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Finiteness property for generalized abelian integrals

FINITENESS PROPERTY
FOR GENERALIZED ABELIAN INTEGRALS

by Rémi SOUFFLET

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1. Introduction and results.

We note $\mathbf{P}_1$ the real projective line with its standard analytic structure. If $n \in \mathbb{N}$, all subsets of $\mathbb{R}^n$ are naturally embedded in the analytic manifold $\mathbf{P}_1^n$.

We assume that the functions log and power functions $x \mapsto x^\gamma$, $\gamma \in \mathbb{R}$, are defined on $\mathbb{R}$ and are equal to 0 out of $]0, +\infty[$. A (real) power map $\Gamma : \mathbb{R}^p \to \mathbb{R}^p$ is the data of p real numbers $(\gamma_1, ..., \gamma_p)$ and is defined by $\Gamma(x_1, ..., x_p) = (x_1^{\gamma_1}, ..., x_p^{\gamma_p})$.

A subset X of $\mathbb{R}^n$ is a *globally semianalytic set* if it is defined, in a neighbourhood of any point of $\mathbf{P}_1^n$, by a finite number of equalities and inequalities satisfied by analytic functions. A *globally subanalytic set* of $\mathbb{R}^n$ is the image of a globally semianalytic set of $\mathbb{R}^n \times \mathbb{R}^m$ by the canonical projection from $\mathbb{R}^{n+m}$ to $\mathbb{R}^n$. A *globally subanalytic map* is a map such that its graph is globally subanalytic. A globally subanalytic function is a globally subanalytic map from $\mathbb{R}^n$ to $\mathbb{R}$.

The functions we will deal with can be defined as follows.

DEFINITION 1.1. — *A x^λ-map $f : \mathbb{R}^n \to \mathbb{R}^p$ is a finite composition of globally subanalytic maps and power maps. A x^λ-function is a x^λ-map from $\mathbb{R}^n$ to $\mathbb{R}$.*

This kind of functions has been studied by Khovanskii [Kh2], Tougeron [To] and Miller [Mi]. The importance of their geometry takes back to Dulac's problem [Du]. Namely, they appear in the study of the Poincaré return map (near a polycycle) associated to an analytic vector field (see [Mo] and [MR]). More recently, Grigoriev and Singer showed that real power functions and series of real power functions appear as solutions of algebraic differential equations [GS]. The understanding of their geometry is thus essential in the theory of differential equations and foliations.

In [LR2], Lion and Rolin study the volume of globally subanalytic sets. They show that the integration of a globally subanalytic function on a globally subanalytic set leads to a function which belongs to the class of $\mathcal{LE}$-*functions*. They can be defined as finite compositions of globally subanalytic maps with the functions exp and log. From [DMM], these functions satisfy the following finiteness property:

$\mathfrak{F}$: *there exists an integer N which bounds the number of connected components of the fibers $f^{-1}(t)$, uniformly with respect to $t \in \mathbb{R}$.*

Such a property is always true in an *o-minimal structure*, a definition of which is the following.

DEFINITION 1.2. — *Let $\mathfrak{A}_n$ be a collection of subsets of $\mathbb{R}^n$. We say that $\mathfrak{A} = \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$ is an o-minimal structure if*

   – it contains the semialgebraic subsets of the spaces $\mathbb{R}^n$, $n \in \mathbb{N}$,

   – for all n, $\mathfrak{A}_n$ is a boolean subalgebra of $\mathcal{P}(\mathbb{R}^n)$,

   – the elements of $\mathfrak{A}$ are stable under cartesian product and linear projection,

   – $\mathfrak{A}_1$ consists of the finite unions of points and intervals.

If $\mathfrak{A}$ is an o-minimal structure, a map $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be *definable* in $\mathfrak{A}$ and is called a $\mathfrak{A}$-function if its graph belongs to the structure $\mathfrak{A}$.

The first examples of o-minimal structures are the semialgebraic sets (it is a result of Tarski [Ta], see also [BCR],[BR]) and the globally subanalytic sets [Ga]. As a consequence of Khovanskii's theory [Kh2], the

property $\mathfrak{F}$ is true for the class of x^λ-functions. Miller specifies this result by showing that the x^λ-sets form an o-minimal structure [Mi]. The works of Wilkie and of van den Dries, Macintyre and Marker extend the o-minimality to the class of $\mathfrak{LE}$-sets [DMM]. Consequently, the results of [LR2] give an other proof of the finiteness properties of abelian integrals established by Varchenko [Va] and Khovanskii [Kh1].

Our aim is to generalize the results of [LR2] to the class of x^λ-functions.

Parameter family of x^λ-functions.— Consider a x^λ-function $f = h_k \circ \Gamma_k \circ ... \circ h_1 \circ \Gamma_1$ where the h_i's are globally subanalytic maps and the Γ_i's are real power maps. We consider that the maps h_i are fixed and the maps Γ_i are taken as parameters. The data of the maps Γ_i is equivalent to the data of a multidimensional parameter $\gamma \in \mathbb{R}^N$. This way, we consider a parameter family of x^λ-functions we denote $(f_\gamma)_{\gamma \in \mathbb{R}^N}$.

If we consider a function f depending on a real parameter γ, then it can happen a change of dependency under integration. Let us take an example to illustrate this fact. Consider the 1-parameter family of functions $(g_\gamma)_\gamma$ defined by $g_\gamma(x) = x^\gamma$. Its integration leads to the functions $G_\gamma(x) = \frac{1}{\gamma+1} x^{\gamma+1}$ if $\gamma \neq -1$ or the function log if $\gamma = -1$. The functions g_γ and G_γ are x^λ-functions but they have not the same dependency with respect to γ. In the case of G_γ, a rational dependency with a pole at -1 occurs. Thus we should better formulate the integration process under the following form: define the functions $G_{\mu,\nu}$ for $\mu, \nu \in \mathbb{R}$ and $\mu \neq -1$ by

$$G_{\mu,\nu}(x) = \frac{1}{\mu+1} x^{\nu+1}.$$

Then $G_{\gamma,\gamma}$ is a primitive of g_γ for all $\gamma \neq -1$. Of course, in this elementary example, the function $(\mu, \nu, x) \mapsto G_{\mu,\nu}(x)$ has an explicit finiteness property. Nevertheless, the case of several variables (at least two) leads to dependencies that can be much wilder. This may happen for instance when considering a convergent sum of functions $G_{\mu,\nu}$ for infinite different values of μ.

