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LENGTH MINIMIZING HAMILTONIAN PATHS FOR SYMPLECTICALLY ASPHERICAL MANIFOLDS

by E. KERMAN and F. LALONDE(*)

1. Introduction.

On a closed symplectic manifold (M, ω), each time-dependent function $H \in \mathcal{H} = C^\infty([0,1] \times M, \mathbb{R})$ defines a time-dependent Hamiltonian vector field X_H via the equation

$$i_{X_H}\omega = -dH.$$

The corresponding flow is denoted by $\phi_H^{t\in[0,1]}$ and the space $\mathrm{Ham}(M, \omega)$ consists of all the time-1 maps, ϕ_H^1, obtained in this manner.

Every path $h_t\colon [0,1] \to \mathrm{Ham}(M,\omega)$ has a family of Hamiltonians $H \in \mathcal{H}$ which satisfy $h_t = \phi_H^t \circ h_0$. In [Ho1], Hofer used these functions to define the length of the path h_t by

$$\begin{aligned} Length\,(h_t) &= \int_0^1 \max_M H(x,t) - \min_M H(x,t)\; dt \\ &= \|H\|^+ + \|H\|^-, \end{aligned}$$

which is independent of the choice of H.

There is a unique Hamiltonian for each path that satisfies the normalization condition

$$\int_M H(x,t)\omega^n = 0, \quad \forall t \in [0,1].$$

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For the normalized Hamiltonian H of the path h_t, both $\|H\|^+$ and $\|H\|^-$ are non-negative and provide different measures of length of h_t called the positive and negative Hofer lengths, respectively.

Remark 1.1. — We will use both normalized and unnormalized Hamiltonians. Therefore, we will mention explicitly when this condition is assumed.

A path is said to *minimize the (positive, negative) Hofer length* if there is no path in $\mathrm{Ham}(M, \omega)$ with the same end points that is shorter (in the appropriate sense).

Let us recall the properties which are necessary for H to generate a path which minimizes the Hofer length.

DEFINITION 1.2. — *A function $H \in \mathcal{H}$ is said to be quasi-autonomous if it has at least one fixed global maximum $P \in M$ and one fixed global minimum $Q \in M$. In other words,*

$$H(t, P) \geqslant H(t, x) \geqslant H(t, Q), \quad \forall t \in [0,1] \text{ and } x \in M.$$

Such fixed global extrema are clearly fixed points of the flow $\phi_H^{t \in [0,1]}$.

DEFINITION 1.3. — *A fixed point $x \in M$ of the flow $\phi_H^{t \in [0,1]}$ is said to be under-twisted if, given any value $T \in [0,1]$, the linearized flow $D\phi_H^{t \in [0,T]}(x) \colon T_xM \to T_xM$ has no non-constant T-periodic orbit.* [1] *It is generically under-twisted if the origin is the only fixed point of the flow $D\phi_H^{t \in (0,1]}$.*

THEOREM 1.4 ([BP], [LM1], [Us]). — *If a Hamiltonian generates a length minimizing path then it must be quasi-autonomous. Moreover, if there are finitely many fixed global extrema, then at least one global maximum and one global minimum must be under-twisted.*

In this paper we consider the converse question: When (for how long) is a path length minimizing if it is generated by a quasi-autonomous Hamiltonian with at least one (generically) under-twisted fixed global maximum and minimum? As described below, our approach relies on Polterovich's natural idea [Po] of applying Floer theoretic method to

[1] We use the terminology T-periodic orbit to mean any orbit that comes back to its initial position at time T, *i.e.*, a closed orbit of period T; it should not be understood as *periodic in time*.

Hofer's geometry (see also Theorem 5.11 and Proposition 5.12 in Schwarz [Sc]). As in [Sc] and [Po], we restrict ourselves here to the case when (M,ω) is symplectically aspherical, *i.e.*, $\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}$. For such manifolds the action functional $\mathcal{A}_H: \mathcal{L}M \to \mathbb{R}$ is well-defined on the space $\mathcal{L}M$ of contractible loops in M. It is given by

$$\mathcal{A}_H(x(t)) = \int_0^1 H(t,x(t))\,dt - \int_{D^2} \bar{x}^*\omega, \tag{1}$$

where $\bar{x}: D^2 \to M$ satisfies $\bar{x}|_{\partial D^2} = x$.

