *and* $\deg(fg) \leq \deg(f)\deg(g)[\deg(f)+\deg(g)+1]$.

(2) *If $f(z,t)$ is a Nash algebraic function on an open subset of $\mathbb{C}^{n'}\times\mathbb{C}^m$ with $n'+m=n$, and $r(z)=(r^{(1)}(z),\ldots, r^{(m)}(z))$ are non zero Nash algebraic functions defined on an open subset $V\subset\mathbb{C}^{n'}$, then $f(z,r(z))$ is a Nash algebraic function satisfying*

$$\deg\Big(f(z,r(z))\Big) \leq m(\deg(f))^2\Big(\max_{j=1}^{m}\deg(r^{(j)})\Big)^3.$$

(3) *If $f=\frac{P}{Q}$ is a Nash algebraic function, then for any index α, $\frac{\partial f}{\partial z^\alpha}$ is an algebraic function with*

$$\deg\Big(\frac{\partial f}{\partial z^\alpha}\Big) \leq (\deg(f))^5.$$

(4) *Let f, g be Nash algebraic functions near p. If $D^\alpha f(p)=D^\alpha g(p)$ for any*

$$|\alpha| \leq \deg(f)\deg(g)[\deg(f)+\deg(g)],$$

then $f\equiv g$.

Proof of Lemma 2.1′. — We only explain (4). The others can be done as in Lemma 2.1.

Assume that $p=0$. Suppose $f-g\not\equiv 0$. Since $f-g$ is algebraic, there is an irreducible polynomial $R(z,X)$ such that $R(z,f(z)-g(z)) = a_n(z)(f-g)^n(z)+\cdots a_1(z)(f-g)(z)+a_0(z)\equiv 0$ with $\deg(R)=\deg(f-g)$ where $a_0(z)\not\equiv 0$. Notice that $\deg(a_0)\leq\deg(R)=\deg(f-g)\leq \deg(f)\cdot\deg(g)(\deg(f)+\deg(g))$ by (1). Applying the differential operator D^α to $R(z,f(z)-g(z))$ for $|\alpha|\leq\deg(R)$ and letting $z=0$, we conclude that $D^\alpha a_0(0)=0$, for any $|\alpha|\leq\deg(a_0)$. Hence, $a_0(z)\equiv 0$. This is a contradiction. □

LEMMA 2.2. — *Let $f=(f_1,\ldots,f_n): U\to f(U)\subset\mathbb{C}^n$ be a Nash algebraic biholomorphic mapping, where $U\subset\mathbb{C}^n$ is an open subset. Then every component f_j^{-1} of its inverse map $f^{-1}=(f_1^{-1},\ldots,f_n^{-1})$ is Nash algebraic on $f(U)$ with*

$$\deg(f_j^{-1}) \leq (\deg\ f)^{5^{n-1}}.$$

Here $\deg(f) := \max_{j=1}^{n} \deg(f_j)$.

Proof of Lemma 2.2. — Consider $w_j = A_j^{(1)}(z) := f_j(z)$, $1 \leq j \leq n$. By the implicit function Theorem for algebraic functions (cf. [BER2, Theorem 5.4.6]), without loss of generality, we can assume that $z_n = A_n(z_1, \ldots, z_{n-1}, w_n)$. By Lemma 2.1 (4), A_n is Nash algebraic and $\deg(A_n) \leq \deg(f)$.

Consider $w_j = A_j^{(2)}(z_1, \ldots, z_{n-1}, w_n) := A_j^{(1)}(z_1, \ldots, z_{n-1}, A_n)$ for $1 \leq j \leq n-1$. By the implicit function Theorem for algebraic function again, we can still assume $z_{n-1} = A_{n-1}(z_1, \ldots, z_{n-2}, w_{n-1}, w_n)$. By Lemma 2.1 (4) and Lemma 2.1′ (2), $\deg(A_{n-1}) \leq \deg(A_{n-1}^{(2)}) \leq (\deg A_{n-1}^{(1)})^2 \cdot (\deg A_n)^3 \leq (\deg f)^2 (\deg f)^3 = (\deg(f))^5$.

Repeating this process, we can assume that $w_1 = f_1(z) = f_1(z_1, A_2, \ldots, A_n)$ with $z_1 = A_1(w_1, w_2, \ldots, w_n)$, where A_1 is a Nash algebraic function and $\deg(A_1) \leq (\deg f)^{5^{n-1}}$. Since $A_1 = f_1^{-1}$, it follows that $\deg(f_1^{-1}) \leq (\deg f)^{5^{n-1}}$. Similarly, we can get the same estimates for $\deg(f_j^{-1})$ for $2 \leq j \leq n$. □

3. Cartan's method on equivalence problems.

• **Cartan's Theorem on equivalence problems.** — We will restrict ourselves here mainly to the real category. However, we emphasize that the results in this section can be stated in a parallel manner in the complex setting.

Let $V, \widetilde{V} \subset \mathbb{R}^n$ (or $\mathbb{C}^n$) be open subsets with $p \in V$ and $\widetilde{p} \in \widetilde{V}$. Let $\theta_V =^t (\theta_V^1, \ldots, \theta_V^n)$ and $\widetilde{\theta}_{\widetilde{V}} =^t (\widetilde{\theta}_{\widetilde{V}}^1, \ldots, \widetilde{\theta}_{\widetilde{V}}^n)$ be coframes on V and $\widetilde{V}$, respectively. (In the complex setting, they are assumed to be holomorphic one forms.) Let $G \subset GL(n, \mathbb{R})$ (or $GL(n, \mathbb{C})$, in the complex setting) be a connected linear subgroup. With the natural left action of G on the product space $V \times G$ (resp. $\widetilde{V} \times G$):

$$C(p, S) = (p, CS), \quad \forall C, S \in G, \ \forall p \in V \quad (\text{resp. } \widetilde{V}), \tag{3.0}$$

we say that $V \times G$ (or $\widetilde{V} \times G$) is a *G-space*. Let $\pi_V : V \times G \to V$ (or $\pi_{\widetilde{V}} : \widetilde{V} \times G \to \widetilde{V}$) be the natural projection. We then obtain 1-forms on $V \times G$ and $\widetilde{V} \times G$ defined by

$$(\omega^1, \ldots, \omega^n)|_{(V,S)} = S\ \pi_V^* \theta_V, \quad (\widetilde{\omega}^1, \ldots, \widetilde{\omega}^n)|_{(\widetilde{V},S)} = S\ \pi_{\widetilde{V}}^* \widetilde{\theta}_{\widetilde{V}}. \tag{3.1}$$

LEMMA 3.1 (Cartan, cf. [Ga, p. 11]). — *There exists a diffeomorphism $\Phi : V \to \widetilde{V}$ satisfying*

$$\Phi^*\widetilde{\theta}_{\widetilde{V}} = \gamma_{V\widetilde{V}}\theta_V, \quad \textit{with} \quad \gamma_{V\widetilde{V}} : V \to G, \ \Phi(p) = \widetilde{p}, \tag{3.2}$$

if and only if there exits a diffeomorphism $\Phi^1 : V \times G \to \widetilde{V} \times G$ such that

$$\Phi^{1*}(\widetilde{\omega}^1, \ldots, \widetilde{\omega}^n) = (\omega^1, \ldots, \omega^n), \quad \textit{with} \quad \Phi^1(P) = \widetilde{P}, \tag{3.3}$$

where P is in $V \times G$ such that $\pi_V(P) = p$ and $\widetilde{P}$ is a certain point in $\widetilde{V} \times G$ such that $\pi_{\widetilde{V}}(\widetilde{P}) = \widetilde{p}$.

Let dim $G = r$. Then $\dim(V \times G) = n+r$. With the forms $\omega_1, \ldots, \omega_n$, we would like to add r more 1-forms $\omega^{n+1}, \ldots, \omega^{n+r}$ on $V \times G$ to form a coframe Ω such that the induced group becomes the trivial group $\{e\}$ from G. Such a Ω is called an *e-structure*. Consequently, there exists Φ satisfying (3.2) if and only if there exists a diffeomorphism $\Phi^1 : V \times G \to \widetilde{V} \times G$ satisfying

$$\Phi^*\widetilde{\omega}^j = \omega^j, \quad 1 \le j \le n+r. \tag{3.4}$$

Suppose the existence of such an Ω and write

$$\Omega := \{\omega^1, \ldots, \omega^{n+r}\} \quad (\widetilde{\Omega} = \{\widetilde{\omega}^1, \ldots, \widetilde{\omega}^{n+r}\}, \text{ respectively}). \tag{3.5}$$

For a differentiable function γ defined on $V \times G$, we define its *covariant partial derivative*: $d\gamma = \sum \gamma_{|i}\omega^i$. From $d\omega^i = \sum C^i_{jk}\omega^j \wedge \omega^k$ with $C^i_{jk} = -C^i_{kj}$, it gives

$$\Phi^{1*}\widetilde{C}^i_{jk} = C^i_{jk}. \tag{3.6}$$

Hence $\{C^i_{jk}\}$ are invariants with respect to such a Φ^1. For each integer s with $1 \le s \le n+r$, we define

$$\Gamma_s(\Omega, V \times G) := \Big\{C^i_{jk}, C^i_{jk|l_1}, \cdots, C^i_{jk|l_1\cdots|l_{s-1}} \mid 1 \le i, j, k, l_1, \ldots, l_{s-1} \le n+r\Big\}, \tag{3.7}$$

which is ordered lexicographically. Define

$$k_s(p) := \operatorname{rank}\{\mathrm{d}\Gamma_s(\Omega, V \times G)\}(p), \quad p \in V \times G, \tag{3.8}$$

to be the dimension of the span of the differentials of functions in the ordered set $\Gamma_s(\Omega, V \times G)$. The *order* of the e-structure at $p \in V \times G$ is the smallest $j_0 = j_0(p)$ such that

$$k_{j_0}(p) = k_{j_0+1}(p). \tag{3.9}$$

In this case, the *rank* of the e-structure at p is defined to be

$$\rho_0 = \rho_0(p) := k_{j_0}(p). \tag{3.10}$$

We say that the e-structure is *regular* of order j_0 and rank ρ_0 at $p \in V \times G$ if there exists a neighborhood U_p of p in $V \times G$ such that $j_0(q) \equiv j_0(p)$ $\rho_0(q) \equiv \rho_0(p)$, $\forall q \in U_p$. Suppose Ω is regular with order j_0 and rank ρ_0. Then we can find ρ_0 functions $\{g_1, \ldots, g_{\rho_0}\} \subset \Gamma_{j_0}(\Omega, V \times G)$, and a certain neighborhood U_p of p in $V \times G$, so that

(3.11)

$$d\,g_1 \wedge \cdots \wedge d\,g_{\rho_0} \neq 0\ ,\ d\,g \wedge d\,g_1 \wedge \cdots \wedge dg_{\rho_0} \equiv 0\ ,\ \text{on } U_p\ , \forall g \in \Gamma_{j_0+1}(\Omega, V \times G).$$

Notice that $1 \leq j_0 \leq n+r$. The case $j_0 = 1$ occurs when the functions $C^i_{jk} \equiv$ *constant* for all i, j and k. And the case $j_0 = n + r$ occurs if and only if one invariant function is added at each jet level.

Notice that $0 \leq \rho_0 \leq n+r$. When $\rho_0 = n+r$, we say that $\Gamma(\Omega, V \times G)$ is of *maximal rank*.

In what follows, we always assume that $\rho_0 \geq 1$.

Notice that these $g_1, \cdots, g_{\rho_0}$ can be extended to a coordinate system in a neighborhood U_p of p in $V \times G$. Namely, we can define a coordinates map at p by adding some new functions $h_{\rho_0+1}, \ldots, h_{n+r}$ (if $\rho_0 < n + r$):

$$h : U_p \to \mathbb{R}^{n+r}\ , q \mapsto x = (x_1, \cdots, x_n) = (g_1(q), \ldots, g_{\rho_0}(q), h_{\rho_0+1}(q), \ldots, h_{n+r}(q)), \tag{3.12}$$

with $h_{\rho_0+j}(p) = 0$ for $j \geq 1$ (if $\rho_0 < n + r$).

THEOREM 3.2 (Cartan, [Ga, p. 59]). — *Let Ω and $\widetilde{\Omega}$ be regular e-structures of order j_0 and rank ρ_0. Let h and $\widetilde{h}$ be defined near p and $\widetilde{p}$, respectively, as in (3.12). Assume that $(\widetilde{g}_1, \ldots, \widetilde{g}_{\rho_0})$ are the corresponding invariant functions with the identical lexicographic order as for $(g_1, \ldots, g_{\rho_0})$ in the corresponding set of invariant functions. Then there exists a diffeomorphism Φ such that (3.2) holds if and only if*

$$\widetilde{\Gamma}_{j_0+1}(\widetilde{\Omega}, \widetilde{V} \times G) \circ \widetilde{h}^{-1} = \Gamma_{j_0+1}(\Omega, V \times G) \circ h^{-1}. \tag{3.13}$$

• **Equivalence between analytic e-structures.** — Suppose that Ω and $\widetilde{\Omega}$ are regular with rank ρ_0 and order j_0 at p and $\widetilde{p}$, respectively. Let us consider the analytic case (*i.e.*, all coframes, $\gamma_{V\widetilde{V}}, \ldots$, are real analytic). Take $g_1, \ldots, g_{\rho_0} \in \Gamma_{j_0}(\Omega, V \times G)$ as in (3.11):

$$dg_1 \wedge \cdots \wedge dg_{\rho_0} \neq 0,\ \ dg \wedge dg_1 \wedge \cdots \wedge dg_{\rho_0} \equiv 0,\ \ \forall g \in \Gamma_{j_0+1}(\Omega, V \times G). \tag{3.14}$$

Notice that

$$g = A_g(g_1, \ldots, g_{\rho_0}) \tag{3.15}$$

for a certain uniquely determined real analytic function A_g near $(g_1(p), \ldots, g_{\rho_0}(p))$, which is called the *relation function* of g with respect to $\{g_1, \ldots, g_{\rho_0}\}$.

LEMMA 3.3. — *Let Ω and $\widetilde{\Omega}$ be analytic regular e-structures of order j_0 and rank ρ_0 at p and $\widetilde{p}$ respectively. Let $g_1, \ldots, g_{\rho_0}$ be as in (3.14). Let $\widetilde{g}_1, \ldots, \widetilde{g}_{\rho_0}$ be the corresponding set with the same lexicographic order as for $g_1, \ldots, g_{\rho_0}$. Then the following statements are equivalent:*

(i) *There exists a C^ω diffeomorphism $\Phi^1 : V \times G \to \widetilde{V} \times G$ with*

$$\Phi^{1*}\widetilde{\Omega} = \Omega, \quad \Phi^1(p) = \widetilde{p}.$$

(ii) *$\Gamma_k(\widetilde{\Omega}, \widetilde{V} \times G)(\widetilde{p}) = \Gamma_k(\Omega, V \times G)(p)$ holds for any k.*

(iii) *$\widetilde{g}_j(\widetilde{p}) = g_j(p)$ holds for $1 \leq j \leq \rho_0$, and for any function $g \in \Gamma_{j_0+1}(\Omega, V \times G)$ and $\widetilde{g} \in \Gamma_{j_0+1}(\widetilde{\Omega}, V \times G)$ with the same lexicographic order, it holds that $A_g \equiv A_{\widetilde{g}}$.*

Proof of Lemma 3.3. — We only explain the implication: (ii) $\Rightarrow$ (i).

For $\{g_1, \ldots, g_{\rho_0}\}$ and $\{\widetilde{g}_1, \ldots, \widetilde{g}_{\rho_0}\}$, we take h and $\widetilde{h}$ as in (3.12), respectively.

