ANNALES

DE

L'INSTITUT FOURIER

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Tome 53, nº 4 (2003), p. 1117-1139.

<http://aif.cedram.org/item?id=AIF_2003__53_4_1117_0>

Ann. Inst. Fourier, Grenoble
53, 4 (2003), 1117–1139

PLANE CURVE SINGULARITIES AND CAROUSELS

by LÊ DŨNG TRÁNG

Introduction.

Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function. Since we are only interested in the topology of the germ of plane curve $(\Gamma, 0)$ defined by f, we assume that f is *reduced*, *i.e.*, if $f = f_1^{a_1} \cdots f_k^{a_k}$ is the decomposition of f into irreducible factors in the ring $\mathbb{C}\{X, Y\}$ of convergent power series in two variables, then $a_1 = \cdots = a_k = 1$. We still denote by f a representative of this germ defined on an open neighbourhood U of the origin 0 in $\mathbb{C}^2$ and $\Gamma = f^{-1}(0) \cap U$.

Since f is reduced at 0, the point 0 is an isolated singular point of Γ, *i.e.*, there is a sufficiently small neighbourhood V of 0 in $\mathbb{C}^2$, such that the space $(\Gamma - \{0\}) \cap V$ is non-singular. In other words the ideal $(f, \partial f/\partial X, \partial f/\partial Y)$ generated in $\mathbb{C}\{X, Y\}$ by f and the partial derivatives of f is primary for the maximal ideal $\mathcal{M}$ of $\mathbb{C}\{X, Y\}$. It can be shown that this is equivalent to the fact that the Jacobian ideal $(\partial f/\partial X, \partial f/\partial Y)$ is $\mathcal{M}$-primary, *i.e.*, the quotient $\mathbb{C}$-algebra:

$$\mathbb{C}\{X, Y\}/(\partial f/\partial X, \partial f/\partial Y)$$

is a finite dimensional vector space over $\mathbb{C}$. This dimension is called the *Milnor number* of f at 0. We shall denote it by $\mu(f, 0)$.

It is known (see [M]) that, for $\varepsilon > 0$ sufficiently small, the real 3-sphere $S_\varepsilon(0)$ of $\mathbb{C}^2$ centered at 0 with radius ε intersects Γ transversally.

Keywords: Plane curves – Singularities – Carousels – Puiseux expansions – Characteristic exponents.
Math. classification: 14J17 – 57M27 – 57R30.

Moreover the smooth type of the link $\Gamma \cap S_\varepsilon(0)$ in $S_\varepsilon(0)$ is an analytic invariant of the germ $(\Gamma, 0)$. For commodity we call *algebraic link* a link which has the smooth type of the link associated to the singularity of a complex plane curve.

The number of components of this link equals the number k of irreducible components of $f = f_1 \cdots f_k$. Each component f_i $(1 \leqslant i \leqslant k)$ defines a *branch* $(\Gamma_i, 0)$ of $(\Gamma, 0)$. It was shown by K. Brauner in [B] that the knot $K_i = \Gamma_i \cap S_\varepsilon(0)$ in $S_\varepsilon(0)$ associated to the branch $(\Gamma_i, 0)$ is an iterated torus knot determined by the Puiseux pairs of f_i at 0 (see [Lê1]). The linking number $L(K_i, K_j)$ of the knot K_i with the knot K_j in $S_\varepsilon(0)$ for $i \neq j$ equals the complex dimension of the complex vector space (see [R]):

$$\mathbb{C}\{X, Y\}/(f_i, f_j)$$

which is the intersection number of Γ_i and Γ_j at 0.

In general a link is not determined by its components and their pairwise linking numbers. However, a consequence of a result by M. Lejeune-Jalabert (see [Lê2] tome 1) and O. Zariski ([Z]) is that algebraic links are determined by their components and the pairwise linking numbers of their components. This result is essentially a consequence of the computation of the intersection number at 0 of the branches Γ_i and Γ_j by using the minimal resolution of the function $f_i f_j$. D. Eisenbud and W. Neumann in [EN] and F. Michel and C. Weber in [MW] also obtained this result and gave an explicit necessary and sufficient condition so that an iterated toric link is an algebraic link (see [MW] (Proposition 5.3.2). The fact that an algebraic link is determined by its components and the pairwise linking numbers of the components is not satisfactory, because this statement does not show directly how the equation of the associated curve gives the embedding of this link in the 3-sphere. A similar problem arises with the topology of the complement in the complex projective plane of a non-singular plane curve of degree d. It is easily seen that this topology is uniquely determined by the degree, but little is known about it except when $d = 2$ (see e.g. [LSV]).

In this paper we aim to describe an algebraic link in the 3-sphere. For this purpose, we shall use the notion of carousel of a germ of plane curve (see [Lê2]) and connect it with the description of the corresponding algebraic link. In fact, we show that by integrating the vector fields which define the carousel, we obtain an explicit Waldhausen decomposition of the 3-sphere (see [LMW]) adapted to the components of the link.

We thank the referee for valuable corrections in an earlier version of this paper.

1. Topological setting.

1.1. — We consider a plane curve $\Gamma \subset U$ as in the introduction. It is convenient to replace the ball $B_\varepsilon(0)$ of $\mathbb{C}^2$ centered at 0 with radius ε by a polydisc $D_{\eta_1}(0) \times D_{\eta_2}(0)$. We choose the polydisc in the following way. There is $r_2 > 0$ small enough, such that

$$(\{0\} \times D_{\eta_2}(0)) \cap \Gamma = \{(0,0)\}$$

for any η_2, $r_2 \geqslant \eta_2 > 0$. Then, for a given $\eta_2 > 0$, there is $r_1 > 0$ such that, for any η_1, $r_1 \geqslant \eta_1 > |t| > 0$, $(\{t\} \times D_{\eta_2/2}(0)) \cap \Gamma$ contains k points, where k is the intersection number of $\{0\} \times \mathbb{C}$ with Γ at the origin $\{(0,0)\}$.

It is known that the ball $B_\varepsilon(0)$ is a manifold with boundary diffeomorphic to the manifold with corner $D_{\eta_1}(0) \times D_{\eta_2}(0)$. In particular the boundary $S_\varepsilon(0)$ of the ball $B_\varepsilon(0)$ is diffeomorphic with the manifold with corners

$$\partial[D_{\eta_1}(0) \times D_{\eta_2}(0)] = [\partial D_{\eta_1}(0) \times D_{\eta_2}(0)] \cup [D_{\eta_1}(0) \times \partial D_{\eta_2}(0)].$$

Then, we have (see [Lê3]):

THEOREM 1.1.1. — *For $1 \gg \varepsilon > 0$ and $1 \gg \eta_2 \gg \eta_1 > 0$, the algebraic link $\Gamma \cap S_\varepsilon(0)$ in $S_\varepsilon(0)$ has the same smooth type as the link $\Gamma \cap \partial(D_{\eta_1}(0) \times D_{\eta_2}(0))$ in $\partial(D_{\eta_1}(0) \times D_{\eta_2}(0))$.*

We have

$$\partial(D_{\eta_1}(0) \times D_{\eta_2}(0)) = (\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \cup (D_{\eta_1}(0) \times \partial D_{\eta_2}(0)).$$

Since $\Gamma \cap [D_{\eta_1}(0) \times \partial D_{\eta_2}(0)] = \emptyset$, we have only to consider

$$\Gamma \cap [\partial D_{\eta_1}(0) \times D_{\eta_2}(0)]$$

to describe the algebraic link $\Gamma \cap S_\varepsilon(0)$ in $S_\varepsilon(0)$.

1.2. — The embedding of $\Gamma \cap [\partial(D_{\eta_1}(0) \times D_{\eta_2}(0)]$ into the solid torus $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$ defines a *closed braid*.

DEFINITION 1.2.1. — A closed braid *is a closed embedded* 1-*manifold L in a solid torus $S_1 \times D_2$, such that the first projection onto the circle S_1 restricted to L is a covering over S_1, and, for any $t \in S_1$, the points of L over t are in the interior of $\{t\} \times D_2$.*

The following theorem is needed to recognize algebraic links.

THEOREM 1.2.2. — *Let L and L' be two closed braids such that L' is contained in a tubular neighbourhood of L in the solid torus $S_1 \times D_2$ and the first projection onto the circle S_1 restricted to L' is a covering of the same degree as the covering of L over S_1. Then, the links of the 3-sphere defined by L and L' are of the same smooth type.*

For instance, let L_0 be the trivial closed braid corresponding to the curve $Y = 0$ at the point 0, then this theorem shows that the closed braid L corresponding to a curve $Y = \sum_{0\neq \ell \in \mathbb{N}} X^\ell$ defines a knot which has the same type as the trivial knot.

