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SEMICLASSICS OF THE QUANTUM CURRENT IN VERY STRONG MAGNETIC FIELDS

by Søren FOURNAIS*

Contents.

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1. Introduction.

1.1. Motivation.

In this article we will study the magnetic response properties of a non-interacting electron gas in a strong, constant, magnetic field. The (local) magnetic moment $\vec{m}$ of the gas is defined as the variational derivative of the energy with respect to the magnetic field, i.e. it is the first order correction to the energy when the magnetic field is slightly perturbed.

In the Pauli Hamiltonian describing the electron gas, it is not the magnetic field $\vec{B}$ itself that appears but the magnetic vector potential $\vec{A}$; $\operatorname{curl} \vec{A} = \vec{B}$. Therefore, it is more convenient to calculate the current $\vec{j} = \delta E/\delta \vec{A}$ instead of $\vec{m} = \delta E/\delta \vec{B}$. From the results on the current, corresponding results on the magnetisation can be derived using that $\operatorname{curl} \vec{m} = \vec{j}$.

The current in strong magnetic fields has already been studied in a number of papers: In [Fou01a] the semiclassical limit of the current was calculated when the magnetic field strength μ and the semiclassical parameter h satisfied the condition that μh remains bounded above as h tends to zero. Furthermore, in [Fou99] the microlocal machinery of Ivrii [Ivr98] and Sobolev [Sob94] was applied to the problem and asymptotic formulae with good error estimates were obtained under conditions of smoothness of the electrostatic potential V. When the microlocal techniques are applied, the condition that μh is bounded can be replaced by the much weaker assumption that μh^{ζ} is bounded for some arbitrary constant ζ.

Though the microlocal techniques permit a much better control of the error terms and were (in the same paper) applied to potentials with a Coulomb singularity, some points are still unsatisfactory: The first is that the assumption that μh^{ζ} be bounded should be superfluous; the second is related to the connection between semiclassics and large atoms. Semiclassical problems of an electron gas in a strong magnetic field appear in the study of large atoms in strong magnetic fields. Here, the drawback of the microlocal approach is that the relevant electrostatic potential - the magnetic Thomas-Fermi potential V_{MTF} - coming from the reduction of the atomic problem to a one-particle problem, does not satisfy the smoothness properties required for the microlocal techniques to work.

It is the objective of the present work to calculate the current at large μh without any recourse to microlocal analysis. In doing so, we will solve the two problems mentioned above.

1.2. Statement of the results.

Let $V(x)$ be a (real-valued) function on $\mathbb{R}^3$. The (Pauli) Hamiltonian that we will work with is the following (where μ and h are positive parameters):

$$H = H(h, \mu, \vec{A}, V) = (-ih\nabla + \mu\vec{A})^2 - \mu h + V(x),$$

where $\vec{A} = (-x_2/2, x_1/2, 0)$ (and therefore $\vec{B} = \operatorname{curl}\vec{A} = (0,0,1)$). We will always work under conditions that assure that H is a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^3)$.

Notice that we have chosen for simplicity of notation to restrict the usual spin-dependent Pauli operator living on $L^2(\mathbb{R}^3, \mathbb{C}^2)$ to the spin-down subspace identified with $L^2(\mathbb{R}^3, \mathbb{C})$. Since the magnetic field is constant, the full Pauli operator splits into a direct sum of operators on the spin–down and spin–up subspaces and therefore there is no loss of generality.

The current operator is the following:

$$J(\vec{a}) = J(h, \mu, \vec{a}) = \vec{a} \cdot (-ih\nabla + \mu\vec{A}) + (-ih\nabla + \mu\vec{A}) \cdot \vec{a} - hb_3, \tag{1.1}$$

where $\vec{a} = (a_1, a_2, a_3) \in C_0^{\infty}(\mathbb{R}^3, \mathbb{R}^3)$ is any test vector potential, and $b_3 = \partial_1 a_2 - \partial_2 a_1$.

With this notation the *energy* of the electron gas is defined as

$$E = E(h, \mu, V) = \operatorname{tr}[H 1_{(-\infty,0]}(H)],$$

and the *current* $\vec{j}$ is defined (as a distribution) by

$$\int \vec{j} \cdot \vec{a} \, dx = \operatorname{tr}[J(\vec{a}) 1_{(-\infty,0]}(H)].$$

Let us first recall the semiclassical results on the energy. In [LSY94a] it was proved that for $\vec{A}$, as above, the following semiclassical formula holds uniformly in the magnetic field strength:

$$\lim_{h \to 0} (E(h,\mu,V)/E_{\text{scl}}(h,\mu,V)) = 1,$$

where

$$E_{\text{scl}}(h,\mu,V) = -\frac{\mu}{3\pi^2 h^2} \int \sum_{n=0}^{\infty} [V(x) + 2n\mu h]_-^{3/2} \, dx. \tag{1.2}$$

Here, we have written $[x]_- = \begin{cases} -x & x \leqslant 0 \\ 0 & x > 0 \end{cases}$.

Formally, $\vec{j} = \frac{\partial E}{\partial \vec{A}}$, and we get by formal differentiation of the expression (1.2) with respect to $\vec{A}$ (remembering that $\mu = \mu|\vec{B}| = \mu|\operatorname{curl} \vec{A}|$) the following formal expression for the semiclassical current:

$$\begin{aligned}\int \vec{j}_{\text{scl}} \cdot \vec{a} \, dx \\ = \frac{-1}{3\pi^2 h^2} \sum_{n=0}^{\infty} \int b_3(x) \Big([2nh\mu + V(x)]_-^{3/2} - 3nh\mu [2nh\mu + V(x)]_-^{1/2} \Big) dx.\end{aligned}$$

In particular, when $2\mu h \geqslant -\inf V$, we get

$$\int \vec{j}_{\text{scl}} \cdot \vec{a} \, dx = \frac{-1}{3\pi^2 h^2} \int b_3(x) [V(x)]_-^{3/2} dx.$$

It indeed turns out that this formal calculation gives the right result:

THEOREM 1.1. — *Suppose $V \in C_0^1(B(0,1))$, $\vec{a} = (a_1, a_2, 0) \in C_0^3(B(0,1))$. Then for all $\epsilon > 0$ there exist $h_0, \nu_0 > 0$ such that if $h < h_0$ and $\mu h > \nu_0$, then*

$$\frac{h^2}{\mu} \left| \operatorname{tr}[J(\mu\vec{a}) 1_{(-\infty,0]}(H)] - \frac{-\mu}{3\pi^2 h^2} \int b_3(x) [V(x)]_-^{3/2} dx \right| \leqslant \epsilon. \tag{1.3}$$

Remark 1.2 (A note on the type of limit). — The asymptotic problem that we study involves a simultaneous limit in the two parameters h (tending to zero) and μh (tending to infinity), i.e. it is the limit $h + (\mu h)^{-1} \to 0$. We will use the standard o, O-notation with respect to that limit. Thus a statement like

$$f(h, \mu) = o(h),$$

(as $\mu h \to \infty$ and $h \to 0$) means that *for all* $\epsilon > 0$ *there exist* $h_0, \nu_0 > 0$ *such that if* $h < h_0$ *and* $\mu h > \nu_0$, *then*

$$|f(h, \mu)/h| \leqslant \epsilon.$$

It is not yet known in complete generality that the parallel (to $\vec{B}$) current j_3 is smaller than the perpendicular current (j_1, j_2). However, we can prove that j_3 is small under certain symmetry assumptions:

THEOREM 1.3. — *We have the following two cases:*

1. *If V satisfies the conditions of Theorem 1.1 and furthermore satisfies the symmetry condition $V(x_1, x_2, x_3) = V(x_1, x_2, -x_3)$, then the conclusion of Theorem 1.1 remains true with $\vec{a} = (a_1, a_2, a_3) \in C_0^4(B(0,1))$.*

2. *If $V(x) = V(r_\perp, x_3)$, with $r_\perp = \sqrt{x_1^2 + x_2^2}$ and $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, then*

$$j_3 = 0.$$

Notice that in Theorem 1.3 part 2, j_3 vanishes; whereas in part 1, it only becomes of lower order than the perpendicular current. Part 1 has been proved in [Fou01a] and the argument will not be repeated here. We wish to point out, however, that one needs Theorem 1.1 in order to prove part 1. In that sense it is a corollary of Theorem 1.1. The proof of part 2 is elementary and will be given in Section 3 below. Furthermore, it is a symmetry argument and as such independent of any knowledge on the perpendicular current (j_1, j_2).

In applications the scalar potentials V under consideration will often not be of the type required by Theorem 1.1. The main application we have in mind is to large atoms in strong magnetic fields, where the mean field potential is known to have (among others) a Coulomb singularity at the origin. With similar (though a bit more technically involved) arguments as for the above one can get Theorem 1.4 below.

THEOREM 1.4 (Current in potentials with Coulomb singularities). *Let $V(x)$ satisfy the following conditions:*

- $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. *(This implies that H is self-adjoint.)*
- $[V]_- \in L^{3/2}(\mathbb{R}^3) \cap L^{5/2}(\mathbb{R}^3)$.
- $V(x) \geqslant \frac{-c}{|x|}$.
- $|x|^2 V(x) \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^3)$, *i.e.* $|x|^2 V(x)$ *and* $\nabla(|x|^2 V(x))$ *are locally bounded.*
- $V(x) = V(r_\perp, x_3)$, *with* $r_\perp = \sqrt{x_1^2 + x_2^2}$.

Let $\vec{a} \in C_0^3(\mathbb{R}^3, \mathbb{R}^3)$. *Then*

$$\frac{h^2}{\mu} \left| \operatorname{tr}[J(\mu\vec{a}) 1_{(-\infty,0]}(H)] - \frac{-\mu}{3\pi^2 h^2} \int b_3(x) [V(x)]_-^{3/2} dx \right| \to 0, \tag{1.4}$$

as $h \to 0$ and $\mu h \to \infty$.

Remark 1.5. — One can apply localisation techniques such as those described in [Sob95] and [Sob94] in order to analyse the situation with more than one Coulomb singularity. It is beyond the scope of the present paper to include this generalisation.

1.3. Results for MTF-theory.

In particular, Theorem 1.4 can be applied to the mean field potentials V_{MTF} and V_{STF} coming from the analysis of large atoms in terms of a Thomas-Fermi type theory. The precise definition of these potentials will be given in Section 2 below, where we will also prove that they both satisfy the assumption of Theorem 1.4.

The difference between V_{MTF} and V_{STF} is that in V_{MTF} all Landau bands are taken into account, whereas in V_{STF} one restricts to the lowest. Since $2\mu h$ is the distance in energy between the Landau bands one would expect the restriction to the lowest Landau band to be admissible in the limit where $\mu h \to \infty$. It was shown in [LSY94b] and [LSY94a] that this is correct for the highest order terms in the energy.

For the highest order term of the current we show that either of the two potentials can be used:

THEOREM 1.6 (Current in the MTF mean field potentials). — *Let $V_1(x) = V_{MTF}(x) = V_{MTF,(h,\mu)}(x)$ be the mean field potential from* *magnetic Thomas-Fermi theory, and let $V_2(x) = V_{STF}(x)$ be the mean field potential from STF-theory. Write*

$$H_j = (-ih\nabla + \mu\vec{A})^2 - \mu h + V_j(x).$$

Let $\vec{a} \in C_0^3(\mathbb{R}^3, \mathbb{R}^3)$, then we have for $j = 1, 2$:

$$\frac{h^2}{\mu}\left|\mathrm{tr}[J(\mu\vec{a})1_{(-\infty,0]}(H_j)] - \frac{-\mu}{3\pi^2h^2}\int b_3(x)[V_j(x)]_-^{3/2}dx\right| \to 0, \tag{1.5}$$

as $h \to 0$ and $\mu h \to \infty$. Furthermore, in the same limit

$$\left\{\int b_3(x)[V_{MTF}(x)]_-^{3/2}dx - \int b_3(x)[V_{STF}(x)]_-^{3/2}dx\right\} \to 0. \tag{1.6}$$

Remark 1.7. — For bounded μh, the semiclassical limit of the current in the MTF-potential was calculated in [Fou01a]. Thus, Theorem 1.6 together with that paper constitute a complete semiclassical analysis of the current in the mean field from magnetic Thomas-Fermi theory.

1.4. An outline of the paper.

The proof of part 2 of Theorem 1.3 is elementary and independent of the general arguments in the paper, so it is given in Section 3. The proofs of Theorems 1.1 and 1.4 will be the main objective of the paper. Both of these theorems depend on a fairly easy analysis carried through in Section 3 and an estimate on the number of electrons living in the second Landau band. It is only in the nature of these estimates that the proofs are different. The estimate for Theorem 1.1 is stated and proved in Section 6 and for Theorem 1.4 in Section 7. The proof of Theorem 1.4 depends on choosing a convenient gauge, and therefore it depends on Theorem 1.3.

Theorem 1.6, which is maybe the most interesting result of the paper, will follow from an analysis of MTF-theory in Section 2. There it will in particular be shown that V_{MTF} and V_{STF} satisfy the assumptions of Theorem 1.4.