Use $x = h$ and $\widetilde{x} = \widetilde{h}$ for the coordinates on $V \times G$ and $\widetilde{V} \times G$ near $h(p)$ and $\widetilde{h}(\widetilde{p})$, respectively. Under the assumption of (ii), it can be verified that the Taylor coefficients of g with respect to the coordinates x at $h(p)$ is equal to the corresponding Taylor coefficients of $\widetilde{g}$ in the coordinates $\widetilde{x}$ at $\widetilde{h}(\widetilde{p})$. This implies $\widetilde{g} \circ \widetilde{h}^{-1} = g \circ h^{-1}$. □

• **Equivalence between two e-structures with algebraic relation functions.** — For any $g \in \Gamma_{j_0+1}(\Omega, V \times G)$, still write $A_g(x_1, \ldots, x_{\rho_0})$ for the relation function of g such that $g = A_g(g_1, \ldots, g_{\rho_0})$. Here j_0 is the same as in (3.14). Let us consider the real algebraic case (*i.e.*, all coframes, $\gamma_{V\widetilde{V}}, \ldots$, are real algebraic). Let l_0 be the smallest integer satisfying

$$\deg(A_g) \leq l_0, \ \forall g \in \Gamma_{j_0+1}(\Omega, V \times G). \tag{3.16}$$

Then by Lemma 3.3 and Lemma 2.1′ (6), we can easily obtain the following:

LEMMA 3.4. — *Let Ω and $\widetilde{\Omega}$ be real algebraic regular e-structures of order j_0 and rank ρ_0. Let l_0 be given as in (3.16) with respect to both Ω*

and $\widetilde{\Omega}$. *Then there exists a real analytic diffeomorphism* $\Phi^1 : V\times G \to \widetilde{V}\times G$ *with*

$$\Phi^{1*}\widetilde{\Omega} = \Omega, \quad \Phi^1(p) = \widetilde{p}$$

if and only if

$$\Gamma_{2l_0^3}(\Omega, V\times G)(p) = \Gamma_{2l_0^3}(\widetilde{\Omega}, \widetilde{V}\times G)(\widetilde{p}). \tag{3.17}$$

We now give a rough estimate of the l_0 in (3.16). Assume, without loss of generality, that $\rho_0 = n+r$. From (3.15), $A_g = A_g(g_1,\ldots, g_{n+r})\circ (g_1,\ldots,g_{n+r})^{-1}$, where we write $(g_1,\ldots,g_{n+r})^{-1} = (g_1^{-1},\ldots,g_{n+r}^{-1})$. By Lemma 2.2, $\deg(g_j^{-1}) \le (\max_k \deg(g_k))^{5^{n+r-1}}$. Applying Lemma 2.1′ (2), we get $\deg(A_g) \le (n+r)\big(\deg\ A_g(g_1,\ldots,g_{n+r})\big)^2\big(\max_k \deg(g_k^{-1})\big)^3$. Hence

$$l_0 \le (n+r)T^2\Big(\max_k\ \deg\ g_k\Big)^{3\cdot 5^{n+r-1}} = (n+r)T^{2+3\cdot 5^{n+r-1}}, \tag{3.18}$$

where $T := \sup\{\deg(g) \mid g\in \Gamma_{j_0+1}(\Omega, V\times G)\}$.

As we mentioned before, Lemma 3.4 and (3.18) hold in the complex algebraic setting, which will be extensively used in the later discussion.

4. Equivalence problems in CR geometry.

• **Equivalence between Segre families.** — Let $\{\theta,\theta^\alpha,\theta_\alpha\}$ be the coframe (1.5) on $\mathcal{M}$. Let

$$\mathcal{G} := \left\{ \begin{pmatrix} u & 0 & 0\\ u^\alpha & u^\alpha_\beta & 0\\ v_\alpha & 0 & v^\beta_\alpha\end{pmatrix} \;\middle|\; u\neq 0,\ \det(u^\alpha_\beta)\neq 0,\ \det(v^\beta_\alpha)\neq 0\right\} \tag{4.1}$$

be a connected linear subgroup of $G(2n+1,\mathbb{C})$. Then $\mathcal{M}\times\mathcal{G}$ is a $\mathcal{G}$-space by (3.0). Similarly, let $\widetilde{\mathcal{M}}$ be the associated Segre family of $\widetilde{M}$ with the corresponding coframe $\{\widetilde{\theta},\widetilde{\theta}^\alpha,\widetilde{\theta}_\alpha\}$ as in (1.5).

It is easy to verify that there exists a biholomorphic mapping $\Phi(z,\zeta) = (f(z), g(\zeta))$ from an open subset in $\mathbb{C}^{n+1}\times\mathbb{C}^{n+1}$ onto an open subset in $\mathbb{C}^{n+1}\times\mathbb{C}^{n+1}$ such that $\Phi(\mathcal{M})\subset\widetilde{\mathcal{M}}$ with $\Phi(0)=0$ if and only if there is a biholomorphic mapping $\Phi : \mathcal{M}\to\widetilde{\mathcal{M}}$, with $\Phi(0)=0$, satisfying

$$\Phi^*\begin{pmatrix}\widetilde{\theta}\\ \widetilde{\theta}^\alpha\\ \widetilde{\theta}_\alpha\end{pmatrix} = \begin{pmatrix} u & 0 & 0\\ u^\alpha & u^\alpha_\beta & 0\\ v_\alpha & 0 & v^\beta_\alpha\end{pmatrix}\begin{pmatrix}\theta\\ \theta^\alpha\\ \theta_\alpha\end{pmatrix} = (\gamma^\alpha_\beta)\begin{pmatrix}\theta\\ \theta^\alpha\\ \theta_\alpha\end{pmatrix} \tag{4.2}$$

where the $(2n+1)\times(2n+1)$ matrix (γ^α_β) defines a holomorphic mapping from $\mathcal{M}$ into $\mathcal{G}$. The mapping Φ satisfying (4.2) is called a *Segre-isomorphism.* Then by Lemma 3.1, there exists such a Segre-isomorphism $\Phi : \mathcal{M} \to \widetilde{\mathcal{M}}$ if and only if there exists a biholomorphic mapping $\Phi^0 : \mathcal{M}\times\mathcal{G} \to \widetilde{\mathcal{M}}\times\widetilde{\mathcal{G}}$ with

$$\Phi^{0*}\widetilde{\omega} = \omega,\ \Phi^{0*}\widetilde{\omega}^\alpha = \omega^\alpha,\ \Phi^{0*}\widetilde{\omega}_\alpha = \omega_\alpha \tag{4.3}$$

where $\omega, \omega^\alpha, \omega_\alpha$ were defined as in (3.1), *i.e.*,

$$\begin{cases} \omega = u\theta, \\ \omega^\alpha = u^\alpha\theta + u^\alpha_\beta\theta^\beta, \\ \omega_\alpha = v_\alpha\theta + v^\beta_\alpha\theta_\beta, \qquad u\neq 0,\ \det(u^\alpha_\beta)\neq 0,\ \det(v^\beta_\alpha)\neq 0, \end{cases} \tag{4.4}$$

where the matrix $\begin{pmatrix} u & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 \\ v_\alpha & 0 & v^\beta_\alpha \end{pmatrix} \in \mathcal{G}$.

By [CM] and [Ch], the lifting $\{\omega, \omega^\alpha, \omega_a\}$ of the coframe $\{\theta, \theta^\alpha, \theta_\alpha\}$ can be expended to an e-structure on $\mathcal{M}\times\mathcal{G}$ (see Theorem 4.1 below). To see that, we shall replace $\mathcal{M}$ by $\mathcal{E}$ and replace $\mathcal{G}$ by another linear subgroup $\mathcal{G}_1$ as follows.

Let

$$\mathcal{E} := \mathcal{M}\times\{u\theta \mid u\in\mathbb{C},\ u\neq 0\}.$$

$\omega = u\theta$ is globally defined on $\mathcal{E}$. We consider $d\omega = u\theta^\alpha\wedge\theta_\alpha + \omega\wedge\phi_0$, where $\phi_0 = -\frac{du}{u}$. Then we get a coframe $(u\theta, \theta^\alpha, \theta_\alpha, \phi_0)$ on $\mathcal{E}$. Let

$$\mathcal{G}_1 := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 & 0 \\ v_\beta & 0 & v^\beta_\alpha & 0 \\ t & iv_\alpha u^\alpha_\beta & -iu^\alpha v^\beta_\alpha & 1 \end{pmatrix} \;\middle|\; u^\gamma_\beta v^\alpha_\gamma = \delta^\alpha_\beta \right\} \tag{4.5}$$

be a connected linear subgroup of $G(2n+1,\mathbb{C})$. As in (3.0), $\mathcal{E}\times\mathcal{G}_1$ is a $\mathcal{G}_1$-space. Now we write $d\omega = i\omega^\alpha\wedge\omega_\alpha + \omega\wedge\phi$. Similarly, let $\widetilde{\mathcal{E}}$ be the associated bundle over $\widetilde{M}$ with the coframe $\{u\widetilde{\theta}, \widetilde{\theta}^\alpha, \widetilde{\theta}_\alpha, \widetilde{\phi}_0\}$ which has the same properties as for $\{u\theta, \theta^\alpha, \theta_\alpha, \phi_0\}$.

Let $\Phi := (f(z), g(\zeta)) : \mathcal{M} \to \widetilde{\mathcal{M}}$ be a Segre-isomorphism. It is easy to verify that Φ induces a unique biholomorphic mapping, still denoted as Φ, from $\mathcal{E}$ to $\widetilde{\mathcal{E}}$, satisfying

$$\Phi^*\begin{pmatrix} \widetilde{\theta} \\ \widetilde{\theta}^\alpha \\ \widetilde{\theta}_\alpha \\ \widetilde{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 & 0 \\ v_\beta & 0 & v^\beta_\alpha & 0 \\ t & iv_\alpha u^\alpha_\beta & -iu^\alpha v^\beta_\alpha & 1 \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\alpha \\ \theta_\alpha \\ \phi \end{pmatrix} = (\gamma^\alpha_\beta)\begin{pmatrix} \theta \\ \theta^\alpha \\ \theta_\alpha \\ \phi \end{pmatrix}$$

where the $(2n+2)\times(2n+2)$ matrix $(\gamma^{\alpha}_{\beta})$ defines a holomorphic mapping from $\mathcal{E}$ into $\mathcal{G}_1$. Then by (3.2) and (3.3), there exists a Segre-isomorphism $\Phi:\mathcal{M}\to\widetilde{\mathcal{M}}$ if and only if there exists a biholomorphic mapping

$$\Phi^1:\mathcal{Y}:=\mathcal{E}\times\mathcal{G}_1\to\widetilde{\mathcal{Y}}:=\widetilde{\mathcal{E}}\times\mathcal{G}_1 \tag{4.6}$$

such that

$$\Phi^{1*}\widetilde{\omega}=\omega,\ \Phi^{1*}\widetilde{\omega}^{\alpha}=\omega^{\alpha},\ \Phi^{1*}\widetilde{\omega}_{\alpha}=\omega_{\alpha},\Phi^{1*}\widetilde{\phi}=\phi. \tag{4.7}$$

$\pi:\mathcal{Y}\to\mathcal{E}$ is called the *projective structure bundle* of $\mathcal{M}$. Any map Φ^1 satisfying (4.7) is called a *$\mathcal{G}_1$-isomorphism.*

THEOREM 4.1 [Ch]. — *Let M be strongly pseudoconvex real analytic hypersurface. From $\omega,\omega^{\alpha},\omega_{\alpha}$ as in (4.4), one can construct holomorphic 1-forms $\phi,\phi^{\alpha}_{\beta},\phi^{\alpha},\phi_{\beta},\psi$ on $\mathcal{Y}$ such that*

$$\Omega:=\{\Omega^j,\ 1\le j\le(n+2)^2-1\}:=\{\omega,\omega^{\alpha},\omega_{\beta},\phi,\phi^{\alpha}_{\beta},\phi^{\alpha},\phi_{\beta},\psi\} \tag{4.8}$$

forms an e-structure on $\mathcal{Y}$ and these 1-forms are uniquely determined by the following structure equations:

$$\begin{aligned}
d\omega&=i\omega^{\alpha}\wedge\omega_{\alpha}+\omega\wedge\phi\\
d\omega^{\alpha}&=\omega^{\beta}\wedge\phi^{\alpha}_{\beta}+\omega\wedge\phi^{\alpha}\\
d\omega_{\alpha}&=\phi^{\beta}_{\alpha}\wedge\omega_{\beta}+\omega_{\alpha}\wedge\phi+\omega\wedge\phi_{\alpha}\\
d\phi&=i\omega^{\alpha}\wedge\phi_{\alpha}+i\phi^{\alpha}\wedge\omega_{\alpha}+\omega\wedge\psi\\
d\phi^{\beta}_{\alpha}&=\phi^{\gamma}_{\alpha}\wedge\phi^{\beta}_{\gamma}+i\omega_{\alpha}\wedge\phi^{\beta}-i\phi_{\alpha}\wedge\omega^{\beta}-i\delta^{\beta}_{\alpha}(\phi_{\sigma}\wedge\omega^{\sigma})-\tfrac{1}{2}\delta^{\beta}_{\alpha}\psi\wedge\omega+\Phi^{\beta}_{\alpha}\\
d\phi^{\alpha}&=\phi\wedge\phi^{\alpha}+\phi^{\beta}\wedge\phi^{\alpha}_{\beta}-\tfrac{1}{2}\psi\wedge\omega^{\alpha}+\Phi^{\alpha}\\
d\phi_{\alpha}&=\phi^{\beta}_{\alpha}\wedge\phi_{\beta}-\tfrac{1}{2}\psi\wedge\omega_{\alpha}+\Phi_{\alpha}\\
d\psi&=\phi\wedge\psi+2i\phi^{\alpha}\wedge\phi_{\alpha}+\Psi
\end{aligned}$$

where $\Phi^{\beta}_{\alpha}=S^{\beta\sigma}_{\alpha\rho}\omega^{\rho}\wedge\omega_{\sigma}+R^{\beta}_{\alpha\gamma}\omega\wedge\omega^{\gamma}+T^{\beta\gamma}_{\alpha}\omega\wedge\omega_{\gamma}$

$$\begin{aligned}
\Phi^{\alpha}&=T^{\alpha\gamma}_{\beta}\omega^{\beta}\wedge\omega_{\gamma}-\tfrac{i}{2}Q^{\alpha}_{\beta}\omega\wedge\omega^{\beta}+L^{\alpha\beta}\omega\wedge\omega_{\beta}\\
\Phi_{\alpha}&=R^{\beta}_{\alpha\gamma}\omega^{\gamma}\wedge\omega_{\beta}+P_{\alpha\beta}\omega\wedge\omega^{\beta}-\tfrac{i}{2}Q^{\beta}_{\alpha}\omega\wedge\omega_{\beta}\\
\Psi&=Q^{\beta}_{\alpha}\omega^{\alpha}\wedge\omega_{\beta}+H_{\alpha}\omega\wedge\omega^{\alpha}+K^{\alpha}\omega\wedge\omega_{\alpha}
\end{aligned}$$

and $S^{\beta\sigma}_{\alpha\rho}=S^{\beta\sigma}_{\rho\alpha}=S^{\alpha\beta}_{\alpha\rho}$, $R^{\beta}_{\alpha\gamma}=R^{\beta}_{\gamma\alpha}$, $T^{\alpha\gamma}_{\beta}=T^{\gamma\alpha}_{\beta}$, $L^{\alpha\beta}=L^{\beta\alpha}$, $P_{\alpha\beta}=P_{\beta\alpha}$, $S^{\beta\sigma}_{\alpha\sigma}=R^{\alpha}_{\alpha\beta}=T^{\alpha\beta}_{\alpha}=Q^{\alpha}_{\alpha}=0$.

Theorem 4.1 was proved in [Ch] over $\mathcal{S}(\mathcal{M})\subset PT\mathbb{C}^{n+1}$, where $\mathcal{S}$ is defined in (1.3). By our choice of the coframes (1.5), all computation over $\mathcal{S}(\mathcal{M})$ can be identically passed back over $\mathcal{M}$. A biholomorphic mapping

$\Phi^1 : \mathcal{Y} \to \widetilde{\mathcal{Y}}$ is a $\mathcal{G}_1$-isomorphism if and only if $\Phi^{1*}\widetilde{\Omega} = \Omega$. When M is, in addition, real algebraic, since all the functions $S_{\alpha\rho}^{\beta\sigma}, R_{\alpha\gamma}^{\beta}, T_{\beta}^{\alpha\gamma}, L^{\alpha\beta}, \ldots$, are obtained by operating the addition, subtraction, multiplication, division, partial differentiation (cf. [Ch], or see the proof of Theorem 6.1) to the defining function, they are all algebraic functions. Hence from Lemma 3.4 and Theorem 4.1, we get the following.