Results.— One can find in [So1] the announcement of these results (without proof). A function of the form $f = P(t_1, ..., t_d, \log t_1, ..., \log t_d)$, where P is a polynomial and the t_i's are x^λ-functions, is called a $\mathfrak{G}$-*function*. As in the case of x^λ-functions, we consider parameter families of $\mathfrak{G}$-functions $(f_\gamma)_{\gamma \in \mathbb{R}^N}$ or $(f_{\gamma,P})$ where γ is the parameter coming from the x^λ-functions t_i. Our main result is the following.

THEOREM 1.3. — Let $(f_\gamma)_{\gamma \in \mathbb{R}^N}$ be a parameter family of $\mathfrak{G}$-functions of $\mathbb{R}^n \times \mathbb{R}$. There exists a subset $D \subset \mathbb{R}^N$ of full Lebesgue measure such that, for all $\gamma \in D$, for all x^λ-functions φ, ψ of $\mathbb{R}^n$, the function $F_\gamma(x) = \int_{\varphi(x)}^{\psi(x)} f_\gamma(x, y) dy$ is a $\mathfrak{G}$-function.

To clarify this result, we can formulate it under the following form: there exists a parameter family $(G_\delta)_{\delta \in \mathbb{R}^M}$ of functions of $\mathbb{R}^n$ such that, if $\gamma \in D$, then $F_\gamma = G_{\delta(\gamma)}$. But the dependency of G_δ with respect to γ is still not well understood.

Theorem 1.3 says that for *almost all value* of the exponent parameter γ, the integration of a $\mathfrak{G}$-function on a x^λ-set belongs to the o-minimal class of $\mathfrak{LE}$-functions. From [DMM], we are then able to derive the following finiteness result. It can be seen as a partial generalization of [Va] and [Kh1].

THEOREM 1.4. — Let $(f_{\gamma, P})$ be a parameter family of $\mathfrak{G}$-functions of $\mathbb{R}^{n+1}$ and let (φ_μ) and (ψ_ν) be parameter families of x^λ-functions of $\mathbb{R}^n$. We note ξ the parameter given by $\gamma \in \mathbb{R}^N, \mu, \nu$ and P. Then the function $F_\xi(x) = \int_{\varphi_\mu(x)}^{\psi_\nu(x)} f_{\gamma, P}(x, y) dy$ satisfies: there exists a subset $D \subset \mathbb{R}^N$ of full Lebesgue measure such that, for all non negative integer m, for all $\gamma \in D$, there exists a constant $K(m, \gamma) \in \mathbb{N}$ which bounds the number of connected components of the level sets $\{F_\xi = c\}$, uniformly with respect to $c \in \mathbb{R}$, the parameters μ and ν, and the polynomial P of degree less or equal to m.

The paper is organized as follows. In Section 2, we prove a first integration result in the case of families of *reduced* x^λ-functions. This is done thanks to the preparation theorem of Parusiński [Pa], Lion and Rolin [LR1] which allows to localize the integration study on suitable x^λ-cylinders. In Section 3, we study the relationships between the local parameters and the global ones and we give a result in the spirit of Theorem 1.3 in the case of x^λ-functions. The proof of Theorem 1.3 is done in Section 4. Section 5 is devoted to the proof of Theorem 1.4.

2. The reduced case.

The first step in the proof of Theorem 1.3 consists in restricting the study to suitable *reduced* functions.

Reduced x^λ-functions and diophantine conditions. — Thanks to the preparation theorem of Parusiński [Pa] (see also [LR1]), a x^λ-function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ can be *reduced* to the following form:

$$f_{|C}(x, y) = y_1^{\delta_0} A(x) U(x, y_1^{\delta_1}, ..., y_1^{\delta_s})$$

on a partition of $\mathbb{R}^{n+1}$ into finitely many x^λ-cylinders C. We refere to [LR1] for the details. In such an expression, the δ_i's are real numbers and $\delta_0 = \sum_{\ell=1}^{s} m_\ell \delta_\ell / q$ is a rational combination of $\delta_1, ..., \delta_s$, $y_1 = |y - \theta(x)|$ where θ is a x^λ-function equal to 0 or comparable with y on C, A is a x^λ-function and U is a x^λ-unit. Under this form, one can give sufficient conditions under which the formal integration of f with respect to y is convergent. This leads to the following definition.

DEFINITION 2.1. — Let $\delta = (\delta_1, ...\delta_s) \in \mathbb{R}^s$ and let c, ν be positive real numbers. We say that δ is of (c, ν)-type if, for all $(n_0, ..., n_s) \in \mathbb{Z}^{s+1} \setminus \{0\}$, we have $|n_0 + n_1\delta_1 + ... + n_s\delta_s| \geqslant c/(\sum_{i=0}^{s} |n_i|)^\nu$.

We denote $\mathfrak{D}_{c,\nu}^s$ the set of (c, ν)-type numbers of $\mathbb{R}^s$ and $\mathfrak{D}_\nu^s = \cup_{c>0} \mathfrak{D}_{c,\nu}^s$. It is a well-known fact that $\mathfrak{D}_\nu^s$ has full Lebesgue measure if $\nu > s + 2$ (see [Ar] section 5.24.C). Moreover, the following lemma is just a consequence of the multi-dimensional Cauchy's criterion for analytic functions.

LEMMA 2.2. — Let $V(X, Y, Z) = \sum_{I=(k,i,j)} a_I X^k Y^i Z^j$ be an analytic function of $\mathbb{R}^{p+2s}$ where $X \in \mathbb{R}^p$, $Y, Z \in \mathbb{R}^s$, and let D be a polydisc of convergence of V. Let $\delta = (\delta_1, ..., \delta_s) \in \mathfrak{D}_\nu^s$ and $b_I = a_I/(1 + \sum_{\ell=1}^{s} \delta_\ell (i_\ell - j_\ell))$ for all $I \in \mathbb{N}^{p+2s}$. Then the series $W(X, Y, Z) = \sum_I b_I X^k Y^i Z^j$ defines an analytic function in D.

