ANNALES

DE

L'INSTITUT FOURIER

Ivan V. ARZHANTSEV

Algebras with finitely generated invariant subalgebras

Tome 53, nº 2 (2003), p. 379-398.

<http://aif.cedram.org/item?id=AIF_2003__53_2_379_0>

Ann. Inst. Fourier, Grenoble
53, 2 (2003), 379–398

ALGEBRAS WITH FINITELY GENERATED INVARIANT SUBALGEBRAS

by Ivan V. ARZHANTSEV (*)

1. Introduction.

It is easy to prove that any subalgebra in the polynomial algebra $K[x]$ is finitely generated. On the other hand, one can construct many non-finitely generated subalgebras in $K[x_1, \dots .x_n]$ for $n \geqslant 2$. More generally, any subalgebra in an associative commutative finitely generated integral algebra $\mathcal{A}$ with unit is finitely generated if and only if $\operatorname{Kdim} \mathcal{A} \leqslant 1$, where $\operatorname{Kdim} \mathcal{A}$ is Krull dimension of $\mathcal{A}$. The aim of this paper is to obtain an equivariant version of this result.

Let $\mathcal{A}$ be an associative commutative finitely generated integral algebra with unit over an algebraically closed field K of characteristic zero, and let G be a connected reductive algebraic group over K acting rationally on $\mathcal{A}$. The latter condition means that there is a homomorphism $G \to \operatorname{Aut}(\mathcal{A})$ such that the orbit Ga of any element $a \in \mathcal{A}$ is contained in a finite-dimensional subspace in $\mathcal{A}$ where G acts rationally. We say in this case that $\mathcal{A}$ is a *G-algebra*.

In Section 2 we introduce three special types of G-algebras. Theorem 1 states that any invariant subalgebra in a G-algebra $\mathcal{A}$ is finitely generated if and only if $\mathcal{A}$ belongs to one of these three types.

(*) The work was supported by CRDF grant RM1-2088, by RFBR grants 01-01-00756, 02-01-06401 and by CNRS.
Keywords: Algebraic groups – Rational G-algebras – Quasi-affine homogeneous spaces – Affine embeddings.
Math. classification: 13A50 – 13E15 – 14L17 – 14L30 –14M17 –14R20.

In Section 3 we consider a geometric method for constructing a non-finitely generated subalgebra in a G-algebra. The proof of Theorem 1 is given in Section 5. It is based on the notion of *an affine embedding* of a homogeneous space defined in Section 4.

An (affine) homogeneous space G/H is said to be *affinely closed* if any affine embedding of G/H coincides with G/H (cf. [AT01]). It was proved by D. Luna [Lu75] that a homogeneous space G/H is affinely closed if and only if H is a reductive subgroup of finite index in its normalizer $N_G(H)$. For convenience of the reader we recall the proof of this result following G. Kempf [Ke78], Cor. 4.5.

In Section 6 some results on affine embeddings are given. In particular, some characterizations of embeddings with a G-fixed point are presented (Propositions 3, 4 and 6). The notion of *the canonical embedding* of a homogeneous space G/H, where H is a Grosshans subgroup of G, is introduced in Section 7. (Let us recall that H is a Grosshans subgroup of G if the homogeneous space G/H is a quasi-affine variety and the algebra of regular functions $K[G/H]$ is finitely generated.) This embedding is a very natural object associated with G/H, and investigation of its properties leads to some characteristics of the pair (G, H).

In section 8 a version of our result over algebraically closed fields of positive characteristic is discussed. Finally, some problems are collected in Section 9.

Acknowledgements. The author is grateful to Michel Brion for numerous discussions and suggestions, to A. Premet for pointing out an error in a preliminary version of this text, to A. J. Parameswaran for a useful remark and to the referee for many comments and corrections.

This paper was written during the staying at Institut Fourier (St. Martin d'Hères, France). The author wishes to thank this institution and especially Michel Brion for invitation and hospitality, and Independent Moscow University for support.

2. Three types of G-algebras.

Type C. Here $\mathcal{A}$ is a finitely generated domain of Krull dimension $\operatorname{Kdim} \mathcal{A} = 1$ (i.e., the transcendence degree of the quotient field $Q\mathcal{A}$ equals one) with any (for example, trivial) G-action. Such algebras may be considered as the algebras of regular functions on irreducible affine curves.

Type HV. Let H be a closed subgroup of G and

$$\mathcal{A}(H) = K[G]^H = K[G/H] = \{f \in K[G] \mid f(gh) = f(g) \text{ for any } g \in G,\ h \in H\}.$$

The left G-action $(l(g')f)(g) := f(g'^{-1}g)$ on $\mathcal{A}(H)$ is rational.

Further we follow notation of the book [Gr97]. Let $B = TU$ be a Borel subgroup of G with the unipotent radical U and a maximal torus T. Here T normalizes U and there is a G-equivariant T-action on $\mathcal{A}(U)$ defined by right translation $(r(t)f)(g) := f(gt)$. For a character $\omega \in X(T)$ consider the G-invariant subspace

$$E(\omega^*) = \{f \in \mathcal{A}(U) \mid r(t)f = \omega(t)f \text{ for all } t \in T\}.$$

The G-module $E(\omega^*)$ is $\{0\}$ unless ω is dominant. Denote by $X^+(T)$ the set of dominant weights. For every $\omega \in X^+(T)$, $E(\omega^*)$ is a simple G-submodule having highest weight denoted by ω^*. The map $\omega \to \omega^*$ is an involution on $X^+(T)$. Since each element in $\mathcal{A}(U)$ is a sum of T-weight vectors (where T acts by right translation), we see that $\mathcal{A}(U)$ is the direct sum of the $E(\omega)$, $\omega \in X^+(T)$. From the definition, it is obvious that if $\omega,\ \omega' \in X^+(T)$, then $E(\omega)E(\omega') \subseteq E(\omega + \omega')$.

Consider the G-algebra

$$\mathcal{A}(\lambda) = \bigoplus_{m \geqslant 0} E(m\lambda) \subset \mathcal{A}(U),$$

where λ is a dominant weight. (More geometrically, the algebra $\mathcal{A}(\lambda)$ may be realized as

$$\mathcal{A}(\lambda) = \bigoplus_{m \geqslant 0} H^0(G/B, L_{m\lambda^*}),$$

where $L_{m\lambda^*} = G *_B K(-m\lambda^*)$ is the G-line bundle on the flag variety G/B corresponding to the character $m\lambda^*$.)