2. Iterated torus knots.

In this paragraph let us assume that the germ of function f is analytically irreducible at the point 0. In this case the algebraic link associated to the curve defined by f has one component and is an algebraic knot.

2.1. — Let us first recall Puiseux Theorem:

THEOREM (Puiseux Theorem). — *Let $f \in \mathbb{C}\{X, Y\}$ be an analytically irreducible germ. Assume the series $f(0, Y) \neq 0$ has valuation k. There is unit u in $\mathbb{C}\{X, Y\}$ and a unique power series $\sigma \in \mathbb{C}\{X^{1/k}\}$, such that*

$$f(X, Y) = u(X, Y) \prod_{\xi^k=1} (Y - \varphi(\xi X^{1/k})),$$

where ξ is a k-th root of unity and $X^{1/k}$ is a k-th root of X.

The series $\varphi(X^{1/k})$ depends on the choice of coordinates and on the choice of a k-th root of X. The set of series $\varphi(\xi X^{1/k})$ with $\xi^k = 1$, is called the *Puiseux expansion* of f in the coordinates (X, Y). Therefore the Puiseux expansion is the set of solutions of $f(X, Y) = 0$ in the ring $\mathbb{C}\{X^{1/k}\}$. Let

$$\varphi(X^{1/k}) = \sum_\ell a_\ell X^{\ell/k}.$$

Now, we define the *Puiseux characteristic exponents* of f relatively to the coordinates (X, Y).

If $k = 1$, there is no Puiseux characteristic exponent.

If $k \geqslant 2$, the set $\{\ell \,|\, a_\ell \neq 0,\ \ell/k \notin \mathbb{N}\}$ is not empty. Let ℓ_1 be the lower bound of this set and let

$$\frac{\ell_1}{k} = \frac{m_1}{k_1}$$

where m_1 and k_1 are relatively prime. If $k_1 = k$, we only have one Puiseux characteristic exponent m_1/k_1. If $k_1 < k$, k_1 divides k and the set

$$\Big\{\ell \,|\, a_\ell \neq 0,\ \frac{\ell}{k} \notin \frac{1}{k_1}\mathbb{N}\Big\}$$

is not empty. Let ℓ_2 be the lower bound of this latter set. There is a unique way to write

$$\frac{\ell_2}{k} = \frac{m_2}{k_1 k_2}$$

where m_2 and k_2 are relatively prime. If $k_1 k_2 = k$, we have two Puiseux characteristic exponents m_1/k_1 and $m_2/k_1 k_2$. If $k_1 k_2 < k$, $k_1 k_2$ divides k. Suppose that we have defined r Puiseux exponents, $r \geqslant 2$, $m_1/k_1, m_2/k_1 k_2, \ldots, m_r/k_1 \cdots k_r$, where m_r and k_r are relatively prime. If $k_1 \cdots k_r = k$, we have r Puiseux characteristic exponents. If $k_1 \cdots k_r < k$, $k_1 \cdots k_r$ divides k and the set

$$\Big\{\ell \,|\, a_\ell \neq 0,\ \frac{\ell}{k} \notin \frac{1}{k_1 \cdots k_r}\mathbb{N}\Big\}$$

is not empty. Let ℓ_{r+1} be the lower bound of this set. There is a unique way to write

$$\frac{\ell_{r+1}}{k} = \frac{m_{r+1}}{k_1 k_2 \cdots k_{r+1}}$$

where m_{r+1} and k_{r+1} are relatively prime.

Since k has a finite number of divisors, there is a finite number g of Puiseux characteristic exponents of f relatively to the coordinates (X, Y).

2.2. — There is a more algebraic way to define the Puiseux characteristic exponents. Consider the field extension $\mathbb{C}\langle\!\langle X^{1/k}\rangle\!\rangle$ generated by k-th root of X over the field of fractions $\mathbb{C}\langle\!\langle X\rangle\!\rangle$ of ring of convergent series $\mathbb{C}\{X\}$. We may consider the Puiseux expansion as elements of $\mathbb{C}\langle\!\langle X^{1/k}\rangle\!\rangle$. Of course, the field $\mathbb{C}\langle\!\langle X^{1/k}\rangle\!\rangle$ is the field of fractions of the ring $\mathbb{C}\{X^{1/k}\}$. Let G be the Galois group of this extension. This Galois group is isomorphic to the cyclic group $\mathbb{Z}/k\mathbb{Z}$. In fact it is generated by the isomorphism of $\mathbb{C}\langle\!\langle X^{1/k}\rangle\!\rangle$ induced by multiplying $X^{1/k}$ by a primitive k-th root of unity.

Let v be the valuation of the field $\mathbb{C}\langle\!\langle X^{1/k}\rangle\!\rangle$ for which $v(X^{1/k}) = 1$. We define:

$$G_i := \{\sigma \in G \ v(Y - \sigma Y) \geqslant i\}.$$

Then $G = G_0$ and $G \supset G_1 \supset \cdots \supset G_\ell \supset \cdots$. Notice that, for any $i \geqslant 0$, G_i is a subgroup of G. In fact, let σ_1 and σ_2 be elements in G_i. Then

$$Y - (\sigma_1\sigma_2)Y = Y - \sigma_1 Y + \sigma_1 Y - (\sigma_1\sigma_2)Y.$$

So

$$v(Y - (\sigma_1\sigma_2)Y) \geqslant \inf(v(Y - \sigma_1 Y), v(\sigma_1 Y - (\sigma_1\sigma_2)Y)).$$

Since

$$v(\sigma_1 Y - (\sigma_1\sigma_2)Y) = v(Y - \sigma_2 Y)$$

we have that $\sigma_1\sigma_2 \in G_i$. Similarly We have

$$v(Y - \sigma_1 Y) = v(\sigma_1^{-1} Y - Y)$$

which yields $\sigma_1^{-1} \in G_i$. Since G is a finite group the above descending sequence of groups $G \supset G_1 \supset \cdots \supset G_\ell \supset \cdots$ is stationary. Moreover $G = \cdots = G_{\ell_1}, \ldots, G_{\ell_{g-1}+1} = \cdots = G_{\ell_g}$ and $G_\ell = \{0\}$ when $\ell > \ell_g$. The Puiseux characteristic exponents k_i/k relatively to (X, Y) are therefore the quotients ℓ_i/k, where $\ell_1, \ldots, \ell_g$ are the indices where the descending sequence $G \supset G_1 \supset \cdots \supset G_\ell \supset \cdots$ is strictly decreasing.

2.3. — Puiseux characteristic exponents give an explicit description of algebraic knots.

Algebraic knots are defined by successive satellisations:

Definition 2.3.1. — *Let K be an oriented knot in the oriented 3-sphere $\mathbf{S}^3$. An oriented knot L is a (p,q)-satellite of K if it has the same smooth type of a torus knot (p,q) on the boundary of a tubular neighbourhood of K on which meridians are chosen to be non-singular closed oriented curves which have a linking number $+1$ with K and parallels are non-singular closed oriented curves which do not link K and have a intersection number $+1$ with a meridian.*

Let m_1/k_1, m_2/k_1k_2, …, $m_g/k_1 \cdots k_g$ be the Puiseux characteristic exponents relatively to the coordinates (X, Y) of a plane branch defined by $f = 0$. We define the following g numbers:

i) $\lambda_1 = m_1$;

ii) for $g \geqslant r \geqslant 2$, $\lambda_r = m_r + (\lambda_{r-1}k_{r-1} - m_{r-1})k_r$.

Then the Theorem of K. Brauner can be formulated in the following way:

THEOREM 2.3.2. — *Let* $(\Gamma, 0)$ *be the germ of a plane branch defined by the germ of function* f *with the Puiseux characteristic exponents* m_1/k_1, m_2/k_1k_2, ..., $m_g/k_1 \cdots k_g$ *relatively to the coordinates* (X, Y). *Let* λ_r, $1 \leqslant r \leqslant g$, *defined as above. Then the local knot associated to* $(\Gamma, 0)$ *has the type of the knot* K_g *defined by induction on* g *by*

i) K_0 *is the trivial knot (oriented positively);*

ii) *for* $g \geqslant r \geqslant 2$, K_r *is the a* (λ_r, k_r)*-satellite of* K_{r-1}.