The following *cutting lemma* will be useful in the sequel.

LEMMA 2.3. — Let $f(x, y) = y_1^{\delta_0} A(x) U(x, y_1^{\delta_1}, ..., y_1^{\delta_s})$ be a reduced x^λ-function on a x^λ-cylinder C. There exist x^λ-functions g_1 of $\mathbb{R}^{n+1}$ and g_2 of $\mathbb{R}^n$ such that $f(x, y) = g_1(x, y) + g_2(x)/y_1$.

Proof. — Recall that $U = V \circ \psi$ where

$$\psi(x, y_1^{\delta_1}, ..., y_1^{\delta_s}) = (c_1(x), ..., c_p(x), y_1^{\delta_1}/a_1(x), ..., y_1^{\delta_s}/a_s(x),$$
$$b_1(x)/y_1^{\delta_1}, ..., b_s(x)/y_1^{\delta_s})$$

is a so-called *morphism of reduction* with values in $[-1, 1]^{p+2s}$ and $V(X, Y, Z) = \sum_{I=(k,i,j)} a_I X^k Y^i Z^j$ is an analytic unit of polyradius of convergence $(2, ..., 2)$. The functions c_k, a_i and b_j are x^λ-functions of $\mathbb{R}^n$. We define $E = \{I \in \mathbb{N}^{p+2s} \mid 1 + \delta_0 + \sum_{\ell=1}^{s} \delta_\ell (i_\ell - j_\ell) = 0\}$ and $F = \mathbb{N}^{p+2s} \setminus E$ and the corresponding analytic functions

$$V_1(X, Y, Z) = \sum_{I \in F} a_I X^k Y^i Z^j \quad , \quad V_2(X, Y, Z) = \sum_{I \in E} a_I X^k Y^i Z^j$$

considering that these functions are identically equal to 0 if the corresponding set E or F is empty. Then we have $g_1(x,y) = y_1^{\delta_0} A(x) V_1(x, y_1^{\delta_1}, ..., y_1^{\delta_s})$ and the function g_2 is given by $g_2(x) = y_1^{\delta_0+1} A(x) V_2(x, y_1^{\delta_1}, ..., y_1^{\delta_s})$. $\qquad\square$

We can then prove some sort of *almost stability* result for the integration of reduced x^λ-functions. This is the following proposition.

PROPOSITION 2.4. — Let $f(x,y) = y_1^{\delta_0} A(x) U(x, y_1^{\delta_1}, ..., y_1^{\delta_s})$ be a reduced x^λ-function on a x^λ-cylinder C. There exists a subset $\mathbb{Q}^s \subset D \subset \mathbb{R}^s$ of full Lebesgue measure such that, if $\delta = (\delta_1, ..., \delta_s) \in D$, then there exist x^λ-functions G_1 of $\mathbb{R}^{n+1}$ and g_2 of $\mathbb{R}^n$ such that $F(x,y) = G_1(x,y) + g_2(x) \log y_1$ is a primitive of f with respect to y.

Proof. — It suffices to prove that such a D contains the sets $\mathbb{Q}^s$ and $\mathfrak{D}_\nu^s$, $\nu > s+2$. From Lemma 2.3, we have $f = g_1 + g_2/y_1$. Hence the problem is to find sufficient conditions under which g_1 admits a x^λ-primitive with respect to y. Up to a sub-decomposition of C into 2 x^λ-cylinders, we can assume that $y - \theta(x)$ has constant sign on C.

– Assume that $\nu > s + 2$ and $\delta \in \mathfrak{D}_\nu^s$. In this case, the function g_2 is identically equal to 0. Then we get

$$g_1(x,y) = y_1^{\delta_0} A(x) V(c_1(x), ..., c_p(x), y_1^{\delta_1}/a_1(x), ..., y_1^{\delta_s}/a_s(x),$$
$$b_1(x)/y_1^{\delta_1}, ..., b_s(x)/y_1^{\delta_s})$$
$$= A(x) \sum_{I=(k,i,j)} a_I c(x)^k \frac{b(x)^j}{a(x)^i} y_1^{\alpha_{i,j}}$$

where $\alpha_{i,j} = \sum_{\ell=1}^s (i_\ell - j_\ell + m_\ell/q) \delta_\ell$. As $\alpha_{i,j} \neq -1$, we put $b_I = a_I/(1+\alpha_{i,j})$ and we define the series $W(X,Y,Z) = \sum_I b_I X^k Y^i Z^j$. From Lemma 2.2, the condition $\delta \in \mathfrak{D}_\nu^s$ implies that W is analytic on the same polydisc of convergence as V. We then consider

$$F(x,y) = y_1^{\delta_0+1} A(x) W(c_1(x), ..., c_p(x), y_1^{\delta_1}/a_1(x), ..., b_s(x)/y_1^{\delta_s}).$$

This is a x^λ-function on C and an elementary computation shows that $\partial_y F = g_1$.

– If we assume that $\delta \in \mathbb{Q}^s$ then we can use the cutting Lemma 2.3 one more time to deduce that g_1 admits a x^λ-primitive with respect to y. The computations being quite the same as in the previous case, we do not give the details. $\qquad\square$

3. Some intermediate results.

When considering a parameter family of x^λ-functions $(f_\gamma)_{\gamma \in \mathbb{R}^N}$, the definition implies that γ is a *global* parameter. Conversely, in the expression of a reduced x^λ-function on a cylinder, the real exponents $(\delta_1, ..., \delta_s)$ are some kind of *local* parameters. To prove Theorem 1.3, it is necessary to understand how the local parameters depend on the global one. This is the aim of the following results.

LEMMA 3.1. — *Let $(f_\gamma)_{\gamma \in \mathbb{R}^N}$ be a parameter family of x^λ-functions of $\mathbb{R}^{n+1}$. If f_γ is reduced to the form $f_\gamma(x, y) = y_1^{\delta_0} A(x) U(x, y_1^{\delta_1}, ..., y_1^{\delta_s})$ on a x^λ-cylinder, then the δ_i's are polynomials in γ with rational coefficients.*

Proof. — We proceed by induction on the number m of globally subanalytic maps in the definition of f_γ.