1.5. Notation and preliminaries.

We will for shortness introduce the notation

$$p_{\vec{A}} = (-ih\nabla + \mu\vec{A}) = (p_{\vec{A},1}, p_{\vec{A},2}, p_{\vec{A},3}).$$

The kinetic energy operator $\hat{K}$ in the variables perpendicular to the magnetic field plays a crucial role in magnetic field problems. It is defined as

$$\hat{K} = p_{\vec{A},1}^2 + p_{\vec{A},2}^2 - \mu h. \tag{1.7}$$

We will think of $\hat{K}$ as an operator on $L^2(\mathbb{R}^3)$ (though it could equally well be defined on $L^2(\mathbb{R}^2)$). It is well known that the spectrum of $\hat{K}$ is a set of (infinitely degenerate) eigenvalues $\{0, 2\mu h, 4\mu h, ...\} = 2\mu h(\mathbb{Z}_+ \cup \{0\})$, the so-called Landau levels. We will often use the term "ν'th Landau level" both to describe the eigenvalue $2\nu\mu h$ and to describe the corresponding eigenspace of $\hat{K}$. The projections onto these Landau levels will be used repeatedly. We will now describe their explicit form (See [LSY94a, p. 95]): Let

$$\begin{aligned} &\Pi_\nu^{(2)}(x_\perp, y_\perp) \\ &\quad = \frac{\mu}{2\pi h} \exp\left\{ i(x_\perp \times y_\perp)\frac{\mu\vec{B}}{2h} - |x_\perp - y_\perp|^2 \frac{\mu}{4h} \right\} L_\nu\left(|x_\perp - y_\perp|^2 \frac{\mu}{2h}\right), \end{aligned} \tag{1.8}$$

where we have written $x \in \mathbb{R}^3$ as $(x_\perp, x_\parallel)$, with $x_\perp \perp \vec{B}$ and $x_\parallel \parallel \vec{B}$. In (1.8), L_ν are Laguerre polynomials normalized by $L_\nu(0) = 1$. The projection on the ν'th Landau level is now given as

$$\Pi_\nu = \Pi_\nu^{(2)} \otimes I,$$

where the tensor product refers to the decomposition

$$L^2(\mathbb{R}^3) = L^2(\mathbb{R}^2_\perp) \otimes L^2(\mathbb{R}^1_\parallel) = L^2(\mathbb{R}^2_{(x_1,x_2)}) \otimes L^2(\mathbb{R}^1_{x_3}).$$

All tensor products in this paper will refer to this decomposition.

The lowest Landau level plays a special role, so we will often use the decomposition $I = \Pi_0 + \Pi_>$, which defines $\Pi_> = \sum_{j=1}^\infty \Pi_j$.

We will also use the following raising and lowering operators[1]:

$$a = p_{\vec{A},1} - ip_{\vec{A},2}, \qquad a^* = p_{\vec{A},1} + ip_{\vec{A},2}. \tag{1.9}$$

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[1] Notice that we will reserve the notation a for the lowering operator. The test vector fields $\vec{a}$ in the definition of $J(\vec{a})$ (see (1.1)) will always have a vector arrow or a tilde (as in $\tilde{a}$).

Then

$$a^*a = \hat{K} \text{ and } [a, a^*] = 2\mu h.$$

It is therefore easy to see that a^* maps $\operatorname{Ran}\Pi_\nu$ to $\operatorname{Ran}\Pi_{\nu+1}$, i.e. raises the Landau level by one, and that a lowers it.

Define

$$H_0 = p_{\vec{A}}^2 - \mu h = \hat{K} - h^2\partial_{x_3}^2. \tag{1.10}$$

Then it is clear that

$$[\Pi_j, H_0] = 0 \text{ for all } j \in \mathbb{N} \cup \{0\}. \tag{1.11}$$

We will use a number of different norms: For *functions* V we will denote by $\|V\|_p$ (with $p \geqslant 1$) the L^p norm of V - with the exception that $\|V\|$ is the L^2 norm. We will also need norms of *operators*: $\|K\|$ denotes the operator (uniform) norm of the operator K, and $\|K\|_p$ ($p \geqslant 1$) the Schatten norm

$$\|K\|_p^p = \operatorname{tr}[|K|^p],$$

in particular $\|K\|_2$ is the Hilbert-Schmidt norm. It will always be clear from the context whether a given object is considered an operator or a function.

Finally, let us fix a number of standard notations: $\Re(z), \Im(z)$ denote the real and imaginary parts of the complex number (or the operator) z. $B(x,r)$ denotes the open ball of radius r around the point x. It will always be clear from the context in which space the ball is taken. Lastly, $D\vec{a}$ denotes the Jacobian of the function $\vec{a}$.

We will also use the standard custom of letting c or C denote arbitrary constants, the value of which may change from line to line or even within a line. We will in general not try to keep track of the numerical value of these constants.

2. The MTF potential.

In this section we will discuss the regularity and decay properties of the MTF potential. It is well known (see [Lie81]) that in usual Thomas Fermi theory (without magnetic field) the effective potential is a smooth function except for a Coulomb singularity at the origin. In MTF-theory this is unfortunately not true, which is one of the reasons why precise asymptotic formulae are difficult to obtain for large atoms in strong magnetic fields. The MTF-potential turns out not only to have a Coulomb singularity at the origin (at the nucleus) but also points of non-smoothness on an infinite number of "surfaces" tending towards the origin as μh tends to infinity.

The MTF-potential comes from the study of large atoms in strong magnetic fields. By means of correlation estimates one can (with a small error) reduce the study of a neutral atom with nuclear charge Z in the constant magnetic field B to a semiclassical problem in a mean field $V_{\rm MTF}$ and parameters μ, h given by

$$\begin{aligned} \beta &= B/Z^{4/3}, \\ \mu &= (B^2/Z)^{1/5} \text{ when } B \gg Z^{4/3}, \\ h &= (B/Z^3)^{1/5} \text{ when } B \gg Z^{4/3}, \\ \mu h &= (B^3/Z^4)^{1/5} = \beta^{3/5}. \end{aligned} \tag{2.1}$$

We will only discuss (scaled) MTF-theory in the case we are interested in - i.e. $\mu h \to \infty$, which corresponds to $\beta = B/Z^{4/3} \to \infty$. Furthermore, we will only describe the results we need for the proof of Theorem 1.6. For a general discussion of MTF-theory see [LSY94a].

The mean field $V_{\rm MTF}$ is found through the Thomas-Fermi equation (2.3) below. However, in order to state that equation we need to introduce some notation.

The Magnetic Thomas-Fermi (MTF) theory that correctly models the behaviour (to leading order) of the energy of atoms in strong magnetic fields is (after a scaling) given by the following functional:

$$\mathcal{E}_\beta^{\rm MTF}[\rho] = \int_{\mathbb{R}^3} \hat{\tau}_\beta(\rho(x))\,dx + \int_{\mathbb{R}^3} V(x)\rho(x)\,dx + D(\rho,\rho). \tag{2.2}$$

Here $V(x)$ is the Coulomb potential $V(x) = \frac{-1}{|x|}$, $D(\rho,\rho)$ is the Coulomb norm or direct Coulomb interaction

$$D(f,g) = \frac{1}{2}\int_{\mathbb{R}^6} \frac{f(x)g(y)}{|x-y|}\,dxdy,$$

and $\hat{\tau}_\beta$ is the kinetic energy density of an electron gas in a magnetic field, given as the Legendre transform of the (scaled) pressure

$$\hat{\tau}_\beta(t) = \sup_{w \geqslant 0}[tw - \hat{P}_\beta(w)],$$

with

$$\hat{P}_\beta(w) = \frac{1}{3\pi^2}\sum_{\nu=0}^{\infty}[2\nu\mu h - w]_-^{3/2} = \frac{1}{3\pi^2}\sum_{\nu=0}^{\infty}[2\nu\beta^{3/5} - w]_-^{3/2}.$$

Notice that the functional only depends on h, μ through the parameter β. For simplicity this will be the only parameter appearing in this section. One can restore the h, μ's using 2.1. The corresponding (scaled) MTF-energy is now given as[2]

$$E^{\mathrm{MTF}}(\beta) = \inf\{\mathcal{E}_\beta^{\mathrm{MTF}}[\rho] \mid \rho \in L^1(\mathbb{R}^3) \cap L^{5/3}(\mathbb{R}^3), \rho \geqslant 0, \int \rho \leqslant 1\}.$$

The (scaled) STF-functional comes from taking formally the limit $\beta = (\mu h)^{5/3} \to \infty$. This gives

$$\hat{P}_\infty(w) = \frac{1}{3\pi^2} w^{3/2},$$

and therefore

$$\hat{\tau}_\infty(t) = \frac{4\pi^4}{3} t^3.$$

Therefore,

$$\mathcal{E}^{\mathrm{STF}}[\rho] \equiv \mathcal{E}_\infty^{\mathrm{MTF}}[\rho] = \int_{\mathbb{R}^3} \hat{\tau}_\infty(\rho(x))\, dx + \int_{\mathbb{R}^3} V(x)\rho(x)\, dx + D(\rho, \rho).$$

Finally, the STF-energy is defined by (note that the domain is different than for MTF-theory)

$$E^{\mathrm{STF}} = \inf\{\mathcal{E}^{\mathrm{STF}}[\rho] \mid \rho \in L^1(\mathbb{R}^3) \cap L^3(\mathbb{R}^3), \rho \geqslant 0, \int \rho \leqslant 1\}.$$

In order not to have to notationally distinguish STF and MTF theory, let us define that $\mathcal{E}^{\mathrm{STF}} = \mathcal{E}_\infty^{\mathrm{MTF}}$.

[2] For simplicity we restrict attention to *neutral* atoms, so the scaled density has to integrate to (less than) one.

From [LSY94a] we get the following results on minimizers of the MTF-functional:

THEOREM 2.1 (Existence and properties of MTF-minimizers). — *For $\beta \in [1, \infty]$ we have*

• *The functional $\mathcal{E}_\beta^{\mathrm{MTF}}$ has a unique minimizer ρ_β.*

• *The minimizer ρ_β has support contained in a fixed (independent of β) compact set.*

• *The minimizer ρ_β satisfies the (scaled)* TF *equation:*

$$\rho_\beta = \hat{P}'_\beta([V_{\mathrm{MTF},\beta}]_-), \tag{2.3}$$

where the MTF*-potential (effective potential) is defined as*

$$V_{\mathrm{MTF},\beta}(x) = \frac{-1}{|x|} + \rho_\beta * |x|^{-1}. \tag{2.4}$$

• *The* TF *equation can be written in the following form, involving only $V_{\mathrm{MTF},\beta}$:*

$$-(4\pi)^{-1}\Delta V_{\mathrm{MTF},\beta}(x) = \delta_0(x) - \hat{P}'_\beta([V_{\mathrm{MTF},\beta}]_-). \tag{2.5}$$

• *If $\beta \to \beta_0 \in [1, \infty]$, then $\rho_\beta \rightharpoonup \rho_{\beta_0}$ weakly in $L^{5/3}_{\mathrm{loc}}(\mathbb{R}^3)$.*

From the above results we easily get

COROLLARY 2.2 (Properties of $V_{\mathrm{MTF},\beta}$).

(1) $V_{\mathrm{MTF},\beta}(x) \geqslant -|x|^{-1}$.

(2) *Let $\mathcal{O} \in SO(3)$, (i.e. $\mathcal{O}$ is an orthogonal matrix with determinant 1), then $V_{\mathrm{MTF},\beta}(\mathcal{O}x) = V_{\mathrm{MTF},\beta}(x)$.*

Proof. — The proof of 1 is simply the observation that ρ is positive and thus $\rho * |x|^{-1}$ as well. The symmetry property 2 follows from the uniqueness of the minimizer ρ_β and the symmetry of the functional $\mathcal{E}^{\mathrm{MTF}}$. □

Furthermore, we get continuity of $V_{\mathrm{MTF},\beta}$ in β:

LEMMA 2.3. — *Let $\beta_0 \in [1, \infty]$. Then $[V_{\mathrm{MTF},\beta}]_-$ tends to $[V_{\mathrm{MTF},\beta_0}]_-$ in $L^{3/2}_{\mathrm{loc}}(\mathbb{R}^3)$ as $\beta \to \beta_0$.*

Proof. — From (2.4) it is clear that $V_{\mathrm{MTF},\beta} \geqslant \frac{-1}{|x|}$. Therefore, using Lebesgue's theorem on dominated convergence, it is enough to prove that $V_{\mathrm{MTF},\beta}(x) \to V_{\mathrm{MTF},\beta_0}(x)$ pointwise.

Now,

$$\begin{aligned} V_{\mathrm{MTF},\beta}(x) - V_{\mathrm{MTF},\beta_0}(x) &= (\rho_\beta - \rho_{\beta_0}) * \frac{1}{|x|} \\ &= \int (\rho_\beta(y) - \rho_{\beta_0}(y)) \frac{1}{|x-y|}\, dy. \end{aligned}$$

From Theorem 2.1 we get that

$$(\rho_\beta(y) - \rho_{\beta_0}(y)) = (\rho_\beta(y) - \rho_{\beta_0}(y)) 1_{[0,R]}(|y|),$$

for some fixed (independent of β) $R > 0$. Furthermore, for fixed $x \in \mathbb{R}^3$, we have

$$1_{[0,R]}(|y|) \frac{1}{|x-y|} \in L^{5/2}(\mathbb{R}^3_y),$$

and — as a function of y — the function $1_{[0,R]}(|y|) \frac{1}{|x-y|}$ has compact support. Since, using Theorem 2.1 again, ρ_β is weakly convergent, this finishes the proof. □

For the analysis of the current it will be very important for us that the singularity of $V_{\mathrm{MTF},\beta}$ at the origin is essentially of Coulomb type ($\approx |x|^{-1}$). The precise thing that we need is that $|x|^2 V_{\mathrm{MTF},\beta}$ is differentiable. This we will prove next:

LEMMA 2.4.— *There exists $\beta_1 > 1$ such that if $\beta \geqslant \beta_1$, then*

$$|x|^2 V_{\mathrm{MTF},\beta} \in C^{0,1}(B(0,1)),$$

and furthermore,

$$\| |x|^2 V_{\mathrm{MTF},\beta} \|_{L^\infty(B(0,1))} + \| \nabla (|x|^2 V_{\mathrm{MTF},\beta}) \|_{L^\infty(B(0,1))}$$

is bounded uniformly for $\beta \in [\beta_1, +\infty]$.

Remark 2.5.— Notice that Lemma 2.3 and Lemma 2.4 make Theorem 1.6 a corollary of Theorem 1.4.

