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TRACES AND THE F. AND M. RIESZ THEOREM FOR VECTOR FIELDS

by S. BERHANU and J. HOUNIE

Dedicated to the memory of Laurent Schwartz.

Introduction.

In this paper we study the existence of boundary values for solutions of smooth, locally integrable complex vector fields in the plane. A nowhere vanishing smooth, complex vector field

$$L = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$$

is said to be locally integrable in an open set D if each $p \in D$ is contained in a neighborhood which admits a smooth function $Z(x,y)$ with the properties that $LZ = 0$ and the differential $dZ \neq 0$. Among other important examples of locally integrable vector fields we note the class of real analytic vector fields and the class of smooth, locally solvable vector fields (Definition 4.1). On the subject of locally integrable vector fields we refer to the treatise [T]. Not every smooth vector field is locally integrable although nondegenerate

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vector fields which are not locally integrable are nontrivial to construct; the first example of such a vector field is due to Nirenberg ([N]).

The central question studied in this work is the existence of a distribution boundary value for continuous solutions of $Lf = 0$. If f is continuous on a rectangle $Q = (-a, a) \times (0, b)$, we say f has a boundary value $bf \in \mathcal{D}'(-a, a)$ (or a trace at $y = 0$) if for every $\psi(x) \in C_c^\infty(-a, a)$,

$$\lim_{y \to 0^+} \int_{-a}^{a} f(x, y)\psi(x)\, dx \ \text{ exists.} \tag{0.1}$$

If (0.1) holds for every $\psi(x) \in C_c^\infty(-a, a)$, a standard category argument shows that there exists a distribution $bf \in \mathcal{D}'(-a, a)$ such that

$$\langle bf, \psi \rangle = \lim_{y \to 0^+} \int_{-a}^{a} f(x, y)\psi(x)\, dx, \quad \psi(x) \in C_c^\infty(-a, a),$$

and that $f(\cdot, y)$ remains bounded in $H^s_{\text{loc}}(-a, a)$ for some $s \in \mathbb{R}$ as $y \searrow 0$. It is remarkable that a qualitative property satisfied by $f(x, y)$, namely that it satisfies a homogeneous PDE, guarantees the existence of the boundary value when what is only a priori known is that the size of $f(\cdot, y)$ blows up in a tempered manner rather than remain bounded as $y \searrow 0$. A well-known instance of this phenomenon is (see e.g., [Hö], Thm. 3.1.14) that if h is holomorphic in a rectangle $Q = (-a, a) \times (0, b)$, then the traces $h(\cdot, y)$ converge as $y \mapsto 0^+$ to a distribution $bh(x)$ if and only if there exists an integer N such that

$$|h(x + iy)| = O(y^{-N}) \tag{0.2}$$

uniformly for x in compact sets. Among other things, this result implies that

(1) The holomorphic functions on Q with boundary values form an algebra, and

(2) if the holomorphic function h has a boundary value at $\{y = 0\}$, then it also has a boundary value on any curve $\gamma \subset \overline{Q}$ through the origin.

In this work we explore extensions of these results for solutions of locally integrable complex vector fields. In Section 1 we present a condition akin to a weaker form of (0.2) that is sufficient for the existence of a boundary value for solutions f of $Lf = g$ where L is a locally integrable vector field and g is a locally integrable function. In Section 2 we show that this condition is necessary for the existence of a boundary value for the solutions of a real analytic vector field on a noncharacteristic piece of the boundary. Lemma 4.3 in Section 4 also establishes the same necessity for a subclass of locally solvable, smooth vector fields. Sections 3, 4 and 5 contain results on the F. and M. Riesz property for vector fields. The celebrated classical F. and M. Riesz theorem may be stated as: if a holomorphic function $f(z)$ defined on a smooth domain of the complex plane has a weak boundary value which is a measure then the measure is in fact absolutely continuous with respect to Lebesgue measure. In a recent paper ([BH1]) we extended the F. and M. Riesz theorem to all locally integrable, smooth complex vector fields L in the plane for smooth domains at the noncharacteristic part of the boundary. In our theorem we assumed that the solutions f grew at a tempered rate near the boundary. For holomorphic functions, as mentioned already, such a growth rate is equivalent to the existence of a boundary value. Moreover, in our recent paper ([BH2], Theorem 1.1) it was shown that tempered growth as in (0.2) implies the existence of a boundary value for any vector field L (not necessarily locally integrable) for which the x-axis is noncharacteristic. For locally integrable vector fields L, Theorem 1.1 here establishes the existence of a boundary value under a growth assumption which is weaker than tempered growth. This growth assumption is expressed in terms of a first integral of L and it specializes to (0.2) in the case of the Cauchy Riemann operator. In Sections 3, 4, and 5, we will prove the F. and M. Riesz property for classes of vector fields under the weaker and more natural assumption on the growth of the solution that is considered in Sections 1 and 2. In its original formulation, the classical F. and M. Riesz theorem states that a complex measure μ defined on the boundary $\mathbb{T}$ of the unit disc Δ all of whose negative Fourier coefficients vanish, *i.e.*,

$$(0.3) \qquad \widehat{\mu}(k) = \int_0^{2\pi} \exp(-2\pi i k\theta)\, d\mu(\theta) = 0, \quad k = -1, -2, \ldots,$$

is absolutely continuous with respect to Lebesgue measure $d\theta$. Condition (0.3) is equivalent to the existence of a holomorphic function $f(z)$ defined on Δ whose weak boundary value is μ. In other words, the theorem asserts that if a holomorphic function f on Δ has a weak boundary value bf that is a measure, then in fact $bf \in L^1(\mathbb{T})$.

The F. and M. Riesz theorem has had numerous applications and it has inspired an extensive generalization in two different directions: i) generalized analytic function algebras, which has as a starting point the fact that (0.3) means that μ is orthogonal to the algebra of continuous functions f on $\mathbb{T}$ that extend holomorphically to F on Δ with $F(0) = 0$; ii) ordered groups, which emphasizes instead the role of the group structure of $\mathbb{T}$ in the classical result. For the generalization along function algebras, we mention the book [BK] by Klaus Barbey and Heinz Konig. A description and survey of the second direction of generalization can be found in [K1] and [K2]. Thus, although absolute continuity with respect to Lebesgue measure is a local property (*i.e.*, if each point has a neighborhood where it holds then it holds everywhere), both directions have focused on global objects. Exceptions are the paper [B] where it is shown that if a CR measure on a hypersurface of $\mathbb{C}^{\mathbb{K}}$ is the boundary value of a holomorphic function defined on a side, then it is absolutely continuous with respect to Lebesgue measure, and the results in [BH1]. In this work we continue to stress the local character of the F. and M. Riesz property. The recent works [RS1] and [RS2] discuss results on boundary values which are hyperfunctions.

1. Sufficiency.

Let L be a smooth locally integrable vector field defined in an open subset of the plane. In appropriate coordinates (x, t) we may assume that L possesses a smooth first integral $Z(x, t) = x + i\varphi(x, t)$ defined on a neighborhood of the closure of the rectangle $Q = (-A, A) \times (-B, B)$. Thus, after multiplication by a nonvanishing factor, L may be written as

$$L = \frac{\partial}{\partial t} - \frac{i\varphi_t}{1 + i\varphi_x} \frac{\partial}{\partial x}$$

and $\varphi(x, t)$ is defined and smooth for $|x| \leqslant A$, $|t| \leqslant B$.

The next theorem gives, in particular, a sufficient condition for the existence of a boundary value of a continuous function f when f is a homogeneous solution $Lf = 0$ of a locally integrable vector field L:

THEOREM 1.1. — *Let L be as above and let f be continuous on $Q^+ = (-A, A) \times (0, B)$. Suppose*

i) $Lf \in L^1(Q^+)$;

ii) *there exists $N \in \mathbb{N}$ such that*

$$\int_0^B \int_{-A}^A |\varphi(x, t) - \varphi(x, 0)|^N |f(x, t)| \, dxdt < \infty.$$

Then $\lim_{t \to 0^+} f(x, t) = bf$ exists in $\mathcal{D}'(-A, A)$ and it is a distribution of order $N + 1$.

Proof. — Let $\Psi \in C_0^\infty(-A, A)$. Fix $0 < T < B$. For each integer $m \geqslant 0$, we will show that there exists $\Psi_m(x, t) \in C^\infty((-A, A) \times [0, T])$ such that

i) $\Psi_m(x,0) = \Psi(x)$ and

ii) $|L\Psi_m(x,t)| \leqslant C|\varphi(x,t) - \varphi(x,0|^m$,

where C depends only on the size of $D^j\Psi(x)$ for $j \leqslant m+1$. To get $\Psi_m(x,t)$ with these properties, let $\Sigma = \{x + i\varphi(x,0)\}$ and choose a smooth function $u = u(x,y)$ defined near $(0,0) \in \Sigma$ and satisfying

a) $u(x,\varphi(x,0)) = \Psi(x)$, and

b) $\left|(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})u(x,y)\right| \leqslant \operatorname{dist}((x,y),\Sigma)^m$.

If we set

$$V = \frac{i}{Z_x(x,0)}\frac{\partial}{\partial x},$$

such a function u can be taken to be

$$u(x,y) = \sum_{k=0}^{m} \frac{V^k\Psi(x)}{k!}(y - \varphi(x,0))^k.$$

It is easy to check that (a) above holds and

$$\Big(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\Big)(x,y) = \frac{1}{m!}\frac{\partial}{\partial x}V^m\Psi(x)(y - \varphi(x,0))^m.$$

Let

$$\Psi_m(x,t) = u(x,\varphi(x,t)).$$

Then $\Psi_m(x,0) = \Psi(x)$ and

$$L\Psi_m(x,t) = \frac{-i\varphi_t(x,t)}{1 + i\varphi_x(x,t)}\Big(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\Big)(x,\varphi(x,t)).$$

Hence Ψ_m satisfies i) and ii). Observe next that if $g(x,t)$ is a C^1 function, the differential of the one form $g(x,t)\,dZ$ where $Z = x + i\varphi(x,t)$ is given by

$$d(gdZ) = Lgdt \wedge dZ.$$

This observation and integration by parts lead to

$$\begin{aligned}(1.1)\quad \int_{-A}^{A} f(x,\epsilon)\Psi_N(x,\epsilon)\,dZ(x,\epsilon) &= \int_{-A}^{A} f(x,T)\Psi_N(x,T)\,dZ(x,T)\\ &\quad + \int_{-A}^{A}\int_{\epsilon}^{T} f(x,t)L\Psi_N(x,t)\,dt\wedge dZ\\ &\quad + \int_{-A}^{A}\int_{\epsilon}^{T} Lf(x,t)\Psi_N(x,t)\,dt\wedge dZ.\end{aligned}$$

Note that the x-support of $\Psi_N(x,t)$ (i.e., the support of $x \mapsto \Psi_N(t,x)$) is contained in the support of $\Psi(x)$. Now by the hypothesis on $f(x,t)$ and property ii) of $\Psi_N(x,t)$, $|f(x,t)L\Psi_N(x,t)| \in L^1$ and so the second integral on the right in (1.1) has a limit as $\epsilon \to 0$. The third integrand on the right is in L^1 since Lf is. Therefore,

$$\lim_{\epsilon\to 0}\int_{-A}^{A} f(x,\epsilon)\Psi_N(x,\epsilon)\,dZ(x,\epsilon) \text{ exists.} \tag{1.2}$$

Let $P(x,t) = \varphi(x,t) - \varphi(x,0)$. For $g(x,t) \in C^\infty((-A,A)\times(-B,B))$ whose x-support is contained in a fixed compact set independent of t, and m a nonnegative integer, set

$$T_m g(x,t) = g_m(x,t) = \sum_{k=0}^{m} \frac{V^k(g)(x,t)}{k!} P(x,t)^k, \quad g_0(x,t) = g(x,t),$$

so with this notation we have $\Psi_m(x,t) = T_m\Psi(x,t)$. It is easy to see that (1.2) also applies to any such $g(x,t)$ so that

$$\lim_{t\to 0}\int_{-A}^{A} f(x,t)T_N g(x,t)\,dZ(x,t) \text{ exists.} \tag{1.3}$$

For $\psi(x,t) \in C^\infty((-A,A)\times(-B,B))$ whose x-support is contained in a fixed compact set independent of t set $g(x,t) = \psi(x,t)P(x,t)^N$ in (1.3). Observe that we may write

$$g_N(x,t) = T_N g(x,t) = P(x,t)^N(1+e^N(x,t))\psi(x,t) + P^{N+1}h^N(x,t), \tag{1.4}$$

where e^N and h^N are smooth, the x-support of $h^N(x,t)$ is contained in a compact set that does not depend on t, and

$$\lim_{t\to 0} D_x^j e^N(x,t) = 0, \quad \forall\, j = 0,1,2,\dots$$

From (1.3) we know that

$$\lim_{t\to 0}\int_{-A}^{A} f(x,t)(P^N(1+e^N)\psi + P^{N+1}h^N)(x,t)\,dZ(x,t) \text{ exists.} \tag{1.5}$$

Observe next that

$$\lim_{t\to 0}\int_{-A}^{A} f(x,t)P(x,t)^{N+1}h^N dZ(x,t) \text{ also exists.} \tag{1.6}$$

Indeed, this follows from applying the integration by parts formula (1.1) to the 1-form $f(x,t)P(x,t)^{N+1}h^N(x,t)\,dZ(x,t)$ and using the hypotheses on f. From (1.5) and (1.6) we conclude that

$$\lim_{t\to 0}\int_{-A}^{A} f(x,t)P(x,t)^N(1+e^N(x,t))\psi(x,t)\,dZ(x,t) \text{ exists.} \tag{1.7}$$

But then since the function $e^N(x,t)$ and its x derivatives go to zero as $t \to 0$, it follows that $(1+e^N(x,t))^{-1}\psi(x,t) \to \psi(x,0)$ in $C_c^\infty(-A,A)$ which implies that

$$\lim_{t\to 0}\int_{-A}^{A} f(x,t)P(x,t)^N\psi(x,t)\,dZ(x,t) \text{ exists} \tag{1.8}$$

for all $\psi(x,t) \in C^\infty((-A,A)\times(-B,B))$ provided its x-support is contained in a fixed compact set independent of t. We now return to a general $g(x,t) \in C^\infty((-A,A)\times(-B,B))$ with x-support contained in a fixed compact set independent of t.