Our first result is the following.

THEOREM 1.5. — *Let $H \in \mathcal{H}$ be quasi-autonomous with a generically under-twisted fixed global maximum at P and a generically under-twisted fixed global minimum at Q. If there are no nonconstant contractible 1-periodic orbits of $\phi_H^{t\in[0,1]}$ with action outside the interval $[\mathcal{A}_H(Q), \mathcal{A}_H(P)]$, then ϕ_H^t is length minimizing over the time interval* $[0,1]$.

In fact, Theorem 1.5 will be shown to be a consequence of the following more general result.

DEFINITION 1.6. — *We say that $H \in \mathcal{H}$ dominates $K \in \mathcal{H}$ at P if $H(t,x) \geqslant K(t,x)$ for all t,x with equality at $x = P$ for all t.*

THEOREM 1.7. — *Let $H \in \mathcal{H}$ have an under-twisted fixed global maximum at P. Assume that H dominates some Hamiltonian K at P, such that P is a generically under-twisted fixed global maximum of K and $\phi_K^{t\in[0,1]}$ has no contractible 1-periodic orbits with action greater than $\mathcal{A}_K(P) = \mathcal{A}_H(P)$. Then the path $\phi_H^{t\in[0,1]}$ generated by H minimizes the positive Hofer length.*

Remark 1.8. — A similar result also holds for the negative Hofer length of a Hamiltonian H with an under-twisted fixed global minimum at Q (see Corollary 6.1).

Note that our criterion depends only on the contractible periodic orbits *with period one*, not on periodic orbits with intermediate periods. In the autonomous case, we get:

COROLLARY 1.9. — *Let $H: M \to \mathbb{R}$ have a nondegenerate global maximum P and minimum Q which are under-twisted. If there are no nonconstant contractible 1-periodic orbits of $\phi_H^{t\in[0,1]}$ with action outside the interval $[H(Q), H(P)]$, then the path $\phi_H^{t\in[0,1]}$ is length minimizing.*

Of course, if there are no nonconstant contractible 1-periodic orbits of $\phi_H^{t\in[0,1]}$, then it is *a fortiori* length minimizing.

The papers [LM2], [En], [McSl] and [Oh3] all include theorems similar to Corollary 1.9, but which hold for more general symplectic manifolds.[2] However, they all require that there be no nonconstant contractible periodic orbits with *period less than or equal* to 1.

Theorem 1.5 also applies to quasi-autonomous Hamiltonians and allows for the existence of nonconstant contractible periodic orbits. This is significant because the existence of such orbits with all periods is a generic property for quasi-autonomous Hamiltonians. More precisely, if H is suitably generic, then the Arnold conjecture implies that there are at least $SB(M)$ contractible T-periodic orbits for each $T \in [0,1]$. Here $SB(M)$ denotes the sum of Betti numbers of M. Hence, even when H is quasi-autonomous, there are, for most manifolds, contractible T-periodic orbits of ϕ_H^T other than the fixed global maximum and minimum of H. To assume that these other orbits are also constant for all $T \in (0,1]$, is clearly quite restrictive.

As an application of Theorem 1.5, we will construct new examples of autonomous Hamiltonian flows which are length minimizing for all times. These constructions are based on the rich geometry of coisotropic submanifolds.

Theorem 1.7 also yields a proof of the following result for the case of symplectically aspherical manifolds.

THEOREM 1.10 ([Mc]). — *Let $H \in \mathcal{H}$ be quasi-autonomous. Then ϕ_H^t is length minimizing over $[0,\epsilon]$ for sufficiently small ϵ.*

In [LM2], Lalonde and McDuff prove this result for any symplectic manifold but for the weaker notion of being length minimizing in the homotopy class of paths with fixed endpoints. Recently, McDuff in [Mc] has extended this to work with no assumption on the homotopy class of paths. We include the proof of the more restrictive case above as a Floer theoretic interpretation of this fact.

The proof of Theorem 1.7 relies on methods from Floer theory which were introduced to the study of Hamiltonian paths by Schwarz in [Sc] and

[2] Similar results were first obtained, for $\mathbb{R}^{2n}$, by Hofer in [Ho2] and Bialy-Polterovich in [BP].

Polterovich in [Po] [3]. In [Oh3], Oh also applies these methods, in a more general context, to study Hamiltonian paths.