A knot of the type K_r is a rth-*iterated torus knot*. The knot K_1 is a (λ_1, k_1)-torus knot. In the original paper of K. Brauner, he uses a parallel P_r' of the boundary ∂T of a neighbourhood of K_r which links K_r. Actually if P_r and M_r are a parallel and a meridian of ∂T, as defined in the definition 2.3.1, and P_r' and M_r' are a parallel and a meridian considered by K. Brauner, in the first homology of ∂T, we have

$$[P_r'] = [P_r] + (\lambda_{r-1}k_{r-1} - m_{r-1})[M_r]$$
$$[M_r'] = [M_r]$$

where $[P]$ is the homology class of P.

3. Carousels of plane branches.

In order to be simple, in this paragraph we first construct carousels for plane branches.

3.1. — Let $(\Gamma, 0)$ be the germ of a plane branch defined by the germ of function f with the Puiseux expansion relatively to the variables (X, Y):

$$\varphi(X^{1/k}) = \sum_{0 \leqslant i < \ell_1} a_i X^{i/k} + a_{\ell_1} X^{m_1/k_1} + \sum_{\ell_1 < i < \ell_2} a_i X^{i/k} + \cdots + a_{\ell_g} X^{m_g/k_1 \cdots k_g} + \sum_{\ell_g < i} a_i X^{i/k}$$

where the terms $a_{\ell_r} X^{m_r/k_1 \cdots k_r}$ $(1 \leqslant r \leqslant g)$, called *Puiseux terms*, correspond to Puiseux characteristic exponents. The non-zero terms of $\sum_{0 \leqslant i < \ell_1} a_i X^{i/k}$ have integral exponents, the non-zero terms of $\sum_{\ell_{r-1} < i < \ell_r} a_i X^{i/k}$ $(1 < r \leqslant g)$ have exponents in $(1/k_1 \cdots k_{r-1})\mathbb{N}$ and the non-zero terms of $\sum_{\ell_g < i} a_i X^{i/k}$ are in $(1/k)\mathbb{N}$.

3.1.1. — Now, let us choose η_1 and η_2, as in Theorem 1.1.1 ($1 \gg \eta_2 \gg \eta_1 > 0$).

Consider the following truncated Puiseux expansions defined by induction on g:

i) $\varphi_0 = 0$;

ii) $\tilde{\varphi}_0 = \sum_{0 \leqslant i < \ell_1} a_i X^{i/k}$;

iii) $\varphi_1 = \sum_{0 \leqslant i < \ell_1} a_i X^{i/k} + a_{\ell_1} X^{m_1/k_1}$;

iv) $\tilde{\varphi}_1 = \sum_{0 \leqslant i < \ell_1} a_i X^{i/k} + a_{\ell_1} X^{m_1/k_1} + \sum_{\ell_1 < i < \ell_2} a_i X^{i/k}$;

v) for $1 < r \leqslant g$, $\varphi_r = \sum_{0 \leqslant i < \ell_1} a_i X^{i/k} + a_{\ell_1} X^{m_1/k_1} + \cdots + a_{\ell_r} X^{m_r/k_1 \cdots k_r}$;

vi) for $1 < r \leqslant g-1$, $\tilde{\varphi}_r = \varphi_r + \sum_{\ell_r < i < \ell_{r+1}} a_i X^{i/k}$;

vii) $\tilde{\varphi}_g = \varphi_g + \sum_{\ell_g < i} a_i X^{i/k}$.

3.1.2. — For η_1 sufficiently small, we can define solid tori $\tilde{T}_j \subset T_j$ ($0 \leqslant j \leqslant g$) inside the solid torus $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$ such that

a) the knot L_j defined by φ_j is the core of T_j and the knot $\tilde{L}_j$ defined by $\tilde{\varphi}_j$ is the core of $\tilde{T}_j$;

b) for any $g \geqslant j_1 > j$, the solid torus T_{j_1} is contained in $\tilde{T}_j$.

We construct the solid tori T_j and $\tilde{T}_j$ by induction on j.

The solid torus T_0 is the interior of $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$. We set $\eta_2 = \varepsilon_0(\eta_1)$.

The solid torus $\tilde{T}_0$ is the open subspace of points (X, Y) of T_0 for which there is a point (u, v) of the trivial knot $\tilde{L}_0$ such that $u = X$ and $|Y - v| < \tilde{\varepsilon}_0(\eta_1)$, where $\tilde{\varepsilon}_0(\eta_1)$ is such that

$$|a_{\ell_1}| {\eta_1}^{m_1/k_1} + \sum_{\ell_1 < i < \ell_2} |a_i| {\eta_1}^{i/k} + \cdots + |a_{\ell_g}| {\eta_1}^{m_g/k_1 \cdots k_g} + \sum_{\ell_g < i} |a_i| {\eta_1}^{i/k} < \tilde{\varepsilon}_0(\eta_1) < \eta_2.$$

Suppose that $j \geqslant 0$ and that we have defined the solid tori T_r, $\tilde{T}_r$ and the positive numbers $\varepsilon_r(\eta_1)$, $\tilde{\varepsilon}_r(\eta_1)$ for $0 \leqslant r \leqslant j$. By definition of $\varepsilon_j(\eta_1)$ the knot L_{j+1} is contained in $\tilde{T}_j$. In fact the knot L_{j+1} lies on a torus surface whose core is the knot $\tilde{L}_j$ and which is the boundary of a tubular neighbourhood of the knot $\tilde{L}_j$ contained in $\tilde{T}_j$. One can observe that L_{j+1}

is a (λ_{j+1}, k_{j+1})-satellite of the knot $\tilde{L}_j$. Let us choose $\varepsilon_{j+1}(\eta_1)$, such that

$$\sum_{\ell_j<i<\ell_{j+1}} |a_i|{\eta_1}^{i/k} + \cdots + |a_{\ell_g}|{\eta_1}^{m_g/k_1\cdots k_g} + \sum_{\ell_g<i} |a_i|{\eta_1}^{i/k} < \varepsilon_{j+1}(\eta_1)$$

and such that the open subspace of points (X, Y) of T_j, for which there is a point (u, v) of the knot L_{j+1} such that $u = X$ and $|Y - v| < \varepsilon_{j+1}(\eta_1)$, is an open tubular neighbourhood of L_{j+1} contained in $\tilde{T}_j$. We denote T_{j+1} this tubular neighbourhood. It is an open solid torus whose core is L_{j+1}. It contains the knot $\tilde{L}_{j+1}$.

Theorem 1.2.2 implies that the knots L_{j+1} and $\tilde{L}_{j+1}$ have the same smooth type. Now we choose $\tilde{\varepsilon}_{j+1}(\eta_1)$ in the following way:

i) If $j + 1 = g$, we choose any positive number such that the subset of the points (X, Y) of T_{g-1}, for which there is a point (u, v) of the knot $\tilde{L}_g$ such that $u = X$ and $|Y - v| < \tilde{\varepsilon}_g(\eta_1)$, is an open tubular neighbourhood $\tilde{T}_g$ of $\tilde{L}_g$ contained in T_g;

ii) If $j+1 < g$, we choose $\tilde{\varepsilon}_{j+1}(\eta_1)$ such that the subset of the points (X, Y) of T_j, for which there is a point (u, v) of the knot $\tilde{L}_{j+1}$ such that $u = X$ and $|Y - v| < \tilde{\varepsilon}_{j+1}(\eta_1)$, is an open tubular neighbourhood $\tilde{T}_{j+1}$ of $\tilde{L}_{j+1}$ contained in T_{j+1} and

$$|a_{\ell_{j+2}}|{\eta_1}^{m_{j+2}/k_1\cdots k_{j+2}} + \cdots + |a_{\ell_g}|{\eta_1}^{m_g/k_1\cdots k_g} + \sum_{\ell_g<i} |a_i|{\eta_1}^{i/k} < \tilde{\varepsilon}_{j+1}(\eta_1).$$

3.1.3. — In the preceding construction, we notice that, for $0 \leqslant j \leqslant g$, the complement, in the closure $\overline{T_j}$ of T_j, of the open solid torus $\tilde{T}_j$ is diffeomorphic to the product of a torus surface $\mathbf{S}^1 \times \mathbf{S}^1$ with the closed interval $[0, 1]$.