Case $m = 1$. — We put $f_\gamma(x, y) = (h_1 \circ \Gamma_1)(x, y) = h_1(x^\mu, y^{\gamma_1})$ where $\mu \in \mathbb{R}^n$, $\gamma_1 \in \mathbb{R}$ and $h_1 = (h_{1,1}, ..., h_{1,p})$ is a globally subanalytic map. It suffices to apply the preparation theorem for globally subanalytic functions [Pa],[LR1]. The functions $h_{1,j}$ can be simultaneously reduced to the form

$$h_{1,j}(x^\mu, y^{\gamma_1}) = y_1^{q_j \gamma_1} A_j(x^\mu) U_j(x^\mu, y^{\gamma_1}).$$

Hence the conclusion holds in the case $m = 1$.

Step of induction. — Assume that, after m compositions of terms of the form $h_i \circ \Gamma_i$, we get p functions $g_1, ..., g_p$ and a x^λ-cylinder C such that the g_i's verify simultaneously the conclusion of the lemma. We can then write

$$g_i(x, y) = y_1^{\delta_{0,i}} A_i(x) U_i(x, y_1^{\delta_1}, ..., y_1^{\delta_s}).$$

Notice that we can assume that the polynomials $\delta_1, ..., \delta_s$ are common to the reduced functions g_i (it suffices to add zero terms in the units U_i). We put $\Gamma_{m+1} = (\lambda_1, ..., \lambda_p) : \mathbb{R}^p \to \mathbb{R}^p$ identifying the power map and the corresponding vector in $\mathbb{R}^p$. We have: $g_i^{\lambda_i} = y_1^{\delta_{0,i} \lambda_i} A_i^{\lambda_i} U_i^{\lambda_i}$. Moreover, U_i being a x^λ-unit, the function $\tilde{U}_i = U_i^{\lambda_i}$ is also a x^λ-unit. It comes

$$g_i^{\lambda_i}(x, y) = y_1^{\delta_{0,i} \lambda_i} B_i(x) \tilde{U}_i(x, y_1^{\delta_1}, ..., y_1^{\delta_s}).$$

If $h_{m+1} = (h_{m+1,1}, ..., h_{m+1,t})$ is a globally subanalytic map, then we have

$$h_{m+1,j}(g_1^{\lambda_1}, ..., g_p^{\lambda_p}) = H_{m+1,j}(a(x), y_1^{\delta_{0,1} \lambda_1}, ..., y_1^{\delta_{0,p} \lambda_p}, y_1^{\delta_1}, ..., y_1^{\delta_s})$$

where the $H_{m+1,j}$'s are globally subanalytic functions for all j and a is x^λ-map. The exponents in the variable y_1 that appear in this expression are all polynomials in the global parameter γ with rational coefficients. The result follows now as in the case $m = 1$ (see Proposition 4 of [LR1]).   □

The second intermediate result is arithmetical: from Lemma 3.1, we must prove that the set of $\gamma \in \mathbb{R}^N$ such that the s-tuple $(P_1(\gamma), ..., P_s(\gamma))$ satisfies a diophantine condition (with P_i polynomial with rational coefficients) has full Lebesgue measure.

Before proving such a statement, we start with more general results which may be of its own interest. In the following, B_1 denotes the closed ball of $\mathbb{R}^N$ centered at the origin and of radius 1.

LEMMA 3.2. — *Let (f_t) be a continuous globally subanalytic family of analytic functions of B_1 where the parameter t varies in a compact subset. Assume that, for all t, f_t does not vanish identically. Then there exist constants $c > 0$ and $\nu > 0$, with $\nu \in \mathbb{Q}$, such that, for all t and for all $\varepsilon \in]0, 1]$, we have*

$$\mathrm{Vol}_N(X_{t,\varepsilon}) \leqslant c\varepsilon^\nu$$

where $X_{t,\varepsilon} = \{x \in B_1 \mid |f_t(x)| < \varepsilon\}$ and Vol_k denotes the k-volume in $\mathbb{R}^N$ for all $k \leqslant N$.

Proof. — We proceed in 4 steps.

Step 1. — Let (Z_s) be a globally subanalytic family of globally subanalytic subsets of B_2 of dimension at most $N - 1$. There exists a constant $K > 0$ which bounds the $(N - 1)$-volumes of the Z_s's uniformly with respect to s.

This is a direct consequence of the finiteness property of globally subanalytic sets (Gabrielov's property) and the Cauchy-Crofton Formula.

Step 2. — Let (Y_u) be a globally subanalytic family of globally subanalytic subsets of B_1 of dimension at most $N - 1$. If $r \in]0, 1]$, we note $Y_{u,r} = \{x \in B_2 \mid d(x, Y_u) < r\}$ and $Z_{u,r}$ the frontier of $Y_{u,r}$. Then there exists $K > 0$ which bounds the $(N - 1)$-volumes of the $Z_{u,r}$'s and such that Kr bounds the N-volumes of the $Y_{u,r}$'s for all $r \in]0, 1]$.

We start with a general result about the volumes. Let A be a compact subanalytic set of dimension at most $N - 1$. We set $A^0 = A$, $A^s = \{x \in \mathbb{R}^N \mid d(x, A^0) = s\}$ for $s > 0$ and $A_r = \{x \in \mathbb{R}^N \mid d(x, A^0) \leqslant r\}$

for $r > 0$. Let us prove that (we use arguments inspired by the study of tubular neighbourhoods made in [BG]):

$$\mathrm{Vol}_N(A_r) = \int_0^r \mathrm{Vol}_{N-1}(A^s)ds.$$

The function distance to A^0 being subanalytic and proper, there exists a finite subanalytic stratification of $\mathbb{R}^N$ such that, if X is one of these strata, we have: either X has dimension less than $N - 1$, either X has dimension equal to N, is open and $d(., A)_{|X}$ is a submersion. The strata of dimension less than $N - 1$ being of no interest for the computation of the volume, let us consider an open stratum X. We note $X^s = A^s \cap X$ and $X_r = A_r \cap X$. The set X^s is a hypersurface and let us choose $x \in X$. Let u be a vector normal to X^s and such that $x - su \in A^0$. We have $x + tu \in X^{s+t}$ for $|t|$ sufficiently small which means that the volume form of $\mathbb{R}^N$ is locally splitted into the product of the $(N - 1)$-volume form on X^s at x by the 1-dimensional volume form of the normal of X^s at x. It is now sufficient to apply Fubini's theorem to get

$$\mathrm{Vol}_N(X_r) = \int_0^r \mathrm{Vol}_{N-1}(X^s)ds$$

which gives easily the formula above.