Let us recall the basic idea as described in [Po](Chapter 13). Given a Hamiltonian H which is suitably generic and has an under-twisted fixed global maximum at P, one considers the role of P in the Floer complex of H. If one can show that P is "homologically essential" for the Floer complex, then certain perturbed pseudo-holomorphic cylinders must exist. An estimate for the energy of these cylinders then implies that the path generated by H minimizes the positive Hofer length. This strategy was used in [Oh3], [Po], [Sc] to prove length minimizing properties of paths generated by autonomous and quasi-autonomous Hamiltonians whose flows had no nonconstant contractible periodic orbits.

In this paper, we simply refine this argument, in the simplest case of symplectically aspherical manifolds, in order to determine a length-minimizing criteria for Hamiltonian paths which allows for the existence of nonconstant contractible closed orbits. No new methods are introduced here. Formally, the only new ingredient is that the notion of an element in the Floer complex "being homologically essential" is generalized to the notion of it being *homologically essential with respect to a filtration*: See the Definition 4.3. This generalization is needed only to pass from Theorem 1.5 to Theorem 1.7, *i.e.*, to prove length minimizing properties for all Hamiltonians H that dominate some other Hamiltonian K satisfying the conditions in Theorem 1.5. As pointed out by Oh, this notion is already present in his earlier article [Oh3], as an important step in his Lemma 7.8 (Non-pushing down Lemma). However, to the knowledge of the authors, it does not seem to be used there for the same purpose.

Finally, we note that the results presented here yield a description of the changes that must occur in the Floer chain complex in order for a generic quasi-autonomous Hamiltonian to stop generating a length minimizing path. Let $H \in \mathcal{H}$ be such a Hamiltonian with an under-twisted fixed global maximum at P and minimum at Q. Assume further that the contractible closed orbits of ϕ_H^t are nondegenerate for all $t \in (0,1]$. We then have the following generic picture described by Floer in [12]. There is a finite set

$$\{t_0 = 0, t_1, \dots, t_{k-1}, t_k = 1\} \subset [0,1]$$

[3] This is closely related to a suggestion made by Viterbo in Remark 1.5B of Bialy-Polterovich [BP]. Later, Oh extended these ideas to the action functional in [Oh1], [Oh2].

such that for $t \in (t_j, t_{j+1})$ the number of contractible closed orbits of ϕ_H^t remains constant, as do the Conley-Zehnder indices of the orbits. As t increases through each t_j, a pair of closed orbits of $\phi_H^{t_j}$, $\{x_j^-, x_j^+\}$, is either created or destroyed. The Conley-Zehnder index of x_j^+ is one greater than that of x_j^- and the action of x_j^+ is greater than the action of x_j^- .

In this picture, the path ϕ_H^t is length minimizing at least until the first t_j at which a pair $\{x_j^-, x_j^+\}$ is created for which *either* the Conley-Zehnder index of x_j^+ is one greater than that of P *or* the Conley-Zehnder index of x_j^- is one less than that of Q. This can only happen at t_j with $j > 0$. Moreover, in the first case it is also necessary that the action of x_j^+ be greater than that of P. Similarly, in the second case, the path is length minimizing until x_j^- has action less than Q.

Remark 1.11. — If the path becomes non-minimal at some time, then it obviously remains non-minimal for all subsequent times. Hence Theorem 1.5 implies when a path becomes non-minimal at some time t', such t-closed orbits must exist for all times $t \geqslant t'$ as long as both global extrema remain generically under-twisted. Since one is free to extend such a path by *any* Hamiltonian that keeps the minimum and maximum under-twisted, this is a surprising result. It is a consequence of the fact that our main theorem is stated in terms of the non-existence of closed orbits of period one only.

What is described in this note extends to more general classes of symplectic manifolds — it will be developed further in the sequel [KL2]. Geometric extensions of these ideas will also be studied in the paper [La].

1.1. Organization of the paper.

In the next section, we recall the construction and properties of the Floer complex of a generic Hamiltonian. Following Schwarz, we describe in Section 3, how to identify the Floer complexes of normalized Hamiltonians which generate the same time-1 map. This identification is then used to prove the existence of a special cycle which represents the fundamental class in the Floer homology of a fixed normalized Hamiltonian. The notion of being "homologically essential with respect to a filtration", which is equivalent to Oh's Non-pushing down Lemma, is given in Section 4. The proof of Theorem 1.7 is contained in Section 5. This result is then used in Section 6 to prove Theorem 1.5 and Theorem 1.10. In Section 7, we construct many new examples of autonomous functions which generate Hamiltonian paths that are length minimizing for all time. Finally, we prove a technical claim (Claim 5.1) in the Appendix.