Applying (1.3) and (1.8) with $\psi = V^N g/k!$ to the identity

$$f(x,t)T_{N-1}g(x,t) = f(x,t)T_N g(x,t) - f(x,t)\frac{V^N g(x,t)}{k!}P^N(x,t)$$

we conclude that

$$\lim_{t\to 0}\int_{-A}^{A} f(x,t)T_{N-1}g(x,t)\,dZ(x,t) \text{ exists} \tag{1.9}$$

for any $g(x,t) \in C^\infty((-A,A)\times(-B,B))$ with x-support contained in a fixed compact set independent of t. We will prove by descending induction that for any such $g(x,t)$ and $0 \leqslant k \leqslant N$,

$$\lim_{t\to 0}\int_{-A}^{A} f(x,t)T_k g(x,t)\,dZ(x,t) \text{ exists,}$$

which for $k=0$ and $g(x,t) = \Psi(x) \in C_c^\infty(-A,A)$ gives us the desired limit. To proceed by induction, suppose $1 \leqslant k \leqslant N$ and assume that the limits

$$\begin{gathered}\lim_{t\to 0}\int_{-A}^{A} f(x,t)P^k(x,t)g(x,t)\,dZ(x,t),\\ \lim_{t\to 0}\int_{-A}^{A} f(x,t)T_{k-1}g(x,t)(x,t)\,dZ(x,t)\end{gathered} \tag{1.10_k}$$

both exist for any $g(x,t) \in C^\infty((-A,A)\times(-B,B))$ with x-support contained in a fixed compact set independent of t. We have already seen that (1.10_k) is true for $k=N$ as follows from (1.8) and (1.9). Take $g(x,t) = \psi(x,t)P(x,t)^{k-1}$ in the limit on the right in (1.10_k) and observe that $g_{k-1} = T_{k-1}g$ may be written as

$$g_{k-1}(x,t) = P(x,t)^{k-1}(1+e^{k-1}(x,t))\psi(x,t) + P(x,t)^k h^{k-1}(x,t), \tag{1.11}$$

where e^{k-1} and h^{k-1} are smooth, the x-support of $h^{k-1}(x,t)$ is contained in a compact set that is independent of t, and

$$\lim_{t\to 0} D_x^j e^{k-1}(x,t) = 0, \quad \forall\, j = 0,1,2,\ldots$$

From the second limit in (1.10_k) we know that

$$(1.12)\quad \lim_{t\to 0}\int_{-A}^{A} f(x,t)(P(x,t)^{k-1}(1+e^{k-1}(x,t))\psi(x,t) \\ +P(x,t)^k h^{k-1}(x,t))\,dZ(x,t)$$

exists. The first limit in (1.10_k) tells us that

$$(1.13)\qquad \lim_{t\to 0}\int_{-A}^{A} f(x,t)P(x,t)^k h^{k-1}(x,t)\,dZ(x,t) \text{ exists.}$$

Hence from (1.12), (1.13) and the fact that $\lim_{t\to 0} D_x^j e^{k-1}(x,t) = 0$, it follows that

$$(1.14)\qquad \lim_{t\to 0}\int_{-A}^{A} f(x,t)P(x,t)^{k-1}\psi(x,t)\,dZ(x,t) \text{ exists}$$

for all $\psi(x,t) \in C^\infty((-A,A)\times(-B,B))$ with x-support contained in a fixed compact set independent of t. Consider now a general $g(x,t)$ and write $fT_{k-2}g = fT_{k-1}g + f\psi P^{k-1}$. Hence, taking account of (1.10_k) and (1.14) we conclude that

$$(1.15)\qquad \lim_{t\to 0}\int_{-A}^{A} f(x,t)T_{k-2}g(x,t)\,dZ(x,t) \text{ exists.}$$

We have thus proved (1.14) and (1.15) which state precisely that (1.10_{k-1}) holds, completing the inductive step. Therefore,

$$(1.16)\qquad \lim_{\epsilon\to 0}\int_{-A}^{A} f(x,\epsilon)\Psi(x)\,dx \text{ exists}$$

and thus $bf = \lim_{t\to 0} f(\cdot,t)$ exists. Moreover, since the functions

$$x \longmapsto \Psi_N(x,\epsilon) - \Psi(x) \text{ and } x \longmapsto Z(x,\epsilon) - Z(x,0)$$

and all their x-derivatives converge to zero as $\epsilon \to 0$, (1.1), (1.2) and (1.16) imply the following formula for bf:

$$(1.17)\qquad \langle Z_x(x,0)bf,\Psi\rangle = \int_{-A}^{A} f(x,T)\Psi_N(x,T) \\ + \int_{-A}^{A}\int_0^T f(x,t)L\Psi_N(x,t)\,dt\wedge dZ \\ + \int_{-A}^{A}\int_0^T Lf(x,t)\Psi_N(x,t)\,dt\wedge dZ.$$

This formula shows that bf is a distribution of order $N+1$. □

Since the finiteness condition ii) in Theorem 1.1 is formulated in terms of a particular first integral Z that was expressed in special coordinates it is appropriate to discuss the invariance of this condition. Observe first, that $\Sigma = \{x + i\varphi(x, 0)\} = Z(\gamma)$ where γ is the initial interval $(-A, A) \times \{0\}$. If $p = x + iy$, $-A < x < A$ is a point of the complex plane lying on $Z(Q)$, we have

$$C_1 \operatorname{dist}(p, \Sigma) \leqslant |y - \varphi(x, 0)| \leqslant C_2 \operatorname{dist}(p, \Sigma)$$

for some constants $C_1, C_2 > 0$. Thus, if we define the weight

$$w_Z(x, t) = \operatorname{dist}(Z(x, t), Z(\gamma)), \quad (x, t) \in Q,$$

the finite integral condition may be written as

$$\int_{Q^+} |f(x, t)|\, w_Z(x, t)^N \, dxdt < \infty.$$

Consider now another first integral W of the vector field L defined in a neighborhood of $\overline{Q}$, *i.e.*, a smooth function satisfying $LW = 0$ and $dW \neq 0$ on its domain of definition. By a standard consequence of the Baouendi-Treves approximation theorem [BT], there is a rectangle $Q_1 \subset Q$ centered at the origin and functions $F \in C^\infty(Z(\overline{Q}_1))$ and $G \in C^\infty(W(\overline{Q}_1))$ such that $W = F \circ Z$ and $Z = G \circ W$ on $\overline{Q}_1$. It turns out that $F\colon Z(\overline{Q}_1)) \longrightarrow W(\overline{Q}_1)$ is a diffeomorphism with inverse $F^{-1} = G$. In particular,

$$\begin{aligned} w_Z(x, t) = \operatorname{dist}(Z(x, t), Z(\gamma)) \sim \operatorname{dist}(W(x, t),\\ W(\gamma)) = w_W(x, t), \quad (x, t) \in Q_1, \end{aligned}$$

which implies that

$$\int_{Q_1^+} |f(x, t)|\, w_Z(x, t)^N \, dxdt < \infty \Longleftrightarrow \int_{Q_1^+} |f(x, t)|\, w_W(x, t)^N \, dxdt < \infty$$

and shows that the finite integral condition is locally independent of the choice of the first integral.

In the appendix we will explore when the existence of the limit in (1.2) above implies the existence of a trace for f without any assumption on Lf.

We finish this section with an example that shows that the finite integral condition that suffices to guarantee the existence of a trace in Theorem 1.1 is less stringent than previously known sufficient conditions for trace existence.

Example 1.2. — Let $g(x, y) \in C^\infty(R)$, $R = (-1, 1) \times (-1, 1)$, $g(x, y) > 0$ for $y > 0$. Set

$$Z(x, y) = x + iyg(x, y) \quad \text{and} \quad L = \frac{\partial}{\partial y} - \frac{Z_y(x, y)}{Z_x(x, y)} \frac{\partial}{\partial x}.$$

The function $Z(x,y)$ is a first integral for L. Define $f(x,y)$ on $R^+ = (-1,1) \times (0,1)$ by

$$f(x,y) = \frac{1}{(Z(x,y))^2}.$$

Clearly f is smooth and $Lf = 0$ in R^+. Note that the function $g(x,y)$ can be chosen so that given $a > 0$ and m an integer, there is a constant $C = C(a,m)$ such that

$$\int_{-a}^{a} |f(x,y)|\, dx > Cy^{-m} \quad \text{as} \quad y \to 0^+.$$

For example, if $g(x,y) = \frac{\exp(-1/y)}{y}$ for $y > 0$, then

$$\int_{-a}^{a} |f(x,y)|\, dx > \exp(1/2y) \quad \text{as} \quad y \to 0^+.$$

Therefore, in general, Theorem 1.1 in ([BH2]) cannot be applied to $f(x,y)$. However, Theorem 1.1 in this paper implies that f always has a trace at $y = 0$ for any choice of $g(x,y)$.

2. Necessity.

We will next consider the necessity of the growth condition in Theorem 1.1 under the additional assumption that the vector field

$$L = \frac{\partial}{\partial t} + a(x,t)\frac{\partial}{\partial x}$$

is real analytic in $Q = (-B,B) \times (-A,A)$. Let $Z(x,t) = x + i\varphi(x,t)$ be a real analytic first integral of L in Q, φ real-valued, $\varphi(0,0) = \varphi_x(0,0) = 0$.