We say that a G-algebra $\mathcal{A}$ is an algebra of type HV if it is G-isomorphic to an invariant subalgebra of $\mathcal{A}(\lambda)$ for some $\lambda \in X^+(T)$. Any G-algebra of type HV is finitely generated, see Lemma 2 below.

The algebra $\mathcal{A}(\lambda)$ may be considered as the algebra of regular functions on the orbit closure of a highest weight vector in the simple G-module with highest weight λ^*. Clearly, any invariant subalgebra in $\mathcal{A}(\lambda)$ has the form

$$\mathcal{A}(P, \lambda) = \bigoplus_{p \in P} E(p\lambda),$$

where P is a subsemigroup in the additive semigroup Z_+ of non-negative integers, cf. [PV72].

Example 1. — Let G be $SL_n(K)$ and $\omega_1, \dots, \omega_{n-1}$ be its fundamental weights. The natural linear action $G : K^n$ induces an action on regular functions

$$G : \mathcal{A} = K[x_1, \dots, x_n], \ (g * f)(v) := f(g^{-1}v).$$

The homogeneous polynomials $K[x_1, \dots, x_n]_m$ of degree m form an (irreducible) isotypic component corresponding to the weight $m\omega_{n-1}$. Hence $\mathcal{A} = \mathcal{A}(\omega_{n-1})$ and any invariant subalgebra in $\mathcal{A}$ is composed of homogeneous components indexed by elements of a subsemigroup $P \subseteq Z_+$.

Type N. Let H be a reductive subgroup of G. Then the algebra $\mathcal{A}(H)$ is finitely generated. Denote by $C_G(H)$ the centralizer of H in G. Consider the following condition:

(*) *H is reductive and for any one-parameter subgroup $\nu : K^* \to C_G(H)$ the image $\nu(K^*)$ is contained in H.*

Let us note that for a reductive subgroup H one has $N_G(H)^0 = H^0 C_G(H)^0$, where L^0 denotes the connected component of unit in an algebraic group L. Hence condition (*) may be reformulated as "H is reductive and the group $W(H)^0$ is unipotent", where $W(H) = N_G(H)/H$. But the normalizer $N_G(H)$ is reductive [LR79], Lemma 1.1 and thus condition (*) is equivalent to the condition

(**) *H is reductive and the group $W(H)$ is finite.*

We say that a G-algebra $\mathcal{A}$ is of type N if there exists a subgroup $H \subset G$ satisfying condition (*) such that $\mathcal{A}$ is G-isomorphic to $\mathcal{A}(H)$. Any G-invariant subalgebra of a G-algebra of type N is finitely generated (Lemma 1).

Example 2. — Let $G = SL_n$ and $H = SO_n$. The group G acts on the space of symmetric $n \times n$-matrices by $(g, s) \to g^T s g$. The stabilizer of the identity matrix E is the subgroup H and the orbit GE is the set X of symmetric matrices with determinant 1. This yields that the algebra $\mathcal{A} = K[X]$ with the G-action $(g * f)(s) := f((g^{-1})^T s g^{-1})$ is an algebra of type N.

A G-algebra $\mathcal{A}$ is a G-algebra of type N if and only if

(***) *$\mathcal{A}$ contains no proper G-invariant ideals and the group of G-equivariant automorphisms of $\mathcal{A}$ is finite* (see Remarks in Section 5).

Now we are able to formulate the main result.

THEOREM 1. — *Let $\mathcal{A}$ be a G-algebra. Then any G-invariant subalgebra of $\mathcal{A}$ is finitely generated if and only if $\mathcal{A}$ is an algebra of one of the types C, HV or N.*

The proof of Theorem 1 is given in Section 5. Now we begin with some auxiliary results.

3. Non-finitely generated subalgebras.

Let X be an irreducible affine algebraic variety and Y be a proper closed irreducible subvariety. Consider the subalgebra

$$\mathcal{A}(X,Y) = \{f \in K[X] \mid f(y_1) = f(y_2) \text{ for any } y_1, y_2 \in Y\} \subset \mathcal{A} = K[X].$$

PROPOSITION 1. — *The subalgebra $\mathcal{A}(X,Y)$ is finitely generated if and only if Y is a point.*

Proof. — If Y is a point, then $\mathcal{A}(X,Y) = K[X]$. Suppose that Y has positive dimension. Consider the ideal $\mathcal{I} = \mathcal{I}(Y) = \{f \in K[X] \mid f(y) = 0 \text{ for any } y \in Y\}$. Then $K[X]/\mathcal{I}$ is an infinite-dimensional vector space. By the Nakayama lemma, we can find $i \in \mathcal{I}$ such that in the local ring of Y the element i is not in $\mathcal{I}^2$. For any $a \in K[X] \setminus \mathcal{I}$ the element ia is in $\mathcal{I} \setminus \mathcal{I}^2$. Hence the space $\mathcal{I}/\mathcal{I}^2$ has infinite dimension.

On the other hand, suppose that $f_1, \ldots, f_n$ are generators of $\mathcal{A}(X,Y)$. Subtracting constants, one may suppose that all f_i are in $\mathcal{I}$. Then $\dim \mathcal{A}(X,Y)/\mathcal{I}^2 \leqslant n+1$, a contradiction. □

PROPOSITION 2. — *Let $\mathcal{A}$ be a finitely generated domain containing K. Then any subalgebra in $\mathcal{A}$ is finitely generated if and only if* $\operatorname{Kdim} \mathcal{A} \leqslant 1$.

Proof. — If $\operatorname{Kdim} \mathcal{A} \geqslant 2$, then the statement follows from the previous proposition. The case $\operatorname{Kdim} \mathcal{A} = 0$ is obvious. It remains to prove that if $\operatorname{Kdim} \mathcal{A} = 1$, then any subalgebra is finitely generated. By taking the integral closure, one may suppose that $\mathcal{A}$ is the algebra of regular functions on a smooth affine curve C_1. Let C be the smooth projective curve such that $C_1 \cong C \setminus \{P_1, \dots, P_k\}$. The elements of $\mathcal{A}$ are the rational functions on C that may have poles only at points P_i. Let $\mathcal{B}$ be a subalgebra in $\mathcal{A}$. By induction on k, we may suppose that the subalgebra $\mathcal{B}' \subset \mathcal{B}$ consisting of functions regular at P_1 is finitely generated, say $\mathcal{B}' = K[s_1, \dots, s_m]$. (Functions that are regular at any point P_i are constants.) Let $v(f)$ be the order of the zero/pole of $f \in \mathcal{B}$ at P_1. The set $V = \{v(f) \mid f \in \mathcal{B}\}$ is an additive subsemigroup of integers. Any such subsemigroup is finitely generated. Let $f_1, \dots, f_n$ be elements of $\mathcal{B}$ such that the $v(f_i)$ generate V. Then for any $f \in \mathcal{B}$ there exists a polynomial $P(y_1, \dots, y_n)$ with $v(f - P(f_1, \dots, f_n)) \geqslant 0$, thus $f - P(f_1, \dots, f_n) \in \mathcal{B}'$. This shows that $\mathcal{B}$ is generated by $f_1, \dots, f_n, s_1, \dots, s_m$. □

4. Affine embeddings.

To go further we need some definitions.