LEMMA 4.2. — *Let Ω and $\widetilde{\Omega}$ be holomorphic algebraic regular e-structures of order j_0 and rank ρ_0 over $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$, respectively. Then there exists a Segre-isomorphism $\Phi : \mathcal{M} \to \widetilde{\mathcal{M}}$ with $\Phi(p) = \widetilde{p}$ if and only if*

$$\Gamma_{2l_0^3}(\Omega, \mathcal{Y})(P) = \Gamma_{2l_0^3}(\widetilde{\Omega}, \widetilde{\mathcal{Y}})(\widetilde{P}), \tag{4.9}$$

where l_0 is as in (3.16), $\pi(P) = p$ and $\pi(\widetilde{P}) = \widetilde{p}$.

• **Equivalence between CR hypersurfaces.** — Let M be as before. For the moment, let $\theta = i\partial r$ and $\theta^\alpha = dz_\alpha$. We have a coframe $\{\theta, \theta^\alpha, \overline{\theta^\alpha}\}$ on M. Let

$$G := \left\{ \begin{pmatrix} u & 0 & 0 \\ u^\alpha & u_\beta^\alpha & 0 \\ \overline{v^\alpha} & 0 & \overline{u_\beta^\alpha} \end{pmatrix} \,\middle|\, u \in \mathbb{R},\ u_\beta^\alpha, u^\alpha \in \mathbb{C},\ u > 0,\ \det(u_\beta^\alpha) \neq 0 \right\}, \tag{4.10}$$

be a connected linear subgroup of $G(2n+1, \mathbb{R})$. $M \times G$ is a G-space. Similarly, let $\widetilde{M}$ be another real analytic hypersurface as in §1 with a similar coframe $\{\widetilde{\theta}, \widetilde{\theta}^\alpha, \overline{\widetilde{\theta}^\alpha}\}$.

It can be verified that there exists a biholomorphic mapping $\Phi(z)$ from an open subset in $\mathbb{C}^{n+1}$ onto an open subset in $\mathbb{C}^{n+1}$ such that $\Phi(M) \subset \widetilde{M}$ if and only if there is a C^ω diffeomorphism $\Phi : M \to \widetilde{M}$ satisfying

$$\Phi^* \begin{pmatrix} \widetilde{\theta} \\ \widetilde{\theta}^\alpha \\ \widetilde{\theta}_\alpha \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ u^\alpha & u_\beta^\alpha & 0 \\ \overline{u^\alpha} & 0 & \overline{u_\beta^\alpha} \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\alpha \\ \theta_\alpha \end{pmatrix} = (\gamma_\beta^\alpha) \begin{pmatrix} \theta \\ \theta^\alpha \\ \theta_\alpha \end{pmatrix} \tag{4.11}$$

where the $(2n+1) \times (2n+1)$ matrix (γ_β^α) defines a real analytic mapping from M into G. Such a mapping Φ is also called a CR isomorphism. By (3.2) and (3.3), there exists a CR isomorphism $\Phi : M \to \widetilde{M}$ if and only if there exists a C^ω diffeomorphism $\Phi^0 : M \times G \to \widetilde{M} \times G$ such that

$$\Phi^{0*}\widetilde{\omega} = \omega,\ \Phi^{0*}\widetilde{\omega}^\alpha = \omega^\alpha,\ \Phi^{0*}\overline{\widetilde{\omega}_\alpha} = \overline{\omega_\alpha}, \tag{4.12}$$

where ω, ω^α are defined as in (4.4).

Define

$$E = M \times \{\omega = u\theta : \omega = \overline{\omega},\ u > 0\}$$

Choose $\theta^\alpha := u^\alpha\theta + u^\alpha_\beta dz^\beta$ for some real analytic functions u^α, u^α_β so that $d\theta = i\theta^\alpha \wedge \overline{\theta^\alpha}$, mod$(\theta)$. We obtain a coframe $(\omega, \theta^\alpha, \overline{\theta^\alpha}, \phi_0)$ on E, where $d\omega = u\theta^\alpha \wedge \overline{\theta^\alpha} + \omega \wedge \phi_0$. Let

(4.13)
$$G_1 := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 & 0 \\ \overline{u^\alpha} & 0 & \overline{u^\alpha_\beta} & 0 \\ s & i\overline{u^\alpha}u^\alpha_\beta & -iu^\alpha\overline{u^\alpha_\beta} & 1 \end{pmatrix} \middle| \; u > 0,\ s\in\mathbb{R}, u^\alpha, u^\alpha_\beta \in \mathbb{C}, u^\gamma_\beta \overline{u^\alpha_\gamma} = \delta^\alpha_\beta \right\}$$

be a connected linear subgroup of $G(2n+1, \mathbb{R})$. $E \times G_1$ is a G_1-space. Let $\widetilde{E}$ be the associated bundle over $\widetilde{M}$ with the corresponding coframe $\{\widetilde{\omega}, \widetilde{\theta}^\alpha, \overline{\widetilde{\theta}^\alpha}, \widetilde{\phi}_0\}$.

Let $\Phi : M \to \widetilde{M}$ be a CR isomorphism. It is easy to verify that Φ induces a unique C^ω diffeomorphism, still denoted as Φ, from E to $\widetilde{E}$ satisfying

$$(4.14)\quad \Phi^* \begin{pmatrix} \widetilde{\theta} \\ \widetilde{\theta}^\alpha \\ \widetilde{\theta}_\alpha \\ \widetilde{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 & 0 \\ \overline{u^\alpha} & 0 & \overline{u^\alpha_\beta} & 0 \\ s & i\overline{u^\alpha}u^\alpha_\beta & -iu^\alpha\overline{u^\alpha_\beta} & 1 \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\alpha \\ \theta_\alpha \\ \phi \end{pmatrix} = (\gamma^\alpha_\beta) \begin{pmatrix} \theta \\ \theta^\alpha \\ \theta_\alpha \\ \phi \end{pmatrix}$$

where the $(2n+2) \times (2n+2)$ matrix (γ^α_β) defines a real analytic mapping from E into G_1, $\theta_\alpha = \overline{\theta^\alpha}$, *etc.* By (3.2) and (3.3), the existence of a CR isomorphism $\Phi : M \to \widetilde{M}$ is equivalent to the existence of a C^ω diffeomorphism $\Phi^1 : Y := E \times G_1 \to \widetilde{Y} := \widetilde{E} \times G_1$ such that

$$(4.15)\qquad \Phi^{1*}\widetilde{\omega} = \omega,\ \Phi^{1*}\widetilde{\omega}^\alpha = \omega^\alpha,\ \Phi^{1*}\widetilde{\omega}_\alpha = \omega_\alpha, \Phi^{1*}\widetilde{\phi} = \phi.$$

(Y, π, E) is called the *CR-structure bundle over M.*

The fundamental Theorem proved by Cartan-Chern-Moser [CM] asserts that from $\omega, \omega^\alpha, \omega_\alpha$, one can construct 1-forms $\phi, \phi^\alpha_\beta, \phi^\alpha, \overline{\phi^\alpha}, \psi$ on Y, with $\omega = \overline{\omega}$, $\phi = \overline{\phi}$, $\psi = \overline{\psi}$, such that

$$(4.16)\quad \Omega := \{\Omega^j,\ 1 \le j \le (n+2)^2 - 1\} := \{\omega, \omega^\alpha, \overline{\omega^\alpha}, \phi, \phi^\alpha_\beta, \phi^\alpha, \overline{\phi^\alpha}, \psi\}$$

forms an e-structure on Y, and they are uniquely determined by certain structure equations. These structure equations are the restriction of those in Theorem 4.1 from $\mathcal{Y}$ to Y, together with several other reality conditions (see [Fa, Theorem 5.5, p. 151] [BS]). Now from Lemma 3.4, we get the following:

LEMMA 4.3. — *Let Ω and $\widetilde{\Omega}$ be real algebraic regular e-structures of order j_0 and rank ρ_0 over Y and $\widetilde{Y}$, respectively. Then there exists a CR isomorphism $\Phi : M \to \widetilde{M}$ with $\Phi(p) = \widetilde{p}$ if and only if*

$$(4.17)\qquad \Gamma_{2l_0^3}(\Omega, Y)(P) = \Gamma_{2l_0^3}(\widetilde{\Omega}, \widetilde{Y})(\widetilde{P}),$$

where l_0 is as in (3.16), and $\pi(P) = p$ and $\pi(\widetilde{P}) = \widetilde{p}$.

Now, we let θ, θ_α be again as defined in (1.5). Since we have the embedding $M \to \mathcal{M}$, by mapping $z \to (z, \overline{z})$, we can regard the bundles E, Y as the subbundles of $\mathcal{E}, \mathcal{Y}$, respectively, as follows (cf. [Fa, (5.9)], [BS]): Let

$$E^* := \left\{(z, \overline{z}, u\theta) \mid z \in M,\ \frac{u}{ir_{n+1}} > 0\right\} \subset \mathcal{E}, \tag{4.18}$$

over M. On E^*, we see $\omega^* = \overline{\omega^*} := u\theta$. Let Y^* be the collection of the frames in $\mathcal{Y}$ restricted to E^* such that $\omega^*_\alpha = \overline{\omega^{*\alpha}}$, $\phi^* = \overline{\phi^*}$ over E^*. Since $\omega^* = \overline{\omega^*}$, $\omega^*_\alpha = \overline{\omega^{*\alpha}}$ and $\phi^* = \overline{\phi^*}$ hold on Y^*, one can check that the structure equations defining Ω^* over Y^* are the same ones defining Ω on Y. Hence, Y and Y^* are G_1-isomorphic. From now on, we identify E and Y with E^* and Y^*, respectively. Then the restriction of a function $g \in \Gamma(\Omega, \mathcal{Y})$ on Y equals to the lexicographically corresponding function $g|_Y \in \Gamma(\Omega|_Y, Y)$.

5. Construction of $\mathcal{V}$ in Theorem 0.1.

From the procedure in Chern's paper [Ch], any $g \in \Gamma_k(\Omega, \mathcal{Y})$, $k > 0$, is a combination of addition, multiplication, division, covariant partial derivations of $p_{\alpha\beta}(z, p_\alpha)$, (defined in (1.6)), with rational functions in the variables $(u, u^\alpha_\beta, u^\alpha, v_\alpha, t)$. (Also see the proof of Theorem 6.1. In passing, we mention that when $n = 1$, the specific formulas for a basis $g \in \Gamma_1(\Omega, \mathcal{Y})$ were given in [HJY, Theorem 3.1]). Since r is a polynomial, it follows that

(5.1) g is the restriction of a rational function

in $Z := (z^\alpha, z^{n+1}, \zeta_\alpha, \zeta_{n+1}, u, u^\alpha_\beta, u^\alpha, v_\alpha, t)$ to the hypersurface given by the equation $r(z^\alpha, z^{n+1}, \zeta_\alpha, \zeta_{n+1}) = 0$.

Recall $\{M^i_j\}_{1 \le i,j \le n+1}$ covers $M_D = \partial D$ and $\{\widetilde{M^i_j}\}_{1 \le i,j \le n+1}$ covers $\widetilde{M_{\widetilde{D}}} = \partial\widetilde{D}$. Fix a point $p_0 \in M_D$ such that (i): $p_0 \in M^i_j$ for any i and j; (ii): Ω is a regular e-structure with order j_0 and rank ρ_0 at a generic point P_0 in the fiber of $Y \subset \mathcal{Y}$ over p_0. We can also assume that j_0 is the largest possible value.

When ∂D (hence $\partial\widetilde{D}$) is spherical, it was shown that D (hence D') must be biholomorphic to the unit ball $B^{n+1} \subset \mathbb{C}^{n+1}$ [HJ 98]. In this case, Theorem 0.1 holds trivially. Hence, in what follows, we shall assume that ∂D and $\partial\widetilde{D}$ are non-spherical.

Step 1. Construction when ∂D is simply connected. — We first construct $\mathcal{V}$ when ∂D is simply connected. Define a subset $\mathcal{V}^*$ in $\widetilde{\mathcal{Y}}$ by

$$\mathcal{V}^* := \{Q \in \widetilde{\mathcal{Y}} : g(P_0) = \widetilde{g}(Q),\ \forall \widetilde{g} \in \Gamma_k(\Omega, \widetilde{\mathcal{Y}}),\ 0 \le k \le 2l_0^3\} \tag{5.2}$$

where $\widetilde{g} \in \Gamma_k(\widetilde{\Omega}, \widetilde{\mathcal{Y}})$ is the lexicographically corresponding function for g, and l_0 is as in (3.18) with $j_0 = (n+2)^2 - 1$. Since D is not spherical, $\mathcal{V}$ is a proper subvariety of $\widetilde{\mathcal{Y}}$. We also define

$$\mathcal{V} := \mathcal{V}^* \cap \widetilde{Y}. \tag{5.3}$$

Notice that all g are holomorphic. $\mathcal{V}^*$ is a complex analytic subvariety. Since D and $\widetilde{D}$ are algebraic, $\mathcal{V}^*$ is an algebraic complex subvariety. Since $Y \subset \mathcal{Y}$ is real algebraic (see (4.19)), $\mathcal{V}$ is a real algebraic subvariety of $\widetilde{Y}$.

(i) We claim that there is a bijective map Λ:

$$\Lambda : \mathrm{Prop}(D, \widetilde{D}) = \{\text{proper holomorphic maps } f : D \to \widetilde{D}\} \to \mathcal{V}. \tag{5.4}$$

In fact, if $f \in \mathrm{Prop}(D, \widetilde{D})$, f extends holomorphically to a neighborhood of $\overline{D}$, for ∂D is strongly pseudoconvex and real analytic. Since both ∂D and $\partial\widetilde{D}$ are strongly pseudoconvex, the restriction of f to ∂D is a local CR equivalent mapping from ∂D onto $\partial\widetilde{D}$. Hence, there is a unique lifting, a local $\mathcal{G}_1$-isomorphism near P_0, $\mathcal{F} : \mathcal{Y} \to \widetilde{\mathcal{Y}}$, such that its restriction $\mathcal{F}|_Y : Y \to \widetilde{Y}$ is a local G_1-isomorphism near P_0 from Y to $\widetilde{Y}$. Define $\Lambda(f) := \widetilde{P}_0 := \mathcal{F}(P_0) \in \widetilde{\mathcal{Y}} \cap \widetilde{Y}$. Then by Lemma 4.2, $\Lambda(f) \in \mathcal{V}$. Conversely, for any point $\widetilde{P}_0 \in \mathcal{V}$. By Lemma 4.2, there exists a local biholomorphic map $\mathcal{F}$ from $\mathcal{Y}$ to $\widetilde{\mathcal{Y}}$ with $\mathcal{F}(P_0) = \widetilde{P}_0$ such that $\mathcal{F}^*\widetilde{\Omega} = \Omega$. By the discussion in §4, the restriction $\mathcal{F}|_Y$ must be a local G_1-isomorphism from Y to $\widetilde{Y}$ with $\mathcal{F}|_Y(P_0) = \widetilde{P}_0$. Therefore, the local isomorphism $\mathcal{F}|_Y$ is uniquely induced by a local CR-isomorphism f from ∂D to $\partial\widetilde{D}$ near the point p_0.

Since ∂D is not spherical and since ∂D is simply connected, we can apply an extension Theorem of Pinchuk ([Pi]) to conclude that the local map f extends into a CR map from ∂D onto $\partial D'$. By the Bochner extension Theorem [Bo], f extends to a holomorphic map from D onto D'. Also, f must be proper. Thus we have well defined $\Lambda^{-1}(\widetilde{P}_0) := f$. Claim (5.4) is proved.

(ii) Let us show that the variety $\mathcal{V}$ is smooth. In fact, suppose not. Take any two points $\widetilde{P}_1, \widetilde{P}_2 \in \mathcal{V}$, there is a unique local $\mathcal{G}_1$-isomorphism $\mathcal{H}$ from an open subset of $\widetilde{Y}$ to $\widetilde{Y}$ such that $\mathcal{H}(\widetilde{P}_1) = P_2'$. Since $\mathcal{V}$ is invariant under the action of $\mathcal{G}_1$-isomorphisms, this implies that if $\widetilde{P}_1 \in \mathcal{V}$ is a singular point, then $\widetilde{P}_2$ (hence every point of $\mathcal{V}$) must be singular, too. This is impossible. Also it implies that $\mathcal{V}$ is purely dimensional.