THEOREM 2.1. — *Let f be continuous in Q^+ and $Lf = 0$ in Q^+. Assume that* $\lim_{t\mapsto 0} f(\cdot,t) = bf$ *exists in the sense of distributions and that bf is a distribution of fixed order N on $(-A,A)$. Then there is an integer n such that for every compact set K in $(-A,A)$ and $0 < T < B$,*

$$(2.1)\quad |f(x,t)||\varphi(x,t) - \varphi(x,0)|^n \leqslant C, \quad \forall\,(x,t) \in K \times (0,T], \quad C = C(K,T).$$

Proof. — Since φ is real analytic, we can arrange so that $\varphi(x,0) \equiv 0$. We may also assume that φ is not identical zero, $\varphi_t(0,0) = 0$ and φ_t not identically zero. By analyticity, we can then write $\varphi(x,t) = x^m\psi(x,t)$, where m is a nonnegative integer, ψ is real analytic and $t \mapsto \psi(0,t)$ is not identically zero. By the Weierstrass Preparation Theorem, there is $\delta > 0$ such that in the rectangle

$$R_\delta = \{(x,t)\colon |x| < \delta, \quad 0 < t < \delta\}$$

the zero set S of the product $\varphi(x,t)\varphi_t(x,t)$ can be written as a disjoint union

$$S = S_0 \cup S_1^+ \cup \ldots \cup S_{n_1}^+ \cup S_1^- \cup \ldots \cup S_{n_2}^-$$

where $S_0 = \{0\} \times [0,\delta)$ if $m > 0$ and it equals $\{(0,0)\}$ if $m = 0$. Each S_j^+ (likewise for S_k^-) is an analytic graph $\{(x, s_j^+(x))\colon 0 < x < \delta\}$ (s_k^- for S_k^-) with

$$\lim_{x\to 0^+} s_j^+(x) = \lim_{x\to 0^-} s_k^-(x) = 0.$$

The description of S shows that we can get $0 < a < \delta$ such that in some interval $(a-\epsilon, a+\epsilon)$, the function $t \mapsto \varphi(x,t)$ is strictly monotonic on $(0,\epsilon]$. The same property holds for $x \in (-a-\epsilon, -a+\epsilon)$. We will first show that in such a rectangle

$$Q_\epsilon(a) = \{(x,t)\colon x \in (a-\epsilon, a+\epsilon), \quad 0 < t < \epsilon)\},$$

there is an integer ℓ and a constant C such that

$$|f(x,t)||\varphi(x,t)|^\ell \leqslant C \quad \text{for} \quad (x,t) \in Q_\epsilon(a)$$

(likewise for $Q_\epsilon(-a)$). The proof of this is basically the same as the one for the Cauchy-Riemann operator (see [Hö]). Indeed, we may assume that for each $x \in (a-\epsilon, a+\epsilon)$, the function $t \mapsto \varphi(x,t)$ is strictly increasing on $[0,\epsilon]$. The map $Z = x + i\varphi(x,t)$ is then a homeomorphism of $Q_\epsilon(a)$ onto $Z(Q_\epsilon(a))$. Therefore, by the Baouendi-Treves approximation theorem ([BT]), there is a holomorphic function F on $Z(Q_\epsilon(a))$ such that

$$f(x,t) = F(Z(x,t)) \quad \text{for} \quad (x,t) \in Q_\epsilon(a).$$

Let

$$X = Z(Q_\epsilon(a)) = \{x + iy\colon |x-a| < \epsilon, 0 < y < \varphi(x,\epsilon)\}$$

and for some $\alpha < \epsilon$, let $J = (a-\alpha, a+\alpha)$. Let $Y = \{x+iy\colon x \in J, 0 < y < \varphi(x, \frac{\epsilon}{2})\}$. Fix a point $\zeta = \xi + i\varphi(\xi,\eta) \in Y$. Let $g \in C_0^\infty(\mathbb{R}^{\not{2}})$, $g \equiv 1$ on a neighborhood of $\overline{Y}$ and $g = 0$ on $\partial X \setminus (a-\epsilon, a+\epsilon)$. For $0 < t < \eta$, let

$$D_t = \{x + iy\colon |x-a| < \epsilon, \quad \varphi(x,t) < y < \varphi(x,\epsilon)\}.$$

We apply the inhomogeneous Cauchy integral formula to $g(x,y)F(x+iy)$ in the domain D_t:

$$\begin{aligned} f(\xi,\eta) &= g(\zeta)F(\zeta) \\ &= \frac{-1}{\pi}\int\!\!\int_{D_t} F(x+iy)\frac{\partial g}{\partial \overline{z}}(x,y)\frac{1}{x+iy-\zeta}\,dxdy \\ &\quad + \frac{1}{2\pi i}\int_{a-\epsilon}^{a+\epsilon} F(x+i\varphi(x,t))\frac{g(x,\varphi(x,t))}{x+i\varphi(x,t)-\zeta}(1+i\varphi_x(x,t))\,dx. \end{aligned}$$

Replacing F by f, we have

$$f(\xi,\eta) = \frac{-1}{\pi}\int_{a-\epsilon}^{a+\epsilon}\int_t^{\epsilon} f(x,s)\frac{\partial g}{\partial \overline{z}}(x,\varphi(x,s))\varphi_s(x,s)\frac{1}{x+i\varphi(x,s)-\zeta}\,dsdx + \frac{1}{2\pi i}\int_{a-\epsilon}^{a+\epsilon} f(x,t)\frac{g(x,\varphi(x,t))}{x+i\varphi(x,t)-\zeta}(1+i\varphi_x(x,t))\,dx.$$

We will first estimate the double integral. Recall that $\zeta = \xi + i\varphi(\xi,\eta) \in Y$, and $0 < t < \eta$. Since $\frac{\partial g}{\partial \overline{z}}(x,y) \equiv 0$ near $\overline{Y}$, in the integrand of the double integral, we may assume there is β independent of ζ such that

$$\frac{1}{|x+i\varphi(x,s)-\zeta|} \geqslant \frac{1}{|x-\xi|} \geqslant \beta > 0.$$

Moreover, since f has a trace at $t = 0$, the distributions $\{f(\cdot,s)\}$ are uniformly bounded and hence there is a constant C independent of t such that

$$\left|\int_t^{\epsilon} f(x,s))\frac{\partial g}{\partial \overline{z}}(x,\varphi(x,s))\varphi_s(x,s)\frac{1}{x+i\varphi(x,s)-\zeta}\,ds\right| \leqslant C.$$

It follows that the double integral is bounded uniformly in t. Consider next the integral

$$\int_{a-\epsilon}^{a+\epsilon} f(x,t)\frac{g(x,\varphi(x,t))}{x+i\varphi(x,t)-\zeta}(1+i\varphi_x(x,t))\,dx.$$

Again we will exploit the fact that $\{f(\cdot,s)\}$ is a family of bounded distributions. Observe also that since $\varphi_x(0,0) = 0$, by decreasing a and ϵ if necessary, we get

$$|x+i\varphi(x,t)-\zeta| \geqslant c|\varphi(\xi,\eta)-\varphi(\xi,t)|.$$

Therefore, using the uniform boundedness of $\{f(\cdot,s)\}$, there exist M and $C > 0$ independent of t such that

$$\left|\int_{a-\epsilon}^{a+\epsilon} f(x,t)\frac{g(x,\varphi(x,t))}{x+i\varphi(x,t)-\zeta}(1+i\varphi_x(x,t))\,dx\right| \leqslant \frac{C}{|\varphi(\xi,\eta)-\varphi(\xi,t)|^M}.$$

After letting t go to 0 in this latter estimate, we conclude that

$$|f(\xi,\eta)||\varphi(\xi,\eta)|^M \leqslant C < \infty \quad \text{for} \quad (\xi,\eta) \in Q_\epsilon(a)$$

and hence also in $Q_\epsilon(-a)$. From now on fix $a, T > 0$ such that the estimate

$$|f(x,t)||\varphi(x,t)|^M \leqslant C \quad \text{holds for} \quad t \in (0,T) \quad x \quad \text{near} \quad a,-a. \tag{2.2}$$

We will show that (2.1) holds on $(-b,b)\times(0,T)$ for some $0 < b < a$. Fix $0 < b < a$. For $y \in (-b,b)$, $y \neq 0$, define the set

$$A_y = \big\{s \in (0,T)\colon \varphi(y,s)(\varphi(y,s)-\varphi(y,T))\varphi_t(y,s) = 0\big\} \cup \big\{s \in (0,T)\colon \varphi(y,s) = \varphi(y,s_0),\ \ 0 < s_0 < T \text{ where } \varphi_t(y,s_0) = 0\big\}.$$

Because of the description of the zero set S, there is an integer n such that for any $s \notin A_y$, there exist $\ell \leqslant n$ distinct points $0 < s_1 = s, s_2, \dots, s_\ell < T$ such that $\varphi(y, s_j) = \varphi(y, s), j \leqslant \ell$. Fix such $s \notin A_y$ and $s_j, j = 1, \dots, \ell$. By definition of A_y, $\varphi_t(y, s_j) \neq 0$, and $\varphi(y, s_j) \notin \{0, \varphi(y, T)\}$. Moreover, there exists $0 < t_1 < s$ such that

$$\varphi(y,t) \neq \varphi(y,s), \quad \forall\, 0 \leqslant t \leqslant t_1.$$

Fix a positive integer k and let $0 < t \leqslant t_1$. Define a closed curve

$$\Gamma_t = \Gamma_t^1 \cup \Gamma_t^2 \cup \Gamma_t^3 \cup \Gamma_t^4$$

where $\Gamma_t^1 = \{(x,t): -a < x < a\}$, $\Gamma_t^2 = \{(a,r): t < r < T\}$, $\Gamma_t^3 = \{(x,T): -a < x < a\}$, and $\Gamma_t^4 = \{-a, r): t < r < T\}$. Consider the integral

$$\int_{\Gamma_t} \frac{(Z(p) - Z(a,t))^k (Z(p) - Z(-a,t))^k}{Z(p) - Z(y,s)} f(p)\, dZ(p) \tag{2.3}$$

where dZ denotes the differential of $Z = x + i\varphi(x,t)$ and p is a variable point in Γ_t. Note that by the choice of $y \in (-b,b), y \neq 0$, $s \notin A_y$, and t, the integrand has no singularity on Γ_t. Let D_t be the domain bounded by Γ_t. Since $\varphi_t(y, s_j) \neq 0$ for each $j = 1, \dots \ell$, the map $p \mapsto Z(p)$ is a diffeomorphism near the points $p_j = y + is_j$. Let $\epsilon > 0$ such that Z is a diffeomorphism on each of the balls $B_\epsilon(p_j)$. Observe that the integrand has singularities in D_t only at the points p_j. Let

$$\Omega_\epsilon = D_t \setminus \bigcup_{j=1}^{l} B_\epsilon(y + is_j).$$

If h is any C^1 function, we can express its differential as

$$dh = (Lh)\, dt + (Mh)\, dZ \quad \text{where} \quad M = \frac{1}{Z_x} \frac{\partial}{\partial x}.$$

It follows that since $Lf = 0$, the differential of the one form in the integrand of (2.3) equals 0 in the region Ω_ϵ. Therefore by Stokes theorem, we get

$$\begin{aligned} &\int_{\Gamma_t} \frac{(Z(p) - Z(a,t))^k (Z(p) - Z(-a,t))^k}{Z(p) - Z(y,s)} f(p)\, dZ(p) \\ &\qquad = \sum_{j=1}^{l} \int_{\partial B_\epsilon(y+is_j)} \frac{(Z(p) - Z(a,t))^k (Z(p) - Z(-a,t))^k}{Z(p) - Z(y,s)} f(p)\, dZ(p). \end{aligned}$$

Consider next each integral in the preceding sum. We may assume that the map $Z = x + i\varphi$ is a diffeomorphism on a neighborhood of the closure of each of the l balls. This means that the vector field L is elliptic in a neighborhood of each $\overline{B_\epsilon(y + is_j)}$ and so there are holomorphic functions

F_j such that $f(p) = F_j(Z(p))$ for each $p \in \overline{B_\epsilon(y+is_j)}$. This allows us to change variables $z = Z(p)$ and write for each of the balls

$$\begin{aligned}(2.4)\qquad &\int_{\partial B_\epsilon(y+is_j)} \frac{(Z(p)-Z(a,t))^k(Z(p)-Z(-a,t))^k}{Z(p)-Z(y,s)} f(p)\,dZ(p)\\ &= \int_{\partial Z(B_\epsilon(y+is_j))} \frac{(z-Z(a,t))^k(z-Z(-a,t))^k}{z-Z(y,s)} F_j(z)\,dz\\ &= 2\pi i(Z(y,s)-Z(a,t))^k(Z(y,s)-Z(-a,t))^k f(y,s),\end{aligned}$$

where in the last equation we applied the Cauchy integral formula to the holomorphic function

$$z \mapsto (z-Z(a,t))^k(z-Z(-a,t))^k F_j(z).$$

We have shown that for $y \in (-b,b)$, $y \neq 0$, $s \notin A_y$ and $0 \leqslant t \leqslant t_1$:

$$\begin{aligned}(2.5)\quad &2\pi li(Z(y,s)-Z(a,t))^k(Z(y,s)-Z(-a,t))^k f(y,s)\\ &= \int_{\Gamma_t} \frac{(Z(p)-Z(a,t))^k(Z(p)-Z(-a,t))^k}{Z(p)-Z(y,s)} f(p)\,dZ(p).\end{aligned}$$

We shall next estimate the integral in (2.5). Write this integral as a sum $\int_{\Gamma_t} = \sum_{j=1}^4 I_j$ where $I_j = \int_{\Gamma_t^j}$. We will estimate each I_j. Let $g_\epsilon(x) \in C_0^\infty(-a,a)$, $0 \leqslant g_\epsilon(x) \leqslant 1$, $g_\epsilon(x) \equiv 1$ on $[-a+\epsilon, a-\epsilon]$ and $|D^l g_\epsilon(x)| \leqslant \frac{c_l}{\epsilon^l}$ for some constants c_l. Observe that

$$\begin{aligned}I_1 &= \lim_{\epsilon\to 0}\int_0^a g_\epsilon(x)\frac{(Z(x,t)-Z(-a,t))^k(Z(x,t)-Z(a,t))^k f(x,t)}{Z(x,t)-Z(y,s)}\,d_xZ(x,t)\\ &= \lim_{\epsilon\to 0} I_\epsilon.\end{aligned}$$