2. Floer homology for symplectically aspherical symplectic manifolds.

Let $\mathcal{H}_{\mathrm{reg}} \subset C^\infty(S^1 \times M, \mathbb{R})$ be the subset of Hamiltonians which are periodic in t and whose contractible 1-periodic orbits are nondegenerate. In this section, we briefly recall the construction of the Floer chain complex associated to each $G \in \mathcal{H}_{\mathrm{reg}}$ when the symplectic manifold (M, ω) is symplectically aspherical. In particular, we focus on the algebraic aspects of these complexes that will be used later. The reader is referred to the sources [Fl1], [Fl2], [Fl3] for the full details.

2.1. The Floer chain complex.

A symplectic manifold (M, ω) is said to be symplectically aspherical if

$$\omega|_{\pi_2(M)} = 0 \quad \text{and} \quad c_1|_{\pi_2(M)} = 0.$$

Let $\mathcal{L}M$ be the space of contractible loops in M and consider the action functional $\mathcal{A}_G \colon \mathcal{L}M \to \mathbb{R}$ defined by

$$\mathcal{A}_G(x(t)) = \int_0^1 G(t, x(t))\, dt - \int_{D^2} \bar{x}^* \omega, \tag{2}$$

where $\bar{x} \colon D^2 \to M$ satisfies $\bar{x}|_{\partial D^2} = x$. The first condition to be symplectically aspherical implies that the action functional is well-defined.

The critical point set of $\mathcal{A}_G$ is equal to the (finite) set of contractible 1-periodic orbits of X_G, *i.e.*,

$$\mathrm{Crit}(\mathcal{A}_G) = \{x \in \mathcal{L}M \mid \dot{x}(t) = X_G(x(t))\}.$$

The assumption that $c_1|_{\pi_2(M)} = 0$ implies that $\mathrm{Crit}(\mathcal{A}_G)$ is graded by the Conley-Zehnder index, μ_G.

Here, we normalize μ_G as in [Se], so that $\mu_G(x)$ is equal to the Morse index of x whenever x is a critical point of a C^2-small time-independent Hamiltonian. For an under-twisted fixed maximum P of G, $\mu_G(P) = 2n$.

The set $\mathrm{Crit}(\mathcal{A}_G)$ forms the basis of the Floer complex

$$CF_*(G) = \bigoplus_{k\in\mathbb{Z}} CF_k(G)$$

where

$$CF_k(G) = \bigoplus_{x\in\mathrm{Crit}(\mathcal{A}_G),\, \mu_G(x)=k} \mathbb{Z}_2 x.$$

We denote the obvious $\mathbb{Z}_2$-valued pairing for this vector space by $\langle\ ,\ \rangle$. The action of a chain $c \in CF(G)$ will be denoted by $\mathcal{A}_G(c)$ and defined as the maximum of the $\mathcal{A}_G$-values of the elements of $\mathrm{Crit}(\mathcal{A}_G)$ which appear in c with a nonzero coefficient.

To construct the Floer boundary operator $\partial^G_{J_t}$ we first choose a family, $J_t = J_{t+1}$, of ω-compatible almost complex structures. For each pair $x, y \in \mathrm{Crit}(\mathcal{A}_G)$, let $\mathcal{M}(x,y) = \mathcal{M}_{J_t}(x,y)$ be the space of solutions $u\colon \mathbb{R}\times S^1 \to M$ of the equation

$$\partial_s u + J_t(u)(\partial_t u - X_G(u)) = 0$$

which satisfy the boundary conditions

$$\lim_{s\to-\infty} u(s,t) = x \quad\text{and}\quad \lim_{s\to+\infty} u(s,t) = y,$$

and have finite energy

$$E(u) = \int_0^1 \int_{-\infty}^{+\infty} |\partial_s u|^2 \, ds\, dt.$$

If J_t is suitably generic, then $\mathcal{M}_{J_t}(x,y)$ is a smooth manifold of dimension $\mu_G(x) - \mu_G(y)$. We denote the set of these generic ω-compatible almost complex structures by $\mathcal{J}_{\mathrm{reg}}(G)$ and note that it is of second category in the set of all ω-compatible almost complex structures.