Since D and $\widetilde{D}$ are not equivalent to the ball, it is known (see [Be]) that a sequence of proper holomorphic maps from D to $\widetilde{D}$ converges uniformly on compact subsets if and only if it converges in the $C^\infty(\overline{D})$-topology. From this fact, it is clear that with the natural topology equipped to $\mathrm{Prop}(D, D')$, Λ and Λ^{-1} are continuous.

(iii) Let $\widetilde{Y}^i_j$ be the CR-structure bundle over $\widetilde{M}^i_j$. We can regard $\widetilde{Y}^i_j$ as one coordinate patch of the bundle $Y_{\widetilde{D}}$. From the construction, we know

$$\begin{aligned}\mathcal{V}^{n+1}_{n+1} &:= \Big\{Q \mid g(P_0) = \widetilde{g}(Q), \forall g \in \Gamma_{2l_0^3}(\Omega, \mathcal{Y})\Big\} \cap \widetilde{Y}^{n+1}_{n+1}\\ &= \Big\{Q = (\widetilde{z},\ \widetilde{\zeta},\ \widetilde{u},\ \widetilde{u}^\alpha_\beta,\ \widetilde{u}^\alpha,\ \widetilde{v}_\alpha, t) \mid \widetilde{r}(\widetilde{z},\ \widetilde{\zeta}) = 0,\ \widetilde{\zeta} = \overline{\widetilde{z}},\\ &\qquad \widetilde{r}_{n+1}(\widetilde{z}, \widetilde{\zeta}) \neq 0,\ \widetilde{r}^{n+1}(\widetilde{z},\ \widetilde{\zeta}) \neq 0,\ \widetilde{\frac{u}{ir_{n+1}}} > 0,\ \widetilde{\omega}_\alpha = \overline{\widetilde{\omega}^\alpha},\\ &\qquad \widetilde{\phi} = \overline{\widetilde{\phi}},\ g(P_0) = \widetilde{g}(Q),\ \forall \widetilde{g} \in \Gamma_{2l_0^3}(\widetilde{\Omega},\ \widetilde{\mathcal{Y}})\Big\}.\end{aligned} \tag{5.5}$$

Making use of Proposition 1.3, (5.1) and (5.5), we get a uniform bound for the sum of Betti numbers for $\mathcal{V}^{n+1}_{n+1}$. Similar procedure works for $\mathcal{V}^j_i$ for any $i, j \leq n+1$. Hence, applying the Mayer-Vietoris sequence [BT, p. 43], we see the uniform boundedness for the sum of the Betti numbers of $\mathcal{V} = \cup_{1 \leq i,j \leq n+1} \mathcal{V}^j_i$.

Step 2. The construction in the general case. — We choose the same $p_0 \in \partial D$ and $P_0 \in Y \cap \mathcal{Y}$ as above. Define the complex variety $\mathcal{V}_0$ and $\mathcal{V}_0^*$ as in (5.2) and (5.3), respectively. Now $\mathcal{V}_0$ gives the set of local CR isomorphisms f near p_0 from ∂D to $\partial \widetilde{D}$. Clearly, $\mathrm{Prop}(D, \widetilde{D})$ is a subset of $\mathcal{V}_0$.

Our argument goes as follows. First, we notice that there is a number c_0 (see [Hu2, Chapter 1]), depending only on the degree of the defining functions of D and $\widetilde{D}$ such that any local CR diffeomorphism between their boundaries is algebraic and has degree bounded by c_0. Hence, its lifting to the algebraic map between their structure bundles also has a fixed degree depending only on the degree of D and $\widetilde{D}$.

We take a Jordan loop $\gamma_1(t) \in \partial D$, $0 \leq t \leq 1$, with $\gamma_1(0) = \gamma_1(1) = p_0$. We want to define the subset $\mathcal{V}_1 \subset \mathcal{V}_0$ which will be bijective to the set of local CR isomorphic maps f that are initially defined at p_0 and then extend holomorphically along the loop γ_1 with $\lim_{t\to 1-} f|_{\gamma_1} D^\alpha(\gamma_1(t)) = D^\alpha f|_{\gamma_1}(\gamma_1(0))$, with $|\alpha|$ bounded by a certain fixed constant c (in fact, we can take $c = 2$). We shall explain that $\mathcal{V}_1$ is a real algebraic variety defined by algebraic functions of degree less than a certain constant depending only the degrees of D, $\widetilde{D}$.

Let $\{\gamma_j\}$ be a finite set of Jordan loops with $\gamma_j(0) = \gamma_j(1) = p_0$, which generates the fundamental group of ∂D. For each γ_j, we shall similarly define a real algebraic subvariety $\mathcal{V}_j$ as above. Hence we have a real algebraic subvariety defined by

$$\mathcal{V} := \cap_j \mathcal{V}_j. \tag{5.6}$$

This $\mathcal{V}$ therefore represents the set of all local CR isomorphism f with $f(p_0) \in \partial\widetilde{D}$ such that $\lim_{t\to 1-} D^\alpha f|_{\gamma_j}(\gamma_j(t)) = D^\alpha f|_{\gamma_j}(\gamma_j(0))$ $|\alpha| \leq c$ for all j. Therefore for any other point $q \in \partial D$ and a path σ connecting p_0 and q, the extension value $f|_\sigma(q)$ will be independent of the choice of σ. Hence, such an f extends into a well defined CR map from ∂D to $\partial\widetilde{D}$. By Bochner's extension Theorem [Bo], f extends into a holomorphic map from D to $\widetilde{D}$. By the same arguments as in Step 1 above, we see that $\mathcal{V}$ is bijective to the set $\mathrm{Prop}(D, \widetilde{D})$. Now, we turn to the construction of $\mathcal{V}_1$. We lift γ_1 into a Jordan loop $\widetilde{\gamma}_1$ in Y with $\widetilde{\gamma}_1(0) = P_0$. Take $Q' \in \mathcal{V}_0$. Then there is a local isomorphic map $\mathcal{F}_{Q'}$ such that $\mathcal{F}_{Q'}(P_0) = Q'$. Notice that Q' is determined by the second jets of the induced underlining CR diffeomorphic map $F_{Q'}$ and the initial fixed point P_0. By slightly modifying the proof of Proposition 3.1 of [BER1], one can see that $\mathcal{F}_{Q'}(Z)$ depends algebraically on Z and Q' and smoothly on (Z, Q') for (Z, Q') near a small neighborhood of $(\widetilde{\gamma}_1(t), Q')$ with $Q' \in \mathcal{V}_0$. (We leave out such a modification). Moreover, the total degree of $\mathcal{F}_{Q'}$ is bounded by the degree of D and D', for it can be obtained with a fixed number of algebraic preserving operations to the defining equations of the domains. By the persistence property of algebraic functions, $D^\alpha\mathcal{F}_{Q'}(\widetilde{\gamma}_1(1)) := \lim_{t\to 1-} D^\alpha\mathcal{F}_{Q'}(\widetilde{\gamma}_1(t))$ is also an algebraic function of Q' whose degree depends only on the total degree of $\mathcal{F}_{Q'}(Z)$. Next, we get a Nash algebraic variety

$$\begin{aligned} \mathcal{V}_1^* :=& \{Q' \in \mathcal{V}_0^* \mid D^\alpha\mathcal{F}_{Q'}(\widetilde{\gamma}_1(0)) - D^\alpha\mathcal{F}_{Q'}(\widetilde{\gamma}_1(1)) = 0,\ |\alpha| \leq c\}, \\ &\text{and } \mathcal{V}_1 = \mathcal{V}_1^* \cap Y. \end{aligned} \tag{5.7}$$

The uniform estimate of the sum of the Betti numbers of $\mathcal{V}_1$ follows from an algebraic version of Proposition 1.3, which can be also done as in the polynomial case in [M]. The rest of the arguments is similar to the simply connected case. □

6. Completion of the proof of Theorem 0.1.

In this section, we use $Y=\{Y_j\}=\{z^\alpha,z^{n+1},\zeta_\alpha,\zeta_{n+1},u,u^\alpha_\beta,u^\alpha,v_\alpha,t\}$ for the coordinates in $\mathbb{C}^{(n+2)^2}$. Use $Z=\{Z^i\}=(z^\alpha,z^{n+1},\zeta_\alpha,u,u^\alpha_\beta,u^\alpha,v_\alpha,t)$ for the coordinates in $\mathcal{Y}^{n+1}_{n+1}=\mathcal{M}^{n+1}_{n+1}\times\mathbb{C}^{(n+1)^2+1}=\{Y\in\mathbb{C}^{(n+2)^2}\mid r(z^\alpha,r^{n+1},\zeta_\alpha,\zeta_{n+1})=0\}$. Here

$$\mathcal{M}=\{(z^\alpha,z^{n+1},\zeta_\alpha,\zeta_{n+1})\mid r(z^\alpha,z^{n+1},\zeta_\alpha,\zeta_{n+1})=0\}$$

is the Segre family with r a polynomial defining function of degree k_0. Let $h(Y)=\frac{H(Y)}{R(Y)}$ be a holomorphic rational function on $\mathbb{C}^{(n+2)^2}$. We denote by $\deg(h)$ the degree of $h=\frac{H}{R}$ as the rational function of Y, and denote by $\deg_{\mathcal{Y}}(h)$ the degree of the restriction of h on $\mathcal{Y}$ as an algebraic function of Z. By Lemma 2.1 (2),

$$\deg_{\mathcal{Y}}(h)\le 2\deg(h)(k_0)^2\le 2\ \deg_*(h)(k_0)^2, \tag{6.1}$$

where $\deg_*(h)=\max\{\deg(H),\deg(R)+1\}$ as in §2. For simplicity of notation, we also use h for its restriction on $\mathcal{Y}$, if there is no confusion from the contexts. We observe that any CR curvature function g over $\mathcal{Y}$ is the restriction of a certain rational function defined in $\mathbb{C}^{(n+2)^2}$.

THEOREM 6.1. — *Let M be algebraic with degree $\le k_0$. Then for any function $g\in\Gamma_{2l_0^3}(\Omega,\mathcal{Y})$ where l_0 is as in (4.9), g is algebraic and satisfies*

$$\deg(g)\le\deg_*(g)\le 3^{3+2l_0^3}(16n+33+2l_0^3)k_0$$

where $l_0\le[(n+2)^2-1]\left[54\cdot 3^{(n+2)^2}(n^2+20n+37)k_0^3\right]^{2+3\cdot 5^{(n+2)^2-1}}$.

For the proof of Theorem 6.1, we can apparently focus on the calculation in the chart $\mathcal{Y}^{n+1}_{n+1}$. Let $\{\Omega^j\}$ be an e-structure on $\mathcal{Y}$. For a function h defined over $\mathcal{Y}^{n+1}_{n+1}$, we have $dh=\sum_j h_{|\Omega^j}\Omega^j=\sum_i h_{|i}dZ^i$. Suppose $dZ^i=\sum_k C^i_k\Omega^k$, where C^i_k is the restriction of a rational function $\frac{E^i_k}{T}$, $\forall i,k$ with E^i_k and T polynomials in the variables Y. The following lemma will be used to compute the covariant derivatives $h|_{\Omega_j}$ of a CR curvature function h.

LEMMA 6.2. — *Let $h=\frac{H}{R},\{\Omega^j\},C^i_k$ be as above. Then*

$$\deg_*(h_{|\Omega^i})\le 2\deg_*(h)+\max_{i,k}\deg_*(C^i_k)+k_0-2,$$

$$\deg_{\mathcal{Y}}(h_{|\Omega^i})\le 2k_0^2\deg_*(h_{|\Omega^i}).$$

Proof. — We have

$$\deg_*(h_{|\Omega^i}) = \deg_*\Big(\sum_k h_{|k} C_i^k\Big) = \max_k \deg_*\Big(h_{|k} C_i^k\Big)$$

by Lemma 2.1 (1).

Let us denote by D_{Y_i} the partial differential operator with respect to Y_i. We have

$$\begin{cases} h_{|\alpha} = D_{z^\alpha}(h) - \dfrac{r_{z^\alpha}}{r^{\zeta_{n+1}}} D_{\zeta_{n+1}}(h), \\ h_{|n+1} = D_{z^{n+1}}(h) - \dfrac{r_{z^{n+1}}}{r^{\zeta_{n+1}}} D_{\zeta_{n+1}}(h), \\ h_{|n+1+\alpha} = D_{\zeta_\alpha}(h) - \dfrac{r^{\zeta_\alpha}}{r^{\zeta_{n+1}}} D_{\zeta_{n+1}}(h), \\ h_{|2n+2} = D_u(h), \\ h_{|2n+2+(\beta-1)n+\alpha} = D_{u_\beta^\alpha}(h), \\ h_{|2n+2+n^2+\alpha} = D_{u^\alpha}(h), \\ h_{|3n+2+n^2+\alpha} = D_{v_\alpha}(h), \\ h_{|(n+2)^2-1} = D_t(h). \end{cases}$$

Consider $k = \alpha \in \{1, \dots, n\}$. Applying Lemma 2.1, we get

$$\begin{aligned} &\deg_*\Big(h_{|\alpha} C_i^\alpha\Big) \\ &= \deg_*\left[\Big(D_{z^\alpha}(h) - \frac{r_{z^\alpha}}{r^{\zeta_{n+1}}} D_{\zeta_{n+1}}(h)\Big) C_i^\alpha\right] \\ &= \deg_*\left(\frac{r^{\zeta_{n+1}}(RD_\alpha H - HD_\alpha R) - r_{z^\alpha}(RD_{2n+2}H - HD_{2n+2}R)}{R^2 r^{\zeta_{n+1}}} \cdot \frac{E_i^\alpha}{T}\right) \\ &\le 2\deg_*(h) + \max_{i,k}\{\deg_*(C_k^i)\} + k_0 - 2. \end{aligned}$$

A similar consideration for the other terms $h_{|k}$ completes the proof of Lemma 6.2. □

From (1.5), (1.6) and (4.4) notice that

$$\begin{cases} \theta = i(dz^{n+1} - p_\alpha dz^\alpha), \\ \theta^\alpha = dz^\alpha, \\ \theta_\alpha = \dfrac{p_\alpha^{n+1}}{p^{n+1}}\theta + q_\alpha^\gamma d\zeta_\gamma, \quad \text{with} \quad q_\alpha^\gamma = p_\alpha^\gamma - \dfrac{p_\alpha^{n+1} p^\gamma}{p^{n+1}}, \\ \omega = u\theta, \\ \omega^\alpha = u_\beta^\alpha \theta^\beta + u^\alpha \theta, \\ \omega_\alpha = v_\alpha \theta + v_\alpha^\beta \theta_\beta, \quad \text{where} \quad u\delta_\alpha^\beta = i u_\alpha^\gamma v_\gamma^\beta. \end{cases} \tag{6.2}$$

Hence

$$\begin{cases} \theta = \dfrac{1}{u}\omega, \\ \theta^\alpha = dz^\alpha = \dfrac{iv^\alpha_\beta}{u}\omega^\beta - \dfrac{iv^\alpha_\beta u^\beta}{u^2}\omega, \\ \theta_\alpha = \dfrac{iu^\beta_\alpha}{u}\omega_\beta - \dfrac{iu^\beta_\alpha v_\beta}{u^2}\omega, \end{cases}$$

Recall that $(Z^j) = (z^\alpha, z^{n+1}, \zeta_\alpha, u, u^\alpha_\beta, u^\alpha, v_\alpha, t)$ are the coordinates in $\mathcal{Y}^{n+1}_{n+1}$. Write $\Omega := (\Omega^j)$ for those forms as in (4.8), and write $dZ^j = C^j_k\Omega^k$. In the following Lemmas 6.3–6.9, we shall estimate the degrees $\deg_{\mathcal{Y}}(C^j_k)$.