Moreover, since, for $j > 1$, the knot $\tilde{L}_j$ has the same type as the knot L_j and L_j is a (λ_j, k_j)-satellite of $\tilde{L}_{j-1}$, one can prove by induction on j that $\tilde{L}_j$ is an iterated torus knot. This provides another proof of the Theorem of K. Brauner which is essentially due to F. Pham (see [P]).

3.2. — In this paragraph we construct a vector field in the solid torus $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$ which lifts the vector field ξ of constant length η_1 of $\partial D_{\eta_1}(0)$ by the projection p of $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$ onto $\partial D_{\eta_1}(0)$ and which is tangent to the closed braid $\Gamma \cap (\partial D_{\eta_1}(0) \times D_{\eta_2}(0))$.

3.2.1. — In each space $\overline{\tilde{T}_j} \setminus T_{j+1}$, for j, $0 \leqslant j \leqslant g - 1$, we do the following construction. By definition the knot $\tilde{L}_j$ is the core of the solid torus $\tilde{T}_j$. Let ξ_j the unique vector field tangent to $\tilde{L}_j$ which lifts the unit vector field ξ of $\partial D_{\eta_1}(0)$ by p. We extend this vector field to a vector field Ξ_j defined in the whole solid torus $\tilde{T}_j$ in the following way. For any (X,Y) in $\overline{\tilde{T}_j}$, there is a unique point (u,v) of $\tilde{L}_i$, such that $X = u$ and $|Y - v| < \varepsilon_j(\eta_1)$. So we set

$$\Xi_j(X,Y) = \xi_j(u,v).$$

Now, define the vector field ρ_j in $\overline{\tilde{T}_j} \setminus T_{j+1}$ by

$$\rho_j(X,Y) = (0, |Y - v|e^{(\lambda_j/k_j)\arg(p(X,Y))})$$

where $\arg(z)$ is the argument of $z \in \mathbb{C}$. On $\overline{\tilde{T}_j} \setminus T_{j+1}$ we consider the vector field v_j defined by

$$v_j(X,Y) := \Xi_j(X,Y) + \rho_j(X,Y)$$

for any $(X,Y) \in \overline{\tilde{T}_j} \setminus T_{j+1}$.

The integral paths of the vector field v_j are knots which are translates of L_{j+1}. These paths define naturally a $\mathbf{S}^1$-foliation of $\overline{\tilde{T}_j} \setminus T_{j+1}$.

Consider $j = g$. The closed solid torus $\overline{\tilde{T}_g}$ is a locally trivial fibration over $\partial D_{\eta_1}(0)$ by the restriction of the projection p. In $\overline{\tilde{T}_g}$, we define any vector field v_g which is tangent to the knot $\tilde{L}_g$ and the boundary $\partial\overline{\tilde{T}_g}$.

Now we patch these vector fields $v_0, \ldots, v_g$ to define a vector field v on the whole $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$.

First we recall (see 3.1.3) that the spaces $\overline{T_j} \setminus \tilde{T}_j$, for $0 \leqslant j \leqslant g$ are diffeomorphic with $\mathbf{S}^1 \times \mathbf{S}^1 \times [0,1]$. On the other hand, on the boundaries $\partial\overline{\tilde{T}_j}$ and $\partial\overline{T_j}$ of the solid tori $\tilde{T}_j$ and $\overline{T_j}$, there are already vector field restrictions of v_j and v_{j-1}. Since the projection p induces a locally trivial fibration over $\partial D_{\eta_1}(0)$, we can extend these vector fields on $\overline{T_j} \setminus \tilde{T}_j$, e.g. using a convenient partition of unity. We obtain the required vector field v on $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$.

By construction the vector field v is tangent to the knot

$$\Gamma \cap (\partial D_{\eta_1}(0) \times D_{\eta_2}(0))$$

and tangent to the boundary $\partial D_{\eta_1}(0) \times \partial D_{\eta_2}(0)$ of $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$.

3.2.2. — Integrating the vector field v we obtain a diffeomorphism of $\{\eta_1\} \times D_{\eta_2}(0)$ onto itself that we have called a *carousel* of the plane curve Γ relatively to the coordinate X (see [Lê3]). In fact the integral paths of the vector field v define a 1-dimensional foliation of the solid torus $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$ which is transverse to the discs $\{\eta_1 e^{i\theta}\} \times D_{\eta_2}(0)$. From this viewpoint the carousel is nothing else than the Poincaré map of the vector field. By construction the knot $\Gamma \cap \partial(D_{\eta_1}(0) \times D_{\eta_2}(0))$ is an integral path, so the carousel leaves the set $\Gamma \cap (\{\eta_1\} \times D_{\eta_2}(0))$ invariant. Since the embedding of the knot $\Gamma \cap \partial(D_{\eta_1}(0) \times D_{\eta_2}(0))$ in the solid torus $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$ defines a braid and this braid is an iterated "toric braid", the carousel is quasi-finite. One can also view the carousel as the braid monodromy of the preceding braid.

In the paragraph 4 we shall indicate how to build a similar vector field in the case of a germ of plane curve with several branches.

3.3. — Using the preceding vector field we shall give a minimal Waldhausen decomposition of the 3-sphere $\mathbf{S}^3$ in which the knot of the branch is a leaf.

3.3.1. — Recall that a *Waldhausen decomposition* of a 3-manifold M (see [W]) is a finite partition:

$$M = \coprod_j V_j \coprod_k \mathcal{T}_k$$

where the V_j's are Seifert manifolds, *i.e.*, 3-manifolds with a given $\mathbf{S}^1$-foliation, and the $\mathcal{T}_k$'s are 2-tori. This Waldhausen decomposition is adapted to a 1-dimensional submanifold, if this submanifold is the leaf of one of the Seifert manifold V_j. This partition is minimal when the number of tori is minimum.

When M is irreducible and sufficiently large, Jaco-Shalen (see [JS1] and [JS2]) and Johansson proved that if M has a Waldhausen decomposition, it has a unique minimal Waldhausen decomposition up to isotopy.

3.3.2. — Now we show how the vector field v gives a Waldhausen decomposition of the 3-sphere $(\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \cup (D_{\eta_1}(0) \times \partial D_{\eta_2}(0))$ adapted to the knot $\Gamma \cap \partial(D_{\eta_1}(0) \times D_{\eta_2}(0))$.

We saw in (3.2.1) that the integral paths of the vector field v define on each $\overline{\tilde{T}_{j-1}} \setminus T_j$ for j, $1 \leqslant j \leqslant g$, a $\mathbf{S}^1$-foliation. Since the spaces $\overline{T_j} \setminus \tilde{T}_j$, for $0 \leqslant j \leqslant g$ are diffeomorphic with $\mathbf{S}^1 \times \mathbf{S}^1 \times [0,1]$, these foliations of $\overline{\tilde{T}_{j-1}} \setminus T_j$, for $1 \leqslant j \leqslant g-1$, extend to $\overline{T_j} \setminus \overline{\tilde{T}_j}$. In this way we have a $\mathbf{S}^1$-foliation on $\overline{\tilde{T}_{j-1}} \setminus \overline{\tilde{T}_j}$. Therefore, the space $M = \overline{\tilde{T}_0} \setminus \tilde{T}_g$ has the following

Waldhausen structure:

$$M = (\overline{\tilde{T}_0} \setminus \overline{\tilde{T}_1}) \coprod_{j=1}^{g-2} (\tilde{T}_j \setminus \overline{\tilde{T}_{j+1}}) \coprod (\tilde{T}_{g-1} \setminus \tilde{T}_g) \coprod_{k=1}^{g-1} \partial\overline{\tilde{T}_k}.$$

We have to extend this structure to the whole

$$(\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \cup (D_{\eta_1}(0) \times \partial D_{\eta_2}(0)).$$

This extension is easy because what is left are solid tori and thickened tori.

First, the boundary of the solid torus $\overline{\tilde{T}}_g$ is foliated, this foliation extends trivially to the whole solid torus $\overline{\tilde{T}}_g$, but eventually the core of $\overline{\tilde{T}}_g$ will be a singular leaf. Note that the core of $\overline{\tilde{T}}_g$ is the knot $\Gamma \cap \partial(D_{\eta_1}(0) \times D_{\eta_2}(0))$.