DEFINITION 1. — *Let H be a closed subgroup of G. We say that an affine variety X with a regular G-action is* an affine embedding *of the homogeneous space G/H if there exists a point $x \in X$ such that the orbit Gx is dense in X and the orbit map $G \to Gx$ defines an isomorphism between G/H and Gx. We denote this as $G/H \hookrightarrow X$. An embedding is* trivial *if $X = Gx$.*

Note that a homogeneous space G/H admits an affine embedding if and only if G/H is quasi-affine (as an algebraic variety), see [PV89], Th.1.6. In this situation, the subgroup H is said to be *observable* in G. For a group-theoretic description of observable subgroups see [Su88] (char $K = 0$) and [Gr97], Th.7.3 (char K is arbitrary). It is known that G/H is affine if and only if H is reductive [Ri77], Th.A, [Gr97], Th.7.2. In particular, any reductive subgroup is observable.

DEFINITION 2. — *A homogeneous space is said to be* affinely closed *if it admits only the trivial affine embedding. (In this case G/H is affine.)*

The following result is due to D. Luna [Lu75].

THEOREM 2. — *A homogeneous space G/H is affinely closed if and only if H is a subgroup satisfying condition (*). Moreover, if G acts on an affine variety X and the stabilizer H' of a point $x \in X$ contains a subgroup*

H satisfying condition (*), then H' is a subgroup satisfying condition (*) and the orbit Gx is closed in X.

Theorem 2 implies that if H is a subgroup satisfying condition (*), $H \subseteq H' \subseteq G$ and H' is observable in G, then G/H' is affinely closed. We shall give a proof of Theorem 2 in Section 6 in terms of Kempf's adapted one-parameter subgroups [Ke78].

5. Proof of Theorem 1.

Let $\mathcal{A}$ be a G-algebra with $\operatorname{Kdim} \mathcal{A} \geqslant 2$ such that any invariant subalgebra in $\mathcal{A}$ is finitely generated. Consider the corresponding affine variety $X = \operatorname{Spec} \mathcal{A}$. The action $G : \mathcal{A}$ induces a regular (algebraic) action $G : X$.

Suppose that there exists a proper irreducible closed invariant subvariety $Y \subset X$ of positive dimension. Then $\mathcal{A}(X, Y)$ is an invariant subalgebra that is not finitely generated. In particular, this is the case if G acts on X without a dense orbit. Hence we may suppose that either

(i) the action $G : X$ is transitive or

(ii) X consists of an open orbit $\mathcal{O}$ and a G-fixed point o.

In case (i), fix a point $x \in X$ and denote by H the stabilizer of x in G. Here H is reductive and if G/H is not affinely closed, then there is a nontrivial affine embedding $G/H \hookrightarrow X'$. The complement of the open affine subset X in X' is a union of irreducible divisors. Let Y be one of these divisors. The algebra $\mathcal{A}(X', Y)$ is a non-finitely generated invariant subalgebra in $K[X']$ and the inclusion $X \subset X'$ defines an embedding $K[X'] \subset K[X] = \mathcal{A}$. We conclude that G/H should be affinely closed. In this case $\mathcal{A}$ is of type N by Theorem 2.

LEMMA 1. — *If $X = G/H$ is affinely closed, then any invariant subalgebra in $\mathcal{A}(H)$ is finitely generated.*

Proof. — Suppose that there exists an invariant subalgebra $\mathcal{B} \subset \mathcal{A}(H)$ that is not finitely generated. Let $f_1, f_2, \dots$ be a system of generators of $\mathcal{B}$. Consider the finitely generated subalgebras $\mathcal{B}_i = K[< Gf_1, \dots, Gf_i >]$, where $< Gf_1, \dots, Gf_i >$ is the linear span of the orbits $Gf_1, \dots, Gf_i$.

Infinitely many of the $\mathcal{B}_i$ are pairwise different. For the corresponding varieties $X_i := \operatorname{Spec} \mathcal{B}_i$ one has natural dominant G-morphisms

$$X_1 \longleftarrow X_2 \longleftarrow X_3 \longleftarrow \dots$$

We claim that the action $G : X_i$ is transitive for any i. In fact, the morphism $G/H \to X_i$ is dominant and, by Theorem 2, the image of G/H is closed in X_i.

One may consider any X_i as a homogeneous variety G/H_i, where H_i is a reductive subgroup of G containing H. The infinite sequence of subgroups

$$H_1 \supset H_2 \supset H_3 \supset \dots$$

leads to a contradiction. □

Remarks. — 1) In the case $K = \mathbb{C}$, Lemma 1 follows also from [La99]. In fact, the article [La99] was the starting point for the present paper.

2) A G-algebra $\mathcal{A}$ contains no proper invariant ideals if and only if the action $G : X = \operatorname{Spec} \mathcal{A}$ is transitive. We have shown that any G-algebra of type N contains no proper invariant ideals. Moreover, the group of equivariant automorphisms of the homogeneous space G/H (and of the algebra $\mathcal{A}(H)$, at least if H is observable) is isomorphic to $W(H)$. Suppose that H is reductive and $W(H)$ is finite. As is obvious from what has been said any invariant subalgebra in $\mathcal{A}(H)$ has the form $\mathcal{A}(H')$, where $H \subseteq H' \subseteq G$, H' is reductive and $W(H')$ is finite, and hence G-algebras of type N are characterized by property (***).