LEMMA 6.3. — *$dz^\alpha = \frac{iv^\alpha_\beta}{u}\omega^\beta - \frac{iv^\alpha_\beta u^\beta}{u^2}\omega = C^\alpha_\beta dz^\beta + C^\alpha\omega$, with $\deg_*(C^\alpha_\beta), \deg_{\mathcal{Y}}(C^\alpha_\beta)$, $\deg_*(C^\alpha)$, $\deg_{\mathcal{Y}}(C^\alpha) \leq n+1$.*

Proof of Lemma 6.3. — The identity can be easily derived from (4.4). Since $\delta^\alpha_\beta u = iu^\alpha_\gamma v^\gamma_\beta$, v^α_β can be written as a rational function

$$v^\alpha_\beta = \frac{uP^\alpha_\beta}{Q}, \tag{6.3}$$

where Q is the Jacobian of the matrix (u^α_β), which has degree n, and P^α_β is a polynomial of u^α_β with degree $n-1$. Applying Lemma 2.1, we obtain the desired inequalities. □

LEMMA 6.4. — *$dz^{n+1} = \left(\frac{1}{u} + \frac{ir_\alpha v^\alpha_\beta u^\beta}{u^2 r_{n+1}}\right)\omega - \frac{ir_\alpha v^\alpha_\beta}{ur_{n+1}}\omega^\beta = C\omega + C_\beta\omega^\beta$, with $\deg_*(C), \deg_*(C_\beta) \leq n+k_0+1$, and $\deg_{\mathcal{Y}}(C)$, $\deg_{\mathcal{Y}}(C_\beta) \leq 2(n+k_0+1)(k_0)^2$.*

Proof of Lemma 6.4. — The identity follows from (4.4), (1.5) and (1.6). By (6.3) and Lemma 2.1, $\deg_*(C) = \deg\left(\frac{r_\alpha P^\alpha_\beta}{Qr_{n+1}}\right) \leq k_0 + n$. Then by (6.1), $\deg_{\mathcal{Y}}(C) \leq 2(k_0+n)(k_0)^2$. Similarly, $\deg_*(C_\beta) = \deg_*\Big(\frac{1}{u} + \frac{ir_\alpha v^\alpha_\beta u^\beta}{u^2 r_{n+1}}\Big) = \deg_*\left(\frac{Qr_{n+1}+ir_\alpha P^\alpha_b u^\beta}{uQr_{n+1}}\right) \leq n+k_0+1$ and hence $\deg_{\mathcal{Y}}(C_\beta) \leq 2(n+k_0+1)(k_0)^2$. □

LEMMA 6.5. — *$d\zeta_\alpha = \frac{iq^{-1}_\alpha{}^\beta u^\gamma_\beta}{u}\omega_\gamma - \frac{iq^{-1}_\alpha{}^\beta u^\gamma_\beta}{u^2}\left(v_\gamma + \frac{v^\rho_\gamma p^{n+1}_\rho}{p^{n+1}}\right)\omega = C_{\alpha\gamma}\omega^\gamma + C_\alpha\omega$, with $\deg_*(C_{\alpha\gamma})$, $\deg_*(C_\alpha) \leq 4nk_0 + 3k_0 - 4n - 1$, and $\deg_{\mathcal{Y}}(C_{\alpha\gamma})$, $\deg_{\mathcal{Y}}(C_\alpha) \leq 3(4nk_0 + 3k_0 - 4n - 1)(k_0)^2$.*

Proof of Lemma 6.5. — The identity can be derived from (4.4), (1.5), (1.6). By the definition of q_α^β (see (6.2)), (1.6) and Lemma 2.1, we have

$$(6.4)\qquad \left(q_\alpha^\beta\right) = \left(p_\alpha^\beta - \frac{p_\alpha^{n+1}p^\beta}{p^{n+1}}\right) = \frac{(P_{2\alpha}^\beta)}{r^{n+1}(r_{n+1})^3},$$

where $P_{2\alpha}^\beta$ is a polynomial of Y of degree $4k_0 - 5$. Hence we get the inverse matrix

$$(6.5)\qquad \left(q_\alpha^{-1\beta}\right) = \left(q_\alpha^\beta\right)^{-1} = r^{n+1}(r_{n+1})^3(P_{2\alpha}^\beta)^{-1} = \left(\frac{P_{3\alpha}^\beta}{Q_3}\right),$$

where $P_{3\alpha}^\beta$ and Q_3 are polynomials of Y with degree $4nk_0 - 5n + 1$ and $4nk_0 - 5n$, respectively. The desired inequalities now follows. □

From Lemmas 6.3, 6.4 and 6.5, for any differentiable function on $\mathcal{M}$, we can write $\mathrm{d}h = h_{|\omega}\omega + h_{|\omega^\alpha}\omega^\alpha + h_{|\omega_\alpha}\omega_\alpha = h_{z^\alpha}\mathrm{d}z^\alpha + h_{z^{n+1}}\mathrm{d}z^{n+1} + h_{\zeta_\alpha}d\zeta_\alpha$, where

(6.6)

$$\begin{cases} h_{|\omega} = -h_{z^\alpha}\dfrac{iv_\beta^\alpha u^\beta}{u^2} + h_{z^{n+1}}\left(\dfrac{1}{u} + \dfrac{ir_\alpha v_\beta^\alpha u^\beta}{u^2 r_{n+1}}\right) - h_{\zeta_\alpha}\dfrac{iq_\alpha^{-1\beta}u_\beta^\gamma}{u^2}\left(v_\gamma + \dfrac{v_\gamma^\rho p_\rho^{n+1}}{ip^{n+1}}\right), \\ h_{|\omega^\alpha} = h_{z^\beta}\dfrac{iv_\alpha^\beta}{u} - h_{z^{n+1}}\dfrac{iv_\beta v_\alpha^\beta}{ur_{n+1}}, \\ h_{|\omega_\alpha} = h_{\zeta_\gamma}\dfrac{iq_\gamma^{-1\beta}u_\beta^\alpha}{u}. \end{cases}$$

Applying (6.2), (6.6) and Lemma 6.5, we obtain the following:

(6.7)

$$\mathrm{d}\theta = \frac{i}{u}\omega^\alpha \wedge \omega_\alpha - \frac{iu^\alpha}{u^2}\omega \wedge \omega_\alpha + i\frac{v_\alpha}{u^2}\omega \wedge \omega^\alpha, \quad d\theta^\alpha = 0,$$

$$\begin{aligned}
\mathrm{d}\theta_\alpha &= \mathrm{d}\Big(\frac{p_\alpha^{n+1}}{p^{n+1}}\Big) \wedge \theta + \frac{p_\alpha^{n+1}}{p^{n+1}}\mathrm{d}\theta + \mathrm{d}(q_\alpha^\gamma) \wedge \mathrm{d}\zeta_\gamma \\
&= \mathrm{d}\Big(\frac{p_\alpha^{n+1}}{p^{n+1}}\Big) \wedge \theta + \frac{p_\alpha^{n+1}}{p^{n+1}}\Big(\frac{i}{u}\omega^\kappa \wedge \omega_\kappa - \frac{iu^\kappa}{u^2}\omega \wedge \omega_\kappa + i\frac{v_\kappa}{u^2}\omega \wedge \omega^\kappa\Big) \\
&\quad + \mathrm{d}(q_\alpha^\gamma) \wedge \Big[\frac{iq_\gamma^{-1\ \beta}u_\beta^\mu}{u}\omega_\mu - \frac{iq_\gamma^{-1\ \beta}u_\beta^\mu}{u^2}\Big(v_\mu + \frac{v_\mu^\rho p_\rho^{n+1}}{p^{n+1}}\Big)\omega\Big] \\
&= \Big[\frac{1}{u}B_1 - \frac{iv_\theta p_\alpha^{n+1}}{u^2 p^{n+1}} - \frac{iB_2 q_\gamma^{-1\beta}u_\beta^\mu}{u^2}\Big(v_\mu + \frac{v_\mu^\rho p_\rho^{n+1}}{p^{n+1}}\Big)\Big]\omega^\theta \wedge \omega \\
&\quad + \Big[\frac{1}{u}B_3 + \frac{iu^\theta p_\alpha^{n+1}}{p^{n+1}u^2} - \frac{iB_4 q_\gamma^{-1\beta}u_\beta^\theta}{u} - \frac{iB_5 q_\gamma^{-1\beta}u_\beta^\mu}{u^2}\Big(v_\mu + \frac{v_\mu^\rho p_\rho^{n+1}}{p^{n+1}}\Big)\Big]\omega_\theta \wedge \omega \\
&\quad + \Big[\frac{ip_\alpha^{n+1}}{up^{n+1}}\delta_\theta^\mu + \frac{iB_2 q_\gamma^{-1\beta}u_\beta^\mu}{u}\Big]\omega^\theta \wedge \omega_\mu + \frac{iB_5 q_\gamma^{-1\beta}u_\beta^\mu}{u}\omega_\theta \wedge \omega_\mu
\end{aligned}$$

where

$$\begin{cases} B_1 := \left(\frac{p_\alpha^{n+1}}{p^{n+1}}\right)|_{\omega^\theta} = \frac{iv_\theta^\sigma}{u}\frac{\partial}{\partial z^\sigma}\left(\frac{p_\alpha^{n+1}}{p^{n+1}}\right) - \frac{iv_\sigma v_\theta^\sigma}{ur_{n+1}}\frac{\partial}{\partial z^{n+1}}\left(\frac{p_\alpha^{n+1}}{p^{n+1}}\right), \\ B_2 := (q_\alpha^\gamma)|_{\omega^\theta} = \frac{iv_\theta^\sigma}{u}\frac{\partial q_\alpha^\gamma}{\partial z^\sigma} - \frac{iv_\sigma v_\theta^\sigma}{ur_{n+1}}\frac{\partial q_\alpha^\gamma}{\partial z^{n+1}}, \\ B_3 := \left(\frac{p_\alpha^{n+1}}{p^{n+1}}\right)|_{\omega_\theta} = \frac{iq_\gamma^{-1\sigma}u_\sigma^\theta}{u}\frac{\partial}{\partial \zeta^\gamma}\left(\frac{p_\alpha^{n+1}}{p^{n+1}}\right), \\ B_4 := (q_\alpha^\gamma)|_\omega \\ \quad = -\frac{iv_\beta^\alpha u^\beta}{u^2}\frac{\partial q_\alpha^\gamma}{\partial z^\alpha} + \left(\frac{1}{u} + \frac{ir_\alpha v_\beta^\alpha u^\beta}{u^2 r_{n+1}}\right)\frac{\partial q_\alpha^\gamma}{\partial z^{n+1}} - \frac{iq_\alpha^{-1\beta}u_\beta^\gamma}{u^2}\left(v_\gamma + \frac{v_\gamma^\rho p_\rho^{n+1}}{ip^{n+1}}\right)\frac{\partial q_\alpha^\gamma}{\partial \zeta^\alpha}, \\ B_5 := (q_\alpha^\gamma)|_{\omega_\theta} = \frac{iq_\mu^{-1\sigma}u_\sigma^\theta}{u}\frac{\partial}{\partial \zeta_\mu}(q_\alpha^\gamma). \end{cases}$$

LEMMA 6.6. — *$du = -u\phi + ut\omega + iv_\alpha\omega^\alpha - iu^\alpha\omega_\alpha$, with* $\deg(u) = \deg_{\mathcal{Y}}(u)$, $\deg(v_\alpha) = \deg_{\mathcal{Y}}(v_\alpha)$, $\deg(u^\alpha) = \deg_{\mathcal{Y}}(u^\alpha) \le 2$.

Proof of Lemma 6.6. — Since

$$d\omega = i\omega^\alpha \wedge \omega_\alpha + \omega \wedge \left(-\frac{du}{u} + \frac{-iu^\alpha\omega_\alpha + iv_\alpha\omega^\alpha}{u}\right),$$

we get

$$\phi = -\frac{du}{u} + t\omega + \frac{iv_\alpha}{u}\omega^\alpha - \frac{iu^\alpha}{u}\omega_\alpha. \tag{6.8}$$

The identity follows. □

In order to prove Lemma 6.7, we need some preparation. We shall follow the procedure in [Ch] to partially determine the forms $\phi_\alpha^\alpha, \phi^\alpha, \phi_\alpha$ and ψ as follows.

From

$$d\omega^\alpha = d(u^\alpha\theta + u_\beta^\alpha\theta^\beta) = \omega^\beta \wedge \left(\frac{iu^\alpha}{u}\omega^\beta - \frac{iu^\alpha v_\beta}{u^2}\omega - \frac{iv_\beta^\gamma du_\gamma^\alpha}{u}\right)$$
$$+\omega \wedge \left(-\frac{du^\alpha}{u} - \frac{iu^\alpha u^\beta}{u^2}\omega_\beta + \frac{iv_\gamma^\beta u^\gamma}{u^2}du_\beta^\alpha\right),$$

we obtain $d\omega^\alpha = \omega^\beta \wedge \phi_\beta^{\alpha *} + \omega \wedge \phi^{\alpha *}$, where

$$\phi_\beta^{\alpha *} = \frac{iu^\alpha}{u}\omega_\beta - \frac{iu^\alpha v_\beta}{u^2}\omega - \frac{iv_\beta^\gamma}{u}du_\gamma^\alpha. \tag{6.9}$$

and

$$\begin{aligned} \phi^{\alpha *} &= -\frac{du^\alpha}{u} - \frac{iu^\alpha u^\beta}{u^2}\omega_\beta + \frac{iv_\gamma^\beta u^\gamma}{u^2}d\, u_\beta^\alpha \\ &= -\frac{du^\alpha}{u} - \frac{u^\gamma}{u}\phi_\gamma^{*\alpha} - \frac{iu^\alpha u^\gamma v_\gamma}{u^3}\omega. \end{aligned} \tag{6.10}$$

These $\phi_\beta^{\alpha *}$ and $\phi^{\alpha *}$ are not uniquely determined yet. Indeed we should (cf. [Ch] or see (6.19) below) add a term $b_\beta^\alpha \omega$ to $\phi_\beta^{\alpha *}$. This same term $b_\beta^\alpha \omega^\beta$ must be added to $\phi^{\alpha *}$, and we should add a term $-a_{\alpha\gamma}^\beta \omega^\gamma$ with $a_{\alpha\gamma}^\beta = a_{\gamma\alpha}^\beta$ to $\phi_\alpha^{*\beta}$, too. Here $a_{\alpha\beta}^\gamma$ are determined by the following equation:

$$-d\omega_\alpha + \phi_\alpha^{*\beta} \wedge \omega_\beta + \omega_\alpha \wedge \phi \equiv a_{\alpha\beta}^\gamma \omega^\beta \wedge \omega_\gamma, \quad \mathrm{mod}(\omega). \tag{6.11}$$

We compute the left hand side of the above equation:

$$-d\omega_\alpha + \phi_\alpha^{*\beta} \wedge \omega_\beta + \omega_\alpha \wedge \phi = -d(v_\alpha \theta + v_\alpha^\beta \theta_\beta) + \phi_\alpha^\beta \wedge \omega_\beta + \omega_\alpha \wedge \phi.$$

Since $\delta_\alpha^\beta u = i u_\alpha^\gamma v_\gamma^\beta$, we have

$$dv_\alpha^\beta = \frac{v_\alpha^\beta}{u} du - \frac{i v_\alpha^s v_\gamma^\beta}{u} du_s^\gamma. \tag{6.12}$$

Notice that du in (6.12) can be expressed in terms of $\phi, \omega, \omega^\alpha, \omega_\alpha$ by (6.8); and that du_s^γ in (6.12) can be expressed in terms of $\phi_\alpha^{\gamma *}$, ω_α and ω by (6.9). Notice that we need only to consider terms involving $\omega^\beta \wedge \omega_\gamma$, we find

(6.13)
$$a_{\alpha\beta}^\gamma = -\frac{i\delta_\beta^\gamma v_\alpha}{u} - \frac{i\delta_\alpha^\gamma v_\beta}{u} - \frac{i v_\alpha^t p_t^{n+1} \delta_\beta^\gamma}{u p^{n+1}} - \frac{i q_\rho^{-1t} u_t^\gamma v_\alpha^\kappa}{u} \Big(\frac{i v_\beta^\theta}{u} \frac{\partial q_\kappa^\rho}{\partial z^\theta} - \frac{i v_\mu v_\beta^\mu}{u r_{n+1}} \frac{\partial q_\kappa^\rho}{\partial z^{n+1}} \Big).$$