The boundary map $\partial^G_{J_t}\colon CF_k(G) \to CF_{k-1}(G)$ is now defined by

$$\partial^G_{J_t} x = \sum_{\mu_G(x)-\mu_G(y)=1} \tau(x,y)y,$$

where $\tau(x,y)$ is the number (mod(2)) of elements in $\mathcal{M}(x,y)/\mathbb{R}$, and $\mathbb{R}$ acts like $a\cdot u(s,t) = u(s+a,t)$.

For $J_t \in \mathcal{J}_{\mathrm{reg}}(G)$, the boundary operator satisfies

$$\partial^G_{J_t} \circ \partial^G_{J_t} = 0.$$

The homology of the complex $(CF_*(G), \partial^G_{J_t})$ does not depend on the choice of $J_t \in \mathcal{J}_{\mathrm{reg}}(G)$ and so we write it as $HF_*(G)$, the Floer homology of G.

2.2. Homotopy chain maps and the action functional.

Let G_s be a smooth homotopy $\mathbb{R} \to C^\infty([0,1] \times M, \mathbb{R})$ such that

$$G_s = \begin{cases} G_0 & \text{for } s < -\tau \\ G_1 & \text{for } s > \tau, \end{cases}$$

for some $\tau > 0$ and $G_0, G_1 \in \mathcal{H}_{\text{reg}}$. Associated to each such homotopy is a homotopy chain map

$$\sigma_{G_s}: CF_*(G_0) \to CF_*(G_1).$$

To construct this map we first choose a homotopy of families of ω-compatible almost complex structures, $s \mapsto J_{s,t}$, such that

$$J_{s,t} = \begin{cases} J_t^0 \in \mathcal{J}_{\text{reg}}(G_0) & \text{for } s < -\tau \\ J_t^1 \in \mathcal{J}_{\text{reg}}(G_1) & \text{for } s > \tau. \end{cases}$$

Then we consider the space $\mathcal{M}^s(x,y) = \mathcal{M}^s_{J_{s,t}}(x,y)$ of finite energy solutions of the equation

$$\partial_s u + J_{s,t}(u)(\partial_t u - X_{G_{s,t}}(u)) = 0$$

which satisfy

$$\lim_{s \to -\infty} u(s,t) = x \in \text{Crit}(\mathcal{A}_{G_0}) \text{ and } \lim_{s \to +\infty} u(s,t) = y \in \text{Crit}(\mathcal{A}_{G_1}).$$

For a suitably generic $J_{s,t}$, the space $\mathcal{M}^s_{J_{s,t}}(x,y)$ is a smooth manifold of dimension $\mu_{G_1}(y) - \mu_{G_0}(x)$. The map σ_{G_s} is then defined by

$$x \mapsto \sum_{\mu_{G_0}(x) = \mu_{G_1}(y)} \tau_s(x,y) y,$$

where $\tau_s(x,y)$ is the number (mod(2)) of maps u in the zero-dimensional compact manifold $\mathcal{M}^s(x,y)$.

The next simple lemma describes how the action changes under a homotopy chain map.

LEMMA 2.1. — *Every $u \in \mathcal{M}^s(x,y)$ has energy*

$$E(u) = \mathcal{A}_{G_0}(x) - \mathcal{A}_{G_1}(y) + \int_0^1 \int_{-\infty}^{+\infty} \partial_s G(s,t,u(s,t))\, ds\, dt. \tag{3}$$

In the case of a linear homotopy, $G_s(x,t) = (1 - \beta(s))G_0 + \beta(s)G_1$, where β is a smooth nondecreasing function from 0 to 1, equation (3) takes the following simple form which will be used extensively later on:

$$E(u) = \mathcal{A}_{G_0}(x) - \mathcal{A}_{G_1}(y) + \int_0^1 \int_{-\infty}^{+\infty} \dot{\beta}(s)(G_1 - G_0)(t,u(s,t))\, ds\, dt. \tag{4}$$

Note that if $x \neq y$, then $E(u) > 0$.