Now consider case (ii). We are going to prove that here $\mathcal{A} = K[X]$ is an algebra of type HV following the proof of [Br89], Lemme 1.2 (see also [Po75], Th.4, [Ak77], Th.1). One may assume that X is contained as a closed G-invariant subvariety in a finite-dimensional G-module V with origin o. Let $\mathbb{P}(V \oplus K)$ be the projective space associated with $V \oplus K$, where G acts trivially on K. Denote by $\overline{X}$ the closure of X in $\mathbb{P}(V \oplus K)$. Then $\overline{X}$ intersects the hyperplane at infinity $\mathbb{P}(V)$. This shows that a maximal unipotent subgroup $U \subset G$ has at least two fixed points in $\overline{X}$. But the set of points fixed by a connected unipotent group on a connected complete variety is connected [Ho69], Th.4.1. This proves that for the open orbit $\mathcal{O} \subset X$ one has $\mathcal{O}^U \neq \emptyset$. Let v be a U-fixed vector in $\mathcal{O}$. The vector v has the form $v = \sum v_i$, where $tv_i = \chi_i(t)v_i$ with $\chi_i \in X^+(T)$ for any i and any $t \in T$. Without loss of generality it can be assumed that the group G is semisimple and hence all χ_i belong to the positive (strictly convex) Weyl chamber. Find a one-parameter subgroup $\theta : K^* \to T$ such that

(1) $<\theta, \chi_i> \geqslant 0$ for any i;

(2) there exists a non-zero χ_k (denote it by λ^*) such that $<\theta, \chi_i>=0$ if and only if χ_i is a multiple of λ^*.

Then $v_1 = \lim_{t\to 0}\theta(t)v = \sum v_j$, where the corresponding χ_j are multiples of λ^*, and v_1 is in X. By assumption on X, one has $X = Gv_1 \cup \{0\}$. Let H be the stabilizer of v_1 in G. The bijective morphism $G/H \to \mathcal{O}$ defines an inclusion $K[\mathcal{O}] \subseteq K[G/H]$. Moreover, the subgroup H contains U and $K[G/H] = \oplus_\omega E(\omega)$, where $\omega^* \ |_{T_1} = 1$ for $T_1 = H \cap T$ [Gr97], p. 98. This shows that $K[G/H] \subseteq \mathcal{A}(\lambda)$ and $\mathcal{A} = K[X] \subseteq K[\mathcal{O}]$ is a G-algebra of type HV.

LEMMA 2. — *Any invariant subalgebra of the algebra $\mathcal{A}(\lambda)$ is finitely generated.*

Proof. — Let $\mathcal{B}$ be an invariant subalgebra of $\mathcal{A}(\lambda)$. It is known that $\mathcal{B}$ is finitely generated if and only if the algebra $\mathcal{B}^U$ of U-invariants is finitely generated [Gr97], Th.16.2. But $\operatorname{Kdim} \mathcal{A}(\lambda)^U = 1$, and, by Proposition 2, $\mathcal{B}^U \subseteq \mathcal{A}(\lambda)^U$ is finitely generated. □

The proof of Theorem 1 is completed. □

6. Some results on affine embeddings.

The next proposition is a modification of a construction due to G. Kempf [Ke78].

PROPOSITION 3. — *Let G/H be a quasi-affine non affinely closed homogeneous space. Then G/H admits an affine embedding with a G-fixed point.*

Proof. — Let $G/H \hookrightarrow X$ be a nontrivial embedding and $Y \subset X$ be a proper closed irreducible invariant subvariety. Denote by $f_1, \ldots, f_k$ generators of $K[X]$ and by $g_1, \ldots, g_s$ generators of the ideal $\mathcal{I}(Y)$. One may suppose that the f_i form a basis of $< Gf_1, \ldots, Gf_k >$ and the g_i form a basis of $< Gg_1, \ldots, Gg_s >$. Consider the G-equivariant morphism

$$\psi : X \to (K^{s(k+1)})^*,$$

$$x \to (g_1(x), .., g_s(x), g_1(x)f_1(x), .., g_s(x)f_1(s), .., g_1(x)f_k(x), .., g_s(x)f_k(x)).$$

Let Z be the closure of $\psi(X)$. It is clear that Z is birationally isomorphic to X and is an affine embedding of G/H. But $\psi(Y) = \{0\}$ is a G-fixed point on Z. □

Proof of Theorem 2. — Suppose that H is a subgroup not satisfying condition (*). Consider the subgroup $H_1 = \nu(K^*)H$. The homogeneous fiber space $G *_{H_1} K$, where H acts on K trivially and H_1/H acts on K by dilation, is a two-orbit embedding of G/H.

Conversely, we need to prove that if G/H_1 is a quasi-affine homogeneous space that is not affinely closed and H is a reductive subgroup contained in H_1, then there exists a one-parameter subgroup $\nu : K^* \to C_G(H)$ such that $\nu(K^*)$ is not contained in H. By Proposition 3, there exists an affine embedding $G/H_1 \hookrightarrow X$ with a G-fixed point o. Denote by x_0 the image of eH_1 in the open orbit on X. Let $\gamma : K^* \to G$ be an adapted (to x_0) one-parameter subgroup. Consider the parabolic subgroup

$$P(\gamma) = \{g \in G \mid \lim_{t\to 0} \gamma(t)g\gamma(t)^{-1} \text{ exists in } G\}.$$

Then $P(\gamma) = L(\gamma)U(\gamma)$, where $L(\gamma)$ is a Levi subgroup that is the centralizer of $\gamma(K^*)$ in G, and $U(\gamma)$ is the unipotent radical of $P(\gamma)$. By [Ke78] (see also [PV89], Th. 5.5), the stabilizer $G_{x_0} = H_1$ is contained in $P(\gamma)$. Hence there is an element $u \in U(\gamma)$ such that $H' = uHu^{-1} \subset L(\gamma)$.

We claim that $\gamma(K^*)$ is not contained in H'. In fact, assume the converse. Then $\gamma(t)ux_0 = ux_0$ for any $t \in K^*$. Denote $\gamma(t)u\gamma(t)^{-1}$ by u_t. Then $u_t\gamma(t)x_0 = ux_0$, so that $\gamma(t)x_0 \in U(\gamma)x_0$. By assumption, $\lim_{t\to 0}\gamma(t)x_0 = o \notin Gx_0$. On the other hand, the orbit $U(\gamma)x_0$ is contained in Gx_0 and is closed in X as an orbit of a unipotent group on an affine variety [PV89], p.151. (The proof of the latter statement is based only on the Lie-Kolchin theorem, which holds in arbitrary characteristic [Hu75], 17.5].) This contradiction shows that $\gamma(K^*)$ is not contained in H' and $\gamma(K^*)$ centralizes H'. The one-parameter subgroup conjugated by $u^{-1} \in U(\gamma)$ to $\gamma(K^*)$, is the desired subgroup $\nu(K^*)$. □

Now we return to some properties of affine embeddings. Let us recall that a subgroup $Q \subset G$ is said to be *quasi-parabolic* if Q is the stabilizer of a highest weight vector v in some finite-dimensional irreducible G-module, say V_{λ^*}. If P_{λ^*} is the parabolic subgroup fixing the line $< v >$, then $Q = Q_{\lambda^*} = \{g \in P_{\lambda^*} \mid \lambda^*(g) = 1\}$.