Proof.— Let us write

$$V_{\mathrm{MTF},\beta} = \frac{-1}{|x|} + \rho_\beta * |x|^{-1} = \frac{-1}{|x|} + V_{\mathrm{cont}}(x).$$

Now, general results (see for instance [LL97, Theorem 10.2]) give that $V_{\mathrm{cont}}(x)$ is continuous and bounded since $\rho_\beta \in L^1 \cap L^p$, with $p = 5/3$

or 3. We need to prove that $|x|^2 V_{\rm cont} \in C^{0,1}(\mathbb{R}^3)$ (since $|x|^2 \frac{-1}{|x|}$ is!). In order to do that we write the TF equation for ρ_β:

$$
\begin{aligned}
\rho_\beta &= \hat{P}'_\beta([-1/|x| + V_{\rm cont}(x)]_-) \\
&= \frac{1}{2\pi} \sum_{\nu=0}^{\infty} \Big[2\nu\beta^{3/5} - \frac{1}{|x|} + V_{\rm cont}(x)\Big]_-^{1/2} \\
&= \frac{1}{2\pi} \Big[\frac{-1}{|x|} + V_{\rm cont}(x)\Big]_-^{1/2} + \frac{1}{2\pi} \sum_{\nu=1}^{\infty} \Big[2\nu\beta^{3/5} - \frac{1}{|x|} + V_{\rm cont}(x)\Big]_-^{1/2} \\
&= \rho_1 + \rho_2.
\end{aligned}
$$

For $\beta = \infty$ we have $\rho_2 = 0$. Notice that ρ_2 and ρ_β have compact support, and therefore ρ_1 as well. Now, $\rho_1 \approx |x|^{-1/2}$ close to the origin, so $\rho_1 \in L^{3+\epsilon}$. Therefore if we define V_1 by $V_1 = \rho_1 * |x|^{-1}$, then (from [LL97, Theorem 10.2] again), we get that V_1 is $C^{0,1}$. So let us look at ρ_2:

$$
\rho_2 = \frac{1}{2\pi} \sum_{\nu=1}^{\infty} \Big[2\nu\beta^{3/5} - \frac{1}{|x|} + V_{\rm cont}(x)\Big]_-^{1/2}.
$$

We will approximate the sum by an integral, so let us denote $u = u(x) = 1 - |x|V_{\rm cont}(x)$, and write

$$
\begin{aligned}
\rho_2(x) &= \frac{1_{[0,1/(2\beta^{3/5})]}(\frac{|x|}{u})}{2\pi|x|^{3/2}2\beta^{3/5}} \\
&\quad \times \Big\{\int_0^u \sqrt{u-t}\,dt + \Big(\sum_{\nu=1}^{\infty} \Big[2\nu\beta^{3/5}|x| - u\Big]_-^{1/2} 2\beta^{3/5}|x| - \int_0^u \sqrt{u-t}\,dt\Big)\Big\} \\
&= \frac{1_{[0,1/(2\beta^{3/5})]}(\frac{|x|}{u})}{2\pi|x|^{3/2}2\beta^{3/5}} u^{3/2}\tfrac{2}{3} + \rho_4 \\
&= \rho_3 + \rho_4.
\end{aligned}
$$

Since $t \mapsto \sqrt{u-t}$ is decreasing (for positive t) we have

$$
\begin{aligned}
\sum_{\nu=1}^{\infty} \Big[2\nu\beta^{3/5}|x| - u\Big]_-^{1/2} 2\beta^{3/5}|x| &\leqslant \int_0^u \sqrt{u-t}\,dt \\
&\leqslant \sum_{\nu=0}^{\infty} \Big[2\nu\beta^{3/5}|x| - u\Big]_-^{1/2} 2\beta^{3/5}|x|,
\end{aligned}
$$

and therefore

$$
0 \leqslant \rho_4 \leqslant \frac{1_{[0,1/(2\beta^{3/5})]}(\frac{|x|}{u})}{2\pi|x|^{1/2}} \sqrt{u},
$$

which implies that ρ_4 is in $L^{3+\epsilon}(\mathbb{R}^3)$ uniformly in β. Thus (by [LL97, Theorem 10.2]) $\rho_4 * |x|^{-1} \in C^{0,1}$, so we only have to consider ρ_3.

It is clear that

$$\rho_3 - \frac{1_{[0,1/(2\beta^{3/5})]}\left(\frac{|x|}{u}\right)}{6\pi|x|^{3/2}\beta^{3/5}} \in L^{3+\epsilon}(\mathbb{R}^3),$$

and has compact support, so it is enough to look at

$$\tilde{\rho}_3 = \frac{1_{[0,1/(2\beta^{3/5})]}\left(\frac{|x|}{u}\right)}{6\pi|x|^{3/2}\beta^{3/5}}.$$

If we now choose $\phi \in C_0^\infty(\mathbb{R}^3)$ with $\phi \equiv 1$ on a nbh. of 0, then we can write

$$\tilde{V}_3 = \tilde{\rho}_3 * |x|^{-1} = \frac{c}{\beta^{3/5}}|x|^{1/2}\phi(x) + \tilde{V}_{3,\mathrm{reg}}.$$

We will argue that $\tilde{V}_{3,\mathrm{reg}} \in C^{0,1}(\mathbb{R}^3)$. We have

$$\begin{aligned}\frac{-1}{4\pi}\Delta\tilde{V}_{3,\mathrm{reg}} = \frac{c}{\beta^{3/5}}\Big(|x|^{1/2}\Delta\phi(x) + 2\nabla|x|^{1/2}\cdot\nabla\phi(x)\Big) \\ + \frac{\phi(x) - 1_{[0,1/(2\beta^{3/5})]}\left(\frac{|x|}{u}\right)}{6\pi|x|^{3/2}\beta^{3/5}}\end{aligned}$$

(if the constant c is chosen properly). Now the right hand side is (uniformly in β) in $L^{3+\epsilon}(\mathbb{R}^3)$, and we can apply [LL97, Theorem 10.2] a final time. □

3. The parallel current.

In this short section we will prove Theorem 1.3 part 2. The idea of the proof is to use the cylinder symmetry of V to prove that we may choose the eigenfunctions to be of the form $\psi(x) = \tilde{\psi}(r_\perp, x_3)e^{im\theta}$, where $\tilde{\psi}$ is real. Once we have obtained that, it is easy to see that the parallel current of such a function vanishes.

Proof. — Let $a_3 \in C_0(\mathbb{R}^3)$. We need to calculate

$$\mathrm{tr}[(a_3(-ih\partial_{x_3}) + (-ih\partial_{x_3})a_3)\,1_{(-\infty,0]}(H)] = 2\Re\left(\mathrm{tr}[a_3(-ih\partial_{x_3})1_{(-\infty,0]}(H)]\right).$$

Due to the cylinder symmetry of V we may choose the eigenfuntions of H to be also eigenfunctions of the angular momentum operator L_{x_3} or in other words, to be of the form

$$\psi(x) = \tilde{\psi}(r_\perp, x_3)e^{im\theta},$$

where $m \in \mathbb{Z}$ and $\tilde{\psi}(r_\perp, x_3)$ is an eigenfunction of

$$\tilde{H} = -h^2\left(\partial^2_{r_\perp} + \frac{1}{r_\perp}\partial_{r_\perp} - \frac{m^2}{r_\perp^2} + \partial^2_{x_3}\right) + \mu^2 r_\perp^2 - \mu h - 2\mu h m + V(r_\perp, x_3).$$

Now, $\tilde{H}$ commutes with complex conjugation, so we may choose the eigenfunctions $\tilde{\psi}(r_\perp, x_3)$ to be real. But if $\tilde{\psi}(r_\perp, x_3)$ is real, then it is easy to see that

$$\Re\left(\langle\tilde{\psi}(r_\perp, x_3)e^{im\theta}|a_3(-ih\partial_{x_3})|\tilde{\psi}(r_\perp, x_3)e^{im\theta}\rangle\right) = 0.$$

This finishes the proof. □

4. Preliminary analysis.

4.1. Commutator formulae.

Two commutator formulae will be very important in the argument. We will in this section repeatedly appeal to the fact that if ψ is an eigenfunction of the self-adjoint operator H, and A is any operator, then (under very general conditions on A, H):

$$\langle\psi; [H, A]\psi\rangle = 0. \tag{4.1}$$

This implies in particular that

$$\mathrm{tr}\left([H, A]1_{(-\infty,0]}(H)\right) = 0.$$

Remark 4.1. — When A, H are unbounded operators the correctness of the 'virial Theorem' i.e. (4.1) is a bit delicate (see for instance [GG99] for a discussion of this issue). However, standard methods can easily be applied to prove that the formal calculations below are justified.

The first commutator formula proves gauge invariance of the current in $\vec{a}$:

LEMMA 4.2 (Gauge invariance of the current). — *Let $\phi \in C^1_0(\mathbb{R}^3)$, then*

$$\mathrm{tr}[J(\nabla\phi)1_{(-\infty,0]}(H)] = 0,$$

where J is defined by (1.1).

Proof. — This follows by integration by parts or the following easily proved identity:

$$(ih)^{-1}[H,\phi] = J(\nabla\phi). \qquad \square$$

The second commutator formula is essentially a virial theorem for Schrödinger Hamiltonians in the presence of a magnetic field. This formula has previously appeared in [Fou01a] and [Fou01b].

LEMMA 4.3 (Magnetic virial theorem). — *Suppose that $\vec{a} = (a_1, a_2, 0) \in C_0^3(\mathbb{R}^3)$ and define $\tilde{a} = (-a_2, a_1, 0)$. Define furthermore $M = -(D\tilde{a} + (D\tilde{a})^t)$, then*

$$\mathrm{tr}[J(\mu\vec{a})1_{(-\infty,0]}(H)] = \mathrm{tr}[(J_{\mathrm{KIN}} + J_{\mathrm{DENS}})1_{(-\infty,0]}(H)],$$

where

$$J_{\mathrm{KIN}} = p_{\vec{A}} M p_{\vec{A}} - \mu h b_3,$$

and

$$J_{\mathrm{DENS}} = \tilde{a}\cdot\nabla V + \frac{1}{2}h^2\Delta \mathrm{div}\,\tilde{a}.$$

Remark 4.4. — Notice that the term $\mathrm{div}\,\tilde{a}$ is equal to $-b_3$.

Proof. — We will only give the main idea of the proof. For further details see [Fou01a]. The proof of this statement also reduces to the calculation of a commutator. This time the required formula is:

$$(-2ih)^{-1}[H, \tilde{a}\cdot p_{\vec{A}} + p_{\vec{A}}\cdot\tilde{a}] = \mu\vec{a}\cdot p_{\vec{A}} + p_{\vec{A}}\cdot\mu\vec{a} - (J_{\mathrm{KIN}} + J_{\mathrm{DENS}}).$$

The proof of this formula is just a calculation using, in particular, that $[p_{\vec{A},j}, p_{\vec{A},k}] = -i\mu h(\partial_j A_k - \partial_k A_j)$, and that $\sum_{k=1}^3(\partial_j A_k - \partial_k A_j)\tilde{a}_k = a_j$.$\square$

4.2. Known results.

In this subsection we will recall some results on semiclassics of the energy and density in a constant magnetic field. These are all taken from [LSY94a].

The result on the semiclassics of the energy in a constant magnetic field is:

THEOREM 4.5. — *Suppose* $[V]_- \in L^{3/2}(\mathbb{R}^3) \cap L^{5/2}(\mathbb{R}^3)$ *and let* $E(h,\mu,V)$ *and* $E_{\mathrm{scl}}(h,\mu,V)$ *be as given in Section 1. Then*

$$\lim_{h\to 0}\left(\frac{E(h,\mu,V)}{E_{\mathrm{scl}}(h,\mu,V)}\right) = 1,$$

uniformly in the magnetic field strength $\mu \in [1,\infty)$.

By the variational principle, one easily gets:

COROLLARY 4.6. — *Let us keep the assumptions from Theorem 4.5. Suppose* $\phi \in L^{5/2}(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$, *then*

$$\frac{h^2}{\mu}\operatorname{tr}[\phi 1_{(-\infty,0]}(H)] = \frac{1}{2\pi^2}\int \sum_{n=0}^{\infty}[2n\mu h + V(x)]_-^{1/2}\phi(x)\,dx + o(1),$$

and

$$\operatorname{tr}[p_{\vec{A},3}^2 1_{(-\infty,0]}(H)] = O(\mu/h^2),$$

as $h \to 0$. *Furthermore*

$$\frac{h^2}{\mu}\operatorname{tr}[\hat{K}1_{(-\infty,0]}(H)] = o(1),$$

as $h + (\mu h)^{-1} \to 0$.

Proof. — Suppose $\mu h \to \nu_0 \in (0,+\infty]$ as $h \to 0$. The variational principle implies that for all $t \in \mathbb{R}$,

$$\begin{aligned} t\operatorname{tr}[\phi 1_{(-\infty,0]}(H)] &= \operatorname{tr}[(H+t\phi)1_{(-\infty,0]}(H)] - \operatorname{tr}[H1_{(-\infty,0]}(H)] \\ &\geqslant E(h,\mu,V+t\phi) - E(h,\mu,V). \end{aligned}$$

Thus, for $t > 0$,

$$\liminf_{h\to 0}\frac{h^2}{\mu}\operatorname{tr}[\phi 1_{(-\infty,0]}(H)] \geqslant t^{-1}\lim \frac{h^2}{\mu}\left(E_{scl}(h,\mu,V+t\phi) - E_{scl}(h,\mu,V)\right).$$

Letting now $t \searrow 0$, we get the inequality,

$$\liminf_{h\to 0}\frac{h^2}{\mu}\operatorname{tr}[\phi 1_{(-\infty,0]}(H)] \geqslant \frac{1}{2\pi^2}\int \sum_{n=0}^{\infty}[2n\nu_0 + V(x)]_-^{1/2}\phi(x)\,dx.$$

If we let $t < 0$, $t \nearrow 0$ instead, we will get the opposite inequality. This proves the result for $\operatorname{tr}[\phi 1_{(-\infty,0]}(H)]$.

The same type of argument can be applied in the other two cases. □

5. Proof of Theorem 1.1 and Theorem 1.4.

5.1. Discussion and preliminaries.

We now pass to the proof of Theorem 1.1 and Theorem 1.4. The proofs of these two theorems are the same until the final step where we need to invoke an estimate on the number of electrons in the second Landau level. This final estimate is different in the two situations.

The first part of the proof is identical to the argument in [Fou01a]. First we apply the 'virial theorem' (Lemma 4.3). Thus we have transformed the question about the current to

$$\mathrm{tr}[J(\mu\vec{a})1_{(-\infty,0]}(H)] = \mathrm{tr}[(J_{\mathrm{KIN}} + J_{\mathrm{DENS}})1_{(-\infty,0]}(H)],$$

where the $\tilde{a}$ that appears in the definition of J_{KIN} and J_{DENS} is given as $(-a_2, a_1, 0)$.