Modifying $\phi_\alpha^{*\beta}$ and $\phi^{*\alpha}$, we get

$$\begin{aligned} \phi_\alpha^{\beta **} &= \phi_\alpha^{\beta *} - a_{\alpha\gamma}^\beta \omega^\gamma \\ &= -\frac{i v_\alpha^\gamma}{u} du_\gamma^\beta + \frac{i u^\beta}{u} \omega_\alpha - \frac{i u^\beta v_\alpha}{u^2} \omega + \Big[\frac{i\delta_\gamma^\beta v_\alpha}{u} + \frac{i\delta_\alpha^\beta v_\gamma}{u} \\ &\quad + \frac{i v_\alpha^t p_t^{n+1} \delta_\gamma^\beta}{u p^{n+1}} + \frac{i q_\rho^{-1t} u_t^\beta v_\alpha^\kappa}{u} \Big(\frac{i v_\gamma^\theta}{u} \frac{\partial q_\kappa^\rho}{\partial z^\theta} - \frac{i v_\mu v_\gamma^\mu}{u r_{n+1}} \frac{\partial q_\kappa^\rho}{\partial z^{n+1}} \Big) \Big] \omega^\gamma, \end{aligned} \tag{6.14}$$

$$\begin{aligned} \phi^{\alpha **} &= \phi^{\alpha *} + \frac{u^\gamma}{u} a_{\beta\gamma}^\alpha \omega^\gamma \\ &= -\frac{du^\alpha}{u} - \frac{u^\gamma}{u} \phi_\gamma^{\alpha **} - \frac{i u^\alpha u^\gamma v_\gamma}{u^3} \omega + \frac{u^\beta}{u} \Big[\frac{i\delta_\gamma^\alpha v_\beta}{u} + \frac{i\delta_\beta^\alpha v_\gamma}{u} \\ &\quad + \frac{i v_\beta^t p_t^{n+1} \delta_\gamma^\alpha}{u p^{n+1}} + \frac{i q_\rho^{-1t} u_t^\alpha v_\beta^\kappa}{u} \Big(\frac{i v_\gamma^\theta}{u} \frac{\partial q_\kappa^\rho}{\partial z^\theta} - \frac{i v_\mu v_\gamma^\mu}{u r_{n+1}} \frac{\partial q_\kappa^\rho}{\partial z^{n+1}} \Big) \Big] \omega^\gamma. \end{aligned} \tag{6.15}$$

Now we compute ϕ_α^{**} from the equation $d\omega_\alpha - \phi_\alpha^{\beta **} \wedge \omega_\beta - \omega_\alpha \wedge \phi = \omega \wedge \phi_\alpha^{**}$. Similarly we only need to consider terms involving ω:

$$\begin{aligned} d\omega_\alpha - &\phi_\alpha^{\beta **} \wedge \omega_\beta - \omega_\alpha \wedge \phi \\ &= d(v_\alpha \theta + v_\alpha^\beta \theta_\beta) + \frac{i u^\beta v_\alpha}{u^2} \omega \wedge \omega_\beta - \omega_\alpha \wedge t\omega \end{aligned}$$

$$= \frac{dv_\alpha}{u} \wedge \omega + v_\alpha d\theta + dv_\alpha^\beta \wedge \theta_\beta + v_\alpha^\beta d\theta_\beta + \frac{iu^\beta v_\alpha}{u^2}\omega \wedge \omega_\beta - \omega_\alpha \wedge t\omega$$
$$= \frac{dv_\alpha}{u} \wedge \omega + v_\alpha\Big(-\frac{iu^\kappa}{u^2}\omega \wedge \omega_\kappa + i\frac{v_\kappa}{u^2}\omega \wedge \omega^\kappa\Big)$$
$$+ \Big\{v_\gamma^\beta \phi_\alpha^{\gamma **} - v_\alpha^\beta \phi + \Big[tv_\alpha^\beta + \frac{iv_\gamma^\beta u^\gamma v_\alpha}{u^2}\Big]\omega + \Big[-\frac{iu^\gamma v_\alpha^\beta}{u} - \frac{iv_\psi^\gamma u^\psi}{u}\Big]\omega_\gamma$$
$$+ \Big[\frac{iv_\psi v_\alpha^\beta}{u} - v_\gamma^\beta\Big(\frac{i\delta_\psi^\gamma v_\alpha}{u} + \frac{i\delta_\alpha^\gamma v_\psi}{u} + \frac{iv_\alpha^t p_t^{n+1}\delta_\psi^\gamma}{up^{n+1}} + \frac{iq_\rho^{-1t}u_t^\gamma v_\alpha^\kappa}{u}$$
$$\cdot\Big(\frac{iv_\psi^\theta}{u}\frac{\partial q_\kappa^\rho}{\partial z^\theta} + \frac{iv_\mu v_\psi^\mu}{ur_{n+1}}\frac{\partial q_\kappa^\rho}{\partial z^{n+1}}\Big)\Big)\Big]\omega^\psi\Big\} \wedge \Big(\frac{iu_\beta^\kappa}{u}\omega_\kappa - \frac{iu_\beta^\kappa v_\kappa}{u^2}\omega\Big)$$
$$+ v_\alpha^\beta\Big[\frac{1}{u}B_1 - \frac{iv_\theta p_\beta^{n+1}}{u^2p^{n+1}} - \frac{iB_2q_\gamma^{-1\psi}u_\psi^\mu}{u^2}\Big(v_\mu + \frac{v_\mu^\rho p_\rho^{n+1}}{p^{n+1}}\Big)\Big]\omega^\theta \wedge \omega$$
$$+ \quad v_\alpha^\beta\Big[\frac{1}{u}B_3 + \frac{iu^\theta p_\beta^{n+1}}{p^{n+1}u^2} - \frac{iB_4q_\gamma^{-1\psi}u_\psi^\theta}{u} - \frac{iB_5q_\gamma^{-1\psi}u_\psi^\mu}{u^2}$$
$$\cdot\Big(v_\mu + \frac{v_\mu^\rho p_\rho^{n+1}}{p^{n+1}}\Big)\Big]\omega_\theta \wedge \omega + \frac{iu^\beta v_\alpha}{u^2}\omega \wedge \omega_\beta + t\omega \wedge \omega_\alpha.$$

Here (6.7) is used. Hence

$$\phi_\alpha^{**} = -\frac{dv_\alpha}{u} + \frac{v_\gamma}{u}\phi_\alpha^{\gamma **} - \frac{v_\alpha}{u}\phi + \frac{iv_\alpha v_\kappa}{u^2}\omega^\kappa$$
$$+ \frac{iu_\beta^\kappa v_\kappa}{u^2}\Big\{\frac{iv_\psi v_\alpha^\beta}{u} - v_\gamma^\beta\Big[\frac{i\delta_\psi^\gamma v_\alpha}{u} + \frac{i\delta_\alpha^\gamma v_\psi}{u} + \frac{iv_\alpha^t p_t^{n+1}\delta_\psi^\gamma}{up^{n+1}}$$
$$(6.16) \quad + \frac{iq_\rho^{-1t}u_t^\gamma v_\alpha^\kappa}{u}\Big(\frac{iv_\psi^\theta}{u}\frac{\partial q_\kappa^\rho}{\partial z^\theta} + \frac{iv_\mu v_\psi^\mu}{ur_{n+1}}\frac{\partial q_\kappa^\rho}{\partial z^{n+1}}\Big)\Big]\Big\}\omega^\psi$$
$$- v_\alpha^\beta\Big[\frac{1}{u}B_1 - \frac{iv_\theta p_\beta^{n+1}}{u^2p^{n+1}} - \frac{iB_2q_\gamma^{-1\psi}u_\psi^\mu}{u^2}\Big(v_\mu + \frac{v_\mu^\rho p_\rho^{n+1}}{p^{n+1}}\Big)\Big]\omega^\theta$$
$$-\frac{iv_\alpha u^\kappa}{u^2}\omega_\kappa + \Big[tv_\alpha^\beta + \frac{iv_\gamma^\beta u^\gamma v_\alpha}{u^2}\Big]\frac{iu_\beta^\kappa}{u}\omega_\kappa + \frac{iu_\beta^\kappa v_\kappa}{u^2}\Big[-\frac{iu^\gamma v_\alpha^\beta}{u} - \frac{iv_\psi^\gamma u^\psi}{u}\Big]\omega_\gamma$$
$$-v_\alpha^\beta\Big[\frac{1}{u}B_3 + \frac{iu^\theta p_\beta^{n+1}}{p^{n+1}u^2} - \frac{iB_4q_\gamma^{-1\psi}u_\psi^\theta}{u} - \frac{iB_5q_\gamma^{-1\psi}u_\psi^\mu}{u^2}\Big(v_\mu + \frac{v_\mu^\rho p_\rho^{n+1}}{p^{n+1}}\Big)\Big]\omega_\theta$$
$$+ \frac{iu^\beta v_\alpha}{u^2}\omega_\beta + t\omega_\alpha.$$

We next compute ψ^{**} from $d\phi - i\omega^\alpha \wedge \phi_\alpha^{**} - i\phi^{\alpha **} \wedge \omega_\alpha = \omega \wedge \psi^{**}$. We consider terms only involving ω:

$$d\phi - i\omega^\alpha \wedge \phi_\alpha^{**} - i\phi^{\alpha **} \wedge \omega_\alpha$$
$$= d\phi = d\Big(-\frac{du}{u} + t\omega + \frac{iv_\alpha}{u}\omega^\alpha - \frac{iu^\alpha}{u}\omega_\alpha\Big)$$

$$= dt\wedge\omega + td\omega + i\frac{udv_\alpha - v_\alpha du}{u^2}\wedge\omega^\alpha + \frac{iv_\alpha}{u}d\omega^\alpha - i\frac{udu^\alpha - u^\alpha du}{u^2}\wedge\omega_\alpha - \frac{iu^\alpha}{u}d\omega_\alpha$$

$$= \omega\wedge\Big[-dt + t\phi - \frac{iv_\alpha t}{u}\omega^\alpha + \frac{iv_\alpha}{u}\phi^{\alpha**} - \frac{u^\alpha u^\gamma v_\gamma}{u^3}\omega_\alpha + i\frac{u^\alpha t}{u}\omega_\alpha - \frac{iu^\alpha}{u}\phi^{**}_\alpha\Big].$$

Therefore we find

$$\psi^{**} = -dt + t\phi - \frac{iv_\alpha t}{u}\omega^\alpha + \frac{iv_\alpha}{u}\phi^{**\alpha} - \frac{u^\alpha u^\gamma v_\gamma}{u^3}\omega_\alpha + i\frac{u^\alpha t}{u}\omega_\alpha - \frac{iu^\alpha}{u}\phi^{**}_\alpha. \tag{6.17}$$

According to [Ch], we put

$$\begin{cases} \phi^\beta_\alpha = \phi^{\beta**}_\alpha + b^\beta_\alpha\omega, \\ \phi^\alpha = \phi^{\alpha**} + b^\alpha_\beta\omega^\beta + c^\alpha\omega, \\ \phi_\alpha = \phi^{**}_\alpha - b^\beta_\alpha\omega_\beta + d_\alpha\omega, \\ \psi = \psi^{**} + i(d_\alpha\omega^\alpha - c^\alpha\omega_\alpha) + e\omega, \end{cases} \tag{6.18}$$

where $b^\beta_\alpha, c^\alpha, d_\alpha$ and e are certain uniquely determined functions, and $\phi^\beta_\alpha, \phi^\alpha, \phi_\alpha$ are the uniquely determined forms as in Theorem 4.1. In other words, we have the following relationships:

$$\begin{cases} \phi^{\beta***}_\alpha = \phi^{\beta**}_\alpha + b^\beta_\alpha\omega, \\ \phi^{\alpha***} = \phi^{\alpha**} + b^\alpha_\beta\omega^\beta, \\ \phi^{***}_\alpha = \phi^{**}_\alpha - b^\beta_\alpha\omega_\beta, \\ \psi^{***} = \psi^{**}\ ; \end{cases} \quad \begin{cases} \phi^{\beta****}_\alpha = \phi^{\beta***}_\alpha, \\ \phi^{\alpha****} = \phi^{\alpha***} + c^\alpha\omega, \\ \phi^{****}_\alpha = \phi^{***}_\alpha + d_\alpha\omega, \\ \psi^{****} = \psi^{***} + i(d_\alpha\omega^\alpha - c^\alpha\omega_\alpha)\ ; \end{cases}$$

$$\begin{cases} \phi^\beta_\alpha = \phi^{\beta****}_\alpha, \\ \phi^\alpha = \phi^{\alpha****}, \\ \phi_\alpha = \phi^{****}_\alpha, \\ \psi = \psi^{****} + e\omega. \end{cases} \tag{6.19}$$

Lemma 6.7.

$$du^\beta_\alpha = -u^\xi_\alpha\phi^\beta_\xi + \frac{iu^\xi_\alpha u^\beta}{u}\omega_\xi - \Big(\frac{iu^\xi_\alpha u^\beta v_\xi}{u^2} - u^\xi_\alpha b^\beta_\xi\Big)\omega$$

$$+u^\xi_\alpha\Big[\frac{i\delta^\beta_\gamma v_\xi}{u} + \frac{i\delta^\beta_\xi v_\gamma}{u} + \frac{iv^t_\xi p^{n+1}_t\delta^\beta_\gamma}{up^{n+1}} + \frac{iq^{-1t}_\rho u^\beta_t v^\kappa_\xi}{u}\Big(\frac{iv^\theta_\gamma}{u}\frac{\partial q^\rho_\kappa}{\partial z^\theta} - \frac{iv_\mu v^\mu_\gamma}{ur_{n+1}}\frac{\partial q^\rho_\kappa}{\partial z^{n+1}}\Big)\Big]\omega^\gamma.$$

If we write the above as $du^\beta_\alpha = C^\beta_{\alpha k}\sigma^k$, *where* $\{\sigma^j\} = \{\omega, \omega^\alpha, \omega_\alpha, \phi^\beta_\alpha\}$, *then*

$$\deg_*(C^\beta_{\alpha\gamma}),\quad \deg_*(b^\beta_\alpha),\quad \deg_*(S^{\beta\sigma}_{\alpha\gamma}) \le 16nk_0 + 30k_0 - 14n - 38,$$

$$\deg_{\mathcal{Y}}(C^{\beta}_{\alpha\beta}),\quad \deg_{\mathcal{Y}}(b^{\beta}_{\alpha}),\quad \deg_{\mathcal{Y}}(S^{\beta\sigma}_{\alpha\gamma}) \le 2(16nk_0+30k_0-14n-38)(k_0)^2.$$

Proof of Lemma 6.7. — The identity follows from (6.14) and (6.19). We want to show

$$\deg_*(b^{\beta}_{\alpha}) \le 16nk_0+33k_0. \tag{6.20}$$

Since we can easily verify that for each $C^{\beta}_{\alpha\kappa}$, $\deg_*(C^{\beta}_{\alpha\kappa}) \le 16nk_0+33k_0$. Thus (6.20) implies the inequalities for $\deg_*(C^{\beta}_{\alpha\gamma})$ and $\deg_{\mathcal{Y}}(C^{\beta}_{\alpha\kappa})$.