To estimate I_ϵ, we will use the fact that $\{f(\cdot,\tau)\}$ is a uniformly bounded set of distributions on a neighborhood of $[-a,a]$. Thus there exists an integer $N \in \mathbb{N}$ and a constant $C > 0$ such that

$$\begin{aligned}(2.6)\quad &|I_\epsilon|\\ &\leqslant C\sum_{m=0}^{N}\Big\|D_x^m\Big(\frac{(Z(x,t)-Z(-a,t))^k(Z(x,t)-Z(a,t))^k}{Z(x,t)-Z(y,s)} g_\epsilon(x)\Big)\Big\|_{L^\infty}\\ &\leqslant C\sum_{m=0}^{N}\sum_{\ell=0}^{m}\Big\|D_x^{m-\ell}\Big(\frac{(Z(x,t)-Z(-a,t))^k(Z(x,t)-Z(a,t))^k}{Z(x,t)-Z(y,s)}\Big)D_x^\ell g_\epsilon(x)\Big\|_{L^\infty}\\ &\leqslant C\sum_{m=0}^{N}\sum_{\ell=0}^{m}\Big\|\sum_{j=0}^{m-\ell}\Big(\frac{\epsilon^{-\ell}}{|Z(x,t)-Z(y,s)|^{m-\ell-j+1}}\Big)\sum_{r=0}^{j}|x+a|^{k-r}|x-a|^{k-j+r}\Big\|\\ &\leqslant C\sup\frac{1}{|Z(x,t)-Z(y,s)|^{N+1}}\end{aligned}$$

where in the latter inequality we have assumed that $k \geqslant N$. Since $\varphi_x(0,0) = 0$, after decreasing T, we have

$$|Z(x,t) - Z(y,s)| \geqslant \frac{1}{2}(|x-y| + |\varphi(x,t) - \varphi(y,s)|) \tag{2.7}$$
$$\geqslant \frac{1}{4}(|x-y| + |\varphi(y,t) - \varphi(y,s)|).$$

From (2.6) and (2.7) we conclude

$$|I_\epsilon| \leqslant \frac{C}{|\varphi(y,s) - \varphi(y,t)|^{N+1}} \tag{2.8}$$

where the constant C is independent of y, s, t and ϵ. It follows that

$$|I_1| \leqslant \frac{C}{|\varphi(y,s) - \varphi(y,t)|^{N+1}}. \tag{2.9}$$

Next we will estimate I_2 which is the integral over the curve Γ_t^2. Observe that on this curve, by inequality (2.2),

$$|f(a,r)|\,|\varphi(a,r)|^M \leqslant C.$$

Moreover, since $|y| < b < a$ and $r \mapsto \varphi(a,r)$ is monotonic on $[0,T]$, we have

(2.10)

$$\begin{aligned}|I_2| &= \left|\int_t^T \frac{(Z(a,r) - Z(-a,t))^k (Z(a,r) - Z(a,t))^k}{Z(a,r) - Z(y,s)} f(a,r)\varphi_r(a,r)\,dr\right| \\ &\leqslant C\int_t^T |\varphi(a,r) - \varphi(a,t)|^k\,|f(a,r)|\varphi_r(a,r)\,dr \\ &\leqslant C\int_t^T |\varphi(a,r) - \varphi(a,0)|^k\,|f(a,r)|\varphi_r(a,r)\,dr \\ &= C\int_t^T |\varphi(a,r)|^k\,|f(a,r)|\varphi_r(a,r)\,dr \\ &\leqslant C\end{aligned}$$

if we take $k \geqslant M$. The integral over Γ_t^4 is also estimated the same way and so we also have

$$|I_4| \leqslant C \text{ for some constant } C. \tag{2.11}$$

Finally we will estimate I_3 which is the integral over Γ_t^3. Observe that $f(x,T)$ is bounded and since φ_x may be assumed to be small, as we saw in (2.7),

$$|Z(x,T) - Z(y,s)| \geqslant \frac{1}{4}(|x-y| + |\varphi(y,T) - \varphi(y,s)|).$$

Hence

$$\begin{aligned}(2.12)\qquad |I_3| &\leqslant C\int_0^a \frac{1}{|x-y|+|\varphi(y,T)-\varphi(y,s)|}\,dx\\ &\leqslant C|\ln|\varphi(y,T)-\varphi(y,s)||.\end{aligned}$$

From the estimates for the $I_j (j=1,\dots,4)$ and (2.5) we get

$$\begin{aligned}(2.13)\qquad &|Z(y,s)-Z(-a,t)|^M|Z(y,s)-Z(a,t)|^M|f(y,s)|\\ &\leqslant C\Big(\frac{1}{|\varphi(y,s)-\varphi(y,t)|^{M+1}}+|\ln(|\varphi(y,T)-\varphi(y,s)|)|\Big)\end{aligned}$$

since we may assume that $N \leqslant M$. Observe next that for some $C > 0$,

$$(2.14)\qquad |Z(y,s)-Z(a,t)|, |Z(y,s)-Z(-a,t)| \geqslant C|\varphi(y,s)-\varphi(y,t)|.$$

Therefore, using (2.13), and (2.14):

$$\begin{aligned}(2.15)\qquad &|\varphi(y,s)-\varphi(y,t)|^{3M+1}|f(y,s)|\\ &\leqslant C(1+(\varphi(y,s)-\varphi(y,t))^{M+1}|\ln|\varphi(y,T)-\varphi(y,s)||).\end{aligned}$$

Letting $t \mapsto 0$ in the latter inequality, and recalling that $\varphi(x,0)\equiv 0$, we arrive at

$$(2.16)\qquad |\varphi(y,s)|^{3M+1}|f(y,s)| \leqslant C+|\varphi(y,s)|^{M+1}|\ln|\varphi(y,T)-\varphi(y,s)||.$$

Recall now that $\varphi(x,t)=x^m\psi(x,t)$ with $\psi(0,t)$ not identically vanishing. We may assume that a and T were chosen so that $|\psi(x,T)|>0$ whenever $|x|\leqslant a$. Moreover, since $\psi(x,0)\equiv 0$ (because $\varphi(x,0)\equiv 0$), there exists $0<T_0<T$ such that

$$(2.17)\qquad |\psi(x,r)| \leqslant |\psi(x,T)-\psi(x,r)| \quad\text{for any}\quad r\in[0,T_0], \text{and}\quad |x|\leqslant a.$$

It follows that if we choose (y,s) so that $0<|y|<b$, $s\notin A_y$ and $0<s\leqslant T_0$, then (2.16) and (2.17) imply

$$|\varphi(y,s)|^{3M+1}|f(y,s)| \leqslant C+|\varphi(y,s)|^{M+1}|\ln|\varphi(y,s)|| \leqslant C_1 < \infty$$

which proves the theorem. □

From Theorems 1.1 and 2.1, we get the following consequences:

COROLLARY 2.2. — *Let L be real analytic and f continuous in Q^+, $Lf=0$ in Q^+. Then the following are equivalent:*

(1) $\lim_{t\to 0^+} f(\cdot,t)=bf$ *exists in the sense of distributions and bf is a distribution of fixed order on $(-A,A)$;*

(2) *there exists N such that for every compact set K in $(-A,A)$:*

$$|f(x,t)||\varphi(x,t)-\varphi(x,0)|^N \leqslant C,\quad \forall\, x\in K$$

for some $C = C(K)$;

(3) *there exists $M \in \mathbb{N}$ such that for any compact set $K \subset (-A, A)$, there exists $C = C(K) > 0$ such that*

$$\int_0^B \int_K |\varphi(x,t) - \varphi(x,0)|^M |f(x,t)|\, dxdt \leqslant C.$$

COROLLARY 2.3. — *Let L be real analytic. Suppose f and g are continuous solutions in Q^+ with boundary values at $t = 0$. Then their product fg also has a boundary value.*

COROLLARY 2.4. — *Let L and f be as in Corollary 2.3 with f having a boundary value at $\{t = 0\}$. Then f has a boundary value on any curve $\gamma \subset \overline{Q}^+$ through the origin.*

Example 2.5: Derivative estimates. — Let L be a real analytic vector field and $Z = x + i\varphi(x,y)$ a first integral. Suppose $f(x,y$ is a C^k solution in Q^+, $k \geqslant 1$ and assume that bf exists. Let $M = \frac{1}{Z_x(x,y)} \frac{\partial}{\partial x}$. Then Mf is also a continuous solution of L in Q^+ with a trace at $y = 0$ and so by Theorem 2.1, for some C and N,

$$|D_x f(x,y)| \leqslant \frac{C}{|\varphi(x,y) - \varphi(x,0)|^N}.$$

But then the latter together with the equation $Lf = 0$ imply that

$$|D_y f(x,y)| \leqslant \frac{C}{|\varphi(x,y) - \varphi(x,0)|^N}.$$

Thus the first derivatives of f satisfy a growth condition similar to that of f. By iterating this argument, we conclude that all derivatives of order no more than k satisfy a similar growth condition.

For a concrete example, consider the vector field

$$L = \frac{\partial}{\partial y} - ix\frac{\partial}{\partial x}$$

for which $Z(x,y) = x\exp(iy)$ is a first integral. If $f(x,y)$ is a smooth solution in Q^+ with a trace at $y = 0$, then for any multi-index α, there are constants C and k depending on α such that

$$|xy|^k \, |D^\alpha f(x,y)| \leqslant C.$$

3. Applications.

We wish to use the result of Section 1 to strengthen the F. and M. Riesz theorem proved in [BH1].

Let
$$L = \frac{\partial}{\partial t} + a(x,t)\frac{\partial}{\partial x}$$
be a smooth vector field on a neighborhood $Q = (-A,A) \times (-T,T)$ of the origin in the plane with a smooth first integral $Z(x,t) = x + i\varphi(x,t)$. Write $L = X + iY$ where
$$X = \frac{\partial}{\partial t} + \operatorname{Re} a(x,t)\frac{\partial}{\partial x} = \frac{\partial}{\partial t} - \left(\frac{\varphi_t\varphi_x}{1+\varphi_x^2}\right)\frac{\partial}{\partial x}$$
and
$$Y = \operatorname{Im} a(x,t)\frac{\partial}{\partial x} = \frac{-\varphi_t}{1+\varphi_x^2}\frac{\partial}{\partial x}.$$
Fix $B < A$. There exists $T_0 > 0$ and a map
$$\Gamma(s,\tau)\colon (-B,B) \times (0,T_0) \to Q^+ = (-A,A) \times (0,T)$$
such that $\Gamma(s,0) = (s,0)$ and
$$\frac{\partial \Gamma}{\partial \tau}(s,\tau) = (\operatorname{Re} a(\Gamma(s,\tau)),1).$$
For each fixed s, $\tau \to \Gamma(s,\tau)$ is a piece of the integral curve of X through $(s,0)$ and it has the form $\Gamma(s,\tau) = (x(s,\tau),\tau)$.

LEMMA 3.1. — *Let f be continuous in Q^+, $Lf = 0$ in Q^+ and set*
$$F = \{s \in (-B,B) : \text{ for some } \delta = \delta(s), X \text{ and } Y \text{ are linearly dependent on } \Gamma(s,\tau), \forall\, \tau \in (0,\delta)\}.$$
Assume that $(\varphi(x,t) - \varphi(x,0))^N f(x,t) \in L^1(Q^+)$ for some integer N and its boundary value $bf(x) = f(x,0)$ is a Radon measure μ. Let $\mu = g + \nu$ where g is locally integrable in $(-B,B)$ and ν is a measure supported on a set whose Lebesgue measure is zero. Then the support of ν is a countable subset of F.

Proof. — We recall from Theorem 3.1 in [BH1] that if $p \notin F$, then bf is in fact microlocally hypoanalytic in some direction (p,ξ) and so by the classical F. and M.Riesz theorem, $p \notin \operatorname{supp}(\nu)$. Observe next that if $0 < \delta_n \leqslant T_0$ is a sequence converging to 0, then we may write $F = \cup_{n=1}^{\infty} F_n$ where each
$$F_n = \{s \in F\colon X \text{ and } Y \text{ are linearly dependent on } \Gamma(s,\tau),\ \forall\, \tau \in (0,\delta_n)\}.$$

We may therefore assume from now on that

$F = \{s \in (-B, B): X$ and Y are linearly dependent on $\Gamma(s,\tau),\ \forall \tau \in (0, T_0)\}$.