Now Corollary 4.6 readily gives (by integration by parts) that

$$\mathrm{tr}[J_{\mathrm{DENS}}1_{(-\infty,0]}(H)] = \frac{-\mu}{3\pi^2h^2}\int b_3(x)[V(x)]_-^{3/2}dx + o(\mu/h^2).$$

Therefore the real task is to prove that

$$\mathrm{tr}[J_{\mathrm{KIN}}1_{(-\infty,0]}(H)] = o(\mu/h^2), \tag{5.1}$$

as $h \to 0$ and $\mu h \to \infty$.

A standard technique for obtaining such a result would be to write $H(t) = H + tJ_{\mathrm{KIN}}$ and then study $E(t) = \mathrm{tr}[H(t)1_{(-\infty,0]}(H(t))]$. The asymptotics of $\mathrm{tr}[J_{\mathrm{KIN}}1_{(-\infty,0]}(H)]$ should then be obtained using the same arguments (Feynman-Hellman) as applied for the density in [LSY94a]. However, this strategy will not work in the present case, since it can be shown that $E(t)$ is of order $(1+\mu h)\frac{\mu}{h^2}$ for $t \neq 0$, whereas $E(0)$ is only of order $\frac{\mu}{h^2}$. This is the reason why the work [Fou01a], which applied this strategy, had to be restricted to bounded μh.

The idea that we will apply below is that the main term of J_{KIN} is an operator that couples the lowest Landau level with the second. Formally, this is clear since the projections Π_j almost commute with functions: $[\Pi_j, \phi] \approx \sqrt{h/\mu}$. This "almost"is made precise in a number of auxiliary lemmas in Appendix A. Thus, if we can obtain a good bound on the number of electrons living in the second Landau level, we can prove the bound (5.1) directly. In this section we will reduce the proof of (5.1) to an estimate on the second Landau level ((5.10) below), the proof of which is technical and will occupy the rest of the paper.

Let us decompose J_{KIN} as follows. The matrix M can be written as

$$\begin{aligned} M &= -\big(D\tilde{a} + (D\tilde{a})^t\big) = M_t + N \\ &= \begin{pmatrix} b_3 & 0 & 0 \\ 0 & b_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} N_{11} & N_{12} & N_{13} \\ N_{12} & -N_{11} & N_{23} \\ N_{13} & N_{23} & 0 \end{pmatrix}. \end{aligned} \tag{5.2}$$

Here $b_3 = \partial_1 a_2 - \partial_2 a_1$ and

$$N = \begin{pmatrix} \partial_1 a_2 + \partial_2 a_1 & \partial_2 a_2 - \partial_1 a_1 & \partial_3 a_2 \\ \partial_2 a_2 - \partial_1 a_1 & -(\partial_1 a_2 + \partial_2 a_1) & -\partial_3 a_1 \\ \partial_3 a_2 & -\partial_3 a_1 & 0 \end{pmatrix}.$$

Notice that $\mathrm{tr}[N] = 0$.

Using this decomposition we write

$$J_{\mathrm{KIN}} = J_{\mathrm{KIN,diag}} + J_{\mathrm{KIN,off}}, \tag{5.3}$$

where

$$J_{\mathrm{KIN,diag}} = p_{\vec{A}} M_t p_{\vec{A}} - \mu h b_3,$$

and

$$J_{\mathrm{KIN,off}} = p_{\vec{A}} N p_{\vec{A}}.$$

The motivation for this decomposition is that $J_{\mathrm{KIN,diag}}$ almost respects the Landau levels (and vanishes on the lowest one!) whereas $J_{\mathrm{KIN,off}}$ (to highest order) couples the j'th and the $(j+2)$'th Landau levels. For $J_{\mathrm{KIN,diag}}$ we could apply the analysis from [Fou01a], but we will not do this, since another more direct approach works, which is more in the spirit of the present paper. The off-diagonal term $J_{\mathrm{KIN,off}}$ is the problematic term that makes the variational technique break down. But since it couples the lowest and the second Landau levels it will suffice to get a good bound on the density of states in the second Landau band, in order to estimate this term.

Remark 5.1.— Below we will often use the following identity:

$$\mathrm{tr}[O 1_{(-\infty,0]}(H)] = \mathrm{tr}[1_{(-\infty,0]}(H) O 1_{(-\infty,0]}(H)], \tag{5.4}$$

for different operators O. For bounded operators this can be seen as the well known cyclicity property of the trace, since $1_{(-\infty,0]}(H)^2 = 1_{(-\infty,0]}(H)$. In our case, O will often be unbounded — however, it will be H-bounded. By definition,

$$\operatorname{tr}[O1_{(-\infty,0]}(H)] = \sum_j \langle e_j, Oe_j\rangle, \tag{5.5}$$

where $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis for $\operatorname{Ran}1_{(-\infty,0]}(H)$. The equation (5.5) clearly implies (5.4).

5.2. Estimate on $J_{\mathrm{KIN,diag}}$.

LEMMA 5.2

$$\operatorname{tr}[J_{\mathrm{KIN,diag}}1_{(-\infty,0]}(H)] = o(\mu/h^2),$$

as $h \to 0$ and $\mu h \to \infty$.

Proof. — By writing

$$\begin{aligned} J_{\mathrm{KIN,diag}} &= (\Pi_0 + \Pi_>)J_{\mathrm{KIN,diag}}(\Pi_0 + \Pi_>) \\ &= \Pi_0 J_{\mathrm{KIN,diag}}\Pi_0 + \Pi_> J_{\mathrm{KIN,diag}}\Pi_> \\ &\quad + (\Pi_> J_{\mathrm{KIN,diag}}\Pi_0 + \Pi_0 J_{\mathrm{KIN,diag}}\Pi_>), \end{aligned}$$

we get a decomposition of $\operatorname{tr}[J_{\mathrm{KIN,diag}}1_{(-\infty,0]}(H)]$ into three terms that we will treat separately.

$\Pi_> J_{\mathrm{KIN,diag}}\Pi_>$:

In this term we replace the matrix M_t with its absolute value and can thereby estimate

$$\pm\operatorname{tr}[\Pi_> J_{\mathrm{KIN,diag}}\Pi_> 1_{(-\infty,0]}(H)] \leqslant c\operatorname{tr}[\hat{K}1_{(-\infty,0]}(H)] = o(\mu/h^2),$$

by the weak localisation to the lowest Landau level; Corrollary 4.6.

For the last two terms we introduce the raising and lowering operators from (1.9). We have $p_{\vec{A},1} = (a + a^*)/2$, $p_{\vec{A},2} = (a - a^*)/(2i)$. Thus an easy calculation gives

$$J_{\mathrm{KIN,diag}} = \frac{1}{2}(a^*b_3a + ab_3a^*) - \mu h b_3.$$

$\Pi_0 J_{\mathrm{KIN,diag}}\Pi_0$:

Let us remember that $a\Pi_0 = 0$. Then

$$\begin{aligned}\Pi_0 J_{\mathrm{KIN,diag}}\Pi_0 &= \Pi_0\left(\frac{1}{2}ab_3a^* - \mu h b_3\right)\Pi_0\\ &= \Pi_0\left(\frac{1}{2}\left([a,b_3]a^* + b_3[a,a^*]\right) - \mu h b_3\right)\Pi_0\\ &= \Pi_0\left(\frac{1}{2}[a,b_3]a^*\right)\Pi_0\\ &= \frac{1}{2}\Pi_0\left[[a,b_3],a^*\right]\Pi_0.\end{aligned}$$

The double commutator gives h^2 times a continuous, compactly supported function ϕ. Choose $f(x_3) \in C_0^\infty(\mathbb{R})$ such that $f\phi = \phi$. Then

$$\begin{aligned}&h^2\mathrm{tr}[\Pi_0\phi\Pi_0 1_{(-\infty,0]}(H)]\\ &\quad= h^2\mathrm{tr}[\left(e^{-|x_\perp|}f(x_3)\right)\left(e^{|x_\perp|}\Pi_0\phi\Pi_0 e^{|x_\perp|}\right)\left(e^{-|x_\perp|}f(x_3)\right)1_{(-\infty,0]}(H)]\\ &\quad\leqslant h^2\|e^{-|x_\perp|}f(x_3)1_{(-\infty,0]}(H)\|_2^2\|e^{|x_\perp|}\Pi_0\phi\Pi_0e^{|x_\perp|}\|.\end{aligned}$$

We now use Lemma A.6 and Corollary 4.6 to get an estimate of order $h^2O(\mu/h^2)$.

$\Pi_> J_{\mathrm{KIN,diag}}\Pi_0 + \Pi_0 J_{\mathrm{KIN,diag}}\Pi_>$:

We calculate:

$$\begin{aligned}&\mathrm{tr}[\Pi_0\left((a^*b_3a + ab_3a^*)/2 - \mu h b_3\right)\Pi_> 1_{(-\infty,0]}(H)]\\ &\quad= \mathrm{tr}[\Pi_0\left(aa^*b_3/2 + a[b_3;a^*]/2 - \mu h b_3\right)\Pi_> 1_{(-\infty,0]}(H)]\\ &\quad= \mathrm{tr}[\Pi_0\left(a^*ab_3/2 + a[b_3;a^*]/2 + ([a;a^*]b_3/2 - \mu h b_3)\right)\Pi_> 1_{(-\infty,0]}(H)]\\ &\quad= \frac{1}{2}\mathrm{tr}[\Pi_0 a[b_3,a^*]\Pi_> 1_{(-\infty,0]}(H)].\end{aligned}$$

Now, $[b_3,a^*] = h\phi$, where ϕ is a continuous compactly supported function. Choose $f(x_3) \in C_0^\infty(\mathbb{R})$ such that $f\phi = \phi$. Then

$$\Pi_0 a[b_3,a^*] = h(f(x_3)e^{-|x_\perp|})e^{|x_\perp|}\Pi_0 a\phi,$$

and we estimate

$$\begin{aligned}&\mathrm{tr}[\Pi_0 a[b_3,a^*]\Pi_> 1_{(-\infty,0]}(H)]\\ &\leqslant h\|1_{(-\infty,0]}(H)f(x_3)e^{-|x_\perp|}\|_2\|e^{|x_\perp|}\Pi_0 a\phi\|\,\|\Pi_> 1_{(-\infty,0]}(H)\|_2\\ &\leqslant ch\sqrt{\mathrm{tr}[f(x_3)^2e^{-2|x_\perp|}1_{(-\infty,0]}(H)]}\|e^{|x_\perp|}\Pi_0 a\phi\|\sqrt{1/(\mu h)\mathrm{tr}[\hat{K}1_{(-\infty,0]}(H)]}.\end{aligned}$$

We now use Lemma A.6 and Corollary 4.6 to get an estimate of order $ho(\mu/h^2)$. □

5.3. First estimates on $J_{\mathrm{KIN,off}}$.

One part of $J_{\mathrm{KIN,off}}$ is easy to handle – this is the part in the third row or column. So let us consider that separately. We write

$$N = \begin{pmatrix} N_{11} & N_{12} & 0 \\ N_{12} & -N_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & N_{13} \\ 0 & 0 & N_{23} \\ N_{13} & N_{23} & 0 \end{pmatrix} = N_2 + N_3, \tag{5.6}$$

and define J_2, J_3 as $J_k = p_{\vec{A}} N_k p_{\vec{A}}$.

Lemma 5.3.

$$\mathrm{tr}[J_3 1_{(-\infty,0]}(H)] = o(\mu/h^2),$$

as $h \to 0$ and $\mu h \to \infty$.

Proof. — Let us notice that $p_{\vec{A},3}$ commutes with the Π_j's. In terms of the raising and lowering operators J_3 becomes a sum of terms of the form $a\phi p_{\vec{A},3} + p_{\vec{A},3}\overline{\phi}a^*$ (here $\phi = (N_{13} - iN_{23})/2$) or $a^*\phi p_{\vec{A},3} + p_{\vec{A},3}\overline{\phi}a$ (here $\phi = (N_{13} + iN_{23})/2$).

Now,

$$\Pi_0 a\phi p_{\vec{A},3}\Pi_0 = \Pi_0[a,\phi]\Pi_0 p_{\vec{A},3} = h\Pi_0\tilde{\phi}\Pi_0 p_{\vec{A},3} = hf(x_3)e^{-|x_\perp|}e^{|x_\perp|}\Pi_0\tilde{\phi}\Pi_0 p_{\vec{A},3},$$

as in the previous proof. We estimate as above

$$\begin{aligned} &\Big|\mathrm{tr}[\Pi_0 a\phi p_{\vec{A},3}\Pi_0 1_{(-\infty,0]}(H)]\Big| \\ &\qquad \leqslant h\|1_{(-\infty,0]}(H)f(x_3)e^{-|x_\perp|}\|_2\|e^{|x_\perp|}\Pi_0\tilde{\phi}\Pi_0\|\,\|p_{\vec{A},3}1_{(-\infty,0]}(H)\|_2 \\ &\qquad = O(\mu/h). \end{aligned}$$

Furthermore

$$\begin{aligned} &\Big|\mathrm{tr}[\Pi_> a\phi p_{\vec{A},3}\Pi_> 1_{(-\infty,0]}(H)]\Big| \\ &\qquad \leqslant \|1_{(-\infty,0]}(H)\Pi_> a\phi\|_2\|p_{\vec{A},3}1_{(-\infty,0]}(H)\|_2 \\ &\qquad \leqslant \sqrt{\mathrm{tr}[\Pi_> a|\phi|^2 a^*\Pi_> 1_{(-\infty,0]}(H)]\mathrm{tr}[p_{\vec{A},3}^2 1_{(-\infty,0]}(H)]} \\ &\qquad \leqslant \sqrt{\mathrm{tr}[\hat{K}1_{(-\infty,0]}(H)]\mathrm{tr}[p_{\vec{A},3}^2 1_{(-\infty,0]}(H)]} \\ &\qquad = o(\mu/h^2). \end{aligned}$$

Finally,

$$\begin{aligned}\operatorname{tr}[\Pi_0 a\phi p_{\vec{A},3}\Pi_> 1_{(-\infty,0]}(H)] &= \operatorname{tr}[p_{\vec{A},3}\Pi_0 a\phi\Pi_> 1_{(-\infty,0]}(H)] \\ &\quad + \operatorname{tr}[\Pi_0 a[\phi, p_{\vec{A},3}]\Pi_> 1_{(-\infty,0]}(H)].\end{aligned}$$

The last term is easily estimated, and the first is estimated as

$$\begin{aligned}&\|1_{(-\infty,0]}(H)p_{\vec{A},3}\|_2\|\Pi_0 a\phi\Pi_> 1_{(-\infty,0]}(H)\|_2 \\ &\leqslant \sqrt{\operatorname{tr}[p_{\vec{A},3}^2 1_{(-\infty,0]}(H)]}\sqrt{\operatorname{tr}[\hat{K}1_{(-\infty,0]}(H)]} \\ &= o(\mu/h^2).\end{aligned}$$

□

5.4. Analysis of J_2.

So we have isolated J_2 as the problematic term. As will be seen in the proof below, this term has as its main component a coupling of the zeroeth Landau level to the second. So we need a very precise estimate on the number of electrons in the second Landau band. In the proof of Lemma 5.4 below we invoke such an estimate in order to finish the proof of Theorem 1.1 and Theorem 1.4.