Now we apply the step 1 to the family $Z_{u,r}$ and we get the constant $K > 0$. For all $r \in]0, 1]$, we then write

$$\mathrm{Vol}_N(Y_{u,r}) = \int_0^r \mathrm{Vol}_{N-1}(Z_{u,s})ds$$

and the conclusion follows from the previous argument.

Step 3. — Let (O_u) be a globally subanalytic family of globally subanalytic open subsets of B_1. Let r_u be the supremum of the radii of the open balls contained in O_u. There exists $K > 0$ such that $\mathrm{Vol}_N(O_u) \leqslant Kr_u$.

Let Y_u be the frontier of O_u. If $r > r_u$ then O_u is contained in $Y_{u,r}$. It is then sufficient to apply step 2.

Step 4. — Let us now turn to the proof of the lemma.

We denote $u = (t, \varepsilon)$ and $O_u = X_{t,\varepsilon}$. If $\varepsilon \in]0, 1]$, we note $R_\varepsilon = \sup_t r_{t,\varepsilon}$ where $r_{t,\varepsilon}$ is defined in step 3. Let us show that R_ε tends to 0 when ε tends to 0.

If R_ε does not tend to 0, there exist sequences (ε_i), (t_i) and a sequence of balls B_i of radius r_{t_i,ε_i} with $B_i \subset X_{t_i,\varepsilon_i}$ such that $\lim_i \varepsilon_i = 0$, $\lim_i t_i = t_\infty$ (as the parameter t varies in a compact subset, t_∞ exists), $\lim_i r_{t_i,\varepsilon_i} = R > 0$ and the Haussdorff's limit of (B_i) is a ball B_∞ of radius $R > 0$. Denote B'_∞ the ball with the same center as B_∞ and of radius $R/2$. For sufficiently large i, we have $B'_\infty \subset B_i$. From the uniform continuity of (f_t), we deduce that the function f_{t_∞} is equal to 0 on B'_∞. Hence f_{t_∞} is identically equal to zero (analytic continuation) which is absurd by assumption.

We have then $\lim_{\varepsilon \to 0_+} R_\varepsilon = 0$. As the function R_ε is globally subanalytic, from Łojasiewicz's inequality, we deduce that there exists some constants $d > 0$ and $\nu > 0$ with $\nu \in \mathbb{Q}$, such that

$$R_\varepsilon \leqslant d\varepsilon^\nu.$$

Taking $c = Kd$, we apply step 3 to complete the proof. □

This lemma admits an o-minimal version. Its proof is very close to the above one (thus it is left to the reader as an easy exercice).

PROPOSITION 3.3. — Let $\mathfrak{A}$ be an o-minimal structure. Let (f_t) be a continuous $\mathfrak{A}$-definable family of $\mathfrak{A}$-definable functions defined on a $\mathfrak{A}$-definable compact K of $\mathbb{R}^N$. Assume that the parameter varies in a $\mathfrak{A}$-definable compact subset and that, for all t, the zero set of f_t has empty interior in $\mathbb{R}^N$. Then there exists a continuous $\mathfrak{A}$-definable function $\theta : [0,1] \to \mathbb{R}$, equal to 0 at 0, such that for all $\varepsilon \in]0,1]$,

$$\mathrm{Vol}_N(X_{t,\varepsilon}) \leqslant \theta(\varepsilon)$$

where $X_{t,\varepsilon}$ is the set of points $x \in K$ such that $|f_t(x)| < \varepsilon$.

In the polynomial case, the following version holds.

LEMMA 3.4. — Let d be an integer. There exist $c > 0$ and $\nu > 0$ with $\nu \in \mathbb{Q}$ such that, for all non constant polynomial P of degree at most equal to d satisfying $\sup_{B_1} |P| \geqslant 1$ and for all $\varepsilon \in]0,1]$, we have

$$\mathrm{Vol}_N(X_{P,\varepsilon}) \leqslant c\varepsilon^\nu$$

where $X_{P,\varepsilon} = \{x \in B_1 \mid |P(x)| < \varepsilon\}$.

Proof. — This time, the parameter is the polynomial P itself. The space of parameters is not compact any more. Nevertheless, the space of

polynomials of degree at most equal to d has finite dimension and all the norms on this space are thus equivalent. It comes:

Fact. — Let W_1 and W_2 be bounded open subsets of $\mathbb{R}^N$. There exists a constant $L > 0$ such that, for all polynomials P of degree at most equal to d, $\sup_{W_1} |P| \leqslant L \sup_{W_2} |P|$.

To prove the lemma, we argue as in Lemma 3.2 in 4 steps. The first 3 steps are completely the same. We modify the 4th one in the following way:

If the function R_ε does not tend to 0 with ε, there exists a sequence (ε_i), there exist a sequence of polynomials (P_i) and a sequence of balls B_i of radius r_i such that $B_i \subset X_{P_i, \varepsilon_i}$, with $\lim_i \varepsilon_i = 0$, $\lim_i r_i = R > 0$ and the sequence (B_i) converges (in Haussdorff's topology) to a ball B_∞ of radius $R > 0$. Denote B'_∞ the ball with the same center as B_∞ and of radius $R/2$. For sufficiently large i, we have $B'_\infty \subset B_i$ and thus $\sup_{B'_\infty} |P| < \varepsilon_i$. But $\sup_{B_1} |P| \geqslant 1$. This fact contradicts the equivalence of the norms (with $W_1 = B_1$ and $W_2 = B'_\infty$). The conclusion follows as in Lemma 3.2. $\square$

Remark 3.5. — One can find stronger versions of this last result in [Yo] and [CY]. The method applied in those references is based on the notion of metric entropy.

From Lemma 3.4, we can deduce the following arithmetical lemma which will be used in the proof of Theorem 1.3. The author would like to thank D. Trotman for pointing out a mistake in the first proof of this result.