By [Ch, (27)], the b^{β}_{α} are determined by

$$b^{\beta}_{\alpha} = \frac{1}{i(n+2)}\Big(S^{\beta}_{\alpha} - \frac{1}{2(n+1)}\delta^{\beta}_{\alpha}S^{\gamma}_{\gamma}\Big), \tag{6.21}$$

where $S^{\beta}_{\alpha} := S^{\beta\rho}_{\alpha\rho}$, and $S^{\beta\sigma}_{\alpha\rho}$ are determined by (see Theorem 4.1)

$$\Phi^{\beta}_{\alpha} = S^{\beta\sigma}_{\alpha\rho}\omega^{\rho}\wedge\omega_{\sigma} + \text{ other terms},$$

where

$$\Phi^{\beta}_{\alpha} = d\phi^{\beta *}_{\alpha} - \phi^{\gamma **}_{\alpha}\wedge\phi^{\beta}_{\gamma} - i\omega_{\alpha}\wedge\phi^{\beta **} + i\phi^{**}_{\alpha}\wedge\omega^{\beta} + i\delta^{\beta}_{\alpha}(\phi^{**}_{\sigma}\wedge\omega^{\sigma}) + \frac{1}{2}\delta^{\beta}_{\alpha}\psi^{**}\wedge\omega.$$

Let us calculate Φ^{β}_{α} in which we only consider the $\omega^{\rho}\wedge\omega_{\sigma}$ terms:

$$\begin{aligned}
\Phi^{\beta}_{\alpha} &= d\phi^{\beta **}_{\alpha} - \phi^{\gamma **}_{\alpha}\wedge\phi^{\beta **}_{\gamma} - i\omega_{\alpha}\wedge\phi^{\beta **} + i\phi^{**}_{\alpha}\wedge\omega^{\beta} + i\delta^{\beta}_{\alpha}(\phi^{**}_{\sigma}\wedge\omega^{\sigma})\\
&\quad + \frac{1}{2}\delta^{\beta}_{\alpha}\psi^{**}\wedge\omega\\
&= d\Big\{-\frac{iv^{\gamma}_{\alpha}}{u}du^{\beta}_{\gamma} + \frac{iu^{\beta}}{u}\omega_{\alpha} - \frac{iu^{\beta}v_{\alpha}}{u^2}\omega + \Big[\frac{i\delta^{\beta}_{\gamma}v_{\alpha}}{u} + \frac{i\delta^{\beta}_{\alpha}v_{\gamma}}{u}\\
&\quad + \frac{iv^{t}_{\alpha}p^{n+1}_{t}\delta^{\beta}_{\gamma}}{up^{n+1}} + \frac{iq^{-1t}_{\rho}u^{\beta}_{t}v^{\kappa}_{\alpha}}{u}\Big(\frac{iv^{\theta}_{\gamma}}{u}\frac{\partial q^{\rho}_{\kappa}}{\partial z^{\theta}} - \frac{iv_{\mu}v^{\mu}_{\gamma}}{ur_{n+1}}\frac{\partial q^{\rho}_{\kappa}}{\partial z^{n+1}}\Big)\Big]\omega^{\gamma}\Big\}\\
&= \Big[\Big(-\frac{iv^{\gamma}_{\alpha}}{u}\Big)|_{\omega^{\theta}}\omega^{\theta} + \Big(-\frac{iv^{\gamma}_{\alpha}}{u}\Big)|_{\omega_{\theta}}\omega_{\theta}\Big]\wedge\Big\{\frac{iu^{\xi}_{\gamma}u^{\beta}}{u}\omega_{\xi} + u^{\xi}_{\gamma}\Big[\frac{i\delta^{\beta}_{\lambda}v_{\xi}}{u} + \frac{i\delta^{\beta}_{\xi}v_{\lambda}}{u}\\
&\quad + \frac{iv^{t}_{\xi}p^{n+1}_{t}\delta^{\beta}_{\lambda}}{up^{n+1}} + \frac{iq^{-1t}_{\rho}u^{\beta}_{t}v^{\kappa}_{\xi}}{u}\Big(\frac{iv^{\theta}_{\lambda}}{u}\frac{\partial q^{\rho}_{\kappa}}{\partial z^{\theta}} - \frac{iv_{\mu}v^{\mu}_{\lambda}}{ur_{n+1}}\frac{\partial q^{\rho}_{\kappa}}{\partial z^{n+1}}\Big)\Big]\omega^{\lambda}\Big\}\\
&\quad + \Big(\frac{iu^{\beta}}{u}\Big)|_{\omega^{\gamma}}\omega^{\gamma}\wedge\omega_{\alpha} + \frac{u^{\beta}v_{\alpha}}{u^2}\omega^{\gamma}\wedge\omega_{\gamma} - \Big[\frac{i\delta^{\beta}_{\gamma}v_{\alpha}}{u} + \frac{i\delta^{\beta}_{\alpha}v_{\gamma}}{u}\\
&\quad + \frac{iv^{t}_{\alpha}p^{n+1}_{t}\delta^{\beta}_{\gamma}}{up^{n+1}} + \frac{iq^{-1t}_{\rho}u^{\beta}_{t}v^{\kappa}_{\alpha}}{u}\Big(\frac{iv^{\theta}_{\gamma}}{u}\frac{\partial q^{\rho}_{\kappa}}{\partial z^{\theta}} - \frac{iv_{\mu}v^{\mu}_{\gamma}}{ur_{n+1}}\frac{\partial q^{\rho}_{\kappa}}{\partial z^{n+1}}\Big)\Big]|_{\omega_{\chi}}\omega^{\gamma}\wedge\omega_{\chi}.
\end{aligned}$$

Hence

$$S^{\beta\sigma}_{\alpha\chi} = \Big(-\frac{iv^{\gamma}_{\alpha}}{u}\Big)|_{\omega^{\sigma}}\frac{iu^{\chi}_{\gamma}u^{\beta}}{u} - \Big(-\frac{iv^{\gamma}_{\alpha}}{u}\Big)|_{\omega_{\chi}}u^{\xi}_{\gamma}\Big[\frac{i\delta^{\beta}_{\sigma}v_{\xi}}{u} + \frac{i\delta^{\beta}_{\xi}v_{\sigma}}{u}$$

$$(6.22)\qquad + \frac{iv_\xi^t p_t^{n+1}\delta_\sigma^\beta}{up^{n+1}} + \frac{iq_\rho^{-1t}u_t^\beta v_\xi^\kappa}{u}\Big(\frac{iv_\sigma^\theta}{u}\frac{\partial q_\kappa^\rho}{\partial z^\theta} - \frac{iv_\mu v_\sigma^\mu}{ur_{n+1}}\frac{\partial q_\kappa^\rho}{\partial z^{n+1}}\Big)\Big]$$
$$+ \delta_\alpha^\chi\Big(\frac{iu^\beta}{u}\Big)_{|\omega^\sigma} + \frac{u^\beta v_\alpha \delta_\sigma^\chi}{u^2} - \Big[\frac{i\delta_\gamma^\beta v_\alpha}{u} + \frac{i\delta_\alpha^\beta v_\gamma}{u}$$
$$+ \frac{iv_\alpha^t p_t^{n+1}\delta_\gamma^\beta}{up^{n+1}} + \frac{iq_\rho^{-1t}u_t^\beta v_\alpha^\kappa}{u}\Big(\frac{iv_\gamma^\theta}{u}\frac{\partial q_\kappa^\rho}{\partial z^\theta} - \frac{iv_\mu v_\gamma^\mu}{ur_{n+1}}\frac{\partial q_\kappa^\rho}{\partial z^{n+1}}\Big)\Big]_{|\omega_\chi}.$$

Here the functions such as $\Big(\frac{iv_\alpha^\beta}{u}\Big)_{|\omega^\rho}$ in (6.22) are covariant partial derivatives with respect to the coframe

$$(6.23)\qquad \{\sigma^j\} = \{\omega, \omega^\alpha, \omega_\alpha, \phi, \phi_\beta^{**\alpha}, \phi^{**\alpha}, \phi_\alpha^{**}, \psi^{**}\} \text{ on } \mathcal{Y}.$$

Recall that we have fixed the coframe $\{dZ^j\} = \{dz^\alpha, dz^{n+1}, d\zeta_\alpha, du, du_\beta^\alpha, du^\alpha, dv_\alpha, dt\}$ on $\mathcal{Y}_{n+1}^{n+1}$. Consider $dZ^j = \sum_k \widetilde{C}_k^j\sigma^k$, where $\{\sigma^j\}$ are as in (6.23). We claim:

$$(6.24)\qquad \begin{aligned}&\deg_*(\widetilde{C}_k^j) \le 8nk_0 + 11k_0 - 8n - 10,\\ &\deg_{\mathcal{Y}}(\widetilde{C}_k^j) \le 2(8nk_0 + 11k_0 - 8n - 10)(k_0)^2.\end{aligned}$$

By Lemmas 6.3–6.6, we have

$$(6.25)\qquad \begin{cases} dz^\alpha = \widetilde{C}_{\alpha k}\sigma^k \ \text{ with } \ \deg(\widetilde{C}_{\alpha k}) \le n+1,\\ dz^{n+1} = \widetilde{C}_{n+1\ k}\sigma^k \ \text{ with } \ \deg(\widetilde{C}_{n+1\ k}) \le n+k_0+1,\\ d\zeta_\alpha = \widetilde{C}_{\alpha k}\sigma^k \ \text{ with } \ \deg(\widetilde{C}_{\alpha k}) \le 4nk_0+3k_0-4n+1,\\ d\,u = \widetilde{C}_k\sigma^k \ \text{ with } \ \deg(\widetilde{C}_k) \le 2.\end{cases}$$

It remains to consider $du_\beta^\alpha, du^\alpha, dv_\alpha$ and dt. By (6.14),

$$du_\sigma^\beta = -u_\sigma^\alpha\phi_\alpha^{\beta**} + u_\sigma^\alpha\frac{iu^\beta}{u}\omega_\alpha - u_\sigma^\alpha\frac{iu^\beta v_\alpha}{u^2}\omega + u_\sigma^\alpha\Big[\frac{i\delta_\gamma^\beta v_\alpha}{u} + \frac{i\delta_\alpha^\beta v_\gamma}{u}$$
$$+ \frac{iv_\alpha^t p_t^{n+1}\delta_\gamma^\beta}{up^{n+1}} + \frac{iq_\rho^{-1t}u_t^\beta v_\alpha^\kappa}{u}\Big(\frac{iv_\gamma^\theta}{u}\frac{\partial q_\kappa^\rho}{\partial z^\theta} - \frac{iv_\mu v_\gamma^\mu}{ur_{n+1}}\frac{\partial q_\kappa^\rho}{\partial z^{n+1}}\Big)\Big]\omega^\gamma.$$

By (6.3), (6.5), (6.19), $-\frac{q_\rho^{-1t}u_t^\beta v_\alpha^\kappa}{u}\cdot\frac{v_\mu v_\gamma^\mu}{ur_{n+1}}\frac{\partial q_\kappa^\rho}{\partial z^{n+1}}$ is equal to the restriction on $\mathcal{Y}$ of a rational function

$$\frac{P}{Q} := -\frac{P_{3\rho}^t u_t^\beta u P_\alpha^\kappa v_\mu u P_\gamma^\mu}{uQ_3QuQr_{n+1}}\ \frac{\partial}{\partial z^{n+1}}\Big(\frac{P_{2\kappa}^\rho}{r^{n+1}(r_{n+1})^3}\Big)$$

where P, Q are polynomials of Y in $\mathbb{C}^{(n+2)^2}$ with degree $\deg(P), \deg(Q) \le 4nk_0 + 9k_0 - 3n - 9$. Then $\deg\Big(\frac{P}{Q}\Big) \le 4nk_0 + 9k_0 - 3n - 8$. If we write $du_\beta^\alpha = \widetilde{C}_{\beta k}^\alpha\sigma^k$, it is easy to check that any other $\widetilde{C}_{\beta k}^\alpha$ is the restriction of a rational function $\frac{P}{Q}$ with $\deg\left(\frac{P}{Q}\right) \le 4nk_0 + 8k_0 - 3n - 8$. Applying Lemma 2.1 (2),

$$\deg_*(\widetilde{C}^\alpha_{\beta k}) \le (4n+9)k_0 - 3n - 8. \tag{6.26}$$

By (6.15), we have

$$du^\alpha = -u\phi^{\alpha**} - u^\gamma\phi^{\alpha**}_\gamma - \frac{iu^\alpha u^\gamma v_\gamma}{u^2}\omega + u^\beta\Big[\frac{i\delta^\alpha_\gamma v_\beta}{u} + \frac{i\delta^\alpha_\beta v_\gamma}{u} + \frac{iv^t_\beta p^{n+1}_t\delta^\alpha_\gamma}{up^{n+1}} + \frac{iq^{-1t}_\rho u^\alpha_t v^\kappa_\beta}{u}\Big(\frac{iv^\theta_\gamma}{u}\frac{\partial q^\rho_\kappa}{\partial z^\theta} - \frac{iv_\mu v^\mu_\gamma}{ur_{n+1}}\frac{\partial q^\rho_\kappa}{\partial z^{n+1}}\Big)\Big]\omega^\gamma.$$

If we write $du^\alpha = C^\alpha_\gamma\sigma^\gamma$, we similarly get

$$\deg(C^\alpha_\gamma) \le (4n+9)k_0 - 3n - 8. \tag{6.27}$$

By (6.16),

$$\begin{aligned}
dv_\alpha = {} & -u\phi^{**}_\alpha + v_\gamma\phi^{\gamma**}_\alpha - v_\alpha\phi + \frac{iv_\alpha v_\kappa}{u}\omega^\kappa \\
& + \frac{iu^\kappa_\beta v_\kappa}{u}\Big\{\frac{iv_\psi v^\beta_\alpha}{u} - v^\beta_\gamma\Big[\frac{i\delta^\gamma_\psi v_\alpha}{u} - \frac{i\delta^\gamma_\alpha v_\psi}{u} - \frac{iv^t_\alpha p^{n+1}_t\delta^\gamma_\psi}{up^{n+1}} \\
& - \frac{iq^{-1t}_\rho u^\gamma_t v^\kappa_\alpha}{u}\Big(\frac{iv^\theta_\psi}{u}\frac{\partial q^\rho_\kappa}{\partial z^\theta} + \frac{iv_\mu v^\mu_\psi}{ur_{n+1}}\frac{\partial q^\rho_\kappa}{\partial z^{n+1}}\Big)\Big]\Big\}\omega^\psi \\
& - v^\beta_\alpha\Big[B_1 - \frac{iv_\theta p^{n+1}_\beta}{up^{n+1}} - \frac{iB_2 q^{-1\psi}_\gamma u^\mu_\psi}{u}\Big(v_\mu + \frac{v^\rho_\mu p^{n+1}_\rho}{p^{n+1}}\Big)\Big]\omega^\theta \\
& - \frac{iv_\alpha u^\kappa}{u}\omega_\kappa + \Big[tuv^\beta_\alpha + \frac{iv^\beta_\gamma u^\gamma v_\alpha}{u}\Big]\frac{iu^\kappa_\beta}{u}\omega_\kappa + \frac{iu^\kappa_\beta v_\kappa}{u}\Big[-\frac{iu^\gamma v^\beta_\alpha}{u} - \frac{iv^\gamma_\psi u^\psi}{u}\Big]\omega_\gamma \\
& - v^\beta_\alpha\Big[B_3 + \frac{iu^\theta p^{n+1}_\beta}{p^{n+1}u} - iB_4 q^{-1\psi}_\gamma u^\theta_\psi - \frac{iB_5 q^{-1\psi}_\gamma u^\mu_\psi}{u^2}\Big(v_\mu + \frac{v^\rho_\mu p^{n+1}_\rho}{p^{n+1}}\Big)\Big]\omega_\theta \\
& + \frac{iu^\beta v_\alpha}{u}\omega_\beta + tu\omega_\alpha.
\end{aligned}$$

Write $dv_\alpha = \widetilde{C}_{\alpha\gamma}\sigma^\gamma$. As we did for (6.26), from the definition of B_j in (6.7) it follows that

$$\deg_*(\widetilde{C}_{\alpha\gamma}) \le 8nk_0 + 11k_0 - 8n - 10. \tag{6.28}$$

By (6.17), $dt = -\psi^{**} + t\phi - \frac{iv_\alpha t}{u}\omega^\alpha + \frac{iv_\alpha}{u}\phi^{\alpha**} - \frac{u^\alpha u^\gamma v_\gamma}{u^3}\omega_\alpha + \frac{iu^\alpha}{u}\omega_\alpha - \frac{iu^\alpha}{u}\phi^{**}_\alpha$. If we write $dt = \widetilde{C}_\gamma\sigma^\gamma$, then

$$\deg_*(\widetilde{C}_\gamma) \le 4. \tag{6.29}$$

Therefore from (6.25) to (6.31), Claim (6.24) follows.