LEMMA 5.4.

$$\operatorname{tr}[J_2 1_{(-\infty,0]}(H)] = o(\mu/h^2),$$

as $h \to 0$ and $\mu h \to \infty$.

Proof. — We write J_2 using the raising and lowering operators

$$\begin{aligned}J_2 &= p_{\vec{A}} \cdot N_2 p_{\vec{A}} \\ &= a\phi a + a^*\overline{\phi}a^* \\ &= aa\phi + \overline{\phi}a^*a^* + a[\phi, a] + [a^*, \overline{\phi}]a^*, \end{aligned}\tag{5.7}$$

where

$$\phi = (N_{11} + iN_{12})/2 = \frac{1}{2}\left(\partial_1 a_2 + \partial_2 a_1 + i(\partial_2 a_2 - \partial_1 a_1)\right).$$

Now, $\Pi_0 J_2 \Pi_0 = 0$, and $\pm\Pi_> J_2 \Pi_> \leqslant c\hat{K}$, so we only have to deal with

$$\Pi_0 J_2 \Pi_> + \Pi_> J_2 \Pi_0.$$

The commutator terms above are easily seen to give negligible contributions, by the methods applied generally in this section, so we will not consider those.

Let us define $\tilde{\Pi}_> = \Pi_> - \Pi_2$ and consider $\Pi_0 J_2 \tilde{\Pi}_>$ (the other term, $\tilde{\Pi}_> J_2 \Pi_0$, can be estimated analogously). Now,

$$\begin{aligned}\Pi_0 J_2 \tilde{\Pi}_> &= \Pi_0 aa\phi\tilde{\Pi}_> \\ &= \Pi_0 aa[\Pi_2, \phi]\tilde{\Pi}_>,\end{aligned}$$

since $\Pi_0 aa = \Pi_0 aa \Pi_2$, and $\Pi_2 \tilde{\Pi}_> = 0$. Let us choose $f \in C_0^\infty(\mathbb{R})$ such that

$$f(x_3)\phi(x) = \phi(x).$$

Then (since $f(x_3)$ commutes with a and Π_0):

$$\begin{aligned}&\operatorname{tr}[\Pi_0 aa\phi\tilde{\Pi}_> 1_{(-\infty,0]}(H)] \\ &\quad = \operatorname{tr}[\Big(1_{(-\infty,0]}(H)]e^{-|x_\perp|}f(x_3)\Big)\Big(e^{|x_\perp|}\Pi_0 aa[\Pi_2,\phi]\Big)\Big(\tilde{\Pi}_> 1_{(-\infty,0]}(H)\Big)] \\ &\quad \leqslant \|1_{(-\infty,0]}(H)]e^{-|x_\perp|}f(x_3)\|_2 \|e^{|x_\perp|}\Pi_0 aa[\Pi_2,\phi]\| \, \|\tilde{\Pi}_> 1_{(-\infty,0]}(H)\|_2.\end{aligned}$$

It is clear from Corollary 4.6 that

$$\begin{aligned}\|1_{(-\infty,0]}(H)]e^{-|x_\perp|}f(x_3)\|_2 &= \sqrt{\operatorname{tr}[f^2(x_3)e^{-2|x_\perp|}1_{(-\infty,0]}(H)]} \\ &= O(\sqrt{\mu/h^2}).\end{aligned}$$

From Lemma A.7 we get that

$$\|e^{|x_\perp|}\Pi_0 aa[\Pi_2,\phi]\| \leqslant C\mu h\sqrt{h/\mu}.$$

Finally, we see from Corollary 4.6 and cyclicity of trace that

$$\begin{aligned}\|\tilde{\Pi}_> 1_{(-\infty,0]}(H)\|_2 &= \sqrt{\operatorname{tr}[\tilde{\Pi}_> 1_{(-\infty,0]}(H)]} \\ &\leqslant \frac{1}{\sqrt{\mu h}}\sqrt{\operatorname{tr}[\hat{K}1_{(-\infty,0]}(H)]} \\ &= \frac{1}{\sqrt{\mu h}}o(\sqrt{\mu/h^2}).\end{aligned}$$

So we get

$$\operatorname{tr}[\Pi_0 aa\phi\tilde{\Pi}_> 1_{(-\infty,0]}(H)] = o(\sqrt{\mu h}\sqrt{h/\mu}\frac{\mu}{h^2}) = o(\mu/h). \tag{5.8}$$

Therefore, we are left with $\operatorname{tr}[\Pi_0 J_2 \Pi_2 1_{(-\infty,0]}(H)]$. Let us here choose $f(x_3)$ as above and $\tilde{\phi} \in C_0^\infty(\mathbb{R}^3)$ such that $\tilde{\phi}\phi = \phi$. Then we calculate as follows:

$$\begin{aligned}
\operatorname{tr}&[\Pi_0 J_2 \Pi_2 1_{(-\infty,0]}(H)] \\
&\approx \operatorname{tr}[\Pi_0 a a \phi \Pi_2 1_{(-\infty,0]}(H)] \\
&= \operatorname{tr}\Big[\Big(1_{(-\infty,0]}(H) f(x_3) e^{-|x_\perp|}\Big)\Big(e^{|x_\perp|}\Pi_0 a a \tilde{\phi}\Big)\left(\phi \Pi_2 1_{(-\infty,0]}(H)\right)\Big] \\
&\leqslant \|1_{(-\infty,0]}(H) f(x_3) e^{-|x_\perp|}\|_2 \|e^{|x_\perp|}\Pi_0 a a \tilde{\phi}\| \, \|\phi \Pi_2 1_{(-\infty,0]}(H)\|_2 \\
&\leqslant C\mu h \sqrt{\mu/h^2} \|\phi \Pi_2 1_{(-\infty,0]}(H)\|_2, \qquad (5.9)
\end{aligned}$$

where we used Lemma A.6 to estimate the operator norm.

Thus, in order to finish the proof, we need the estimate

$$\|\phi \Pi_2 1_{(-\infty,0]}(H)\|_2^2 = o\left(\frac{1}{(\mu h)^2}\frac{\mu}{h^2}\right). \qquad (5.10)$$

Under the conditions in Theorem 1.1 this (without the ϕ which can be estimated by $\|\phi\|_\infty$) is the result of Lemma A.6 below.

Under the conditions in Theorem 1.4, i.e. in the case where $V(x)$ has a Coulomb type singularity, we will apply Lemma 6.1 below. Notice that if

$$\phi(0, x_3) = \nabla_{x_\perp}\phi(0, x_3) = 0, \qquad (5.11)$$

then we can write $\phi(x) = \phi_1(x)\phi_2(x_\perp)$, with ϕ_1 bounded and $\phi_2 \in C_0^2(\mathbb{R}^2_{x_\perp})$, $\phi_2(0) = \nabla\phi_2(0) = 0$. Therefore, we only need to prove that it is enough to consider ϕ satisfying (5.11). This will be accomplished by a change of gauge. Remember from Lemma 4.2 that the current is gauge invariant in $\vec{a}$. Remember furthermore that

$$\phi(x) = \frac{1}{2}\left(\partial_1 a_2 + \partial_2 a_1 + i(\partial_2 a_2 - \partial_1 a_1)\right).$$

So changing $\vec{a} \mapsto \vec{a} + \nabla\Phi$ will change

$$\phi \mapsto \phi + \partial_{12}^2\Phi + i((\partial_2^2 - \partial_1^2)\Phi).$$

We choose

$$\begin{aligned}
\Phi(x) =& g(x_\perp)\Big(\left(\alpha_{11}(x_3)(x_2^2 - x_1^2) + \alpha_{12}(x_3)x_1x_2\right) \\
&+ \left(\alpha_{111}(x_3)x_1^3 + \alpha_{112}(x_3)x_1^2x_2 + \alpha_{122}(x_3)x_1x_2^2 + \alpha_{222}(x_3)x_2^3\right)\Big),
\end{aligned}$$

where $g(x_\perp) \equiv 1$ on a neighborhood of 0, and the α's are continuous functions of x_3 which can be matched against a Taylor expansion of ϕ in $x_\perp$. Thereby it is clear that we can assure that (5.11) is satisfied. Notice that the change of gauge will also affect a_3, but since (using Theorem 1.3) $j_3 = 0$ this is of no importance. □

6. Estimate on the second Landau level.

In this section we will prove a bound on the density of states living in the second Landau level.

LEMMA 6.1. — *Let $V \in C_0^1(B(1))$, and let $g \in C_0^\infty(\mathbb{R})$. Then there exist constants c, h_0 and ν_0 such that if $h < h_0$ and $\mu h > \nu_0$ then*

$$\operatorname{tr}[\Pi_2 g(H) \Pi_2] \leqslant c \frac{\mu}{h^2} (\mu h)^{-5/2}.$$

Proof. — For the proof we will use the following integral representation for a function of a selfadjoint operator H in terms of its resolvent, valid for all n and functions $g \in C_0^\infty(\mathbb{R})$ ([AdMBG91]):

$$\begin{aligned} g(H) = &\sum_{j=1}^{n-1} \frac{1}{\pi j!} \int_{\mathbb{R}} \partial^j g(\lambda) \Im[i^j (H - \lambda - i)^{-1}] \, d\lambda \\ &+ \frac{1}{\pi(n-1)!} \int_0^1 \tau^{n-1} \int_{\mathbb{R}} \partial^n g(\lambda) \Im[i^n (H - \lambda - i\tau)^{-1}] \, d\lambda \, d\tau. \end{aligned} \tag{6.1}$$

For brevity, we will during the proof just think of the right hand side as an integral transformation of the resolvent and abbreviate the above formula as

$$g(H) = \int (H - z)^{-1} \, d\mu_{g,n}(z).$$

Notice, that by integration by parts in (6.1), we get

$$\begin{aligned} -g'(H) = &\sum_{j=1}^{n-1} \frac{1}{\pi j!} \int_{\mathbb{R}} \partial^j g(\lambda) \Im[i^j (H - \lambda - i)^{-2}] \, d\lambda \\ &\frac{1}{\pi(n-1)!} \int_0^1 \tau^{n-1} \int_{\mathbb{R}} \partial^n g(\lambda) \Im[i^n (H - \lambda - i\tau)^{-2}] \, d\lambda \, d\tau. \\ = &\int (H - z)^{-2} \, d\mu_{g,n}(z). \end{aligned} \tag{6.2}$$

In the calculations below we will repeatedly use (1.11), which in particular implies that

$$\Pi_2(H_0 - z)^{-1} = \Pi_2(H_0 - z)^{-1}\Pi_2 = \Pi_2(\Pi_2 H_0 \Pi_2 - z)^{-1}. \tag{6.3}$$

Let us now pass to the main part of the proof: By linearity we have to study $\Pi_2(H-z)^{-1}\Pi_2$, where z is either $\lambda + i$ or $\lambda + i\tau$. By application of the resolvent identity $(H-z)^{-1} = (H_0 - z)^{-1} - (H-z)^{-1}V(H_0-z)^{-1}$ repeatedly, we get

$$\begin{aligned}
&\int \Pi_2(H-z)^{-1}\Pi_2 d\mu_{g,n}(z) \\
&= \int \Pi_2(H_0-z)^{-1}\Pi_2 d\mu_{g,n}(z) - \int \Pi_2(H_0-z)^{-1}V(H_0-z)^{-1}\Pi_2 d\mu_{g,n}(z) \\
&+ \int \Pi_2(H_0-z)^{-1}V(H_0-z)^{-1}V(H_0-z)^{-1}\Pi_2 d\mu_{g,n}(z) \\
&- \int \Pi_2(H_0-z)^{-1}V(H-z)^{-1}V(H_0-z)^{-1}V(H_0-z)^{-1}\Pi_2 d\mu_{g,n}(z)
\end{aligned} \tag{6.4}$$

which is to be understood as an identity of bounded operators, i.e. the integrals converge in the space of bounded operators.

The first term in (6.4) satisfies

$$\begin{aligned}
\int \Pi_2(H_0-z)^{-1}\Pi_2 d\mu_{g,n}(z) &= \int \Pi_2(\Pi_2 H_0 \Pi_2 - z)^{-1} d\mu_{g,n}(z) \\
&= \Pi_2 g(\Pi_2 H_0 \Pi_2) \\
&= 0,
\end{aligned}$$

since $\Pi_2 H_0 \Pi_2 = 4\mu h - h^2\partial_{x_3}^2 \geqslant 4\mu h$.

We will prove that the remaining terms are trace class and that the integrals converge in trace norm. This is easily seen using Lemma A.2. For instance, we bound the last integrand in (6.4) by

$$\|\Pi_2(H_0-z)^{-1}V\|_2^2 \, \|V\|_\infty^2 \, \frac{1}{|\Im z|^2}.$$

Suppose now we take the parameter n in (6.1) sufficiently big (bigger than 4), then the integrals converge in trace norm.

Thus $\Pi_2 g(H)\Pi_2$ is trace class and we have the following identity:

$$
\begin{aligned}
(6.5)\quad & \operatorname{tr}[\Pi_2 g(H)\Pi_2] \\
&= \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}V(H_0-z)^{-1}\Pi_2]d\mu_{g,n}(z) \\
&+ \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}V(H_0-z)^{-1}V(H_0-z)^{-1}\Pi_2]d\mu_{g,n}(z) \\
&+ \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}V(H-z)^{-1}V(H_0-z)^{-1}V(H_0-z)^{-1}\Pi_2]d\mu_{g,n}(z).
\end{aligned}
$$

Let us first look at the first term in (6.5). We get from (6.3) and cyclicity of trace

$$\operatorname{tr}[\Pi_2(H_0-z)^{-1}V(H_0-z)^{-1}\Pi_2] = \operatorname{tr}[V\Pi_2(H_0-z)^{-2}],$$

and by taking the integrals, we get $\operatorname{tr}[V\Pi_2 g'(\Pi_2 H_0)]$, (using the representation (6.2)) which vanishes for large μh. Thus the first term in (6.5) vanishes.