PROPOSITION 4. — *A homogeneous space G/H admits an affine embedding $G/H \hookrightarrow X$ such that $X = G/H \cup \{o\}$, where o is a G-fixed* *point if and only if H is a quasi-parabolic subgroup of G.*

Proof. — If H is quasi-parabolic, then $X = \overline{Gv} \subset V_{\lambda^*}$ is the desired embedding.

Conversely, as was shown in the proof of Theorem 1, the subgroup H (up to conjugation) is the stabilizer of a sum of highest weight vectors with proportional weights. This shows that H is a quasi-parabolic subgroup. □

Remarks. — 1) Proposition 4 was proved by V. L. Popov [Po75], Th. 4 and Cor. 5. For a description of complete embeddings with an isolated fixed point over the field $\mathbb{C}$ see [Ak77], Th. 2.

2) The assumption that G is reductive is not essential in Proposition 4, see [Po75], Th. 3.

PROPOSITION 5. — *Let H be an observable subgroup of G.*

(1) *If either G/H is affinely closed or H is a quasi-parabolic subgroup of G, then G/H admits only one normal affine embedding (up to G-isomorphisms);*

(2) *if $G = K^*$ and H is finite, then there exist only two normal affine embeddings, namely K^*/H and K/H;*

(3) *in all other cases there exists an infinite sequence*

$$X_1 \xleftarrow{\phi_1} X_2 \xleftarrow{\phi_2} X_3 \xleftarrow{\phi_3} \dots$$

of pairwise nonisomorphic normal affine embeddings X_i of G/H and equivariant dominant morphisms ϕ_i.

Proof. — Here we give characteristic-free arguments.

(1) The statement is obvious for affinely closed spaces. If H is quasi-parabolic, then consider the subalgebra $\mathcal{B}$ in $\mathcal{A} = K[G/H]$ corresponding to a normal affine embedding of G/H. We claim that $\mathcal{A}^U = \mathcal{B}^U$. Indeed, $\mathcal{A}^U \cong K[x]$ is isomorphic to the polynomial algebra in one variable and $\mathcal{B}^U$ is a graded integrally closed subalgebra. Hence $\mathcal{B}^U = K[x^d]$. But $Q\mathcal{A} = Q\mathcal{B}$ implies $Q\mathcal{A}^U = Q\mathcal{B}^U$ and $d = 1$.

Any element of $\mathcal{A}$ is contained in $Q\mathcal{B}$. On the other hand, the algebra $\mathcal{A}$ is integral over $G\mathcal{A}^U$ [Gr97], Th. 14.3 and $G\mathcal{A}^U = G\mathcal{B}^U \subseteq \mathcal{B}$. But $\mathcal{B}$ is integrally closed and finally $\mathcal{A} = \mathcal{B}$.

(2) is obvious.

(3) In this case $K[G/H]$ contains a non-finitely generated subalgebra of type $\mathcal{A}(X,Y)$. One may suppose that X is normal. Then $\mathcal{A}(X,Y)$ is an integrally closed subalgebra in $K[G/H]$. Fix an element $g \in \mathcal{I}(Y)$ and generators $f_1, \dots, f_n$ of $K[X]$. Extend the sequence $g_0 = g, g_1 = gf_1, \dots, g_n = gf_n$ to an (infinite) generating set $g_0, g_1, \dots, g_n, g_{n+1} \dots$ of $\mathcal{A}(X,Y)$. Let $\mathcal{A}_k$ be the integral closure of $K[< Gg_0, \dots, Gg_{n+k} >]$ in $\mathcal{A}(X,Y)$. The varieties $X_i = \operatorname{Spec} \mathcal{A}_i$ are birationally isomorphic to X and $G/H \hookrightarrow X_i$. Infinitely many of X_i are pairwise nonisomorphic. Renumbering, one may suppose that all X_i are pairwise nonisomorphic. The chain

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \dots$$

corresponds to the desired chain

$$X_1 \xleftarrow{\phi_1} X_2 \xleftarrow{\phi_2} X_3 \xleftarrow{\phi_3} \dots$$

□

7. The canonical embedding.

It is easy to check that the intersection of a family of observable subgroups is again an observable subgroup. Hence, one may define *the observable hull* of a subgroup H as the intersection of all observable subgroups containing H, cf. [PV89], 3.7. It is the minimal observable subgroup containing H. Another (but equivalent) approach to the observable hull may be found in [Gr97], page 6.

DEFINITION 3. — *Let H be a subgroup of G. We say that a reductive subgroup L is* a reductive hull *of H if L is a minimal (with respect to inclusions) reductive subgroup of G containing H.*

The intersection of reductive subgroups in general is not reductive, thus a reductive hull may be not unique (see Example 3 below). Any reductive hull contains the observable hull.

Let us recall that an observable subgroup H of G is said to be a *Grosshans subgroup* if the algebra $K[G/H]$ is finitely generated. The famous Nagata counter-example to Hilbert's fourteenth problem provides an example of a unipotent subgroup in SL_{32}, which is not a Grosshans subgroup, see [Gr97].

DEFINITION 4. — *Let H be a Grosshans subgroup of G. Let us call $G/H \hookrightarrow X = \operatorname{Spec} K[G/H]$ the canonical embedding of G/H and denote it as $CE(G/H)$.*

It is well-known that the codimension of the complement of the open orbit in $CE(G/H)$ is $\geqslant 2$ and $CE(G/H)$ is the only normal affine embedding of G/H with this property [Gr97], Th.4.2. If H is reductive, then $CE(G/H)$ is the trivial embedding. For non-reductive subgroups $CE(G/H)$ is an interesting object canonically associated with the pair (G, H).

Fix some notation. There exists a canonical decomposition $K[G/H] = K \oplus K[G/H]_G$, where the first term corresponds to the constant functions and $K[G/H]_G$ is the sum of all nontrivial simple G-submodules in $K[G/H]$.