With (6.24) and (6.22), by Lemma 6.2, we see that $\deg_*\Big(-\frac{iv_\alpha^\gamma}{u}\Big)|_{\omega^\theta}$, $\deg_*\Big(\frac{iu^\beta}{u}\Big)|_{\omega_\theta}$, and $\deg_*\bigg(\Big[\frac{i\delta_\gamma^\beta v_\alpha}{u}+\frac{i\delta_\alpha^\beta v_\gamma}{u}+\frac{iv_\alpha^t p_t^{n+1}\delta_\gamma^\beta}{up^{n+1}}+\frac{iq_\rho^{-1t}u_t^\beta v_\alpha^\kappa}{u}\Big(\frac{iv_\gamma^\theta}{u}\frac{\partial q_\kappa^\rho}{\partial z^\theta}-\frac{iv_\mu v_\gamma^\mu}{ur_{n+1}}\frac{\partial q_\kappa^\rho}{\partial z^{n+1}}\Big)\Big]|_{\omega_\chi}\bigg)$ are less than $16nk_0+30k_0-14n-38$. Hence $\deg_*(S_{\alpha\chi}^{\beta\sigma})$, $\deg_*(b_\alpha^\beta)\le 16nk_0+30k_0-14n-38$. The last inequality in Lemma 6.7 follows from (6.1). □

With a similar calculation as in the proof of Lemma 6.7, we also obtain the following:

Lemma 6.8. — *We have*

$$\begin{aligned}
du^\alpha &= -u\phi^\alpha - u^\gamma\phi_\gamma^\alpha - \Big(\frac{iu^\alpha u^\gamma v_\gamma}{u^2} - uc^\alpha - ub_\alpha^\beta\Big)\omega \\
&\quad + \Big\{ub_\gamma^\alpha + u^\beta\Big[\frac{i\delta_\gamma^\alpha v_\beta}{u} + \frac{i\delta_\beta^\alpha v_\gamma}{u} + \frac{iv_\beta^t p_t^{n+1}\delta_\gamma^\alpha}{up^{n+1}} + \frac{iq_\rho^{-1t}u_t^\alpha v_\beta^\kappa}{u} \\
&\quad \cdot\Big(\frac{iv_\gamma^\theta}{u}\frac{\partial q_\kappa^\rho}{\partial z^\theta} - \frac{iv_\mu v_\gamma^\mu}{ur_{n+1}}\frac{\partial q_\kappa^\rho}{\partial z^{n+1}}\Big)\Big]\Big\}\omega^\gamma. \\
dv_\alpha &= -u\phi_\alpha + v_\gamma\phi_\alpha^\gamma - v_\alpha\phi + \frac{iv_\alpha v_\kappa}{u}\omega^\kappa \\
&\quad + \frac{iu_\beta^\kappa v_\kappa}{u}\Big\{\frac{iv_\psi v_\alpha^\beta}{u} - v_\gamma^\beta\Big[\frac{i\delta_\psi^\gamma v_\alpha}{u} + \frac{i\delta_\alpha^\gamma v_\psi}{u} + \frac{iv_\alpha^t p_t^{n+1}\delta_\psi^\gamma}{up^{n+1}} \\
&\quad + \frac{iq_\rho^{-1t}u_t^\gamma v_\alpha^\kappa}{u}\Big(\frac{iv_\psi^\theta}{u}\frac{\partial q_\kappa^\rho}{\partial z^\theta} + \frac{iv_\mu v_\psi^\mu}{ur_{n+1}}\frac{\partial q_\kappa^\rho}{\partial z^{n+1}}\Big)\Big]\Big\}\omega^\psi \\
&\quad - uv_\alpha^\beta\Big[\frac{1}{u}\Big(\frac{p_\beta^{n+1}}{p^{n+1}}\Big)_{|\omega^\theta} - \frac{iv_\theta p_\beta^{n+1}}{u^2p^{n+1}} - \frac{i(q_\beta^\gamma)_{|\omega^\theta}q_\gamma^{-1\psi}u_\psi^\mu}{u^2}\Big(v_\mu + \frac{v_\mu^\rho p_\rho^{n+1}}{p^{n+1}}\Big)\Big]\omega^\theta \\
&\quad - \frac{iv_\alpha u^\kappa}{u}\omega_\kappa + \Big[tv_\alpha^\beta + \frac{iv_\gamma^\beta u^\gamma v_\alpha}{u^2}\Big]iu_\beta^\kappa\omega_\kappa + \frac{iu_\beta^\kappa v_\kappa}{u}\Big[-\frac{iu^\gamma v_\alpha^\beta}{u} - \frac{iv_\psi^\gamma u^\psi}{u}\Big]\omega_\gamma \\
&\quad - uv_\alpha^\beta\Big[\frac{1}{u}\Big(\frac{p_\beta^{n+1}}{p^{n+1}}\Big)_{|\omega_\theta} + \frac{iu^\theta p_\beta^{n+1}}{p^{n+1}u^2} - \frac{i(q_\beta^\gamma)_{|\omega_\theta}q_\gamma^{-1\psi}u_\psi^\theta}{u} - \frac{i(q_\beta^\gamma)_{|\omega_\theta}q_\gamma^{-1\psi}u_\psi^\mu}{u^2} \\
&\quad \cdot\Big(v_\mu + \frac{v_\mu^\rho p_\rho^{n+1}}{p^{n+1}}\Big)\Big]\omega_\theta + \frac{iu^\beta v_\alpha}{u}\omega_\beta + tu\omega_\alpha - ub_\alpha^\beta\omega_\beta + ud_\alpha\omega - v_\gamma b_\alpha^\gamma\omega.
\end{aligned}$$

Write the above as $du^\alpha = C_j^\alpha\sigma^j$ *and* $dv_\alpha = C_{\alpha j}\sigma^j$, *where* $\{\sigma^j\}=\{\omega,\omega^\gamma,\omega_\gamma,\phi^\alpha,\phi_\gamma^\alpha,\phi_\alpha\}$. *Then*

$$\deg_*(C_j^\alpha),\ \deg_*(C_{\alpha j}),\ \deg_*(c^\alpha),\ \deg_*(d_\alpha),\ \deg_*(R_{\beta\gamma}^\alpha),$$

$$\deg_*(T_\alpha^{\beta\gamma}) \le 3(16nk_0+31k_0-14n-38),\ \deg_{\mathcal{Y}}(C_j^\alpha),\ \deg_{\mathcal{Y}}(C_{\alpha j}),\ \deg_{\mathcal{Y}}(c^\alpha),$$

$$\deg_{\mathcal{Y}}(d_\alpha),\ \deg_{\mathcal{Y}}(R_{\beta\gamma}^\alpha),\ \deg_{\mathcal{Y}}(T_\alpha^{\beta\gamma}) \le 6(16nk_0+31k_0-14n-38)(k_0)^2.$$

LEMMA 6.9. — *We have*

$$dt = -\psi + \frac{iv_\alpha}{u}\phi^\alpha - \frac{iu^\alpha}{u}\phi_\alpha + \Big(t\phi - \frac{iv_\alpha t}{u} - \frac{iv_\beta b^\beta_\alpha}{u} + id_\alpha\Big)\omega^\alpha$$
$$+ \Big(-\frac{u^\alpha u^\gamma v_\gamma}{u^3} + \frac{iu^\alpha}{u} - ic^\alpha - \frac{iu^\beta b^\alpha_\beta}{u}\Big)\omega_\alpha + \Big(e - \frac{iv_\alpha c^\alpha}{u} + \frac{iu^\alpha}{u}d_\alpha\Big)\omega.$$

Write $dt = \sum C_k\sigma^k$, *where* $\{\sigma^k\} = \{\omega, \omega^\alpha, \omega_\alpha, \phi, \phi^\alpha_\beta, \phi^\alpha, \phi_\alpha, \psi\}$. *Then*

$$\deg_*(C_k),\ \deg_*(Q^\beta_\alpha) \le 9(16nk_0 + 32k_0 - 14n - 38),$$
$$\deg_{\mathcal{Y}}(C_k),\ \deg_{\mathcal{Y}}(Q^\beta_\alpha) \le 18(16nk_0 + 32k_0 - 14n - 38)(k_0)^2.$$

COROLLARY 6.10. — *Let* $(dZ^j) := (dz^\alpha, dz^{n+1}, d\zeta_\alpha, du,\ du^\alpha_\beta, du^\alpha, dv_\alpha, dt)$ *and* $(\sigma^j) := (\omega, \omega^\alpha,\ \omega_\alpha, \phi, \phi^\alpha_\beta, \phi^\alpha, \phi_\alpha, \psi)$ *be two coframes on* $\mathcal{Y}^{n+1}_{n+1}$. *Write* $dZ^j = C^j_k\sigma^k$. *Then*

$$\deg_*(C^j_k) \le 9(16nk_0 + 32k_0 - 14n - 38).$$

LEMMA 6.11. — *On* $\mathcal{Y}$ *we have*

$$\deg_*(S^{\beta\sigma})_{\alpha\gamma} \le 16nk_0 + 30k_0 - 14n - 38,$$
$$\deg_*(R^\alpha_{\beta\gamma}),\ \deg(T^{\alpha\gamma}_\alpha) \le 3(16nk_0 + 31k_0 - 14n - 38),$$
$$\deg_*(Q^\beta_\alpha), \deg(L^{\alpha\beta}) \le 9(16nk_0 + 32k_0 - 14n - 38),$$
$$\deg_*(P_{\alpha\beta}) \le 27(16nk_0 + 33k_0 - 14n - 38),\ \deg(H_\alpha),$$
$$\deg_*(K^\alpha) \le 81(16nk_0 + 34k_0 - 14n - 38).$$

Proof of Lemma 6.11. — The first five inequalities follow from Lemmas 6.6, 6.8 and 6.9. To study $P_{\alpha\beta}$, notice that $\Phi_\alpha = R^\beta_{\alpha\gamma}\omega^\gamma \wedge \omega_\beta + P_{\alpha\beta}\omega \wedge \omega^\beta - \frac{i}{2}Q^\beta_\alpha\omega \wedge \omega_\beta$, where $\Phi_\alpha = d\phi_\alpha - \phi^\beta_\alpha \wedge \phi_\beta + \frac{1}{2}\psi \wedge \omega_\alpha$. Then, by Corollary 6.10 and by the same argument as in Lemma 6.9, we find $\deg_*(P_{\alpha\beta}) \le 27(16n+32)k_0$. To study H_α and K^α, we consider the formula: $\Psi = Q^\beta_\alpha\omega^\alpha \wedge \omega_\beta + H_\alpha\omega \wedge \omega^\alpha + K^\alpha\omega \wedge \omega_\alpha$, where $\Psi = d\psi - \phi \wedge \psi - 2i\phi^\alpha \wedge \phi_\alpha$. Similarly, we can get the desired estimates. □

Proof of Theorem 6.1. — From Lemma 6.11, for any $g \in \Gamma_1(\Omega, \mathcal{Y})$, we have $\deg_*(g) \le T_1 := 81(16nk_0 + 34k_0 - 14n - 38)$. By Lemma 6.2, for any $g \in \Gamma_2(\Omega, \mathcal{Y})$, we have $\deg_*(g) \le T_2 := 3^5(16nk_0 + 35k_0 - 14n - 38)$. By induction on k, for any $g \in \Gamma_k(\Omega, \mathcal{Y})$, we have $\deg_*(g) \le T_k := 3^{3+k}[16nk_0 + (33 + k)k_0 - 14n - 38]$. Assume $j_0 + 1 = \dim \mathcal{Y}$. When

$k = j_0 + 1 = (n+2)^2$, for any $g \in \Gamma_{j_0+1}(\Omega, \mathcal{Y})$, $\deg_*(g) \leq T_{j_0+1} := 3^{3+(n+2)^2}\Big[16nk_0 + (33 + (n+2)^2)k_0 - 14n - 38\Big]$. By (6.1), $\deg_{\mathcal{Y}}(g) \leq 2 \cdot 3^{3+(n+2)^2}\Big[16nk_0 + (33+(n+2)^2)k_0 - 14n - 38\Big](k_0)^2$. Then by (3.18),

$$l_0 \leq [(n+2)^2 - 1]\Big[54 \cdot 3^{(n+2)^2}(n^2 + 20n + 37)k_0^3\Big]^{2+3\cdot 5^{(n+2)^2-1}}.$$

Therefore, $\forall g \in \Gamma_{2l_0^3}(\Omega, \mathcal{Y})$, $\deg(g) \leq T_{2l_0^3}$. The proof of Theorem 6.1 is complete. □

Proof of Theorem 0.1. — It remains only to prove Theorem 0.1 (iv). We assume that ∂D is simply connected and algebraic.

By (5.5), each $\mathcal{V}_j^i$ is sub-algebraic. To find a bound for the sum of the Betti numbers of the smooth manifold $\mathcal{V}$, by an inductive use of the Mayer-Vietoris sequence [BT], it suffices to bound the Betti numbers of $\mathcal{V}_j^j := \mathcal{V} \cap \widetilde{Y}_j^j \subset \widetilde{Y}_j^j$ and those of their intersections. Making use of (5.5), Theorem 6.1, Proposition 1.3 and the Mayer-Vietoris sequence [BT, p. 22], it can be easily seen that an upper bound C_{n,k_0} of the sum of the Betti numbers of $\mathcal{V}$ can be taken as the one in Theorem 0.1 (iv). □

BIBLIOGRAPHY

[BER1] M.S. Baouendi, P. Ebenfelt and L.P. Rothschild, Parametrization of local biholomorphisms of real analytic hypersurfaces, Asian J. Math., Vol 1 (1997), 1–16.

[BER2] M. S. Baouendi, P. Ebenfelt and L. Rothschild, Real Submanifolds in Complex Spaces and Their Mappings, Princeton Univ. Mathematics Series 47, Princeton University, New Jersey, 1999.

[BER3] M.S. Baouendi, P. Ebenfelt and L.P. Rothschild, Local geometric properties of real submanifolds in complex spaces, Bull. AMS, 37 (2000), 309–336.

[Be] S. Bell, Compactness of families of holomorphic mappings up to the boundary, Lecture Notes in Math., 1268 (S. Krantz Ed.), Springer-Verlag, 29–43.

[Bo] S. Bochner, Analytic and meromorphic continuation by means of Green's formula, Ann. of Math., 44 (1943), 652–673.

[BT] R. Bott and L. W. Tu, Differential Forms in Algebraic Topology, Graduate Texts in Mathematics, Springer-Verlag, 1982.

[BS] D. Jr Burns and S. Shnider, Projective connections in CR geometry, Manuscripta Math., 33 (1980), 1–26.

[Ch] S.-S. Chern, On the projective structure of a real hypersurface in C_{n+1}, Math. Scand., 36 (1975), 74–82.

[CJ1] S.-S. Chern and S. Ji, Projective geometry and Riemann's mapping problem, Math Ann., 302 (1995), 581–600.

[CJ2] S.-S. Chern and S. Ji, On the Riemann mapping theorem, Ann. of Math., 144 (1996), 421–439.

[CM] S. S. CHERN and J. K. MOSER, Real hypersurfaces in complex manifolds, Acta Math., 133 (1974), 219–271.

[ES] S. EILENBERG and N. STEENROD, Foundations of algebraic topology, Princeton Univ. Press, Princeton, N.J., 1952.

[Fa] J. FARAN, Segre families and real hypersurfaces, Invent. Math,. 60 (1980), 135–172.

[Ga] R. GARDNER, The method of equivalence and its applications, CBMS-NSF, regional conference series in applied mathematics, 1989.

[H1] X. HUANG, On the mapping problem for algebraic real hypersurfaces in the complex spaces of different dimensions, Ann. Inst. Fourier, Grenoble, 44-2 (1994), 433–463.

[H2] X. HUANG, Geometric Analysis in Several Complex Variables, Ph. D. Thesis, Washington University, August, 1994.

[H3] X. HUANG, On some problems in several complex variables and Cauchy-Riemann Geometry, Proceedings of ICCM (edited by L. Yang and S. T. Yau), AMS/IP Stud. Adv. Math., 20 (2001) 383–396.

[HJ] X. HUANG and S. JI, Global holomorphic extension of a local map and a Riemann mapping Theorem for algebraic domains, Math. Res. Lett., 5 (1998), 247–260.

[HJY] X. HUANG, S. JI and S.S.T. YAU, An example of real analytic strongly pseudoconvex hypersurface which is not holomorphically equivalent to any algebraic hypersurfaces, Ark. Mat., 39 (2001), 75–93.

[M] J. MILNOR, On the Betti numbers of real varieties. Proc. Amer. Math. Soc., 15 (1964), 275–280.

[Pi] S. PINCHUK, On holomorphic maps or real-analytic hypersurfaces, Mat. Sb., Nov. Ser., 105 (1978), 574–593.

[V] A.G. VITUSHKIN,, Holomorphic mappings and geometry of hypersurfaces, Encyclopaedia of Mathematical Sciences, Vol. 7, Several Complex Variables I, Springer-Verlag, Berlin (1985), 159–214.

[We] S.M. WEBSTER, On the mapping problem for algebraic real hypersurfaces, Invent. Math., 43 (1977), 53–68.

Manuscrit reçu le 3 décembre 2001,
accepté le 19 mars 2002.

Xiaojun HUANG,
Rutgers University
Department of Mathematics
New Brunswick, NJ 08903 (USA).
huangx@math.rutgers.edu
&
Shanyu JI,
Houston University
Department of Mathematics
Houston, TX 77204 (USA).
shanyuji@math.uh.edu