The last two terms in (6.5) will be the main terms. The first of these we write as

$$
\begin{aligned}
&\int \operatorname{tr}[\Pi_2(H_0-z)^{-1}V(H_0-z)^{-1}V(H_0-z)^{-1}\Pi_2]d\mu_{g,n}(z) \\
&\quad= \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}V\Pi_2(H_0-z)^{-1}V(H_0-z)^{-1}\Pi_2]d\mu_{g,n}(z) \\
&\quad+ \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}[\Pi_2,V](H_0-z)^{-1}\Pi_2 V(H_0-z)^{-1}\Pi_2]d\mu_{g,n}(z) \\
&\quad+ \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}V\Pi_2(H_0-z)^{-1}[V,\Pi_2](H_0-z)^{-1}\Pi_2]d\mu_{g,n}(z) \\
&\quad+ \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}[\Pi_2,V](H_0-z)^{-1}[V,\Pi_2](H_0-z)^{-1}\Pi_2]d\mu_{g,n}(z) \\
&\quad= E_1+E_2+E_3+E_4.
\end{aligned}
$$

We will use that when z varies over a compact set and $\mu h \to \infty$, then

$$(6.6)\qquad \|\Pi_2(H_0-z)^{-1}\| = \|\Pi_2 \otimes (-h^2\partial_{x_3}^2 + 4\mu h - z)^{-1}\| \leqslant c\frac{1}{\mu h}.$$

Thus we estimate the E_i's, using (6.6), Lemmas A.2 and A.4 and the fact that z varies over a compact set, as

$$|E_1| \leqslant \int \|\Pi_2(H_0 - z)^{-1}V\|_2^2 \|\Pi_2(H_0 - z)^{-1}\| \leqslant c\frac{\mu}{h^2}\frac{1}{(\mu h)^{5/2}},$$

$$|E_2|+|E_3| \leqslant c\int \|\Pi_2(H_0 - z)^{-1}[\Pi_2, V]\|_2 \|(H_0 - z)^{-1}\Pi_2 V\|_2 \|(H_0 - z)^{-1}\Pi_2\|$$

$$\leqslant c\frac{\mu}{h^2}\frac{1}{(\mu h)^{5/2}}\sqrt{h/\mu},$$

$$|E_4| \leqslant \int \|\Pi_2(H_0 - z)^{-1}[\Pi_2, V]\|_2^2 \|(H_0 - z)^{-1}\| \leqslant c\frac{\mu}{h^2}\frac{h}{\mu}\frac{1}{(\mu h)^{3/2}}.$$

Finally we need to estimate the last term in (6.5). This is done similarly

$$\int \mathrm{tr}[\Pi_2(H_0 - z)^{-1}V(H - z)^{-1}V(H_0 - z)^{-1}V(H_0 - z)^{-1}\Pi_2]d\mu_{g,n}(z)$$

$$= \int \mathrm{tr}[\Pi_2(H_0 - z)^{-1}V(H - z)^{-1}V(H_0 - z)^{-1}\Pi_2 V(H_0 - z)^{-1}\Pi_2]d\mu_{g,n}(z)$$

$$+ \int \mathrm{tr}[\Pi_2(H_0 - z)^{-1}V(H-z)^{-1}V(H_0 - z)^{-1}[V, \Pi_2](H_0 - z)^{-1}\Pi_2]d\mu_{g,n}(z).$$

Now we take the trace norm under the integral sign and estimate using (6.6) and Lemmas A.2 and A.4:

$$|\cdot| \leqslant \int \|\Pi_2(H_0 - z)^{-1}V\|_2^2 \|(H - z)^{-1}\| \|V\|_\infty \|(H_0 - z)^{-1}\Pi_2\|$$

$$+ \|\Pi_2(H_0 - z)^{-1}V\|_2 \|(H - z)^{-1}\| \|V(H_0 - z)^{-1}[V, \Pi_2]\|_2 \|(H_0 - z)^{-1}\Pi_2\|$$

$$\leqslant \frac{\mu}{h^2}\frac{1}{(\mu h)^{5/2}} + \frac{\mu}{h^2}\frac{1}{(\mu h)^{5/2}}\sqrt{h/\mu}. \qquad \square$$

7. An estimate on the second Landau level with a singular potential.

The aim of this section is to prove the following:

LEMMA 7.1. — *Let*

$$H = H_0 + V(x),$$

where $V(x) \geqslant \frac{-c}{|x|}$, *for some constant* $c > 0$ *and* $|x|^2 V(x) \in C^{0,1}_{\mathrm{loc}}$. *Suppose* $\phi \in C_0^2(\mathbb{R}^2)$, $\phi(0) = \nabla\phi(0) = 0$. *Then for all* $\epsilon > 0$ *there exist* h_0, ν_0 *(only* *depending on c, ϕ, $\||x|^2V(x)\|_{L^\infty(B(0,1))}$ and $\|\nabla(|x|^2V(x))\|_{L^\infty(B(0,1))}$) such that if $h < h_0$ and $\mu h > \nu_0$, then*

$$\operatorname{tr}[\phi(x_\perp)\Pi_2 1_{(-\infty,0]}(H)\Pi_2\phi(x_\perp)] \leqslant \epsilon\frac{\mu}{h^2}\frac{1}{(\mu h)^2}.$$

The idea of the proof is as follows: We write $g(H)$ (for a smooth g) in terms of the resolvent. Then we apply the resolvent identity as much as we can. Whenever Π_2 "hits" $(H_0 - z)^{-1}$ we win a power of $(\mu h)^{-1}$ (essentially). If we can get ϕ to multiply V, then the product ϕV becomes differentiable, so we can commute Π_2 through ϕV - since $[\Pi_2, \phi V] \approx \sqrt{h/\mu}$. This will result in more terms where Π_2 hits a resolvent and therefore in an improved estimate. We continue to play this game until we have collected enough powers of $(\mu h)^{-1}$ for our purpose.

Proof. — The proof of this lemma is fairly long, so we divide it into a number of steps.

Preliminaries:

Let $M(h) = -\inf \operatorname{Spec} H$. We want to have an idea of the size of $M(h)$. Now, since $V(x) \geqslant -c|x|^{-1}$, we can use scaling and the known results on the asymptotics of the groundstate energy of hydrogen in a strong magnetic field ([AHS78], [FW94]) to conclude that $M(h) \leqslant ch^{-2}(\max\{1, \log \mu h^3\})^2$. See Appendix B for details.

Let us now choose $g \in C_0^\infty([-M(h)-1, 1])$ such that $g(H)1_{(-\infty,0]}(H) = 1_{(-\infty,0]}(H)$ and such that $\partial^n g \leqslant c$ for all n and where the constant is independent of h, μ. We now have the obvious estimate

$$0 < \operatorname{tr}[\phi\Pi_2 1_{(-\infty,0]}(H)\Pi_2\phi] \leqslant \operatorname{tr}[\phi\Pi_2 g(H)\Pi_2\phi].$$

As in Section 6 we use the integral representation (6.1) for $g(H)$. We once again use the resolvent identity repeatedly and, since ϕ is only a function of $x_\perp$, the terms which of order 1 or 2 in $(H_0 - z)^{-1}$ vanish - just as in the earlier section. Thus we are left with

$$\begin{aligned}(7.1)\quad & \operatorname{tr}[\phi\Pi_2 g(H)\Pi_2\phi] \\ &= \int \operatorname{tr}[\phi\Pi_2(H_0 - z)^{-1}V(H - z)^{-1}V(H_0 - z)^{-1}\Pi_2\phi]\,d\mu_{g,n}.\end{aligned}$$

For shortness we will often leave out the measure $d\mu_{g,n}$ in the integrals below.

Estimation of integrals:

There will be a number of error terms in the calculations below. These terms will be estimated by taking the trace norm under the integral sign. The trace norms will always be estimated by expressions of the form

$$c(\mu, h)|\tau|^{-k}|4\mu h - \lambda|^{-l},$$

where $k = 0, 1$ and $l = 3/2, 5/2$ or $7/2$, and where $c(\mu, h)$ is an (important!) expression in μ, h that is independent of the integration variables τ, λ. It therefore easily follows from the integral representation (6.1) that we can estimate the integrals by

$$\begin{aligned} &c(\mu, h)C(g, n) \int_{-M(h)-1}^{1} |4\mu h - \lambda|^{-l} d\lambda \\ &= c(\mu, h)C(g, n)(\mu h)^{1-l} \int_{(-M(h)-1)/(\mu h)}^{1/(\mu h)} |4 - \nu|^{-l} d\nu \\ &\leqslant c(\mu, h)C \min\left\{ \frac{M(h)}{(\mu h)^l}, (\mu h)^{1-l} \right\} \\ &= c(\mu, h)(\mu h)^{1-l} \min\left\{ \frac{M(h)}{\mu h}, 1 \right\}. \end{aligned} \tag{7.2}$$

Commuting ϕ through Π_2:

In order to be able to commute Π_2 through V, we need to get ϕ to multiply V. Therefore we will first commute ϕ through Π_2. We write

$$r(z) = (-h^2\partial_{x_3}^2 + 4\mu h - z)^{-1}, \tag{7.3}$$

acting in $L^2(\mathbb{R}_{x_3})$ and

$$R_0(z) = \Pi_2(H_0 - z)^{-1} = \Pi_2^{(2)} \otimes r(z) \tag{7.4}$$

acting in $L^2(\mathbb{R}^3) = L^2(\mathbb{R}^2_{x_\perp}) \otimes L^2(\mathbb{R}_{x_3})$. With this notation we have using (7.1):

$$\begin{aligned} \operatorname{tr}[\phi\Pi_2 g(H)\Pi_2\phi] = &\int \operatorname{tr}[R_0(z)\phi V(H - z)^{-1}V\phi R_0(z)] \\ &+ \int \operatorname{tr}[\left([\phi, \Pi_2^{(2)}] \otimes r(z)\right) V(H - z)^{-1}V\phi R_0(z)] \\ &+ \int \operatorname{tr}[R_0(z)\phi V(H - z)^{-1}V \left([\Pi_2^{(2)}, \phi] \otimes r(z)\right)] \\ &+ \int \operatorname{tr}[\left([\phi, \Pi_2^{(2)}] \otimes r(z)\right) V(H - z)^{-1}V \left([\Pi_2^{(2)}, \phi] \otimes r(z)\right)] \\ &= E_1 + E_2 + E_3 + E_4. \end{aligned}$$

To estimate E_4 we use Lemma A.3 to write

$$\begin{aligned}\| \left([\phi,\Pi_2^{(2)}]\otimes r(z)\right) V(H-z)^{-1}V\left([\Pi_2^{(2)},\phi]\otimes r(z)\right) \|_1 & \\ &\leqslant \| \left([\phi,\Pi_2^{(2)}]\otimes r(z)\right) V\|_2^2 \,\|(H-z)^{-1}\| \\ &\leqslant c\frac{1}{|\Im z|}\,\frac{\mu}{h^2}\,\frac{h}{\mu}|4\mu h-z|^{-3/2}.\end{aligned}$$

Therefore, we get, using (7.2)

$$\begin{aligned}|E_4| &\leqslant \frac{\mu}{h^2}\,\frac{h}{\mu}(\mu h)^{-1/2}\min\left\{\frac{M(h)}{\mu h},1\right\}\\ &\leqslant \frac{\mu}{h^2}\,\frac{1}{(\mu h)^2}(h^{3/2}M(h)/\sqrt{\mu}).\end{aligned}$$

Now, $h^{3/2}M(h)/\sqrt{\mu}\to 0$ if $\mu h\to\infty$, so this term satisfies the conclusion of Lemma 7.1.

E_2 and E_3 are similar to each other and can be treated in the same way, so we will only deal explicitely with E_2. Here we commute Π_2 through ϕV (notice that ϕV is differentiable!).

$$\begin{aligned}E_2 &= \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}\phi V\Pi_2(H-z)^{-1}V\left([\Pi_2^{(2)},\phi]\otimes r(z)\right)]\\ &\quad+\int \operatorname{tr}[\Pi_2(H_0-z)^{-1}[\Pi_2,\phi V](H-z)^{-1}V\left([\Pi_2^{(2)},\phi]\otimes r(z)\right)]\\ &= a+b.\end{aligned}$$

In the last term b we write

$$\begin{aligned}\|\Pi_2(H_0-z)^{-1}[\Pi_2,\phi V](H-z)^{-1}V\left([\Pi_2^{(2)},\phi]\otimes r(z)\right)\|_1 & \\ &\leqslant \|\Pi_2(H_0-z)^{-1}[\Pi_2,\phi V]\|_2\,\|(H-z)^{-1}\|\,\|V\left([\Pi_2^{(2)},\phi]\otimes r(z)\right)\|_2.\end{aligned}$$

This can be estimated using (7.2) and Lemmas A.2 and A.3 as

$$|b|\leqslant \frac{\mu}{h^2}\,\frac{1}{(\mu h)^2}(h^{3/2}M(h)/\sqrt{\mu}),$$

which is the same estimate as we had for E_4.