The space $(\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \setminus \tilde{T}_0$ is by definition $\overline{T_0} \setminus \tilde{T}_0$. We saw that it is diffeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1 \times [0,1]$. Then, the foliation defined on $\partial\tilde{T}_0$ extends trivially to $(\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \setminus \tilde{T}_0$. This gives us a foliation on the boundary $\partial D_{\eta_1}(0) \times \partial D_{\eta_2}(0)$ which extends trivially to the solid torus $D_{\eta_1}(0) \times \partial D_{\eta_2}(0)$, the core of this solid torus being a singular leaf.

This ends our construction of a Waldhausen structure on the 3-sphere

$$(\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \cup (D_{\eta_1}(0) \times \partial D_{\eta_2}(0))$$

adapted to the knot $\Gamma \cap (\partial D_{\eta_1}(0) \times D_{\eta_2}(0))$.

3.4. — By using the unicity theorem of Jaco-Shalen and Johansson, we prove the following theorem:

THEOREM 3.4.1. — *The above construction gives the minimal Waldhausen decomposition of the* 3*-sphere*

$$(\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \cup (D_{\eta_1}(0) \times \partial D_{\eta_2}(0))$$

adapted to the knot

$$\Gamma \cap \partial(D_{\eta_1}(0) \times D_{\eta_2}(0))$$

and to the trivial knot $\{X = 0\} \cap \partial(D_{\eta_1}(0) \times D_{\eta_2}(0))$.

The proof of this result uses the fact that none of the different foliations on the Seifert pieces considered in (3.3) can be deleted.

In the case that the first Puiseux exponent is $1/k_1$, the Waldhausen structure of Theorem 3.4.1 is not minimal if we do not require that it is adapted to the trivial knot

$$\{X = 0\} \cap \partial(D_{\eta_1}(0) \times D_{\eta_2}(0)).$$

The complement of $\tilde{T}_1$ in $(\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \cup (D_{\eta_1}(0) \times \partial D_{\eta_2}(0))$ is a solid torus, since the knot $\tilde{L}_1$ is a trivial knot. To obtain the minimal Waldhausen adapted to the knot $\Gamma \cap \partial(D_{\eta_1}(0) \times D_{\eta_2}(0))$, we endow this complement with the foliation trivially extended from the one on the boundary of $\tilde{T}_1$ and we keep the other Seifert manifolds of $\overline{\tilde{T}_2}$.

Then, one can obtain that the minimal Waldhausen structure of a 3-sphere adapted to the algebraic knot of the plane branch $(\Gamma, 0)$ is uniquely determined by the knot if the knot is not the trivial knot, *i.e.*, if $(\Gamma, 0)$ is not non-singular.

3.4.2. — The main consequence of this construction is that it is equivalent to know the iterated torus knot associated to the plane branch or to know the minimal Waldhausen structure of the 3-sphere adapted to the local knot of the branch.

This explains the viewpoint of what follows as we consider plane curve singularities with several branches. The natural structure which generalizes the iterated torus knot structure of an algebraic knot will be the minimal Waldhausen structure of the 3-sphere adapted to the components of the algebraic link. Except in the case of non-singular curves or ordinary double points this latter structure is a topological invariant of the plane curve singularity (see [LMW]).

4. Plane curves with several branches.

In this paragraph we shall consider a plane curve singularity $(\Gamma, 0)$ with several branches $\Gamma_1, \ldots, \Gamma_r$. We shall assume that $r \geqslant 2$.

4.1. — Consider the Puiseux expansions of the branches $(\Gamma_i, 0)$, $0 \leqslant i \leqslant r$, in the given coordinates (X, Y):

$$Y_1 = \sum_{\ell} b_{1,\ell} X^{\ell/k_1}$$

$$\vdots$$

$$Y_r = \sum_{\ell} b_{r,\ell} X^{\ell/k_r}.$$

It will be convenient to set $k = k_1 \cdots k_r$ and write the Puiseux expansions in the following way:

$$Y_1 = \sum_\ell a_{1,\ell} X^{\ell/k}$$
$$\vdots$$
$$Y_r = \sum_\ell a_{r,\ell} X^{\ell/k},$$

with $a_{i,\ell} = b_{i,\ell/\kappa_i}$, whenever ℓ is divisible by $\kappa_i := k_1 \cdots k_{i-1} k_{i+1} \cdots k_r$, and $a_{i,\ell} = 0$, whenever ℓ not divisible by κ_i, for $1 \leqslant i \leqslant r$.

We can consider a series $Y = \sum_\ell b_\ell X^{\ell/k}$ as an element of the ring $\mathbb{C}\{X^{1/k}\}$ and therefore of the field $\mathbb{C}\langle\langle X^{1/k}\rangle\rangle$ (see §2.2 above).

Let $Y_i = \sum_\ell a_{i,\ell} X^{\ell/k}$ and $Y_j = \sum_\ell a_{j,\ell} X^{\ell/k}$, with $1 \leqslant i, j \leqslant r$. Let G be the Galois group of the extension $\mathbb{C}\langle\langle X^{1/k}\rangle\rangle$ of $\mathbb{C}\langle\langle X\rangle\rangle$. We call *contact exponent* of these two series the following

$$\sup(v(Y_i - \sigma(Y_j)), \sigma \in G)$$

where v is the valuation of $\mathbb{C}\langle\langle X^{1/k}\rangle\rangle$ which is 1 for $X^{1/k}$. Note that the contact exponent is $+\infty$ if and only if $i = j$. Otherwise it is an integer.

Let $k_{i,j}$ be the contact exponent of Y_i and Y_j. Then the G-orbits of the truncated series of both Y_i and Y_j at $k_{i,j}$ are the same. In that case, whenever there will be no confusion, we shall say that the truncated series at $k_{i,j}$ coincide.

We define the characteristic exponents of the set of Puiseux series

$$Y_1 = \sum_\ell a_{1,\ell} X^{\ell/k}$$
$$\vdots$$
$$Y_r = \sum_\ell a_{r,\ell} X^{\ell/k},$$

in the following way:

Definition 4.1.1. — *The characteristic exponents of the series $Y_1, \ldots, Y_r$ are the set of Puiseux characteristic exponents of each Y_i, $1 \leqslant i \leqslant r$, and the quotients $k_{i,j}/k$ by k of the pairwise contact exponents $k_{i,j}$, for $1 \leqslant i < j \leqslant r$.*

Note that characteristic exponents of any subset of series among $Y_1, \ldots, Y_r$ are characteristic exponents of $Y_1, \ldots, Y_r$.

4.2. — We use the notations of 4.1. Let ℓ_j/k, $1 \leqslant j \leqslant s$, be the characteristic exponents of the set of Puiseux expansions:

$$Y = \sum_{\ell} a_{1,\ell} X^{\ell/k}$$
$$\vdots$$
$$Y = \sum_{\ell} a_{r,\ell} X^{\ell/k}.$$

We assume that $\ell_1 < \ell_j$, for $2 \leqslant j \leqslant s$.

We order the characteristic exponents by induction in the following way.

In the case of one branch, *i.e.*, one Puiseux expansion, we consider the natural ordering. So the first Puiseux characteristic exponent is called the characteristic exponent of order one, the second Puiseux exponent is called the characteristic exponent of order two and so on.

In the case of $r > 0$ branches, we assume that we have defined the order of the characteristic exponents for r' branches when $r > r'$. Suppose that all the series truncated at ℓ_1 coincide. Then ℓ_1/k is not an integer and ℓ_1/k is the unique characteristic exponent of order one. We shall write

$$\ell_1 = \ell_1^1.$$

If all the series truncated at ℓ_1 do not coincide, the exponent $\ell_1/k = \ell_1^1/k$ is one of the characteristic exponent of order one. Call σ_i, $1 \leqslant i \leqslant r_1$, be the distinct truncated Puiseux series at ℓ_1 for which the coefficient of order ℓ_1 is $\neq 0$. We are left with Puiseux series for which the term of order ℓ_1 vanishes. The other characteristic exponents of order one ℓ_j^1/k $(2 \leqslant j \leqslant s_1)$ are the ones of the set of these remaining Puiseux series. In this case the characteristic exponents of order $\geqslant 2$ are the characteristic exponents of order $\geqslant 2$ of each bunch of Puiseux series with σ_i, $1 \leqslant i \leqslant r_1$, as truncated series and the characteristic exponents of order $\geqslant 2$ of the set of remaining Puiseux series.

It remains to define the characteristic exponents of order $\geqslant 2$ in the case the truncated series at ℓ_1 of all Puiseux series are the same. Let $\ell_1/k = m_1/n_1$, where m_1 and n_1 are relatively prime. Consider ℓ_2 the smallest among $\ell_2, \dots, \ell_s$. If the truncated Puiseux series at ℓ_2 are again equal, ℓ_2/k is not in $(1/n_1)\mathbb{N}$, and ℓ_2/k is the unique characteristic exponent of order 2.