LEMMA 3.6. — *Let $P_1, ..., P_s$ be non constant polynomials of N variables with rational coefficients. There exists $\nu > 0$ such that the set $\mathfrak{B}$ of $\gamma \in \mathbb{R}^N$ such that $(P_1(\gamma), ... P_s(\gamma)) \in \mathfrak{D}_\nu^s$ has full Lebesgue measure in $\mathbb{R}^N$.*

Proof. — We fix a relatively compact ball B_r of radius $r > 0$ of $\mathbb{R}^N$. Let us estimate the measure of the complement of $\mathfrak{B}$ in B_r.

For all $\bar{n} = (n_0, n_1, ..., n_s) \in \mathbb{Z}^{s+1} \setminus \{0\}$, we note

$$P_{\bar{n}}(x) = n_0 + n_1 P_1(x) + ... + n_s P_s(x).$$

Then $P_{\bar{n}} = 0$ defines an algebraic hypersurface $\mathfrak{H}_{\bar{n}}$ of $\mathbb{R}^N$ (or eventually an empty set but the estimation is then straightforward). The polynomials $P_{\bar{n}}$ have degrees bounded by an integer d which does not depend on $\bar{n}$. Moreover, there exists an integer $q > 0$ such that, if $P_{\bar{n}}$ is not constant,

then $\sup_{B_r} |P_{\bar{n}}| \geqslant 1/q$. This comes from the fact that the coefficients of the polynomials P_i are rational numbers whose denominators are bounded.

We denote $\mathfrak{B}^c$ the complement of $\mathfrak{B}$. Then we have

$$\mathfrak{B}^c \cap B_r = \bigcap_{c>0} \bigcup_{\bar{n}} \mathfrak{V}_{c,\bar{n},r}$$

where $\mathfrak{V}_{c,\bar{n},r} = \{\gamma \in B_r \mid |P_{\bar{n}}(\gamma)| < c/(\sum_i |n_i|)^{\nu}\}$. Assume that $c < 1$ (this is not a restriction), from Lemma 3.4 we deduce that there exist $K > 0$ and $\mu > 0$ (μ rational number) such that, for all $\bar{n}$,

$$m(\mathfrak{V}_{c,\bar{n},r}) \leqslant K \frac{c^{\mu}}{(\sum_{i=0}^{s} |n_i|)^{\nu+\mu}}$$

hence

$$m\left(\bigcup_{\bar{n}} \mathfrak{V}_{c,\bar{n},r}\right) \leqslant K c^{\mu} \sum_{\bar{n}} \frac{1}{(\sum_{i=0}^{s} |n_i|)^{\nu+\mu}}.$$

We sum over $\bar{n} \in \mathbb{Z}^{s+1} \setminus \{0\}$ at $\sum_i |n_i| = n$ with $n > 0$ fixed. We get:

$$m\left(\bigcup_{\bar{n}} \mathfrak{V}_{c,\bar{n},r}\right) \leqslant K c^{\mu} \sum_{n>0} \frac{n^{s+1}}{n^{\nu+\mu}}.$$

If ν is large enough, the above sum is convergent. We deduce that the measure of $\cup_{\bar{n}} \mathfrak{V}_{c,\bar{n},r}$ is smaller than $K' c^{\mu}$ with a constant $K' > 0$ depending only on r, s and the polynomials P_i. As c tends to 0, it follows that the measure of $\mathfrak{B}^c \cap B_r$ is equal to 0. This completes the proof. $\square$

From Proposition 2.4, Lemmas 3.1 and 3.6, we can derive the following partial result:

PROPOSITION 3.7. — Let $(f_{\gamma})_{\gamma \in \mathbb{R}^N}$ be a parameter family of x^{λ}-functions of $\mathbb{R}^{n+1}$. There exist an integer n_0 and a subset $D \subset \mathbb{R}^N$ of full Lebesgue measure such that, for all $\gamma \in D$, there exists a finite cover of $\mathbb{R}^{n+1}$ with n_0 x^{λ}-cylinders on which f admits a primitive with respect to the last variable in the class of $\mathfrak{G}$-functions.

Proof. — The only thing to show is that the number of cylinders which appear in a preparation of f with respect to the last variable y does not depend on the global parameter γ. This fact is a consequence of the preparation theorem for globally subanalytic functions (see for instance [LR1]) and we leave the proof to the reader. $\square$

4. Proof of Theorem 1.3.

From the Preparation Theorem for x^λ-functions, the previous lemma and Proposition 3.7, it is clear that we can restrict the study to the following situation: the support of f is a x^λ-cylinder $C = \{(x, y) \mid x \in B, \varphi(x) < y < \psi(x)\}$ where φ and ψ are continuous x^λ-functions on the base B of the cylinder. On this cylinder, the $\mathfrak{G}$-function f is a finite sum of monomials of the form

$$t(x, y) = L(x)g(x, y)(\log h_1(x, y))^{k_1}...(\log h_d(x, y))^{k_d}$$

where

- L is a finite product of logarithms of x^λ-functions of the base B,

- $g, h_1, ..., h_d$ are x^λ-functions on C,

- $k_1, ..., k_d$ are integers,

- the functions $g, h_1, ..., h_d$ are simultaneously reduced on the cylinder C:

$$g(x, y) = y_1^{\delta_0} A(x) U(x, y_1^{\delta_1}, ..., y_1^{\delta_s})$$
$$h_i(x, y) = y_1^{\mu_i} A_i(x) U_i(x, y_1^{\delta_1}, ..., y_1^{\delta_s})$$

where A and the A_i's are x^λ-functions on B, the δ_j's and the μ_i's are real numbers, U and the U_i's are x^λ-units (we can always assume that the morphism of reduction is common to the x^λ-functions considered).

We have then

$$(**) \qquad \prod_{i=1}^{d} (\log h_i)^{k_i} = \prod_{i=1}^{d} (\mu_i \log(y_1 A_i) + \log U_i)^{k_i}.$$

For all i, the function log is analytic in a neighbourhood of the closure of the image of U_i (for U_i is a unit) and thus we have

$$\log U_i = V_i$$

where $V_i(x, y) = W_i \circ \psi(x, y_1^{\delta_1}, ..., y_1^{\delta_s})$ and W_i is analytic with the same polyradius as U_i (we take for instance $(2, ..., 2)$).