For the term a we use the resolvent identity before we apply Lemmas A.2 and A.3 to get

$$\begin{aligned}
a &= \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}\phi V\Pi_2(H_0-z)^{-1}V\left([\Pi_2^{(2)},\phi]\otimes r(z)\right)]\\
&\quad+\int \operatorname{tr}[\Pi_2(H_0-z)^{-1}\phi V\Pi_2(H_0-z)^{-1}V(H-z)^{-1}V\left([\Pi_2^{(2)},\phi]\otimes r(z)\right)]\\
&\leqslant \int \|R_0(z)\phi V\|_2\|R_0(z)\|\|V\left([\Pi_2^{(2)},\phi]\otimes r(z)\right)\|_2\\
&\quad+\int \frac{1}{|\Im z|}\|R_0(z)\|\cdot\|\phi V\|_\infty\|\Pi_2(H_0-z)^{-1}V\|_2\|V\left([\Pi_2^{(2)},\phi]\otimes r(z)\right)\|_2\\
&\leqslant c\frac{\mu}{h^2}\sqrt{h/\mu}\,(\mu h)^{-3/2}\min\left\{\frac{M(h)}{\mu h},1\right\}\\
&= \frac{\mu}{h^2}\frac{1}{(\mu h)^2}h\min\left\{\frac{M(h)}{\mu h},1\right\}.
\end{aligned}$$

Commuting Π_2 through ϕV:

In the remaining term E_1 we have ϕ multiplying V, so we can commute Π_2 through ϕV:

$$\begin{aligned}
E_1 = \int &\operatorname{tr}[\Pi_2(H_0-z)^{-1}\phi V\Pi_2(H-z)^{-1}\Pi_2V\phi(H_0-z)^{-1}\Pi_2]\\
&+\operatorname{tr}[\Pi_2(H_0-z)^{-1}[\Pi_2,\phi V](H-z)^{-1}\Pi_2V\phi(H_0-z)^{-1}\Pi_2]\\
&+\operatorname{tr}[\Pi_2(H_0-z)^{-1}\phi V\Pi_2(H-z)^{-1}[V\phi,\Pi_2](H_0-z)^{-1}\Pi_2]\\
&+\operatorname{tr}[\Pi_2(H_0-z)^{-1}[\Pi_2,\phi V](H-z)^{-1}[V\phi,\Pi_2](H_0-z)^{-1}\Pi_2]\\
&=\alpha+\beta+\gamma+\delta.
\end{aligned}$$

The terms β,γ and δ are estimated similarly to the terms above – making in particular use of Lemma A.4:

$$\begin{aligned}
|\delta| &\leqslant \int \|\Pi_2(H_0-z)^{-1}[\Pi_2,\phi V]\|_2^2\|(H-z)^{-1}\|\\
&\leqslant \frac{\mu}{h^2}\frac{h}{\mu}\frac{M(h)}{(\mu h)^{3/2}}\\
&= \frac{\mu}{h^2}\frac{1}{(\mu h)^2}\frac{h^{3/2}M(h)}{\sqrt{\mu}}.
\end{aligned}$$

For β (and γ which is similar) we have to work a bit more - namely use the resolvent identity once in order to profit from the Π_2 which hits $(H-z)^{-1}$.

Then these terms are also estimated using Lemmas A.2 - A.4 and (7.2):

$$\begin{aligned}
\beta &= \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}[\Pi_2,\phi V](H_0-z)^{-1}\Pi_2 V\phi(H_0-z)^{-1}\Pi_2] \\
&\quad + \operatorname{tr}[\Pi_2(H_0-z)^{-1}[\Pi_2,\phi V](H-z)^{-1}V(H_0-z)^{-1}\Pi_2 V\phi(H_0-z)^{-1}\Pi_2] \\
&\leqslant \int \|\Pi_2(H_0-z)^{-1}[\Pi_2,\phi V]\|_2\|(H_0-z)^{-1}\Pi_2 V\phi\|_2\|(H_0-z)^{-1}\Pi_2\| \\
&\quad + \int \|\Pi_2(H_0-z)^{-1}[\Pi_2,\phi V]\|_2\|(H-z)^{-1}\|\|V(H_0-z)^{-1}\Pi_2\|_2 \\
&\quad \times \|V\phi\|_\infty\|(H_0-z)^{-1}\Pi_2\| \\
&\leqslant c\frac{\mu}{h^2}\sqrt{h/\mu}(\mu h)^{-3/2}\min\left\{\frac{M(h)}{\mu h},1\right\} \\
&= c\frac{\mu}{h^2}\frac{1}{(\mu h)^2}h\min\left\{\frac{M(h)}{\mu h},1\right\}.
\end{aligned}$$

Using the resolvent identity again:

In order to get a good estimate on α we need to apply the resolvent identity twice and furthermore compare with the operator $H_0+\Pi_2\phi V\Pi_2$. Let us write:

$$\begin{aligned}
\alpha &= \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}\phi V\Pi_2(H_0-z)^{-1}\Pi_2 V\phi(H_0-z)^{-1}\Pi_2] \\
&\quad + \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}\phi V\Pi_2(H_0-z)^{-1}V(H_0-z)^{-1}\Pi_2 V\phi(H_0-z)^{-1}\Pi_2] \\
&\quad + \int \operatorname{tr}[\Pi_2(H_0-z)^{-1}\phi V\Pi_2(H_0-z)^{-1}V(H-z)^{-1}V \\
&\quad \times (H_0-z)^{-1}\Pi_2 V\phi(H_0-z)^{-1}\Pi_2] \\
&= \alpha_1+\alpha_2+\alpha_3.
\end{aligned}$$

The last two terms α_2 and α_3 are easily estimated using Lemmas A.2 - A.4 and (7.2):

$$\begin{aligned}
|\alpha_2| &\leqslant \int \|\Pi_2(H_0-z)^{-1}\|^2\|\phi V\|_\infty\|\Pi_2(H_0-z)^{-1}V\|_2\|(H_0-z)^{-1}\Pi_2 V\phi\|_2 \\
&\leqslant c\frac{\mu}{h^2}\min\left\{\frac{M(h)}{(\mu h)^{7/2}},(\mu h)^{-5/2}\right\} \\
&= c\frac{\mu}{h^2}\frac{1}{(\mu h)^2}(\mu h)^{-1/2}\min\left\{\frac{M(h)}{\mu h},1\right\}.
\end{aligned}$$

The term α_3 is estimated similarly - the main difference being the term $(H-z)^{-1}$ which is estimated in operator norm by $1/|\Im z|$.

Comparing with $H_0+\Pi_2\phi V\Pi_2$:

So finally we have to consider α_1. It is easy to see that if we estimate the trace by the trace norm as we have done for all the other terms then we would get the estimate $\frac{\mu}{h^2}\frac{1}{(\mu h)^2}\frac{M(h)}{(\mu h)^{1/2}}$ for this term. In order for that estimate to be acceptable we would need $\mu h^5\to\infty$. However, it turns out that we can do much better by comparing with the operator $H_0+\Pi_2\phi V\Pi_2$. This operator is self adjoint and maps the second Landau band to itself. Furthermore,

$$\Pi_2\left(H_0+\Pi_2\phi V\Pi_2\right)\Pi_2\geqslant 4\mu h-c,$$

so therefore we get

$$\begin{aligned}
0 &= \mathrm{tr}[\Pi_2 g(H_0+\Pi_2\phi V\Pi_2)\Pi_2]\\
&= \int \mathrm{tr}[\Pi_2(H_0-z)^{-1}\Pi_2]\\
&+\int \mathrm{tr}[\Pi_2(H_0-z)^{-1}\Pi_2\phi V\Pi_2(H_0-z)^{-1}\Pi_2]\\
&+\int \mathrm{tr}[\Pi_2(H_0-z)^{-1}\Pi_2\phi V\Pi_2(H_0+\Pi_2\phi V\Pi_2-z)^{-1}\Pi_2\phi V\Pi_2(H_0-z)^{-1}\Pi_2]\\
&=\beta_1+\beta_2+\beta_3.
\end{aligned}$$

The first two terms β_1,β_2 vanish as we have seen in the proof of Lemma 6.1. Therefore only β_3 is left. So we can now compare α_1 and β_3:

$$\begin{aligned}
\alpha_1 &= \alpha_1-\beta_3\\
&= \int \mathrm{tr}[R_0(z)\phi V\Pi_2(H_0-z)^{-1}\Pi_2\phi V\Pi_2(H_0+\Pi_2\phi V\Pi_2-z)^{-1}\Pi_2\phi V R_0(z)].
\end{aligned}$$

Now we can take the trace norm under the integral and get

$$|\alpha_1|\leqslant\frac{\mu}{h^2}\frac{1}{(\mu h)^2}(\mu h)^{-1/2}\min\left\{\frac{M(h)}{\mu h},1\right\}. \qquad \square$$

A. Auxiliary results on the Π_j's.

If $\phi(x)$ is differentiable, then $[\Pi_j,\phi]\approx\sqrt{h/\mu}$, i.e. the projections onto Landau levels "almost" commute with the multiplication by functions.

We will repeatedly need qualitative estimates on what "almost" means. This gives a number of technical lemmas which have been collected in this appendix.

The basic lemma is the following:

LEMMA A.1. — *There exists a constant c such that for all $\phi \in C^{0,1}(\mathbb{R}^3)$, we have*

$$\| [\Pi_j, \phi] \| \leqslant c \|\nabla \phi\|_\infty \sqrt{h/\mu}.$$

We will not prove this lemma since the proof is similar to (and easier than) the proofs of Lemma A.6 and Lemma A.7.

First we have a number of estimates of Hilbert-Schmidt norms of quantities involving a resolvent and sometimes also a commutator. These are Lemmas A.2 to A.4.

LEMMA A.2. — *There exists $c > 0$ such that for all $V \in L^2(\mathbb{R}^3)$, we have*

$$\| \big(\Pi_2^{(2)} \otimes r(z)\big) V \|_2^2 \leqslant c \|V\|^2 \frac{\mu}{h^2} E^{-3/2},$$

where $E = |4\mu h - z|$ and $r(z)$ was defined in (7.3).

Proof. — The operator $\big(\Pi_2^{(2)} \otimes r(z)\big) V$ has integral kernel

$$K(x,y) = \Pi_2^{(2)}(x_\perp, y_\perp) \frac{e^{-\sqrt{4\mu h - z}|x_3 - y_3|/h}}{2\sqrt{4\mu h - z}h} V(y),$$

where the kernel $\Pi_2^{(2)}(x_\perp, y_\perp)$ is known explicitely - see (1.8). The important feature is that $|\Pi_2(x_\perp, y_\perp)|$ is a function F (gaussian times polynomial) of $|x_\perp - y_\perp|\sqrt{\mu/h}$ (notice also the normalisation constant μh). Thus, by a change of variables we easily get

$$\begin{aligned} \int |K(x,y)|^2 \, dx dy &\leqslant \|V\|^2 \int |F(|w_\perp| \sqrt{\mu/h})|^2 \frac{e^{-2\sqrt{|4\mu h - z|}\,|w_3|/h}}{4|4\mu h - z| h^2} \, dw \\ &= c \frac{\mu}{h^2} E^{-3/2}. \end{aligned}$$

□

LEMMA A.3. — *There exists a constant c such that for all $V \in L^2(\mathbb{R}^3)$, $\phi \in C^{0,1}(\mathbb{R}^2_{x_\perp})$ the following estimate holds:*

$$\| \Big([\phi, \Pi_2^{(2)}] \otimes r(z) \Big) V \|_2^2 \leqslant c \|\nabla \phi\|_\infty^2 \|V\|^2 \frac{\mu}{h^2} \frac{h}{\mu} E^{-3/2},$$

where $E = |4\mu h - z|$ and where $r(z)$ was defined in (7.3).

Proof. — As in the previous proof we can explicitely write down the integral kernel $K(x,y)$ of the operator in question i.e. $\Big([\phi,\Pi_2^{(2)}]\otimes r(z)\Big)V$:

$$K(x,y)=\Pi_2^{(2)}(x_\perp,y_\perp)(\phi(x_\perp)-\phi(y_\perp))\frac{e^{-\sqrt{4\mu h-z}|x_3-y_3|/h}}{2\sqrt{4\mu h-z}h}V(y).$$

Now, $|\phi(x_\perp)-\phi(y_\perp)|\leqslant|x_\perp-y_\perp|\|\nabla\phi\|_\infty$, and we get (with notation as in the previous proof) that

$$\begin{aligned}
&\int|K(x,y)|^2\,dxdy\\
&\quad=c\|\nabla\phi\|_\infty^2\|V\|^2\int|F(|w_\perp|\sqrt{\mu/h})|^2|w_\perp|^2\frac{e^{-2\sqrt{|4\mu h-z|}|w_3|/h}}{4|4\mu h-z|h^2}\,dw\\
&\quad=c\|\nabla\phi\|_\infty^2\|V\|^2\frac{\mu}{h^2}\frac{h}{\mu}E^{-3/2}.
\end{aligned}$$

□

Lemma A.4. — *Suppose $\psi\in C_0^1(\mathbb{R}^3)$. Then there exists a constant c (depending on ψ) such that*

$$\|\Pi_2(H_0-z)^{-1}[\Pi_2,\psi]\|_2^2\leqslant c\frac{\mu}{h^2}\frac{h}{\mu}E^{-3/2},$$

where $E=|4\mu h-z|$.