2.3. Homotopy chain maps and $HF_*(G)$.

Every homotopy chain map induces an isomorphism in Floer homology and so $HF_*(G)$ is independent of G. Moreover, when G is a C^2-small Morse function, the Floer complex of G (for a generic time-independent J) coincides with the Morse complex of G (see [HS]). Hence, for all $G \in \mathcal{H}_{\mathrm{reg}}$, the Floer homology $HF_*(G)$ is isomorphic to the singular homology $H_*(M, \mathbb{Z}_2)$ via a map which preserves the grading.

3. Comparing Floer complexes of normalized Hamiltonians generating the same time-1 map.

Given $\phi \in \mathrm{Ham}(M, \omega)$, let $\mathcal{F}(\phi)$ be the set of normalized Hamiltonians in $\mathcal{H}_{\mathrm{reg}}$ whose time-1 map equals ϕ. For each pair of functions $F, G \in \mathcal{F}(\phi)$, there is a natural chain isomorphism between the Floer complexes of F and G. These maps were studied by Seidel in [Se] for a very general class of symplectic manifolds and by Schwarz in [Sc] for the symplectically aspherical case. Let us recall the relevant details.

For $F, G \in \mathcal{F}(\phi)$, consider the loop $g = \phi_G^t \circ (\phi_F^t)^{-1}$. This loop represents an element in $\pi_1(\mathrm{Ham}(M, \omega), \mathrm{id})$ and acts on $\mathcal{L}M$ by

$$g(z(t)) = g_t(z(t)).$$

It is a straightforward consequence of the Arnold conjecture that the orbit of g on any point $x \in M$ is a contractible loop; thus g takes $\mathcal{L}M$ back to itself. Clearly, g takes the 1-periodic orbits of X_F to the 1-periodic orbits of X_G. Since $\mathrm{Crit}(\mathcal{A}_F)$ and $\mathrm{Crit}(\mathcal{A}_G)$ are bases for the respective complexes, the induced map $g: CF(F) \to CF(G)$ is an isomorphism of $\mathbb{Z}_2$-vector spaces.

If we set

$$J_G = Dg_t^{-1} \circ J_F \circ Dg_t,$$

then the moduli space $\mathcal{M}_{J_F}^F(x, y)$ gets mapped bijectively, by g, to $\mathcal{M}_{J_G}^G(g(x), g(y))$ for every $x, y \in \mathrm{Crit}(\mathcal{A}_F)$, (see [Se], Lemma 4.3). Thus, g is also a chain complex isomorphism from $(CF(F), \partial_{J_F}^F)$ to $(CF(G), \partial_{J_G}^G)$.

Moreover, for symplectically aspherical manifolds, the chain complex isomorphism g preserves both the grading by the Conley-Zehnder index and the filtration by the action functional,(see [Sc], Theorem 1.1).

Let $G \in \mathcal{H}_{\text{reg}}$ be a normalized Hamiltonian. The fact that the Floer complex of any other function $F \in \mathcal{F}(\phi_G^1)$ is identical to $(CF(G), \partial_{J_G}^G)$ can be used to prove the following result.

LEMMA 3.1. — *For any $F \in \mathcal{F}(\phi_G^1)$ there is a cycle $v \in CF_{2n}(G)$ which generates $HF_{2n}(G)$ and satisfies $\mathcal{A}_G(v) \leqslant \|F\|^+$.*

Proof. — Let f be a Morse function. For a sufficiently small $\epsilon > 0$ and a suitably generic time-independent almost complex structure, the Floer chain complex of ϵf is well-defined and equal to the Morse complex of ϵf. In particular, the cycle $\sum_i p_{\max}^i$, given by the sum of the local maxima of ϵf, generates $HF_{2n}(\epsilon f)$.

The linear homotopy between ϵf and F yields the homotopy chain map

$$\sigma\colon CF(\epsilon f) \to CF(F).$$

Hence, the cycle $v^F = \sigma(\sum_i p_i^{\max})$ represents the fundamental class in $HF_{2n}(F)$.

Let $v_i^F \in \operatorname{Crit}(\mathcal{A}_F)$ appear in v^F with nonzero coefficient. Then, for some j, there must be an element $u \in \mathcal{M}^s(p_j^{\max}, v_i^F)$. Equation (4) then yields the inequality

$$\mathcal{A}_F(v_i^F) \leqslant \mathcal{A}_{\epsilon f}(p_j^{\max}) + \int_0^1 \max_M \{F(x,t) - \epsilon f(x)\}\, dt.$$

For sufficiently small $\epsilon > 0$ this implies that $\mathcal{A}_F(v_i^F) \leqslant \|F\|^+$. Therefore, $\mathcal{A}_F(v^F) \leqslant \|F\|^+$.