PROPOSITION 6. — *Let H be an observable subgroup in G. The following conditions are equivalent:*

(1) *a reductive hull of H in G coincides with G;*

(2) *any affine embedding of G/H contains a G-fixed point;*

(3) *$K[G/H]_G$ is a subalgebra in $K[G/H]$.*

If H is a Grosshans subgroup, then conditions (1)-(3) *are equivalent to*

(4) *$CE(G/H)$ contains a G-fixed point.*

Proof. — (1) $\Rightarrow$ (2). Suppose that $G/H \hookrightarrow X$ is an affine embedding without G-fixed point and the closed G-orbit in X is isomorphic to G/L. Then L is reductive and by the slice theorem [Lu73] H is contained in a subgroup conjugated to L.

(2) $\Rightarrow$ (1). If $H \subseteq L$, where L is a proper reductive subgroup in G, then H is observable in L and for any affine embedding $L/H \hookrightarrow Y$ the homogeneous fiber space $G *_L Y$ is an affine embedding of G/H without G-fixed point.

(3) $\Rightarrow$ (2). For any affine embedding $G/H \hookrightarrow X$ we have $K[X] = K \oplus K[X]_G$, where $K[X]_G = K[G/H]_G \cap K[X]$. Hence $K[X]_G$ is a maximal G-invariant ideal in $K[X]$ corresponding to a G-fixed point in X.

(2) $\Rightarrow$ (3). Suppose that there are $a, b \in K[G/H]_G$ such that $ab \notin K[G/H]_G$. Let $G/H \hookrightarrow X$ be any affine embedding with $K[X] = K[f_1, \ldots, f_n]$. Consider the subalgebra $\mathcal{S}$ of $K[G/H]$ generated by $f_1, \ldots, f_n$,

$< Ga >, < Gb >$. Then $G/H \hookrightarrow \operatorname{Spec} \mathcal{S}$ and $\mathcal{S}_G$ is not a subalgebra. But $\mathcal{S}_G$ is the only candidate for a maximal G-invariant ideal in $\mathcal{S}$.

(2) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3) are obvious. □

Let G be a connected semisimple group and $P \subset G$ be a parabolic subgroup containing no simple component of G. Denote by U_P the unipotent radical of P.

PROPOSITION 7. — *The homogeneous space G/U_P satisfies conditions (1)-(4) of Proposition 6.*

Proof. — It is known that U_P is a Grosshans subgroup of G [Gr97], Th. 16.4. We shall check that $K[G/U_P]_G$ is a subalgebra in $K[G/U_P]$. For this it is sufficient to find a nonnegative grading on $K[G/U_P]$ with $K[G/U_P]_G$ as the positive part.

Let $B = TU$ be a Borel subgroup in G with $B \subseteq P$ and let $P = LU_P$, where L is the Levi subgroup such that $T \subseteq L$ and $U = (U \cap L)U_P$. Denote by $T_L \subset T$ the center of L. Then $T_L = \{t \in T \mid \alpha_i(t) = 1 \, \forall i\}$, where $\{\alpha_i\}$ is the set of simple roots corresponding to P. Let $\pi : X(T) \longrightarrow X(T_L)$ be the restriction homomorphism of the groups of characters, and $X^+(T) \subset X(T)$ be the set of dominant weights (with respect to B). It is easy to check that $\pi(X^+(T))$ generates a strictly convex cone in $X(T_L) \otimes \mathbb{Q}$. Fix a one-parameter subgroup $\theta : K^* \longrightarrow T_L$ so that $< \theta, \chi >$ is positive for any $\chi \in \pi(X^+(T))$.

Note that L acts on $K[G/U_P]$ as $(l * f)(gU_P) := f(glU_P)$ and this action commutes with the G-action. The L-module $K[G/U_P]_G$ contains no trivial L-submodules because of $K[G/U_P]^L = K[G/P] = K$. On any nontrivial irreducible L-submodule T_L acts by multiplication by $\chi(t)$, $t \in T_L$, for some non-zero $\chi \in \pi(X^+(T))$. The restriction of the T_L-action to $\theta(K^*)$ defines the desired grading. □

Remark. — Let us recall that a subgroup H in G is called *epimorphic* if $K[G/H] = K$. The following generalization of Proposition 7 (and another way to prove it) was kindly communicated to us by F. D. Grosshans: if H is a Grosshans subgroup of G normalized by a maximal torus T and TH is epimorphic in G, then properties (1)-(4) of Proposition 6 hold for G/H. Conversely, for a subgroup C of G containing T the observable hull is reductive. Hence C is epimorphic if and only if C is not contained in a proper reductive subgroup of G. A criterion (in terms of roots) for C to be epimorphic may be found in [BB92], Prop. 2.

Suppose that the observable hull H_1 of a subgroup H is a Grosshans subgroup. Denote by L a reductive hull of H. Then $H_1 \subseteq L$ and the natural map $G/H_1 \to G/L$ defines a map $CE(G/H_1) \to G/L$. This shows that the closed orbit in $CE(G/H_1)$ is isomorphic to G/L. Therefore for any two reductive hulls L_1 and L_2 of H there is an element $g \in G$ such that $L_2 = g^{-1}L_1 g$. In fact, a reductive hull is not unique.

Example 3. — Let $G = SL_n$, $L = SO_n$, and H be a maximal unipotent subgroup of L. It is clear that L is a reductive hull of H. One has $H \subset U$ for some maximal unipotent subgroup U in G. There exists a subgroup H_1 such that $H \subset H_1 \subseteq U$, $\dim H_1 = \dim H + 1$ and H is a normal subgroup of H_1. Consider an element $h_1 \in H_1 \setminus H$. Then $h_1^{-1} L h_1$ is another reductive hull of H.

8. The case of positive characteristic.

If we follow the proof of Theorem 1 over any algebraically closed field K, also the two cases, (i) and (ii), will appear. The consideration of case (ii) and the proof of Lemma 2 go on without any changes. On the other hand, for every $\omega \in X^+(T)$ the submodule $E(\omega)$ contains a simple G-submodule having highest weight ω, but $E(\omega)$ may be not simple, and a G-algebra of type HV is not determined by the semigroup P.