The expansion of $(**)$ yields to the case $f = t.v$ where

$$t(x, y) = L(x)A(x) \prod_{i=1}^{d} (\log(y_1 A_i(x)))^{k_i}$$
$$v(x, y) = y_1^{\delta_0} V(x, y)$$

and $V(x,y) = W \circ \psi(x, y_1^{\delta_1}, ..., y_1^{\delta_s})$, W being analytic of polyradius $(2, ..., 2)$. From Proposition 2.4, we can split v into the form $v(x,y) = v_1(x,y) + w_2(x)/y_1$, v_1 and w_2 being x^λ-functions and v_1 admits a x^λ-primitive with respect to y in the class of x^λ-functions for almost all values of the local parameter $(\delta_1, ..., \delta_s)$. Moreover, from Lemma 3.6, this is also true for almost all values of the global parameter.

Let us put $v_2(x,y) = w_2(x)/y_1$. We now compute a primitive for the functions $t.v_i$, $i = 1, 2$. Following the ideas of [LR2], we proceed by induction on the multi-indices $\nu = (d, k_1, ..., k_d) \in \mathbb{N}^{d+1}$ with the lexicographic order.

Case $\nu = 0$. — No logarithm appears in the expression of $t.v_i$, $i = 1, 2$. Hence this is just a consequence of Proposition 3.7.

Step of induction. — There are two cases:

$i = 1$. – Let w_1 be a x^λ-primitive of v_1 with respect to y. We integrate by parts choosing $H = LAw_1$ as a primitive of LAv_1. It comes

$$\int^y tv_1(x,u)du = \left[LA \prod_{i=1}^d (\log y_1 A_i(x))^{k_i} w_1 \right]^y$$
$$- \int^y H \frac{d}{du} \left(\prod_{i=1}^d (\log y_1 A_i(x))^{k_i} \right) du.$$

The first term is clearly a $\mathfrak{G}$-function and the second term is a primitive of a sum of $\mathfrak{G}$-functions whose multi-indices are strictly less than $(d, k_1, ..., k_d)$. Thanks to the hypothesis of induction, we can choose such a primitive in the class of $\mathfrak{G}$-functions.

$i = 2$. – We still integrate by parts choosing the function

$$H(x,y) = \frac{LAw_2}{k_d + 1} (\log y_1 A_d)^{k_d+1}$$

as a primitive of $LAv_2 (\log y_1 A_d)^{k_d}$. Hence we get

$$\int^y tv_2(x,u)du = \left[H \prod_{i=1}^{d-1} (\log y_1 A_i(x))^{k_i} \right]^y$$
$$- \int^y H \frac{d}{du} \left(\prod_{i=1}^{d-1} (\log y_1 A_i(x))^{k_i} \right) du.$$

The conclusions are now the same as in the previous case.

To finish the proof, it is sufficient to note that if F is a $\mathfrak{G}$-function and φ is a x^λ-map, then the composition $F \circ \varphi$ is a $\mathfrak{G}$-function.   □

If we take into account a dependency with respect to a polynomial parameter P, we have the following result.

THEOREM 4.1. — Let $(f_{\gamma,P})$ be a parameter family of $\mathfrak{G}$-functions of $\mathbb{R}^n \times \mathbb{R}$ and let $(\varphi_\mu), (\psi_\nu)$ be parameter families of x^λ-functions of $\mathbb{R}^n$. There exists a subset $D \subset \mathbb{R}^N$ of full Lebesgue measure such that, for all $\gamma \in D$, for all polynomial $P \in \mathbb{R}[X_1, ..., X_{2d}]$ and for all μ, ν, the function $F_{\gamma,P,\mu,\nu}(x) = \int_{\varphi_\mu(x)}^{\psi_\nu(x)} f_{\gamma,P}(x,y)dy$ is a $\mathfrak{G}$-function.

Proof. — This is a direct consequence of the proof of Theorem 1.3. □

5. Proof of Theorem 1.4.

Let us recall some basic facts about the class of the so-called $\mathfrak{L}\mathfrak{E}$-functions (see [DMM], [DM], [LR1]).

DEFINITION 5.1. — A $\mathfrak{L}\mathfrak{E}$-map $f : \mathbb{R}^n \to \mathbb{R}^p$ is a finite composition of globally subanalytic maps with the functions exp and log. A $\mathfrak{L}\mathfrak{E}$-function is a $\mathfrak{L}\mathfrak{E}$-map from $\mathbb{R}^n$ to $\mathbb{R}$.

This class of functions gives rise to the $\mathfrak{L}\mathfrak{E}$-subsets of the spaces $\mathbb{R}^n$, $n \in \mathbb{N}$. In [DMM], van den Dries, Macintyre and Marker solve the Tarski's conjecture related to these sets by showing that the $\mathfrak{L}\mathfrak{E}$-subsets form an o-minimal structure. Their proof is based on Model Theory. In [LR2], Lion and Rolin give a more geometric proof of this result based on a preparation theorem for $\mathfrak{L}\mathfrak{E}$-functions.

The following result is a consequence of the o-minimality of the collection of $\mathfrak{L}\mathfrak{E}$-subsets.

PROPOSITION 5.2. — Let $(f_{\gamma,P})$ be a parameter family of $\mathfrak{G}$-functions of $\mathbb{R}^n$. For all integer m, there exists an integer $A(m)$ which bounds the number of connected components of the level sets $\{f_{\gamma,P} = c\}$, uniformly with respect to $\gamma \in \mathbb{R}^N$, $c \in \mathbb{R}$ and to the polynomial $P \in \mathbb{R}[X_1, ..., X_{2d}]$ of degree less or equal to m.

Proof. — We fix an integer $m \in \mathbb{N}$. The vector space E_m of polynomials P of $\mathbb{R}[X_1, ..., X_{2d}]$ whose degree is less or equal to m is isomorphic

to $\mathbb{R}^{\delta(m)}$ ($\delta(m)$ depends on d). Hence, if we restrict the parameter family $(f_{\gamma,P})$ to $\mathbb{R}^N \times E_m$, we get a family of $\mathfrak{G}$-functions parametrized by $\mathbb{R}^{N+\delta(m)}$.