Let $s \in F$. Since X and Y are linearly dependent on $\{\Gamma(s,\tau): 0 \leqslant \tau < T_0\}$, we get $\operatorname{Im} a(x(s,\tau),\tau) = 0$ for $0 \leqslant \tau$. It follows that $\operatorname{Re} a(x(s,\tau),\tau) = 0$ for $0 \leqslant \tau$, and hence $\Gamma(s,\tau) = (s,\tau)$, for all $\tau \in [0, T_0]$. Thus $L = \frac{\partial}{\partial t}$ on $\{s\} \times [0, T_0]$ and

$$\varphi(x,t) = \varphi(x,0), \quad \forall\, t \in [0, T_0]. \tag{3.1}$$

Define next

$$F^{i} = \{x \in F: x \ \text{ is an isolated point of } \ F\}$$

and

$$F^{a} = \{x \in F: x \ \text{ is an accumulation point of } \ F\}.$$

Note that F^i is a countable set and F^a is a closed subset of F. We will first show that the restriction μ_{F^a} of μ to F^a is absolutely continuous with respect to Lebesgue measure. For any $y \in F^a$, observe that

$$|\varphi(x,t) - \varphi(x,0)| \leqslant C|x-y|^2 \leqslant C d(x, F^a)^2 \tag{3.2}$$

where $d(x,S)$ denotes the distance from x to a set S.

Recall from Theorem 1.1 that for any $\psi \in C_0^\infty(-B,B)$, and for any integer $m \geqslant N$, we have

$$\begin{aligned}\langle Z_x(x,0)\mu, \psi\rangle = &\int_{-B}^{B} f(x,T_0)\psi_m(x,T_0)\, dZ \\ &+ \int_0^{T_0}\int_{-B}^{B} f(x,t)\, L\psi_m(x,t)\, dt \wedge dZ\end{aligned} \tag{3.3}$$

where

$$\psi_m(x,t) = \sum_{k=0}^{m} (\varphi(x,t) - \varphi(x,0))^k \, \frac{V^k \psi(x)}{k!}$$

and

$$V = \frac{i}{Z_x(x,0)}\, \frac{\partial}{\partial x}.$$

Hence

$$\Big|\int_{-B}^{B} f(x,T_0)\psi_m(x,T_0)\, dZ(x,T_0)\Big| \leqslant C \sum_{k=0}^{m} \int_{-B}^{B} d(x,F^a)^k \, |D_x^k \psi(x)|\, dx. \tag{3.4}$$

Recall next that

$$L\psi_m(x,t) = \frac{-i\varphi_t(x,t)}{1 + i\varphi_x(x,t)} V^{m+1}\psi(x)(\varphi(x,t) - \varphi(x,0))^m$$

which implies that

$$(3.5)\quad \left|\int_0^{T_0}\int_{-B}^{B} f(x,t)\, L\psi_m(x,t)\, dt\wedge dZ\right| \leqslant C\int_0^{T_0}\int_{-B}^{B} |\varphi_t(x,t)V^{m+1}\psi(x)f(x,t)(\varphi(x,t)-\varphi(x,0))^m|\, dxdt.$$

Fix now a compact set $K\subset F^a$ with Lebesgue measure $|K|=0$ and choose a sequence $0\leqslant \phi_\epsilon(x)\leqslant 1\in C_0^\infty(-B,B)$, $\epsilon\to 0$, such that: i) $\phi_\epsilon(x)=1$ for all $x\in K$; ii) $\phi_\epsilon(x)=0$ if $d(x,K)>\epsilon$; iii) $|D_x^j\phi_\epsilon(x)|\leqslant C_j\epsilon^{-j}$. Note that $\phi_\epsilon(x)$ converges pointwise to the characteristic function of K as $\epsilon\to 0$ while $D^j\phi_\epsilon(x)\to 0$ pointwise if $j>0$. Let $\eta\in C_0^\infty(-B,B)$ and apply (3.3), (3.4) and (3.5) to $\psi=\phi_\epsilon\eta$ keeping in mind the trivial estimate $d(x,F^a)\leqslant d(x,K)$. By the dominated convergence theorem, $\langle\mu,\phi_\epsilon\eta\rangle\to\int_K\eta\, d\mu$ while $\|d(x,K)^j\, D_x^j(\phi_\epsilon(x)\eta(x))\|_{L^1}\to 0$ as $\epsilon\to 0$ (when $j=0$, use the fact that $|K|=0$). Thus by (3.4) we conclude that the first integral on the right in (3.3) goes to 0 as $\epsilon\to 0$. That is,

$$(3.6)\qquad \lim_{\epsilon\to 0}\int_{-B}^{B} f(x,T_0)(\phi_\epsilon\eta)_m(x,T_0)\, dZ(x,T_0)=0.$$

Consider next the second integral on the right in (3.3) with $\psi=\phi_\epsilon\eta$. Observe that

$$|V^{m+1}(\phi_\epsilon\eta)|\leqslant\frac{C}{d(x,K)^{m+1}}$$

and so using (3.2) and choosing $m=2N+2$, we have

$$(3.7)\quad |\varphi_t(x,t)V^{m+1}\psi(x)f(x,t)(\varphi(x,t)-\varphi(x,0))|^m \leqslant Cd(x,K)|f(x,t)||\varphi(x,t)-\varphi(x,0)|^N\chi_\epsilon(x)$$

where $\chi_\epsilon(x)$ is the characteristic set of $\{x\colon d(x,K)<\epsilon\}$.

Since $f(x,t)(\varphi(x,t)-\varphi(x,0))^N\in L^1(Q^+)$, we conclude that

$$(3.8)\qquad \lim_{\epsilon\to 0}\int_0^{T_0}\int_{-B}^{B} f(x,t)\, L(\phi_\epsilon\eta)_{2N+2}(x,t)\, dt\wedge dZ=0.$$

We have thus shown that

$$\int_K\psi\, d\mu=0,\quad \psi\in C_0^\infty(-B,B),$$

which implies that the same conclusion holds for any continuous function ψ on K (first extend ψ to a compactly supported function on $(-B,B)$ and then approximate the extension by test functions). Thus the total variation $|\mu|(K)$ of μ on K is zero and by the regularity of μ it follows that $|\mu|(F')=0$ whenever $F'\subset F$ is a Borel set with $|F'|=0$. This proves that μ_{F^a} is absolutely continuous with respect to Lebesgue measure. □

4. The locally solvable case.

We recall first the class of locally solvable vector fields (see [NT], [T]):

DEFINITION 4.1. — *Let L be a smooth vector field defined on an open set $\Omega \subseteq \mathbb{R}^{\not k}$, $p \in \Omega$. L is said to be locally solvable at p if there exists a neighborhood $U = U(p)$ such that for every $f \in C^\infty(\Omega)$ there exists $u \in \mathcal{D}'(\Omega)$ such that the equation $Lu = f$ holds in U. If L is locally solvable at every point $p \in \Omega$ we say that L is locally solvable in Ω.*

We will consider a vector field

$$L = \frac{\partial}{\partial y} + a(x, y)\frac{\partial}{\partial x}$$

which is locally solvable on a neighborhood of the rectangle $R = (-a, a) \times (-b, b)$. Since our point of view is local and locally solvable vector fields are known to be locally integrable [T], we will assume without loss of generality that there is a smooth real-valued function $\varphi(x, y)$ defined on a neighborhood of R such that $Z(x, y) = x + i\varphi(x, y)$ is a first integral of L, i.e., $LZ = 0$ or, equivalently, $a(x, y) = -i\varphi_y(x, y)/(1 + i\varphi_x(x, y))$. Furthermore, we may assume that $\varphi(0, 0) = \varphi_x(0, 0) = 0$. It is well known that the local solvability of L is equivalent to the fact that L satisfies the Nirenberg-Treves condition $(\mathcal{P})$ ([NT], [T]) and this reflects on the behavior of φ in the following way:

- for every $x \in [a, a]$,
- the map $[b, b] \ni y \mapsto \varphi(x, y)$ is monotone.

Our goal here is to prove a strengthened version of the F. and M. Riesz theorem of ([BH1]) for the class of locally solvable complex vector fields. For such vector fields, unlike ([BH1]), we will make a milder assumption on the growth of $f(x, t)$ as in Sections 1 and 2.

THEOREM 4.2. — *Suppose L is locally solvable, $f(x, t)$ is a continuous solution of $Lf = 0$ in $Q^+ = (-A, A) \times (0, T)$ and it has a boundary value $bf(x)$ which is a measure. Assume that $(\varphi(x, t) - \varphi(x, 0))^N f(x, t) \in L^1(Q^+)$. Then $bf(x)$ is absolutely continuous with respect to Lebesgue measure.*

Before we prove the theorem, we will use the result of Section 3 to make some reductions. Write $L = X + iY$ where

$$X = \frac{\partial}{\partial t} + \operatorname{Re} a(x, t)\frac{\partial}{\partial x} = \frac{\partial}{\partial t} - \Big(\frac{\varphi_t \varphi_x}{1 + \varphi_x^2}\Big)\frac{\partial}{\partial x}$$

and

$$Y = \operatorname{Im} a(x,t)\frac{\partial}{\partial x} = \frac{-\varphi_t}{1+\varphi_x^2}\frac{\partial}{\partial x}.$$

In view of Theorem 3.1 in [BH1], the main problem in proving the F. and M. Riesz property comes from vertical segments on which $\varphi_t(x,t)$ vanishes. If φ_t vanishes of order at least two on one of these segments, the arguments given in Lemma 3.1 show that it causes no trouble. The difficulty comes from segments on which this does not happen, *i.e.*, they contain a point on which $\varphi_{tx} \neq 0$. We will show in this section that when L is locally solvable, this difficulty can be circumvented. The local solvability of L together with the vanishing of φ_t on the $t-$ axis to order one (but not two) allow us to make the following assumptions in addition to the usual normalizations $\varphi(0,0) = \varphi_x(0,0) = 0$. Namely,

i) $\varphi(x,t) = x\psi(x,t)$;

ii) $\psi_t(x,t) \geqslant 0$ for $|x| < A$ and $|t| < T$;

iii) $\psi(0,T) > 0$.

iv) $\varphi(x,0) = \psi(x,0) = 0$ for $|x| \leqslant A$.

Condition iv) is not essential in the ensuing arguments. We include it here for simplicity (see Remark 4.8). Actually this condition can be achieved for a locally solvable vector field, at least with a $Z(x,t)$ defined for $t \geqslant 0$, which is the region of relevance here. Choosing a convenient smaller value of T and shrinking A if necessary we may assume that $\psi(x,T) > 0$ and $\psi_t(x,T) > 0$ for $|x| \leqslant A$. Since for $0 \leqslant x \leqslant A$ the function $[0,T] \ni t \mapsto \varphi(x,t)$ is non decreasing it follows that $Z([0,A) \times [0,T])$ is mapped to the closed "triangle"

$$\overline{R}^+ = \{(x,y) \colon \varphi(x,0) \leqslant y \leqslant \varphi(x,T), 0 \leqslant x \leqslant A\}$$

whose interior is

$$R^+ = \{(x,y) \colon \varphi(x,0) < y < \varphi(x,T), 0 < x < A\}.$$

Since $\varphi_x(0,0) = 0$ and $\varphi_x(0,T) = \psi(0,T) > 0$ we see that the curves $\gamma_{\min}(x) = (x,\varphi(x,0)) = (x,0)$ and $\gamma_{\max}(x) = (x,\varphi(x,T))$ that bound R^+ meet at a positive angle at the origin forming a corner with opening $\pi\beta$, where $0 < \beta < 1/2$. We may assume that the initial values of T and A had been chosen small so that the Baouendi-Treves scheme for f converges to f in a neighborhood of $[-A,A] \times (0,T]$. By the Baouendi-Treves approximation theorem [BT] there is a holomorphic function F^+ defined on R^+ such that $f = F^+ \circ Z$ on $(0,A) \times (0,T)$. We begin now with a lemma which shows that if f has a trace, then it has to satisfy a pointwise growth condition:

LEMMA 4.3. — *If L and the first integral $Z(x,t) = x + i\varphi(x,t)$ are as above, and f is a continuous solution of $Lf = 0$ in Q^+ with a weak distribution boundary value at $t = 0$, then after decreasing A and T, there is an integer N and a constant $C > 0$ such that*

$$|f(x,t)\varphi(x,t)^N| \leqslant C \quad \text{for} \quad (x,t) \in Q^+.$$

Proof. — Since f has a trace at $t = 0$, there exists an integer m and a constant $C > 0$ such that for every $\Psi \in C_0^m(-A, A)$,

$$|\langle f(\cdot,t), \Psi\rangle| \leqslant C \sum_{k \leqslant m} \|D^k\Psi\|_{L^\infty}, \quad \forall\, t \in (0,T]. \tag{4.1}$$

It follows that if we set

$$f_{m+1}(x,t) = \begin{cases} (x + i\varphi(x,t))^{m+1} f(x,t), & \text{for } x > 0 \\ 0, & \text{for } x \leqslant 0, \end{cases}$$

then for every $\Psi \in C_0^m(-A, A)$,

$$|\langle f_{m+1}(\cdot,t), \Psi\rangle| \leqslant C \sum_{k \leqslant m} \|D^k\Psi\|_{L^\infty}, \quad \forall\, t \in (0,T]. \tag{4.2}$$