If all the truncated Puiseux series at ℓ_2 are not equal, $\ell_2/k := \ell_1^2/k$ is one of the characteristic exponents of order two. The other characteristic exponents of order two are the characteristic exponents of order two of the bunch of Puiseux series for which the term of order ℓ_2 is 0. Then, the characteristic exponents of order $\geqslant 3$ are the characteristic exponents of order $\geqslant 3$ of each bunch of Puiseux series having the same truncated Puiseux series at ℓ_2 and the characteristic exponents of order $\geqslant 3$ of the set of remaining Puiseux series for which the term of order ℓ_2 is 0.

Then, we proceed by induction on the number of exponents for which the corresponding truncated series of all the Puiseux series are the same. Since these Puiseux series are distinct, this process ends.

Now the order on the set of the exponents is the following:

Let ℓ^i_j/k, $1 \leqslant j \leqslant s_i$, be the characteristic exponents of order i. We say that

$$\ell^i_j/k \prec \ell^m_l/k$$

if $i < m$ and there is a subset of the set of the given Puiseux series such that ℓ^i_j/k and ℓ^m_l/k are both Puiseux characteristic exponents respectively of order i and m of that subset of Puiseux series.

4.3. — In this paragraph we construct a family of solid tori or thickened tori by induction on the number of branches and the number of characteristic exponents:

i) We define the solid torus $T_{0,1} = \partial D_{\eta_1}(0) \times D_{\eta_2}(0)$; for the coherence of the notations, we shall also write $T_{0,i,1} := T_{0,1}$, for any i, $1 \leqslant i \leqslant s$.

ii) As above, $\ell_1 = \ell^1_1$ is the smallest of the exponents of order one. Let η_1 small enough. Then, we can choose $\varepsilon_{0,1}(\eta_1)$ such that

$$\sup_{1\leqslant i\leqslant s}\Big(\sum_{\ell} |a_{i,\ell}|\eta_1^{\ell/k}\Big) < \varepsilon_{0,1}(\eta_1) < \eta_2/2$$

and the solid torus $\tilde{T}_{0,1}$, subspace of points (X,Y) of T_0 for which there is a point (u,v) of the trivial knot $\tilde{L}_0$ defined by the Puiseux expansion

$$Y = \sum_{\ell<\ell_1} a_{1,\ell} X^{\ell/k}$$

such that $u = X$ and $|Y - v| < \varepsilon_{0,1}(\eta_1)$ is a tubular neighbourhood of $\tilde{L}_0$ contained in the interior of $T_{0,1}$ $(T_{0,i,1})$. As in i), we write $\tilde{T}_{0,i,1} := \tilde{T}_{0,1}$ and also $\tilde{L}_{0,i} := \tilde{L}_0$, for any i, $1 \leqslant i \leqslant s$.

iii) If there are several characteristic exponents of order one, say ℓ^1_j, $1 \leqslant j \leqslant s_1$, we order these exponents:

$$\ell^1_1 < \ell^1_2 < \cdots < \ell^1_{s_1}.$$

Then, let $\alpha_1^1 > 0$ such that

$$\alpha_1^1 < \inf_i \{|a_{i,\ell_1^1}| \neq 0\}.$$

Call $\tilde{T}'_{0,1}$ the solid torus, subspace of points (X,Y) of $T_{0,1}$ for which there is a point (u,v) of the trivial knot $\tilde{L}_0$ such that $u = X$ and $|Y - v| < \alpha_1^1 \eta_1^{\ell_1^1}$. For η_1 small enough, the geometric braids corresponding to the Puiseux series for which the coefficient $a_{i,\ell_1^1} \neq 0$ are all contained in the interior of $\tilde{T}_{0,1} \setminus \tilde{T}'_{0,1}$ and the geometric braids of the other Puiseux series for which $a_{i,\ell_1^1} = 0$ are contained in the interior of $\tilde{T}'_{0,1}$. For any i, such that $a_{i,\ell_1^1} \neq 0$, we shall write $\tilde{T}_{0,i,1} := \tilde{T}_{0,1}$ and $\tilde{T}'_{0,i,1} := \tilde{T}'_{0,1}$ Now, let the numbers $\alpha_j^1 > 0$ be such that

$$\alpha_j^1 < \inf_i \{|a_{i,\ell_j^1}| \neq 0\}.$$

We construct by induction on j, $1 \leqslant j \leqslant s_1$, the trivial knots $\tilde{L}_j$, and the sequence of solid tori $\tilde{T}_{0,j}$ and $\tilde{T}'_{0,j}$ such that the interior of $\tilde{T}_{0,j} \setminus \tilde{T}'_{0,j}$ contains all the geometric braids corresponding to the Puiseux series for which $a_{i,\ell_t^1} = 0$ for $t < j$ and $a_{i,\ell_j^1} \neq 0$, while $\tilde{T}'_{0,j}$ is the solid torus, subspace of points (X,Y) of $T_{0,1}$ for which there is a point (u,v) of the trivial knot $\tilde{L}_j$ such that $u = X$ and $|Y - v| < \alpha_j \eta_1^{\ell_j^1}$, which contains the geometric braids of all the Puiseux series such that $a_{i,\ell_t^1} = 0$ for $t \leqslant j$, in its interior. Suppose that, for a given j, $1 \leqslant j < s_1$, we did this construction. By definition of the characteristic exponents, one of the Puiseux series, such that $a_{i,\ell_t^1} = 0$ for $t \leqslant j$, has a truncation at ℓ_{j+1}^1:

$$\sum_{\ell \leqslant \ell_{j+1}^1} a_{i,\ell} X^{\ell/k}$$

such that

$$a_{i,\ell_{j+1}^1} \neq 0.$$

The truncated series

$$\sum_{\ell < \ell_{j+1}^1} a_{i,\ell} X^{\ell/k}$$

define a trivial knot $\tilde{L}_{j+1}$. For η_1 small enough, there is $\varepsilon_{0,j+1}(\eta_1)$ such that

$$\sup_{1 \leqslant i \leqslant s} \Big(\sum_{\ell_{j+1}^1 < \ell} |a_{i,\ell}| \eta_1^{\ell/k} \Big) < \varepsilon_{0,j+1}(\eta_1) < \alpha_j^1 \eta_1^{\ell_j^1}/2$$

and the solid torus $\tilde{T}_{0,j+1}$, subspace of points (X,Y) of T_0 for which there is a point (u,v) of the trivial knot $\tilde{L}_{j+1}$ such that $u = X$ and

$|Y-v|<\varepsilon_{0,j+1}(\eta_1)$ is a tubular neighbourhood of $\tilde{L}_{j+1}$ contained in the interior of $\tilde{T}'_{0,j}$. We call $\tilde{T}'_{0,j+1}$ the subspace of points (X,Y) of $T_{0,1}$ for which there is a point (u,v) of the trivial knot $\tilde{L}_{j+1}$ such that $u=X$ and $|Y-v|<\alpha_{j+1}\eta_1^{\ell^1_{j+1}}$. For η_1 small enough, the interior of $\tilde{T}_{0,j+1}\setminus\tilde{T}'_{0,j+1}$ contains all the geometric braids corresponding to the Puiseux series for which $a_{i,\ell^1_t}=0$ for $t<j+1$ and $\tilde{T}'_{0,j+1}$ contains all the geometric braids of the Puiseux series such that $a_{i,\ell^1_t}=0$ for $t\leqslant j+1$. As above, for the coherence of the notations, for any i such that $a_{i,\ell^1_t}=0$ for $t<j+1$, we shall write $\tilde{L}_{0,i,j+1}:=\tilde{L}_{j+1}$ and, for any i, such that $a_{i,\ell^1_t}=0$ for $t<j+1$ and $a_{i,\ell^1_{j+1}}\neq 0$, we also write $\tilde{T}_{0,i,j+1}:=\tilde{T}_{0,j+1}$, $\tilde{T}'_{0,i,j+1}:=\tilde{T}'_{0,j+1}$ and $\varepsilon_{0,i,j+1}(\eta_1):=\varepsilon_{0,j+1}(\eta_1)$.