We now consider the function f as a function of $x \in \mathbb{R}^n$ and of the parameters $\gamma \in \mathbb{R}^N$ and $P \in E_m$. This is equivalent to consider the function f as a function of x, γ and $a \in \mathbb{R}^{\delta(m)}$ where a is the vector of the coefficients of the monomials of P.

Let $g : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{\delta(m)} \times \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x, \gamma, a, c) = f_{\gamma,P}(x) - c = f(x, \gamma, a) - c$. This is clearly a $\mathcal{LE}$-function of $\mathbb{R}^{n+N+\delta(m)+1}$. Consequently, the set $g^{-1}(\{0\})$ is a $\mathcal{LE}$-subset and has the property of uniform bounds for the number of connected components of its fibers [DMM], [DM]. Hence there exists an integer $A(m)$ which bounds the number of connected components of the sets

$$\{x \in \mathbb{R}^n \mid (x, \gamma, a, c) \in g^{-1}(\{0\})\}$$

uniformly with respect to γ, a and c. As $\{x \in \mathbb{R}^n \mid (x, \gamma, a, c) \in g^{-1}(\{0\})\} = \{f_{\gamma,P} = c\}$, the proof is completed. $\square$

Let us now turn to the proof of Theorem 1.4.

Proof. — We put $\xi = (\gamma, P, \mu, \nu)$ where $\gamma \in \mathbb{R}^N$, $\mu \in \mathbb{R}^M$ and $\nu \in \mathbb{R}^L$, and we put $\zeta = (P, \mu, \nu)$. From Theorem 4.1, we know that there exists a subset $D \subset \mathbb{R}^N$ of full Lebesgue measure such that, for all $\gamma \in D$, the function

$$F_\xi(x) = F_{\gamma,\zeta}(x) = \int_{\varphi_\mu(x)}^{\psi_\nu(x)} f_{\gamma,P}(x,y)\,dy$$

is a $\mathfrak{G}$-function for all values of the parameter ζ.

We now choose $\gamma \in D$, we fix an integer m and we take the parameter P in the space E_m of polynomials of $\mathbb{R}[X_1, ..., X_{2d}]$ whose degree is less or equal to m. From Proposition 5.2, the theorem will be proved if we show that, when ζ varies in $E_m \times \mathbb{R}^M \times \mathbb{R}^L$, the function $F_{\gamma,\zeta}$ varies in a parameter family of $\mathfrak{G}$-functions parametrized by $E'_m \times \mathbb{R}^M \times \mathbb{R}^L$ where E'_m is the space of polynomials of $\mathbb{R}[X_1, ..., X_{2d(m)}]$ whose degree is less or equal to $q(m)$.

A finite sum of $\mathfrak{G}$-functions with polynomials in $2d'(m)$ variables whose degree is less than $q'(m)$ has also this property that is: it can be written as a $\mathfrak{G}$-function with a polynomial whose degree and number of variables are bounded by integers depending only on m. This is true

for a product too. Thus we only have to show that the computation of $F_{\gamma,\zeta}$ involves finitely many operations as finite sums and products of $\mathfrak{G}$-functions.

From the proof of Theorem 1.3, in order to compute a $\mathfrak{G}$-primitive of $f_{\gamma,P}$ with respect to y on a x^λ-cylinder, we write the function f as a finite sum of functions $t.v$ where

$$t(x,y) = L(x)A(x)\prod_{i=1}^{d}(\log(y_1 A_i(x)))^{k_i}$$

$$v(x,y) = y_1^{\delta_0}V(x,y)$$

and the number of functions in this sum is bounded by an integer depending only on m. Such a function $t.v$ can be written as a $\mathfrak{G}$-function with a polynomial whose degree and number of variables are bounded by integers depending on m. Now, the integration process described in the proof of Theorem 1.3 involves finitely many operations of sum and product of $\mathfrak{G}$-functions. Hence, each primitive of f on each x^λ-cylinder can be written as a $\mathfrak{G}$-function with a polynomial whose degree and number of variables are bounded by integers depending on m. To complete the computation of $F_{\gamma,\zeta}$, it suffices to write

$$F_{\gamma,\zeta}(x) = \sum_{k=1}^{K}(F_k(x,\psi_k(x)) - F_k(x,\varphi_k(x)))\mathbf{1}_{C_k}$$

where the C_k's are the x^λ-cylinders of the partition whose boundaries are given by the x^λ-functions ψ_k and φ_k, F_k is a $\mathfrak{G}$-primitive of f on C_k and $\mathbf{1}_{C_k}$ is the characteristic function of C_k (hence is a x^λ-function). Moreover, the number of x^λ-cylinders C_k does not depend on m. $\square$

Remark 5.3. — Theorem 1.4 can be seen as a result of non-oscillation of certain generalized abelian integrals. Indeed, consider two parameter families of $\mathfrak{G}$-functions (F_γ) and (G_δ) of $\mathbb{R}^2$ and a x^λ-function Φ of $\mathbb{R}^2$. If we put $\omega_{\gamma,\delta} = F_\gamma(x,y)dx + G_\delta(x,y)dy$ and $\Gamma_t = \Phi^{-1}(t)$, then the levels Γ_t are x^λ-subsets of $\mathbb{R}^2$ and Theorem 1.4 implies that, for almost all values of the parameters γ and δ, the function $t \mapsto I(t) = \int_{\Gamma_t}\omega_{\gamma,\delta}$ is a $\mathfrak{G}$-function of $\mathbb{R}$. In particular, the number of its roots is uniformly bounded with respect to Φ.

Remark 5.4. — From the statements of Theorems 1.3 and 1.4, the following question naturally arises: what about the remaining set D^c in

the space of the parameter γ? We are able to give a partial answer to this question: one can find "very bad" values of the parameter γ such that the integrated function does not belong to the class of $\mathcal{LE}$-functions. This result uses a precise preparation theorem for $\mathcal{LE}$-functions and is presented in an other paper (see [So2]). Unfortunately, we are not able to say anything about the oscillating properties of the integrated function in this case.

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