For $\epsilon > 0$ small, let $Q_\epsilon = (0, A) \times (\epsilon, T)$ and $\Omega_\epsilon = Z(Q_\epsilon)$. Observe that the function F^+ is continuous on $\overline{\Omega_\epsilon}$. Let $g \in C^\infty([0, A))$, $g(x) \equiv 1$ on $[0, A - 2\delta]$, and $g \equiv 0$ on $(A - \delta, A)$ where δ is a small positive number. We apply the inhomogeneous Cauchy Integral Formula to the function $u(x,y) = g(x)(x + iy)^{m+1}F^+(x + iy)$ on the domain Ω_ϵ:

$$\begin{aligned} &g(x)(x+iy)^{m+1}F^+(x+iy) \\ &\quad = \frac{1}{\pi}\int\int_{\Omega_\epsilon} \frac{Dg(x')(x'+iy')^{m+1}F^+(x'+iy')}{x - x' + i(y - y')}\, dx'dy' \\ &\quad - \frac{1}{2\pi i}\int_{\partial\Omega_\epsilon} \frac{g(x')(x'+iy')^{m+1}F^+(x'+iy')}{x - x' + i(y-y')}\, dx' + idy'. \end{aligned} \tag{4.3}$$

We estimate first the double integral which after the change of variable $y' = \varphi(x', t)$ becomes

$$\int_\epsilon^T \int_0^A \frac{Dg(x')f_{m+1}(x',t)\varphi_t(x',t)}{x - x' + i(y - \varphi(x',t))}\, dx'\, dt.$$

Since the support of $Dg \subseteq (A - 2\delta, A - \delta)$, by (4.2), there is $C > 0$ independent of x and t such that for any $x \in (0, A - 3\delta)$,

$$\left|\int_0^A \frac{Dg(x')f_{m+1}(x',t)\varphi_t(x',t)}{x - x' + i(y - \varphi(x',t))}\, dx'\right| \leqslant C.$$

It follows that

$$\left|\int_\epsilon^T \int_0^A \frac{Dg(x')f_{m+1}(x',t)\varphi_t(x',t)}{x-x'+i(y-\varphi(x',t))}\,dx'dt\right| \leqslant C. \tag{4.4}$$

Consider next the second integral on the right in (4.3) which equals

$$\int_0^A \frac{g(x')f_{m+1}(x',\epsilon)}{x-x'+i(y-\varphi(x',\epsilon))}\,dZ(x',\epsilon) - \int_0^A \frac{g(x')f_{m+1}(x',T)}{x-x'+i(y-\varphi(x',T))}\,dZ(x',T). \tag{4.5}$$

Recall that $\varphi(x,t) = x\psi(x,t), \psi(x,0) \equiv 0$ and $\psi(0,T) > 0$. After decreasing A if necessary, we may therefore assume that for some $t_1 > 0$, and $0 \leqslant x \leqslant A$,

$$3\psi(x,t) \leqslant \psi(x,T), \quad \forall\, 0 \leqslant t \leqslant t_1,\ 0 \leqslant x \leqslant A. \tag{4.6}$$

Fix $(x,t) \in (0,A) \times (0,t_1]$. If $\varphi(x,t) = 0$, then $f(x,t)\varphi(x,t) = 0$ and so we may assume that $\varphi(x,t) > 0$. This assumption together with (4.6) imply that $x + i\varphi(x,t) \in \Omega_\epsilon$ for ϵ small enough. We will estimate the integrals in (4.5) at $x + iy = x + i\varphi(x,t)$ with (x,t) as described. For the second integral in (4.5) with $y = \varphi(x,t)$ we have

$$\begin{aligned}
|x-x'+i(\varphi(x,t)-\varphi(x',T))| &\geqslant \frac{1}{2}(|x-x'| + \varphi(x',T) - \varphi(x,t)) \\
&\geqslant \frac{1}{2}(\varphi(x,T) - \varphi(x,t)) \\
&\geqslant \varphi(x,t) \quad \text{by} \quad (4.6).
\end{aligned}$$

Since $f_{m+1}(x',T)$ is continuous and so bounded on $[0,A]$, it follows that

$$\left|\int_0^A \frac{g(x')f_{m+1}(x',T)}{x-x'+i(\varphi(x,t)-\varphi(x',T))}\,dZ(x',T)\right| \leqslant \frac{C}{\varphi(x,t)} \tag{4.7}$$

where the constant C is independent of (x,t). Consider next the second integral in (4.5) with $x + iy = x + i\varphi(x,t) \in \Omega_\epsilon$ as above. Note that $\varphi(x,\epsilon) < \varphi(x,t)$ and so using (4.2), we get

$$\left|\int_0^A \frac{g(x')f_{m+1}(x',\epsilon)}{x-x'+i(\varphi(x,t)-\varphi(x',\epsilon))}\,dZ(x',\epsilon)\right| \leqslant \frac{C}{|\varphi(x,t)-\varphi(x,\epsilon)|^m} \tag{4.8}$$

where C is independent of (x,t) and ϵ. Since $\varphi_t(x,t) \geqslant 0$, we can let $\epsilon \to 0$ in (4.8) to get

$$\left|\int_0^A \frac{g(x')f_{m+1}(x',\epsilon)}{x-x'+i(\varphi(x,t)-\varphi(x',\epsilon))}\,dZ(x',\epsilon)\right| \leqslant \frac{C}{|\varphi(x,t)|^m}. \tag{4.9}$$

From (4.4), (4.7) and (4.9) we conclude that $g(x)Z(x,t)^{2m+1}f(x,t)$ is bounded for $(x,t) \in (0,A) \times (0,t_1]$, and so since $g(x) \equiv 1$ near 0, we have shown that for some integer N, the function $f(x,t)\varphi(x,t)^N$ is bounded on $(0,A)\times(0,t_1]$ as desired. □

LEMMA 4.4. — *Let f be as in Theorem 4.1 and F^+ the holomorphic function such that $f(x,t)=F^+(Z(x,t))$. Then bF^+ is a measure on $(0,A)$.*

Proof. — Recall that F^+ is holomorphic on the region

$$R^+=\{(x,y)\colon \varphi(x,0)=0<y<\varphi(x,T),\quad 0<x<A\}.$$

Under the assumption that bf is a measure, we wish to show that bF^+ is a measure on $(0,A)$. Fix $\delta>0$ and let $\Psi\in C_0^\infty(\delta,A)$. From (1.1) of Section 1 we have

$$\int_\delta^A f(x,\epsilon)u(x,\varphi(x,\epsilon))\,dZ(x,\epsilon)=\int_\delta^A f(x,T)u(x,\varphi(x,T))\,dZ(x,T) \\ +2i\int_\delta^A\int_\epsilon^T f(x,t)\varphi_t(x,t)\overline{\partial}u(x,\varphi(x,t))\;dxdt. \tag{4.10}$$

By Lemma 4.3, there is an integer N such that $|f(x,t)\varphi(x,t)|^N\leqslant C$. Recall that $u(x,y)$ is a smooth function satisfying

a) $u(x,0)=\Psi(x)$, and

b) $|(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})u(x,y)|\leqslant Cy^N$.

Since $f(x,t)=F^+(x+i\varphi(x,t))$, (4.10) can be written as

$$\int_\delta^A F^+(x+i\varphi(x,\epsilon))u(x,\varphi(x,\epsilon))\,dZ(x,\epsilon) \\ =\int_\delta^A F^+(x+i\varphi(x,T))u(x,\varphi(x,T))\,dZ \\ -2i\int_\delta^A\int_{\varphi(x,\epsilon)}^{\varphi(x,T)} F^+(x+iy)\overline{\partial}u(x+iy)\;dydx. \tag{4.11}$$

Note that $F^+(x+iy)\overline{\partial}u(x+iy)\in L^\infty(R^+)$, and so from (4.10) and (4.11) we get

$$(4.12\quad \langle Z_x(x,0)bf,\Psi\rangle=\langle bf,\Psi\rangle \\ =\int_\delta^A F^+(x+i\varphi(x,T))u(x,\varphi(x,T))\,dZ(x,T) \\ -2i\int_\delta^A\int_0^{\varphi(x,T)} F^+(x+iy)\overline{\partial}u(x+iy)\;dydx.$$

On the other hand, since $F^+(x+iy)y^N$ is bounded, F^+ has a trace on $(0,A)$ and $\langle bF^+,\Psi\rangle$ is also given by the right hand in (4.12). Thus bF^+ is also a measure. □

Proof of Theorem 4.2. — By Lemma 4.3,

$$F^+(z)\ \mathrm{dist}(z,\gamma_{\min})^N = F^+(x+iy)y^N$$

is bounded on R^+. Thus, $F^+(z)$ has a trace bF^+ on $\gamma_{\min}\setminus\{0\}$ and by Lemma 4.4 it must be a bounded measure. Now, any open subarc of $\gamma_{\min}\setminus\{0\}$ can be embedded in a closed smooth curve bounding a region $D\subset R^+$ that can be mapped to the unit disk by a conformal map Φ that extends smoothly to the closure as a homeomorphism. By the classical F. and M. Riesz theorem, the boundary value of $(F^+/D)\circ\Phi$ is an integrable function and this implies that bF^+ is locally integrable in $\gamma_{\min}\setminus\{0\}$. The continuity of $f(x,t)$ for $t>0$ and the Baouendi-Treves approximation formula imply that there are polynomials $p_k(\zeta)$ such that $p_k\circ Z(x,t)$ converge uniformly to $f(x,t)$ on $[-A,A]\times[\epsilon,T]$ for $\epsilon>0$. This shows that F^+ can be extended continuously up to $\gamma_{\max}$. Observe that the curves $[0,A]\ni x\mapsto(x,\varphi(x,t)$ stem from the origin and have slope $\psi(0,t)$ which is a strictly increasing function of t if $t<T$ is close to T. Hence, the segment $[0,A]\ni x\mapsto(x,\lambda x)$ remains between the graphs of $\varphi(x,T)$ and $\varphi(x,t_1)$ for small x if we choose $\psi(0,t_1)<\lambda<\psi(0,T)$. Shrinking A we may assume that

$$\varphi(x,t_1)<\lambda x<\varphi(x,T),\quad 0<x\leqslant A.$$

Since φ is bounded on $[0,A]\times[\epsilon,T]$ for $\epsilon>0$ we see that $F^+(x,\lambda x)\in L^\infty[0,A]$. Now we consider the triangle bounded by the straight line $y=\lambda x$, the horizontal line $y=0=\varphi(x,0)$ and the vertical line $x=A$. We may smoothen up the 2 corners of this triangle that are distinct from the origin and obtain a domain with only one corner with opening $\pi\beta$ located at the origin and call this region R_1. Off this corner the function F^+ has a trace on the boundary of R_1 which is a locally integrable function. We consider now the conformal map $\Phi(z)=z^\beta$ from the upper half plane to the sector $0<\arg z<\beta\pi$, choosing the branch that is real on the positive real axis and set $R_2=\Phi^{-1}(R_1)$, so R_2 is a smooth domain.

LEMMA 4.5. — *The holomorphic function $F_2=F^+\circ\Phi(z)$ grows temperedly at the boundary of R_2.*

Proof. — Since F^+ grows temperedly at the boundary of R_1 and Φ is smooth up to the boundary off the origin, we may only concern ourselves with the growth of F_2 in a neighborhood of the origin. We must show that $|F_2(x+iy)y^m|\leqslant C$ for $|x|<\epsilon$, $0<y<\epsilon$ for some $0<\epsilon<1$ and $m>0$ or, in polar coordinates, $|F_2(re^{i\theta})r^m\sin^m\theta|\leqslant C$, $0<r<\epsilon$, $0<\theta<\pi$. Since $|F^+(x+iy)y^N|\leqslant C$ and $F_2(re^{i\theta})=F^+(r^\beta\exp(i\beta\theta))$ we get

$$|F_2(z)|\leqslant\frac{C}{r^{N\beta}\sin^N\beta\theta}\leqslant\frac{C'}{r^N\sin^N\theta}$$

where we have used the fact that $\sin\theta/\sin\beta\theta$ is bounded on $(0,\pi)$ and $r \leqslant r^\beta$ on $(0,1)$ because $\beta \in (0,1)$. □

Thus, F_2 has a trace at the boundary of R_2 that, off the origin, is the finite measure given by the pull-back of bF_1 where we have written $F_1 \doteq F^+/R_1$. We want to conclude that F_2 is in the Hardy space $E^1(R_2)$ and the next lemma will serve this purpose. We recall first

DEFINITION 4.6. — *A holomorphic function H on a bounded domain D with rectifiable boundary is said to be in the Hardy space $E^p(D)$ $(1 \leqslant p < \infty)$ if there exists a sequence of rectifiable curves C_j in D tending to bD in the sense that the C_j eventually surround each compact subdomain of D, such that*

$$\int_{C_j} |H(z)|^p |dz| \leqslant M < \infty.$$

DEFINITION 4.7. — *Suppose for a bounded region Ω in the plane there is $\alpha = \alpha(\Omega) > 0$ with the property that almost every point p on the boundary admits a nonempty nontangential approach subregion*

$$\Gamma_\alpha(p) = \{z \in \Omega : |z-p| \leqslant (1+\alpha)\operatorname{dist}(z,\partial\Omega)\}$$

that is, for a.e. $p \in \partial\Omega$, $\Gamma_\alpha(p)$ is open and p is in the closure of $\Gamma_\alpha(p)$. For $1 \leqslant p < \infty$ the Hardy space $H^p(\Omega)$ is defined by

$$H^p(\Omega) = \{G \in \mathbb{O}(\nleq) : \mathbb{G}^* \in \mathbb{L}^{\prime}(\partial\nleq)\}$$

where $\mathbb{O}(\nleq)$ denotes the holomorphic functions on Ω and G^* denotes the nontangential maximal function defined using the $\Gamma_\alpha(p)$.