Example 4. — Suppose that $\operatorname{char} K = 2$, $G = SL_2(K)$ and G acts on $\mathcal{A} = K[x_1, x_2]$ as in Example 1. Then the invariant subalgebras $K[x_1^2, x_2^2]$, or $K[x_1^2, x_2^2, x_1^3 x_2, x_1 x_2^3]$, are not of the form $\mathcal{A}(P, \lambda)$.

The author does not know a "constructive" description of G-algebras of type HV in the case $\operatorname{char} K > 0$.

For case (i), we need to find an analog of affinely closed spaces in positive characteristic. Suppose that G acts on an affine variety X. The orbit Gx of a point $x \in X$ is not determined (up to G-isomorphism) by the stabilizer $H = G_x$, and it is natural to consider the isotropy subscheme H' at x, with H as the reduced part, identifying Gx and G/H'. There is a natural bijective purely inseparable and finite morphism $\pi : G/H \to G/H'$ [Hu75], 4.3, 4.6.

PROPOSITION 8. — *The homogeneous space G/H is affinely closed if and only if G/H' is affinely closed.*

Proof. — 1) Note that $K(G/H)^{p^s} \subseteq \pi^* K(G/H')$ and $K[G/H]^{p^s} \subseteq \pi^* K[G/H']$ for some $s \geqslant 0$, where $p = \operatorname{char} K$ if $\operatorname{char} K > 0$ and $p = 1$ otherwise. If G/H is not affinely closed, then there is a nontrivial affine embedding $G/H \hookrightarrow X$. The algebra $\mathcal{C} := K[X] \cap \pi^* K(G/H')$ is finite over $K[X]^{p^s}$. Hence $\mathcal{C}$ is finitely generated, and $X' := \operatorname{Spec} \mathcal{C}$ contains G/H' as an open subset:

$$\begin{array}{ccc} G/H & \hookrightarrow & X \\ \downarrow \pi & & \downarrow \pi' \\ G/H' & \hookrightarrow & X'. \end{array}$$

On the other hand, the morphism $\pi' : X \to X'$ defined by the inclusion $\mathcal{C} \subset K[X]$ is finite. This shows that $G/H' \neq X'$.

2) Suppose that G/H' admits a non-trivial affine embedding $G/H' \hookrightarrow X'$. Consider the integral closure $\mathcal{B}$ of $K[X']$ in the field $K(G/H)$. The variety $X = \operatorname{Spec} \mathcal{B}$ carries a G-action with an open G-orbit isomorphic to G/H, and the finite morphism $X \to X'$ is surjective, hence X is a nontrivial embedding of G/H. □

Definition 5. — *A reductive subgroup H of the group G is* strongly affinely closed *if for any affine G-variety X and any point $x \in X$ fixed by H the orbit Gx is closed in X.*

Below we list some results concerning case (i). It follows from the proof of Theorem 1 that:

(1) if H is reductive and any invariant subalgebra in $K[G/H]$ is finitely generated, then G/H is affinely closed;

(2) if G/H is strongly affinely closed, then any invariant subalgebra in $K[G/H]$ is finitely generated.

The following notion was introduced by Serre, cf. [LS03].

Definition 6. — *A subgroup $D \subset G$ is called G-completely reducible (G-cr for short) if whenever D is contained in a parabolic subgroup P of G, it is contained in a Levi subgroup of P.*

For $G = SL(V)$ this notion agrees with the usual notion of complete reducibility. In fact, if G is any of the classical groups then the notions coincide, although for symplectic and orthogonal groups this requires the assumption that $\operatorname{char} K$ is a good prime for G. The class of G-cr subgroups is wide. Some conditions which guarantee that certain subgroups satisfy the G-cr condition may be found in [McN98], [LS03].

The proof of Theorem 2 implies:

(3) if H is not contained in any parabolic subgroup of G, then G/H is strongly affinely closed;

(4) if H does not satisfy (*), then G/H is not affinely closed;

(5) if H is a G-cr subgroup, then G/H is affinely closed iff G/H is strongly affinely closed iff H satisfies (*).

Example 5. — The following example kindly produced by George J. McNinch on our request shows that the group $W(H) = N_G(H)/H$ may be unipotent even for reductive H. Let L be the space of $(n \times n)$-matrices and H be the image of SL_n in $G = SL(L)$, acting on L by conjugations.

If $p = \operatorname{char} K \mid n$, then L is an indecomposable SL_n-module with 3 composition factors, cf. [McN98], Prop. 4.6.10 a). It turns out that $C_G(H)^0$ is a one-dimensional unipotent group consisting of operators of the form $\operatorname{Id} + aT$, where $a \in K$, and T is a nilpotent operator on L defined by $T(X) = \operatorname{tr}(X)E$.

For example, in the simplest case $p = 2$, we have that $H \cong PSL_2 \subset SL_4$, $N_G(H) = HC_G(H)$ (because H does not have outer automorphisms), $C_G(H)$ is connected, and $W(H) \cong (K, +)$. In this case H is contained in a quasi-parabolic subgroup, hence G/H is not strongly affinely closed.

9. Problems.

In this section we collect some problems that follow naturally from the discussion above.

PROBLEM 1. — *Suppose that* $\operatorname{char} K 0$. *Classify all affinely closed homogeneous spaces. Is it true that any affinely closed space is strongly affinely closed?*

PROBLEM 2. — *Let G be a linear algebraic group. Characterize all G-algebras $\mathcal{A}$ such that any invariant subalgebra in $\mathcal{A}$ is finitely generated.*

This class of algebras seems to be much wider than in the reductive case.

PROPOSITION 9. — *Let G be a reductive group, S be a unipotent group, $H \subset G$ be a subgroup satisfying condition (*), and $F \subset S$*

be any closed subgroup. Then any $G \times S$-invariant subalgebra in $\mathcal{A} = K[(G \times S)/(H \times F)]$ is finitely generated.

Proof. — Fix the notation: $\mathcal{A}_1 = K[G/H]$, $\mathcal{A}_2 = K[S/F]$, $\mathcal{B}$ is an invariant subalgebra in $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. It is clear that $\mathcal{A}^S = \mathcal{A}_1 \otimes K = \mathcal{A}_1$.

It is sufficient to prove that $\mathcal{B}$ contains no proper invariant ideals. (After this we complete the proof following the proof of Lemma 1.)