By the discussion above, the chain isomorphism from $(CF(F), \partial_{J_F}^F)$ to $(CF(G), \partial_{J_G}^G)$, induced by the loop $g = \phi_G^t \circ (\phi_F^t)^{-1}$, preserves indices and actions. Hence, the cycle $v = g(v^F)$ also generates $HF_{2n}(G)$ and satisfies $\mathcal{A}_G(v) \leqslant \|F\|^+$. □

4. Homologically essential with respect to a filtration.

Let (C, ∂) be a differential complex over $\mathbb{Z}_2$ which is freely generated by a basis $\mathcal{B}$.

DEFINITION 4.1. — *Let $\gamma \in H_*(C, \partial)$ be a nontrivial homology class. An element $b \in \mathcal{B}$ is said to be homologically essential for the class γ if it appears with nonvanishing coefficient in every representative of γ.*

In this section we refine this notion in the presence of the following additional structure.

DEFINITION 4.2. — *A filtration on* (C, ∂) *is a map* $\mathcal{V}: \mathcal{B} \to \mathbb{R}$ *such that*

$$\mathcal{V}(b) > \mathcal{V}(b')$$

whenever $b, b' \in \mathcal{B}$ *satisfy* $\langle b', \partial b \rangle \neq 0$. *For an element* $c \in C$, *we define* $\mathcal{V}(c)$ *to be the maximum of the values of* $\mathcal{V}$ *over the elements of* $\mathcal{B}$ *that appear with non-vanishing coefficient in* c.

A Morse function together with its Morse complex, and an action functional together with its Floer complex, are obvious examples of filtrations.

DEFINITION 4.3. — *Let* (C, ∂) *be a complex with filtration* $\mathcal{V}$, *and let* $\gamma \in H_*(C, \partial)$ *be a nontrivial homology class. An element* $b \in \mathcal{B}$ *is said to be homologically essential for* γ *with respect to* $\mathcal{V}$ *if the following two conditions are satisfied:*

1) *there is an element representing* γ *of the form* $b + v$ *such that* $\langle b, v \rangle = 0$ *and* $\mathcal{V}(v) < \mathcal{V}(b)$;

2) *whenever* $\langle b, \partial d \rangle \neq 0$, *then* $\mathcal{V}(\partial d - b) \geqslant \mathcal{V}(b)$.

To better understand the origin of these chain-level conditions, consider a Hamiltonian $H \in \mathcal{H}_{\rm reg}$ with an under-twisted fixed global maximum at P. We will show (in Proposition 5.2) that, under a generic nondegeneracy assumption, P always satisfies the first condition to be homologically essential for the generator of $HF_{2n}(H)$ with respect to $\mathcal{A}_H$. This will follow from the fact that H can be dominated at P by a function G^H for which P satisfies the first condition to be essential for the generator of $HF_{2n}(G^H)$ with respect to $\mathcal{A}_{G^H}$.

On the other hand, assume that H dominates a function $G_H \in \mathcal{H}_{\rm reg}$ for which P is under-twisted and not in the image of the boundary operator ∂^{G_H} (e.g. G_H has no 1-periodic orbits with $\mathcal{A}_{G_H}$-value greater than $\mathcal{A}_{G_H}(P)$). We will also show that this implies that P satisfies the second criteria of being homologically essential for $HF_{2n}(H)$ with respect to $\mathcal{A}_H$ (see Proposition 5.3).

Note that if H satisfies the assumptions of G_H itself, then P is homologically essential for $HF_{2n}(H)$ in the sense of Definition 4.1.

The next proposition is the main result of this section.