LEMMA 4.8. — *Let $F(z)$ be a holomorphic function on the unit disc $\Delta = \{|z| < 1\}$ that grows temperedly at its boundary S^1. Assume that $bF(z) = \mu + \nu$ where μ is a finite measure and ν is a distribution supported at $z = 1$. Then $\nu = 0$ and μ is absolutely continuous with respect to the Lebesgue measure $d\theta$ on the circle. Thus $F \in H^1(\Delta) = E^1(\Delta)$.*

Taking the lemma for granted and using a conformal mapping from R_2 onto Δ we conclude that $F_2 \in E^1(R_2)$, in particular, by [Du], Thm. 10.4

$$\int_{\partial R_2} F_2(z) z^k \, dz = 0, \quad k = 0, 1, 2, \ldots \tag{4.13}$$

On the other hand, the change of variables $\partial R_2 \ni z \mapsto \zeta = z^\beta \in R_1$ gives

$$\int_{\partial R_1} F^+(\zeta)\zeta^k \, d\zeta = \beta \int_{\partial R_2} F_2(z) z^{k\beta} z^{(\beta-1)} \, dz \tag{4.14}$$

so we want to know if the integral on the right is zero for all $k = 0, 1, \ldots$, since in that case, using again [Du], Thm. 10.4, we will conclude that $F_1 \in E^1(R_1)$. Assume first that $k = 0$ in which case we need to show that

$$\int_{\partial R_1} F^+(\zeta)\, d\zeta = \beta \int_{\partial R_2} F_2(z) z^{\beta-1}\, dz = 0. \tag{4.15}$$

Note that F_2 is not a generic element of $E^1(R_2)$ because it satisfies the restriction

$$\int_{\partial R_2} |F_2(z)|\, |z|^{\beta-1}\, |dz| < \infty$$

since by Lemma 4.4, $bF_1 = bf$ on $(0, A)$ and hence $bF_1 \in L^1(0, A)$. Note that for any $\epsilon > 0$,

$$\int_{\partial R_2} F_2(z)(z + i\epsilon)^{(1/\delta)-1}\, dz = 0$$

and letting $\epsilon \searrow 0$ we see that (4.15) holds by the dominated convergence theorem. The same reasoning shows that (4.14) holds for all $k = 1, 2, \ldots$. Thus, $F^+ \in E^1(R_1)$ as we wanted to show. This allows us to estimate the integrals

$$\int_0^A f(x,t)\psi(x)\, dx = \int_0^A F^+(x, \varphi(x,t))\psi\, dx, \quad 0 < t < t_1, \psi \in C_c^\infty(-A, A),$$

by dominating $f(x,t)$ by $m^+(x) = \max(F_*^+(x,0), F_*^+(x, \lambda x))$, where $F_*^+(p)$, $p \in \partial R_1$, is the nontangential maximal function of $F_1 = F^+/R_1$. Reasoning in a similar way for $-A < x < 0$ we conclude that

$$|f(x,t)| \leqslant m^+(x), \quad 0 < t < t_1, \quad 0 < x < A,$$
$$|f(x,t)| \leqslant m^-(x), \quad 0 < t < t_1, \quad -A < x < 0.$$

Thus, an application of Fatou's lemma yields

$$|\langle bf, \psi\rangle| \leqslant \int m(x)|\psi(x)|\, dx, \quad \psi \in C_c^\infty(-A, A),$$

with $m(x) \in L^1(-A, A)$ which shows that bf is absolutely continuous with respect to Lebesgue measure. □

Proof of Lemma 4.8. — Since ν is supported at $\theta = 0$ we know that $\nu = \sum_{k=0}^{L} c_k D_\theta^k \delta$ where $\delta(\theta)$ is the Dirac mass, the c_k's are complex numbers and we may assume that c_L is real by changing F if necessary by a convenient multiple of F. Then F belongs to $H^p(\Delta)$ for any $p < 1/L$. Let p_0 denote the supremum of the set $\{0 < p \leqslant 1, \quad F \in H^p\}$. Since $b(\operatorname{Im} F) = \operatorname{Im} \mu + \sum_{k=0}^{L-1} \operatorname{Im} c_k D_\theta^k \delta$ we see by expressing $\operatorname{Im} F$ as the Poisson integral of its boundary value $b(\operatorname{Im} F)$ that $\operatorname{Im} F$ is in the real Hardy space $\operatorname{Re} H^p$ for any $p < 1/(L-1)$. Since the Hilbert transform maps continuously $\operatorname{Re} H^p$ into itself we conclude that $\operatorname{Re} F \in \operatorname{Re} H^p$ for any $p < 1/(L-1)$, so $p_0 > 1/L$ and this implies that $c_L = 0$. Iterating this argument we see that $p_0 = 1$ which means that $bF = \nu + \mu = c_0\delta + \mu$ and by the F. and M. Riesz theorem $c_0 = 0$ and $\mu \in L^1(S^1)$. □

Remark 4.9. — The restriction iv) that $\varphi(x, 0) \equiv 0$ is not essential and can be removed as we briefly describe now. The main difference without this assumption is that $\gamma_{\min}$ would be a smooth curve rather than a straight segment, but we could use a conformal map from a region bounded from below by $\gamma_{\min}$ and containing $Z([0, A] \times [0, T])$ onto a region contained in the upper half plane so that $\gamma_{\min}$ is mapped into a segment of the real axis. This map would be smooth up to the boundary and we could carry out our reasoning as before with the new region that has a straight lower side.

5. A second application.

Assume now that L is locally integrable with a smooth first integral $Z = x + i\varphi(x, t)$.

THEOREM 5.1. — *Let f be continuous on $Q^+ = (-B, B) \times (0, T)$, and suppose*

i) *$Lf = 0$ in Q^+;*

ii) *there exists $C > 0$ such that*

$$|\varphi(x, t) - \varphi(x, 0)||f(x, t)| \leqslant C, \quad \forall\ (x, t) \in Q^+.$$

Assume that the boundary value $bf = \mu$ is a measure. Then μ is absolutely continuous with respect to Lebesgue measure.

Proof. — Without loss of generality, let $B = 1$. By the reductions in Section 3 and Remark 3.2, we may assume that $\mu = g + c\delta_0$ where g is integrable, c is a number and δ_0 is the Dirac delta distribution at 0. Recall from (1.17) that for any $\psi \in C_0^\infty(-1, 1)$, we have

(5.1)
$$\langle Z_x(x, 0)\mu, \psi\rangle = \int_{-1}^{1} f(x, T_0)\psi_1(x, T_0)\, dZ + \int_0^{T_0} \int_{-1}^{1} f(x, t)\, L\psi_1(x, t)\, dt \wedge dZ$$

where

$$\psi_1(x,t) = \sum_{k=0}^{1} (\varphi(x,t) - \varphi(x,0))^k \, \frac{V^k \psi(x)}{k!}$$

and

$$V = \frac{i}{Z_x(x,0)} \frac{\partial}{\partial x}.$$

Recall next that

$$L\psi_1(x,t) = \frac{-i\varphi_t(x,t)}{1 + i\varphi_x(x,t)} V^2\psi(x)(\varphi(x,t) - \varphi(x,0))$$

which implies that

$$(5.2)\left|\int_0^{T_0}\int_{-1}^{1} f(x,t)\, L\psi_1(x,t)\, dt \wedge dZ\right| \leqslant C\int_0^{T_0}\int_{-1}^{1} |\varphi_t(x,t)V^2\psi(x)|\, dxdt.$$

By Theorem 3.1 in [BH1], we may assume that $\varphi(0,t) = 0$ for $t \in [0, T_0]$ and hence we have

$$(5.3)\qquad \left|\int_0^{T_0}\int_{-1}^{1} f(x,t)\, L\psi_1(x,t)\, dt \wedge dZ\right| \leqslant CT_0 \int_{-1}^{1} |xV^2\psi(x)|\, dx.$$

We will apply formula (5.1) to $\psi = \psi^n(x) = (1 - x^2)^n \psi_0(x)$, where $\psi_0 \in C_c^\infty(-1,1)$ is fixed, $\psi_0(0) = 1$ and n is a positive integer which will go to infinity. Observe that from the form $\mu = g + c\delta_0$, we get

$$(5.4)\qquad \lim_{n\to\infty} \langle \mu, \psi^n \rangle = c.$$

To estimate the second integral on the right in (5.1), by (5.3), we only have to estimate

$$(5.5)\qquad \int_{-1}^{1} |xV^2\psi^n(x)|\, dx.$$

We can write

$$(5.6)\qquad \begin{aligned} V^2\psi^n(x) &= (1-x^2)^n V^2\psi_0 + a_1 n x (1-x^2)^{n-1} V\psi_0 \\ &\quad + (a_2 n(1-x^2)^{n-1} + a_3 n(n-1)x^2(1-x^2)^{n-2})\psi_0 \end{aligned}$$

where the a_i are constants independent of n. In view of (5.5) and (5.6), we need to estimate the integrals

$$\int_0^1 x^3(1-x^2)^{n-2}\, dx, \quad \int_0^1 x(1-x^2)^{n-1}\, dx \quad \text{and} \quad \int_0^1 x^2(1-x^2)^{n-1}\, dx.$$

After using the variable $y = 1 - x^2$, we see that

$$(5.7)\qquad \int_0^1 x^3(1-x^2)^{n-2}\, dx = \frac{1}{2n(n-1)}.$$

We also have

$$\int_0^1 x(1-x^2)^{n-1}\,dx = \frac{1}{2n} \tag{5.8}$$

and so

$$\int_0^1 x^2(1-x^2)^{n-1}\,dx \leqslant \frac{1}{2n}. \tag{5.9}$$

It now follows that there is a constant C independent of n such that the second integral on the right in (5.1) satisfies the estimate

$$\left|\int_0^{T_0}\int_{-1}^1 f(x,t)\,L\psi_1^n(x,t)\,dt\wedge dZ\right| \leqslant CT_0. \tag{5.10}$$

We consider next the first integral on the right in (5.1). We have to estimate

$$\int_{-1}^1 f(x,T_0)\psi_1^n(x,T_0)\,dZ(x,T_0)$$

where

$$\psi_1^n(x,T_0) = \psi^n(x) + (\varphi(x,T_0)-\varphi(x,0))V\psi^n(x) \tag{5.11}$$

and

$$V\psi^n(x) = (1-x^2)^n V\psi_0(x) - 2inx(1-x^2)^{n-1}\psi_0(x). \tag{5.12}$$

Clearly,

$$\lim_{n\to\infty}\int_{-1}^1 f(x,T_0)\psi_1^n(x)\,dZ(x,T_0) = 0 \tag{5.13}$$

and likewise,

$$\lim_{n\to\infty}\int_{-1}^1 f(x,T_0)(\varphi(x,T_0)-\varphi(x,0))(1-x^2)^n V\psi_1^n(x)\,dZ(x,T_0) = 0. \tag{5.14}$$

We therefore need only estimate

$$n\int_{-1}^1 f(x,T_0)x(\varphi(x,T_0)-\varphi(x,0))(1-x^2)^{n-1}\psi_0(x)\,dx. \tag{5.15}$$