Let $\mathcal{I} \subset \mathcal{B}$ be an invariant ideal. By the Lie-Kolchin theorem, $\mathcal{I}^S \neq 0$. Hence $\mathcal{I}^S$ is a non-zero ideal in $\mathcal{B} \cap \mathcal{A}_1$. But any invariant subalgebra in $\mathcal{A}_1$ contains no proper G-invariant ideals. Therefore, we have $\mathcal{I}^S = \mathcal{B} \cap \mathcal{A}_1$ and $\mathcal{I}^S$ contains constants, thus $\mathcal{I} = \mathcal{B}$. □

This proof shows that $(G \times S)/(H \times F)$ is an affinely closed homogeneous space.

PROBLEM 3. — *Characterize all affinely closed homogeneous spaces of a linear algebraic group G.*

The last problem concerns canonical embeddings. Let us recall that the modality of a G-variety X is the maximal number of parameters in a continuous family of G-orbits on X, or, more formally,

$$\mathrm{mod}_G(X) = \max_{Y \subseteq X} \ \mathrm{tr}\,.\deg K(Y)^G,$$

where Y runs through all closed irreducible invariant subvarieties in X.

PROBLEM 4. — *Let H be a Grosshans subgroup of a reductive group G. Find the modality of $CE(G/H)$. In particular, characterize Grosshans subgroups H of G such that $CE(G/H)$ contains a finite number of G-orbits.*

One may suppose that a reductive hull of H is G. Indeed, if a reductive hull of H is L, then, by the slice theorem, $CE(G/H) = G *_L CE(L/H)$ and $\mathrm{mod}_G(CE(G/H)) = \mathrm{mod}_L(CE(L/H))$.

Example 6. — Let $G = SL_n$ and H be the unipotent radical of the maximal parabolic subgroup in G corresponding to the first $(n-2)$ simple roots. It is clear that $CE(G/H) \cong K^n \times \ldots \times K^n$ ($(n-1)$ copies) with the diagonal G-action. This space is covered by finitely many locally closed G-invariant subsets $S_{i_1,\ldots,i_k}$, where $S_{i_1,\ldots,i_k}$ is the set of $(n \times (n-1))$-matrices of rank k with linearly independent columns $i_1 \ldots, i_k$. An orbit in $S_{i_1,\ldots,i_k}$ depends on $k(n-1-k)$ parameters, which are the coefficients of linear expressions of the remaining $n-1-k$ columns in terms of the columns $i_1, \ldots, i_k$. Hence the maximal number of parameters is

$$\mathrm{mod}_G(CE(G/H)) = s^2 \text{ for } n = 2s+1$$

and

$$\mathrm{mod}_G(CE(G/H)) = s^2 - s \text{ for } n = 2s.$$

BIBLIOGRAPHY

[Ak77] D. N. AKHIEZER, Dense orbits with two ends, Izv. Akad. Nauk SSSR, Ser. Mat., 41-2 (1977), 308–324 (in Russian); English trans.: Math. USSR-Izv., 11-2 (1977), 293–307.

[AT01] I. V. ARZHANTSEV and D. A. TIMASHEV, Affine embeddings with a finite number of orbits, Transformation Groups, 6-2 (2001), 101–110.

[BB92] F. BIEN and A. BOREL, Sous-groupes épimorphiques de groupes linéaires algébriques I, C. R. Acad. Sci. Paris, 315, Série I (1992), 649–653.

[Br89] M. BRION, Groupe de Picard et nombres caractéristiques des variétés sphériques, Duke Math. J., 58-2 (1989), 397–424.

[Gr97] F. D. GROSSHANS, Algebraic Homogeneous Spaces and Invariant Theory, LNM 1673, Springer-Verlag Berlin, 1997.

[Ho69] G. HORROKS, Fixed point schemes of additive group actions, Topology, 8 (1969), 233–242.

[Hu75] J. E. HUMPHREYS, Linear Algebraic Groups, Grad. Texts in Math., 21, Springer-Verlag, New-York, 1975.

[Ke78] G. KEMPF, Instability in invariant theory, Ann. of Math., 108-2 (1978), 299–316.

[La99] I. A. LATYPOV, Homogeneous spaces of compact connected Lie groups which admit nontrivial invariant algebras, Journal of Lie Theory, 9 (1999), 355–360.

[LS03] M. W. LIEBECK and G. M. SEITZ, Variations on a theme of Steinberg, Journal of Algebra, 260 (2003), 261–297.

[Lu73] D. LUNA, Slices étales, Bull. Soc. Math. France, Paris, Mémoire 33 (1973), 81–105.

[Lu75] D. LUNA, Adhérences d'orbite et invariants, Invent. Math., 29 (1975), 231–238.

[LR79] D. LUNA and R. W. RICHARDSON, A generalization of the Chevalley restriction theorem, Duke Math. J., 46-3 (1979), 487–496.

[McN98] G. J. MCNINCH, Dimensional criteria for semisimplicity of representations, Proc. London Math. Soc. (3), 76 (1998), 95–149.

[Po75] V. L. POPOV, Classification of three-dimensional affine algebraic varieties that are quasihomogeneous with respect to an algebraic group, Izv. Akad. Nauk SSSR, Ser. Mat., 39:3 (1975), 566-609 (in Russian); English trans.: Math. USSR-Izv., 9 (1975), 535–576.

[PV72] V. L. POPOV and E. B. VINBERG, A certain class of quasihomogeneous affine algebraic varieties, Izv. Akad. Nauk SSSR, Ser. Mat., 36 (1972), 749–764 (in Russian); English trans.: Math. USSR-Izv., 6 (1972), 743–758.

[PV89] V. L. Popov and E. B. Vinberg, Invariant Theory, Itogy Nauki i Tekhniki, Sovr. Problemy Mat. Fund. Napravlenia, vol. 55, VINITI, Moscow, 1989, 137–309 (in Russian); English trans.: Algebraic Geometry IV, Encyclopaedia of Math. Sciences, vol. 55, Springer-Verlag Berlin, 1994, 123–278.

[Ri77] R. W. Richardson, Affine coset spaces of reductive algebraic groups, Bull. London Math. Soc., 9 (1977), 38–41.

[Su88] A. A. Sukhanov, Description of the observable subgroups of linear algebraic groups, Mat. Sbornik, 137:1 (1988), 90–102 (in Russian); English trans.: Math. USSR-Sb., 65-1 (1990), 97–108.

Manuscrit reçu le 30 mai 2002,
accepté le 3 octobre 2002.

Ivan V. ARZHANTSEV,
Moscow State University
Department of Mathematics and Mechanics
Chair of Higher Algebra
Vorobievy Gory
119992, GSP-2, Moscow (Russia).
arjantse@mccme.ru