iv) Now for each Puiseux series such that $a_{i,\ell^1_t}=0$ for $t<j$, and $a_{i,\ell^1_j}\neq 0$, we consider the knot $L_{1,i,j}$ defined by the truncated series at ℓ^1_j:

$$\sum_{\ell\leqslant\ell^1_j} a_{i,\ell}X^{\ell/k}.$$

Then, there is $\varepsilon_{1,i,j}(\eta_1)$ such that

$$\sup_{1\leqslant i\leqslant s}\Big(\sum_{\ell^1_j<\ell} a_{i,\ell}X^{\ell/k}\Big)<\varepsilon_{1,i,j}(\eta_1)$$

and, for η_1 small enough and for all i such that $a_{i,\ell^1_t}=0$ for $t<j$ and $a_{i,\ell^1_j}\neq 0$, the solid tori $T_{1,i,j}$, subspaces of points (X,Y) of $T_{0,1}$ for which there is a point (u,v) of the knots $L_{1,i,j}$ such that $u=X$ and $|Y-v|<\varepsilon_{1,i,j}(\eta_1)$ are mutually disjoint and contained in the interior of $\tilde{T}_{0,i,j}\setminus\tilde{T}'_{0,i,j}$.

v) If we only have characteristic exponents of order one, we have finished the construction of the desired solid tori by induction on the number of characteristic exponents of order one. Otherwise, we have to proceed by induction on the order of characteristic exponents. Let

$$\ell^1_{j_1}/k\prec\ell^2_{j_2}/k\prec\cdots\prec\ell^n_{j_n}/k,$$

with $n\geqslant 2$, be a chain of characteristic exponents of our set of Puiseux series. Let

$$\sum_{\ell} a_{i,\ell}X^{\ell/k}$$

be a Puiseux series having these characteristic exponents, so the coefficients $a_{i,\ell^m_{j_m}}\neq 0$, for $1\leqslant m\leqslant n$. Consider the truncated series at $\ell^n_{j_n}$:

$$\sum_{\ell\leqslant\ell^n_{j_n}} a_{i,\ell}X^{\ell/k}.$$

This defines a knot $L_{n,i,j_1,\dots,j_n}$. We have assume by induction that we have constructed the solid torus $T_{m-1,i,j_1,\dots,j_{m-1}}$ which contains the solid tori $\tilde{T}_{m-1,i,j_1,\dots,j_m}$ and $\tilde{T}'_{m-1,i,j_1,\dots,j_m}$, with $1 \leqslant m \leqslant n$. The interior of the space

$$\tilde{T}_{m-1,i,j_1,\dots,j_m} \setminus \tilde{T}'_{m-1,i,j_1,\dots,j_m}$$

contains all the geometric braids associated to the Puiseux series having the same truncated series

$$\sigma_m = \sum_{\ell<\ell^m_{j_m}} a_{i,\ell} X^{\ell/k}$$

and the coefficient of order $\ell^m_{j_m} \neq 0$, while the interior of $\tilde{T}'_{m-1,i,j_1,\dots,j_m}$ contains all the geometric braids associated to the Puiseux series having that same truncated series, but with the coefficient of order $\ell^m_{j_m} = 0$. We can find $\varepsilon_{n,j_1,\dots,j_n}(\eta_1)$ such that

$$\sup_{1\leqslant i\leqslant s}\Big(\sum_{\ell^n_{j_n}<\ell} a_{i,\ell} X^{\ell/k}\Big) < \varepsilon_{n,j_1,\dots,j_n}(\eta_1)$$

and, for η_1 small enough and for all i corresponding to the indices of the Puiseux series having the truncated series σ_{n-1} and such that $a_{i,\ell^n_j} = 0$ for $j < j_n$ and $a_{i,\ell^n_{j_n}} \neq 0$, the solid tori $T_{n,i,j_1,\dots,j_n}$, subspaces of points (X,Y) of $T_{0,1}$ for which there is a point (u,v) of the knots $L_{n,i,j_1,\dots,j_n}$ such that $u = X$ and $|Y - v| < \varepsilon_{n,j_1,\dots,j_n}(\eta_1)$ are mutually disjoint and contained in the interior of $\tilde{T}_{n-1,i,j_1,\dots,j_n} \setminus \tilde{T}'_{n-1,i,j_1,\dots,j_n}$. If the considered chain of characteristic exponents is maximal, then, we have finished. If there are characteristic exponenta ℓ^{n+1}_j/k of order $n+1$ and $\succ \ell^n_{j_n}/k$, then, we construct the solid tori $\tilde{T}_{n,i,j_1,\dots,j_n,j}$ and $\tilde{T}'_{n,i,j_1,\dots,j_n,j}$ by induction on the number of characteristic exponents of order $n+1$. Let us order these exponents such that

$$\ell^{n+1}_1 < \ell^{n+1}_2 < \cdots < \ell^{n+1}_r.$$

There is $\varepsilon_{n,i,j_1,\dots,j_n,1}(\eta_1)$ such that

$$\sup_{1\leqslant i\leqslant s}\Big(\sum_{\ell^{n+1}_1\leqslant\ell} |a_{i,\ell}|\eta_1^{\ell/k}\Big) < \varepsilon_{n,i,j_1,\dots,j_n,1}(\eta_1)$$

and the solid torus $\tilde{T}_{n,i,j_1,\dots,j_n,1}$, subspace of points (X,Y) of $T_{0,1}$ for which there is a point (u,v) of the knot $L_{n,i,j_1,\dots,j_n}$ such that $u = X$ and $|Y - v| < \varepsilon_{n,j_1,\dots,j_n,1}(\eta_1)$ is in the interior of $T_{n,i,j_1,\dots,j_n}$; for this purpose it is enough that

$$\varepsilon_{n,i,j_1,\dots,j_n,1}(\eta_1) < \varepsilon_{n,i,j_1,\dots,j_n}(\eta_1)/2.$$

Let $\alpha_1^n > 0$ be such that

$$\alpha_1^n < \inf_i\{|a_{i,\ell_1^{n+1}}| \neq 0\}.$$

We define $\tilde{T}'_{n,i,j_1,\dots,j_n,1}$ as the subspace of points (X,Y) of $T_{0,1}$ for which there is a point (u,v) of the knot $L_{n,i,j_1,\dots,j_n}$ such that $u = X$ and $|Y-v| < \alpha_1^{n+1}\eta_1^{\ell_1^{n+1}}$. We notice that the interior of $\tilde{T}_{n,i,j_1,\dots,j_n,1}\setminus\tilde{T}'_{n,i,j_1,\dots,j_n,1}$ contains all the geometric braids corresponding to the Puiseux series such that the truncated series at $\ell_{j_n}^n$ is σ_n and the term of order ℓ_1^{n+1} is $\neq 0$. While the other series with the same truncated series are contained in the interior of $\tilde{T}'_{n,i,j_1,\dots,j_n,1}$. Let us suppose that for j, $1 \leqslant j < j_{n+1}$, the solid tori $\tilde{T}_{n,i,j_1,\dots,j_n,j}$ and $\tilde{T}'_{n,i,j_1,\dots,j_n,j}$ have been defined, such that all the geometric braids corresponding to the Puiseux series such that, the truncated series at $\ell_{j_n}^n$ is σ_n and the terms of order $\ell_{j'}^{n+1} = 0$, for $j' < j$, and ℓ_j^{n+1} is $\neq 0$, are contained in $\tilde{T}_{n,i,j_1,\dots,j_n,j} \setminus \tilde{T}'_{n,i,j_1,\dots,j_n,j}$, while all the geometric braids corresponding to the Puiseux series having the truncated series σ_n and the terms $\ell_{j'}^{n+1} = 0$, for $j' \leqslant j$ are contained in $\tilde{T}'_{n,i,j_1,\dots,j_n,j}$. As above, let $\varepsilon_{n,i,j_1,\dots,j_n,j_{n+1}}(\eta_1)$ be chosen such that the solid torus $\tilde{T}_{n,i,j_1,\dots,j_n,j_{n+1}}$, subspace of points (X,Y) of $T_{0,1}$ for which there is a point (u,v) of the knot $L_{n,i,j_1,\dots,j_n}$ such that $u = X$ and $|Y - v| < \varepsilon_{n,i,j_1,\dots,j_n,j_{n+1}}(\eta_1)$, is contained in $\tilde{T}'_{n,i,j_1,\dots,j_n,j_{n+1}-1}$ and

$$\sup_{1\leqslant i\leqslant s}\Big(\sum_{\ell_{j_{n+1}}^{n+1}<\ell} |a_{i,\ell}|\eta_1^{\ell/k}\Big) < \varepsilon_{n,i,j_1,\dots,j_n,j_{n+1}}(\eta_1).$$

If $j_{n+1} = r$, we finish here. If $j_{n+1} < r$, we choose $\alpha_{j_{n+1}}^{n+1} > 0$ such that

$$\alpha_{j_{n+1}}^{n+1} < \inf_i\{|a_{i,j_{n+1}{}^{n+1}}| \neq 0\}.$$

Then, $\tilde{T}'_{n,i,j_1,\dots,j_n,j_{n+1}}$ is the subspace of points (X,Y) of $T_{0,1}$ for which there is a point (u,v) of the knot $L_{n,i,j_1,\dots,j_n}$ such that $u = X$ and $|Y - v| < \alpha_{j_{n+1}}^{n+1}\eta_1^{\ell_{j_{n+1}}{}^{n+1}}$.