Proof. — Choose $\tilde\psi\in C_0^1(\mathbb{R}^3)$ of the form $\tilde\psi(x)=\psi_1(x_\perp)\psi_2(x_3)$ satisfying $\tilde\psi\psi=\psi$. Then

$$\begin{aligned}
&\|\Pi_2(H_0-z)^{-1}[\Pi_2,\psi]\|_2\\
&\quad=\|\Pi_2(H_0-z)^{-1}[\Pi_2,\tilde\psi\psi]\|_2\\
&\quad\leqslant\|\Pi_2(H_0-z)^{-1}\tilde\psi\|_2\|[\Pi_2,\psi]\|+\|\Pi_2(H_0-z)^{-1}[\Pi_2,\tilde\psi]\psi\|_2.
\end{aligned}$$

Here the first term is readily estimated using Lemmas A.1 and A.2. The second term is

$$\|\big(\Pi_2^{(2)}\otimes r(z)\big)\big([\Pi_2^{(2)},\psi_1(x_\perp)]\otimes\psi_2(x_3)\big)\psi(x)\|_2,$$

which can be estimated using Lemma A.3. □

Finally, we will need to prove bounds on some operator norms. The idea is that Π_0 is essentially a local operator, so therefore $e^{|x_\perp|}\Pi_0 f$ $(=e^{|x_\perp|}\circ\Pi_0\circ f)$ is a bounded operator if the function f has compact support. We will actually need this for $e^{|x_\perp|}\Pi_0 a^2 f$, so we will first find the integral kernel of $\Pi_0 a^2$ and then state and prove Lemmas A.6 and A.7:

PROPOSITION A.5. — *The operator $a^*a^*\Pi_0^{(2)}$ has integral kernel*

$$(a^*a^*\Pi_0^{(2)})(x_\perp, y_\perp) = \mu h\left(\sqrt{\mu/h}[(y_2 - x_2) - i(y_1 - x_1)]\right)^2 \Pi_0^{(2)}(x_\perp, y_\perp).$$

Proof. — The proof is just a calculation. Remember from (1.8) and (1.9) that

$$a^* = p_{\vec{A},1} + ip_{\vec{A},2} = -ih(\partial_{x_1} + i\partial_{x_2}) + \mu/2(-x_2 + ix_1),$$

$$\Pi_0^{(2)}(x_\perp, y_\perp) = \frac{\mu}{2\pi h} e^{i(x_1y_2 - x_2y_1)\mu/(2h)} e^{-|x_\perp - y_\perp|^2\mu/(4h)}.$$

Therefore,

$$\begin{aligned}
&(a^*\Pi_0^{(2)})(x_\perp, y_\perp)\\
&= \frac{\mu}{2\pi h}\left\{(-ih(\partial_{x_1} + i\partial_{x_2}) + \mu/2(-x_2 + ix_1))\, e^{i(x_1y_2 - x_2y_1)\frac{\mu}{2h}}\right\} e^{-|x_\perp - y_\perp|^2\frac{\mu}{4h}}\\
&+ \frac{\mu}{2\pi h} e^{i(x_1y_2 - x_2y_1)\frac{\mu}{2h}}\,(-ih(\partial_{x_1} + i\partial_{x_2}))\, e^{-|x_\perp - y_\perp|^2\frac{\mu}{4h}}\\
&= \frac{\mu}{2\pi h}\Big\{\,[-ih(iy_2 + i(-iy_1))\mu/(2h) + \mu/2(-x_2 + ix_1)]\\
&+ ih\,[(x_1 - y_1)\mu/(2h) + i(x_2 - y_2)\mu/(2h)]\,\Big\} e^{i(x_1y_2 - x_2y_1)\frac{\mu}{2h}}\, e^{-|x_\perp - y_\perp|^2\frac{\mu}{4h}}\\
&= \mu\,[(y_2 - x_2) - i(y_1 - x_1)]\,\Pi_0^{(2)}(x_\perp, y_\perp).
\end{aligned}$$

Thus,

$$\begin{aligned}
&(a^*a^*\Pi_0^{(2)})(x_\perp, y_\perp)\\
&\qquad= \Big\{\,(\mu\,[(y_2 - x_2) - i(y_1 - x_1)])\\
&\qquad\quad+ (-ih(\partial_{x_1} + i\partial_{x_2})\mu\,[(y_2 - x_2) - i(y_1 - x_1)]\,\Big\}(a^*\Pi_0^{(2)})(x_\perp, y_\perp)\\
&\qquad= \Big\{\mu\,[(y_2 - x_2) - i(y_1 - x_1)]\\
&\qquad\qquad+ (-ih)\mu i + (-ih)i\mu(-1)\Big\}(a^*\Pi_0^{(2)})(x_\perp, y_\perp)\\
&\qquad= \mu h\left(\sqrt{\mu/h}[(y_2 - x_2) - i(y_1 - x_1)]\right)^2 \Pi_0^{(2)}(x_\perp, y_\perp). \qquad \square
\end{aligned}$$

Lemma A.6. — *Suppose $f \in C_0(\mathbb{R}^3)$, then we have the following bounds on the operator norms:*

$$\|e^{|x_\perp|}\Pi_0 f\| \leqslant C,$$
$$\|e^{|x_\perp|}\Pi_0 a f\| \leqslant C\sqrt{\mu h},$$
$$\|e^{|x_\perp|}\Pi_0 a^2 f\| \leqslant C\mu h,$$

where C is uniformly bounded in when μ is sufficiently big and h is sufficiently small.

Proof. — We will only prove the last estimate since the first two are similar. The operator $K = e^{|x_\perp|}\Pi_0 a^2 f$ has integral kernel

$$K(x,y) = e^{|x_\perp|}G(x_\perp, y_\perp)\delta(x_3 - y_3)f(y),$$

where $G(x_\perp, y_\perp) = \overline{(a^* a^* \Pi_0^{(2)})(y_\perp, x_\perp)}$. From Schur's Lemma, we can estimate the operator norm of the kernel K as

$$\|K\| \leqslant \max\left\{\sup_x \int |K(x,y)|dy, \sup_y \int |K(x,y)|dx\right\}.$$

So we look at

$$\int |K(x,y)|dy = e^{|x_\perp|}\int |f(y_\perp, x_3)||G(x_\perp, y_\perp)|dy_\perp.$$

Now, since f has compact support and $|G|$ only depends on $x_\perp - y_\perp$, we can estimate the last integral as

$$\begin{aligned}\int |K(x,y)|dy &\leqslant c\mu h\|f\|_\infty e^{|x_\perp|}\int_{\max\{|x_\perp|-c,0\}}^{\infty} \frac{\mu}{h} e^{-r^2\mu/(4h)}(\sqrt{\mu/h}\, r)^2 r\, dr \\ &= c\mu h\|f\|_\infty e^{|x_\perp|}\int_{\sqrt{\mu/h}\max\{|x_\perp|-c,0\}}^{\infty} e^{-t^2/4}t^3\, dt.\end{aligned}$$

It is easy to get the following estimate:

$$\int_l^\infty e^{-t^2/4}t^3\, dt \leqslant ce^{-\frac{l^2}{8}},$$

so we see that

$$\int |K(x,y)|dy \leqslant c\mu h\|f\|_\infty e^{|x_\perp|}e^{-\frac{\mu}{8h}(\max\{|x_\perp|-c,0\})^2}.$$

This finishes the estimate on $\sup_x \int |K(x,y)|dy$.

For the other part $\sup_y \int |K(x,y)|dx$ a similar argument works. □

LEMMA A.7. — *Let $\phi \in C_0^1(\mathbb{R}^3)$. Then the operator $K = e^{|x_\perp|}\Pi_0 a^2[\Pi_2;\phi]$ satisfies the bound*

$$\|K\| \leqslant C\mu h\sqrt{h/\mu},$$

where $C = C(\|\nabla\phi\|_\infty, \operatorname{diam}\operatorname{supp}\phi)$ is uniformly bounded when μ is sufficiently big and h is sufficiently small.

Proof. — The operator has the following integral kernel:

$$K(x,z) =$$
$$e^{|x_\perp|}\int (\Pi_0^{(2)}aa)(x_\perp,y_\perp)\delta(x_3-y_3)\Pi_2^{(2)}(y_\perp,z_\perp)\delta(y_3-z_3)(\phi(y)-\phi(z))\,dy.$$

We have to estimate

$$\sup_x \int |K(x,z)|dz \text{ and } \sup_z \int |K(x,z)|dx.$$

Let us write the kernels of $(\Pi_0^{(2)}aa)$ and $\Pi_2^{(2)}$ as

$$\begin{aligned} |(\Pi_0^{(2)}aa)(x_\perp,y_\perp)| &\leqslant \frac{\mu}{h}\mu h P_1((x_\perp-y_\perp)\sqrt{\mu/4h})e^{-|x_\perp-y_\perp|^2\frac{\mu}{4h}} \\ |\Pi_2^{(2)}(y_\perp,z_\perp)| &\leqslant \frac{\mu}{h}P_2((y_\perp-z_\perp)\sqrt{\mu/4h})e^{-|y_\perp-z_\perp|^2\frac{\mu}{4h}}, \end{aligned}$$

where P_1, P_2 are polynomials. Furthermore,

$$|\phi(y_\perp,x_3)-\phi(z_\perp,x_3)| \leqslant |y_\perp-z_\perp|G(y_\perp,|y_\perp-z_\perp|),$$

where

$$G(y_\perp,r) = \sup_{|y_\perp-z_\perp|\leqslant r} |\nabla\phi(z_\perp,x_3)|.$$

Thus

$$\begin{aligned} \int |K(x,z)|dz &\leqslant c\mu h\sqrt{h/\mu}e^{|x_\perp|}\int \frac{\mu}{h}P_1((x_\perp-y_\perp)\sqrt{\mu/4h})e^{-|x_\perp-y_\perp|^2\frac{\mu}{4h}} \\ &\times\Big(\int P_2(|w_\perp|)|w_\perp|e^{-|w_\perp|^2}G(y_\perp,\sqrt{4h/\mu}|w_\perp|)\,dw_\perp\Big)\,dy_\perp. \end{aligned}$$

Now, since ϕ has compact support, $\operatorname{supp}(r \mapsto G(y_\perp, r)) \subseteq [|y_\perp| - c_0, +\infty) \cap \mathbb{R}_+$, so

$$
\begin{aligned}
\int P_2(|w_\perp|)|w_\perp| e^{-|w_\perp|^2} G(y_\perp, \sqrt{4h/\mu}|w_\perp|)\, dw_\perp
&\leqslant \|\nabla\phi\|_\infty \int_{\max\{0, \sqrt{\mu/4h}(|y_\perp| - c_0)\}}^{\infty} e^{-r^2} P_2(r) r^2 dr \\
&\leqslant c\|\nabla\phi\|_\infty e^{-\frac{1}{2}(\max\{0,\sqrt{\mu/4h}(|y_\perp|-c_0)\})^2} \int_0^\infty e^{-r^2/2} P_2(r) r^2 dr \\
&= c\|\nabla\phi\|_\infty e^{-\frac{\mu}{8h}(\max\{0,|y_\perp|-c_0\})^2}.
\end{aligned}
$$

Let us now split the $y_\perp$ integral into integrations over the sets $\{|y_\perp| \leqslant 2c_0\}$ and $\{|y_\perp| \geqslant 2c_0\}$. For the integration over the bounded set $\{|y_\perp| \leqslant 2c_0\}$ we estimate as follows:

$$
\begin{aligned}
e^{-\frac{\mu}{8h}(\max\{0,|y_\perp|-c_0\})^2} &\leqslant 1, \\
P_1((x_\perp - y_\perp)\sqrt{\mu/4h}) e^{-|x_\perp - y_\perp|^2 \frac{\mu}{4h}} &\leqslant c e^{-|x_\perp - y_\perp|^2 \frac{\mu}{8h}}.
\end{aligned}
$$

So we have to bound

$$
\begin{aligned}
&e^{|x_\perp|} \int_{\{|y_\perp| \leqslant 2c_0\}} \frac{\mu}{h} e^{-|x_\perp - y_\perp|^2 \frac{\mu}{8h}}\, dy_\perp \\
&\leqslant \left(\sup_{\{|y_\perp| \leqslant 2c_0\}} \left\{ e^{|x_\perp|} e^{-|x_\perp - y_\perp|^2 \frac{\mu}{16h}} \right\} \right) \left(\int_{\mathbb{R}^2} \frac{\mu}{h} e^{-|x_\perp - y_\perp|^2 \frac{\mu}{16h}}\, dy_\perp \right) \\
&\leqslant c.
\end{aligned}
$$

When $|y_\perp| \geqslant 2c_0$ we can estimate

$$
\begin{aligned}
e^{-\frac{\mu}{8h}(\max\{0,|y_\perp|-c_0\})^2} &\leqslant e^{-\frac{\mu}{32h}|y_\perp|^2}, \\
P_1((x_\perp - y_\perp)\sqrt{\mu/4h}) e^{-|x_\perp - y_\perp|^2 \frac{\mu}{4h}} &\leqslant c e^{-|x_\perp - y_\perp|^2 \frac{\mu}{8h}}.
\end{aligned}
$$

Therefore, we can use completion of the square to get

$$
\begin{aligned}
&e^{|x_\perp|} \int_{\{|y_\perp| \geqslant 2c_0\}} \frac{\mu}{h} P_1((x_\perp - y_\perp)\sqrt{\mu/4h}) e^{-|x_\perp - y_\perp|^2 \frac{\mu}{4h}} e^{-\frac{\mu}{8h}(\max\{0,|y_\perp|-c_0\})^2} \\
&\qquad \leqslant c e^{|x_\perp|} \int_{\mathbb{R}^2} \frac{\mu}{h} e^{-|x_\perp - y_\perp|^2 \frac{\mu}{8h}} e^{-\frac{\mu}{32h}|y_\perp|^2}\, dy_\perp \\
&\qquad = c e^{|x_\perp|} e^{-\frac{\mu}{32h}\cdot\frac{4}{5}|x_\perp|^2} \int_{\mathbb{R}^2} \frac{\mu}{h} e^{-|y_\perp - \frac{4}{5}x_\perp|^2 \frac{5\mu}{32h}} \\
&\qquad \leqslant c.
\end{aligned}
$$

Hereby,

$$\int |K(x,z)|dz \leqslant c\mu h\sqrt{h/\mu}.$$

The other integral, $\int |K(x,z)|dx$, is treated similarly. □

B. The lower bound of hydrogen atoms in magnetic fields.

In this appendix we recall results from [AH81] on how the bottom of the spectrum of a hydrogen-like Hamiltonian in a constant magnetic field depends on the strength of the magnetic field. Let

$$H(h,\mu) = (-ih\nabla + \mu\vec{A})^2 - \mu h - \frac{1}{|x|},$$

with $\operatorname{curl}\vec{A} = (0,0,1)$. The result is the following:

Proposition B.1. — *There exists a (positive) constant c such that*

$$\inf \operatorname{Spec} H(h,\mu) \geqslant -ch^{-2}\left(\max\{1, \log(\mu h^3)\}\right)^2.$$

Proof. — Using the scaling $x \mapsto h^2 x$ we have that $H(h,\mu)$ is unitarily equivalent to the operator

$$h^{-2}H(1,\mu h^3).$$

Let us notice that the function $b \mapsto \inf \operatorname{Spec} H(1,b)$ is continuous for $b \in [0,\infty)$: For $b > 0$ this follows from perturbation theory (see for instance Reed and Simon [RS78, Chapter XII.2]). Continuity for $b = 0$ is a well-known fact (Zeeman effect) and a rigorous proof is contained in [AH81, Theorem 5.1].

Therefore, we only have to consider the asymptotics as b tends to infinity. From [AH81] we know that

$$\inf \operatorname{Spec} H(1,b) \approx c(\log(b))^2,$$

as $b \to \infty$. This finishes the proof. □

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