Recall that $|\varphi(x,t)-\varphi(x,0)| \leqslant C|x|$, and so the integral in (5.15) is dominated by a constant multiple of

$$n\|f(\cdot,T_0)\|_{L^\infty}\int_0^1 x^2(1-x^2)^{n-1}\,dx. \tag{5.16}$$

We will show that (5.16) goes to zero as $n\to\infty$. Indeed, note first that on the interval $[0,1]$, the function $x^2(1-x^2)^{n-1}$ is bounded by $\frac{1}{n}$ and so for

any $\epsilon > 0$, if n is large enough,
(5.17)

$$\begin{aligned}
n\int_0^1 x^2(1-x^2)^{n-1}\,dx &= n\Big(\int_0^\epsilon x^2(1-x^2)^{n-1}\,dx + \int_\epsilon^1 x^2(1-x^2)^{n-1}\,dx\Big)\\
&\leqslant n\Big(\frac{\epsilon}{n} + \int_\epsilon^1 x^2(1-x^2)^{n-1}\,dx\Big)\\
&\leqslant 2\epsilon.
\end{aligned}$$

From (5.13)–(5.17), we conclude that as $n \to \infty$, the first integral on the right in (5.1) converges to zero. This observation together with (5.4) and estimate (5.10) show that the constant c in (5.4) equals 0 and hence μ is absolutely continuous with respect to the Lebesgue measure on the real line. □

6. Appendix.

We will now explore whether the existence of the limit in (1.2) of Section 1 implies the existence of a trace for f without any assumption on Lf. As before, set $Z(x,t) = x + i\varphi(x,t)$,

$$V = \frac{i}{Z_x(x,0)}\frac{\partial}{\partial x}, \quad \text{and} \quad P(x,t) = \varphi(x,t) - \varphi(x,0). \tag{6.1}$$

For any smooth function $\Psi(x)$ of compact support, and m a nonnegative integer, recall that

$$\Psi_m(x,t) = \sum_{k=0}^{m} \frac{V^k(\Psi)}{k!} P(x,t)^k, \quad \Psi_0(x,t) = \Psi(x). \tag{6.2}$$

Assume $f(x,t)$ is a continuous function on $(-A,A)\times(0,T)$ and consider the limit, which may or may not exist,

$$B_N f(\Psi) = \lim_{\epsilon\to 0}\int_{-A}^{A} f(x,\epsilon)\Psi_N(x,\epsilon)\,dZ(x,\epsilon), \quad \Psi \in C_c^\infty(-A,A), \tag{6.3}$$

for any fixed integer $N \geqslant 0$. Assume we are in the simplest possible case associated to the Cauchy-Riemann vector field, *i.e.*, $\varphi(x,t) = t$, so $V = i\partial/\partial x$, $P(x,t) = t$ and the pull-back of dZ to any horizontal line t =constant is just dx. Thus,

$$\Psi_N(x,t) = \sum_{k=0}^{N} \frac{\Psi^{(k)}(x)}{k!}(it)^k, \quad \Psi_0(x,t) = \Psi(x) \tag{6.4}$$

and

(6.5) $$B_N f(\Psi) = \lim_{\epsilon\to 0}\int_{-A}^{A} f(x,\epsilon)\Psi_N(x,\epsilon)\,dx, \quad \Psi\in C_c^\infty(-A,A).$$

If the limit (6.5) exists for $N = 0$ and every Ψ we easily see that it also does for every N and furthermore, $B_1 = B_2 = \cdots = B_N$. The question is whether the converse holds. Consider the function $f(x,t) = \exp(1/t)\exp(-ix/t)$, which is smooth for $t>0$. Then $B_1 f(\Psi)$ is the limit as $\epsilon \searrow 0$ of

$$\int f(x,\epsilon)(\Psi(x)+i\epsilon\Psi'(x))\,dx = \int (f(x,\epsilon)-i\epsilon f_x(x,\epsilon))\Psi(x)\,dx.$$

Since f satisfies the equation $f - itf_x = 0$ it is clear that $B_1 f(\Psi)$ exists and is zero for any Ψ. On the other hand,

$$\lim_{\epsilon\searrow 0}\int f(x,\epsilon)\Psi(x)\,dx = \lim_{\epsilon\searrow 0} e^{1/\epsilon}\int e^{-ix/\epsilon}\Psi(x)\,dx = \lim_{\epsilon\searrow 0} e^{1/\epsilon}\widehat{\Psi}(1/\epsilon)$$

so the existence of the limit implies that the Fourier transform $\widehat{\Psi}(\xi)$ of Ψ has exponential decay as $\xi \to +\infty$ which cannot be true for a function with compact support unless $\Psi\equiv 0$. The conclusion is that the existence of $B_1 f$ does not imply, in general, the existence of $B_0 f$. Note that $\exp(1/t)\exp(-ix/t)$ blows up non temperedly as $t\searrow 0$ in any neighborhood of the origin. On the other hand, we will now show that if we *assume a priori* that the growth of $f(x,t)$ is tempered as $t\searrow 0$ then the existence of $B_N f$ implies the existence of $B_0 f$, in particular $B_0 f = B_N f$. Hence, the non tempered growth in the example above in which $B_0 f$ does not exist while $B_1 f$ does is an essential feature.

THEOREM 6. — *Assume that $f(x,t)\in C^1((-A,A)\times(0,t))$ satisfies for some $C, m>0$ and $bg\in\mathcal{D}'(-A,A)$:*

(1) $\int_{-A}^{A}|f(x,t)|\,dx \leqslant \frac{C}{t^m}$;

(2) $g(x,t) = \sum_{k=0}^{N}\frac{(-i)^k}{k!}t^k D_x^k f(x,t) \to bg(x)$ *in the sense of distributions as* $t\searrow 0$.

Then as $t\searrow 0$, $f(\cdot,t)$ converges in the sense of distributions on $(-A,A)$.

Proof. — Item (2) above means that the limit $B_N f(\Psi)$ exists for every $\Psi\in C_c^\infty(-A,A)$. Suppose that for some $a_k\in\mathbb{C}$, $k=1,\dots,N$,

$$Tf(x,t) = f(x,t) + \sum_{k=1}^{N} a_k t^k D_x^k f(x,t)$$

has a limit in the sense of distributions as $t \searrow 0$. We will show that there are constants $b_k \in \mathbb{C}$, $k = 1, \dots, N-1$ such that

$$Sf(x,t) = f(x,t) + \sum_{k=1}^{N-1} b_k t^k D_x^k f(x,t)$$

has a limit in t he distribution sense as $t \searrow 0$. Define

$$p(x,t) = Sf(x,t) + btD_x Sf(x,t)$$

where we choose the b_k and b so that $p(x,t) = Tf(x,t)$. Then b and the b_k have to satisfy

$$b_1 + b = a_1, \quad b_j + bb_{j-1} = a_j \quad \text{for} \quad 2 \leqslant j \leqslant N-1, \quad \text{and} \quad bb_{N-1} = a_N.$$

This system of equations leads to a polynomial equation of degree N in b with coefficients depending on the a_j. Since we may assume that $a_N \neq 0$, the polynomial equation has a nonzero solution b which then leads to solutions b_k for $1 \leqslant k \leqslant N-1$. Thus with these choices, we have

$$p(x,t) = Tf(x,t)$$

and hence $p(x,t) = Sf(x,t) + btD_x Sf(x,t)$ has a limit as $t \searrow 0$. By applying this reduction successively, we may assume that for some $\alpha, \beta \in \mathbb{R}$,

$$g(x,t) = f(x,t) - (\alpha + i\beta)tf_x(x,t) \to bg(x)$$

in the distribution sense as $t \searrow 0$.

Fix a small $\epsilon > 0$ and choose $\chi(x) \in C_c^\infty(-A, A)$ such that $\chi(x) = 1$, $|x| \leqslant A - \epsilon$, and set $g_1 = \chi g$. Then, $g_1(x,t)$ converges weakly in $\mathcal{E}'(-A, A)$, considered as the dual of $C^\infty(-A, A)$, to $bg_1 = \chi bg$ and by a standard application of the uniform boundedness principle, there exist $R > 0$ and $s < 0$ such that $\|g_1(\cdot, t)\|_s \leqslant R$, where $\| \;\; \|_s$ denotes the norm in the Sobolev space $H^s(\mathbb{R})$. A density argument shows that $g_1(x,t) \to bg_1$ weakly in H^s as $t \searrow 0$ and by Rellich lemma, replacing s by $s-1$ we may assume that $g_1(x,t) \to bg_1$ strongly in H^s as $t \searrow 0$. Assume first that $\alpha \neq 0$. Set

$$f_1(x,t) = (E_t * g_1(\cdot, t))(x)$$

where $E_t(x) = t^{-1}E(x/t)$, $E(x) = H(-x)\frac{1}{\alpha + i\beta} e^{\frac{x}{\alpha + i\beta}}$, if $\alpha > 0$, $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x \leqslant 0$. If $\alpha < 0$, define $E(x) = -H(x)\frac{1}{\alpha + i\beta} e^{\frac{x}{\alpha + i\beta}}$. Since $E \in L^1(\mathbb{R})$ $E_t * h \to h$ in norm for any $h \in H^s(\mathbb{R})$. Thus, $f_1(x,t) \to bg_1$ in $H^s(\mathbb{R})$, $t \searrow 0$, so f_1 has a weak boundary value. It is easily verified that $f_1 - (\alpha + i\beta)t\partial_x f_1 = g_1$ and since $g_1 = g$ on $I_\epsilon = (-A + \epsilon, A - \epsilon)$ we conclude that $f_1(x,t) - f(x,t) = c(t)e^{\frac{x}{(\alpha + i\beta)t}}$ on $I_\epsilon \times (0, T)$. Pick a test function $\psi \in C_c^\infty(I_\epsilon)$ and observe that

$$\begin{aligned} |c(t)| \left| \int e^{\frac{x}{(\alpha + i\beta)t}} e^{\frac{i\beta x}{(\alpha^2 + \beta^2)t}} \psi(x)\, dx \right| &\leqslant \left| \int (f_1(x,t) - f(x,t)) e^{\frac{i\beta x}{(\alpha^2 + \beta^2)t}} \psi \, dx \right| \\ &\leqslant \frac{C_\psi}{t^m} \end{aligned}$$

in view of (1) and the fact that $[0,T) \ni t \mapsto f_1(\cdot,t) \in H^s(\mathbb{R})$ is continuous. If $\alpha > 0$, choose $\psi \geqslant 0$ satisfying $\psi(x) = 1$ for $x \in [0, A-2\epsilon]$ while if $\alpha < 0$, let $\psi \geqslant 0$ satisfy $\psi(x) = 1$ on $[-A+2\epsilon, 0]$. Without loss of generality, assume that $\alpha > 0$. We see that the integral on the left hand side may be estimated from below yielding for small $t > 0$,

$$|c(t)| \leqslant \frac{Ce^{\frac{\alpha}{\alpha^2+\beta^2}(-A+2\epsilon)/t}}{t^{m+1}}.$$

This implies that $\langle c(t)e^{\frac{x}{(\alpha+\imath\beta)t}}, \psi\rangle \to 0,\ t \searrow 0$, for any $\psi \in C_c^\infty(-A, A-2\epsilon)$ and we conclude that $f(x,t)$ has a weak boundary value defined on $(-A, A-2\epsilon)$ for any $\epsilon > 0$. This shows that $B_0 f$ exists when $\alpha \neq 0$. If $\alpha = 0$, define

$$f_1(x,t) = \frac{i}{\beta t}\int_{-A}^{x} e^{\frac{x-y}{\imath\beta t}} g_1(y,t)\, dy.$$

Note that $f_1 - i\beta t\partial_x f_1 = g_1$ and so $f_1(x,t) - f(x,t) = c(t)e^{\frac{x}{\imath\beta t}}$.

For any $\psi \in C_c^\infty(-A,A)$, we have

$$\int f_1(x,t)\psi(x)e^{\frac{-x}{\imath\beta t}}\, dx = \frac{i}{\beta t}\int_{-A}^{A}\Big(\int_{y}^{A} e^{\frac{-y}{\imath\beta t}}\psi(x)\, dx\Big) g_1(y,t)\, dy.$$

Since $g_1(x,t) \to bg_1$ in H^s as $t \searrow 0$, there exist $C > 0$ and an integer n such that for any smooth $\eta(x)$,

$$\Big|\int g_1(x,t)\eta(x)\, dx\Big| \leqslant C\sum_{j=0}^{n}\|D_x^j\eta\|_{L^\infty}.$$

Thus we get

$$\Big|\int f_1(x,t)e^{\frac{\imath x}{\beta t}}\psi(x)\, dx\Big| \leqslant \frac{C(\psi)}{t^{n+1}}.$$

It therefore follows that

$$|c(t)| \leqslant \frac{C(\psi)}{t^k}, \quad k = \max(m, n+1),$$

and hence $\langle c(t)e^{\frac{x}{\imath\beta t}}, \psi\rangle \to 0$ as $t \searrow 0$ since the Fourier transform of ψ decays rapidly at infinity. We conclude that $B_0 f$ exists. □

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