Finally, in this way, we have obtained the desired family of solid tori.

4.4. — In this paragraph we build a vector field in $\partial D_{\eta_1}(0)\times D_{\eta_2}(0)$ tangent to the algebraic link $\Gamma\cap\partial(D_{\eta_1}(0)\times D_{\eta_2}(0))$ whose integration will give the carousel relatively to X.

Similarly to what we have done for a plane branch, we first construct the vector field in a subspace of each thickened tori and solid tori which contain points of the link $\Gamma \cap (\partial D_{\eta_1}(0) \times D_{\eta_2}(0))$. Then we shall glue our construction by using a partition of unity.

4.4.1. — For simplicity in the notations, we let $\mathbf{m}$ be the index $(n, i, j_1, \ldots, j_n, j_{n+1})$. We first do the construction in a thickened torus like $A_{\mathbf{m}} = \overline{\tilde{T}_{\mathbf{m}}} \setminus \tilde{T}'_{\mathbf{m}}$. Remember that the core of $\tilde{T}_{\mathbf{m}}$ is the algebraic knot $\tilde{L}_{\mathbf{m}}$ defined by a Puiseux expansion

$$Y = \sum_{\ell < \ell^{n+1}_{j_{n+1}}} a_{i,\ell} X^{\ell/k}.$$

As in §3, let ξ be the positive vector field of constant length η_1 of $\partial D_{\eta_1}(0)$. Let $\xi_{\mathbf{m}}$ the unique vector field tangent to $\tilde{L}_{\mathbf{m}}$ which lifts the unit vector field ξ of $\partial D_{\eta_1}(0)$ by p. We extend this vector field to a vector field $\Xi_{\mathbf{m}}$ defined in the whole solid torus $\overline{\tilde{T}_{\mathbf{m}}}$ in the following way. For any (X, Y) in $\overline{\tilde{T}_{\mathbf{m}}}$, there is a unique point (u, v) of $\tilde{L}_{\mathbf{m}}$, such that $X = u$ and $|Y - v| < \varepsilon_{\mathbf{m}}(\eta_1)$. So we set

$$\Xi_{\mathbf{m}}(X, Y) = \xi_{\mathbf{m}}(u, v).$$

Now, define the vector field $\rho_{\mathbf{m}}$ in $\overline{\tilde{T}_{\mathbf{m}}} \setminus \tilde{T}'_{\mathbf{m}}$ by

$$\rho_{\mathbf{m}}(X, Y) = \big(0, |Y - v| e^{(\ell^{n+1}_{j_{n+1}}/k) \arg(p(X,Y))}\big)$$

where $\arg(z)$ is the argument of z. On the subspace $\overline{\tilde{T}_{\mathbf{m}} \setminus (\tilde{T}'_{\mathbf{m}} \cup_t T_{\mathbf{m}_t})}$ of $\overline{\tilde{T}_{\mathbf{m}}} \setminus \tilde{T}'_{\mathbf{m}}$, where $\mathbf{m}_t$ is the multi-index $(n+1, t, j_1, \ldots, j_n, j_{n+1})$ and t ranges in the set of indices of Puiseux series having the truncated Puiseux series $Y = \sum_{\ell < \ell^{n+1}_{j_{n+1}}} a_{i,\ell} X^{\ell/k}$ and $a_{t, \ell^{n+1}_{j_{n+1}}} \neq 0$, we consider the vector field $v_{\mathbf{m}}$ defined by

$$v_{\mathbf{m}}(X, Y) := \Xi_{\mathbf{m}}(X, Y) + \rho_{\mathbf{m}}(X, Y)$$

for any $(X, Y) \in \overline{\tilde{T}_{\mathbf{m}}} \setminus (\tilde{T}'_{\mathbf{m}} \cup_t T_{\mathbf{m}_t})$. Notice that the integral curves of this vector field are satellites of $\tilde{L}_{\mathbf{m}}$.

This construction for all the thickened tori $\tilde{T}_{\mathbf{m}}$.

Since the other thickened tori like

$$A'_{\mathbf{m}} = \tilde{T}'_{(n,i,j_1,\ldots,j_n,j_{n+1})} \setminus \tilde{T}_{(n,i,j_1,\ldots,j_n,j_{n+1}+1)}$$

do not contain components of the link $\Gamma \cap (\partial D_{\eta_1}(0) \times D_{\eta_2}(0))$, we can extend the preceding vector fields in these thickened tori by vector fields which lift the vector field ξ by p by using a proper partition of unity. In this way we obtain a vector field v on $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$ which lifts ξ.

4.5. — Integrating the vector field v, we define a diffeomorphism of $\partial D_{\eta_1}(0) \times \{\eta_1\}$ onto itself which is the carousel (see [Lê3] and [Lê4]) of the germ $(\Gamma, 0)$ relatively to X.

As already said in §3.2.2 the integral paths of v define a 1-dimensional foliation of the plain torus $\partial D_{\eta_1}(0) \times \{\eta_1\}$ whose leaves are transverse to the meridian discs. The carousel is the Poincaré map associated to the vector field v. Therefore the carousel map is the braid monodromy of the braid defined by the embedding of the knot in the solid torus $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$.

As above, we also obtain a Waldhausen decomposition of the 3-sphere

$$(\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \cup (D_{\eta_1}(0) \times \partial D_{\eta_2}(0))$$

in the following way:

The spaces like $A_{\mathbf{m}} \setminus \cup_t T_{\mathbf{m}_t}$ are foliated by satellite of an algebraic knot. We can extend to the thickened tori A'_m that foliation. In the union of solid tori $\cup_t T_{\mathbf{m}_t}$ we make a decomposition by induction on the number of branches and number of characteristic exponents. In this way we have a Waldhausen decomposition of $\partial D_{\eta_1}(0) \times D_{\eta_2}(0)$ which extends to $D_{\eta_1}(0) \times \partial D_{\eta_2}(0)$ with eventually a singular leaf $\{0\} \times \partial D_{\eta_2}(0)$. Then, this gives:

THEOREM 4.5.1. — *The preceding Waldhausen decomposition is the minimal Waldhausen decomposition of the* 3*-sphere* $(\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \cup (D_{\eta_1}(0) \times \partial D_{\eta_2}(0))$ *adapted to the components of* $\Gamma \cap (\partial D_{\eta_1}(0) \times D_{\eta_2}(0))$ *and* $\{0\} \times \partial D_{\eta_2}(0)$.

As in the case of branches, we can extract from this the minimal Waldhausen decomposition of $(\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \cup (D_{\eta_1}(0) \times \partial D_{\eta_2}(0))$ adapted to the components of $\Gamma \cap \partial(D_{\eta_1}(0) \times D_{\eta_2}(0))$. We shall omit details about this latter assertion. It is enough to know that, except for non-singular plane branches or the germ of an ordinary double plane singularity, the minimal Waldhausen decomposition of

$$(\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \cup (D_{\eta_1}(0) \times \partial D_{\eta_2}(0))$$

adapted to the components of $\Gamma \cap (\partial D_{\eta_1}(0) \times D_{\eta_2}(0))$ obtained in this way is unique up to isotopy.

This minimal Waldhausen decomposition of

$$(\partial D_{\eta_1}(0) \times D_{\eta_2}(0)) \cup (D_{\eta_1}(0) \times \partial D_{\eta_2}(0))$$

adapted to the components of $\Gamma \cap \partial(D_{\eta_1}(0) \times D_{\eta_2}(0))$ is therefore a natural generalisation of the iterated knot structure of algebraic knots.

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