PROPOSITION 4.4. — *Let $G \in \mathcal{H}_{\text{reg}}$ be a normalized Hamiltonian with an under-twisted fixed global maximum at P. If P is homologically essential for the unique generator of $HF_{2n}(G)$ with respect to $\mathcal{A}_G$, then the path generated by G minimizes the positive Hofer length, i.e.,*

$$\|G\|^+ \leqslant \|F\|^+ \quad \text{for all} \quad F \in \mathcal{F}(\phi_G^1).$$

Proof. — By our hypothesis on P, there is a chain $w \in CF_{2n}(G)$ such that $[P + w]$ generates $HF_{2n}(G)$ and

$$\mathcal{A}_G(w) < \mathcal{A}_G(P).$$

If there is an $F \in \mathcal{F}(\phi_G^1)$ such that $\|F\|^+ < \|G\|^+$, then by Lemma 3.1 there is a cycle v that generates $HF_{2n}(G)$ and satisfies

$$\mathcal{A}_G(v) \leqslant \|F\|^+ < \|G\|^+ = \mathcal{A}_G(P).$$

The two cycles $P + w$ and v represent the same class, therefore $P + (w - v)$ is exact. The second criteria for P to be homologically essential for this class with respect to $\mathcal{A}_G$ then implies that $\mathcal{A}_G(w - v)$ must be larger than or equal to $\mathcal{A}_G(P)$. This is a contradiction. □

5. Proof of Theorem 1.7.

In proving Theorem 1.7, we may assume that the functions H and K have some additional properties.

First, we may assume that H is normalized. This is clear, since the positive Hofer length of the path ϕ_H^t is independent of the normalization of H. (When H is normalized we just know that the positive Hofer length of ϕ_H^t equals $\|H\|^+$.)

We may also assume that H and K satisfy certain generic nondegeneracy assumptions. Here is the precise statement.

CLAIM 5.1. — *It suffices to prove Theorem 1.7 for H and K with the following additional properties:*

1. $H, K \in \mathcal{H}_{\text{reg}}$,

2. *For all $t \in [0, 1]$, P is the unique global maximum of both $H(t, x)$ and $K(t, x)$, and is nondegenerate (as a critical point).*

The proof of this claim is technical and more or less standard. It is therefore deferred to an appendix.

Since we can assume that H and K are in $\mathcal{H}_{\text{reg}}$, we can consider their Floer complexes. By Proposition 4.4, we will then be done if we can show that P is homologically essential for the generator of $HF_{2n}(H)$ with respect to $\mathcal{A}_H$.

Let's begin with the first condition to be homologically essential.

PROPOSITION 5.2. — *Let $H \in \mathcal{H}_{\text{reg}}$ have a fixed under-twisted global maximum at P such that P is the unique global maximum of $H(t,x)$ and is nondegenerate, for all $t \in [1,0]$. Then P satisfies the first condition to be homologically essential for the unique generating class of $HF_{2n}(H)$ with respect to the filtration $\mathcal{A}_H$. That is to say: $HF_{2n}(H)$ is generated by $[P+w]$ for some $w \in CF_{2n}(H)$ with $\langle w, P\rangle = 0$ and $\mathcal{A}_H(w) < \mathcal{A}_H(P)$.*

Proof. — First we construct a function G^H which dominates H at P. Let $f_P\colon M \to \mathbb{R}$ be an autonomous Morse function for which P is the only critical point with Morse index $2n$ and $f_P(P) = 0$. For sufficiently small $\epsilon > 0$ the function

$$G^H(t,x) = H(t,P) + \epsilon f_P(x)$$

satisfies $G^H(t,x) \geqslant H(t,x)$, with equality only at P. This follows from the uniqueness and nondegeneracy conditions on P. Note that P is also an under-twisted fixed global maximum of G^H and $\mathcal{A}_{G^H}(P) = \mathcal{A}_H(P)$.

The Floer complex of G^H is equal to the Floer complex of $\epsilon f_P(x)$ with the actions shifted upward by $\int_0^1 H(t,P)\,dt$. For sufficiently small ϵ, and a suitably generic time-independent almost complex structure, the Floer complex of ϵf_P coincides with its Morse complex. By our choice of the function f_P, this means that $HF_{2n}(G^H)$ is generated by the class $[P]$.

The linear homotopy from G^H to H yields the homotopy chain map

$$\sigma_{G^H}\colon CF_*(G^H) \to CF_*(H).$$

Since this induces an isomorphism in homology, the class $[\sigma_{G^H}(P)]$ generates $HF_{2n}(H)$.

Now, P is an under-twisted fixed maximum of H, so $\mu_H(P) = 2n$. Consider an element $u \in \mathcal{M}^s(P,P)$. For each such u, equation (4) yields the inequality

$$E(u) = \mathcal{A}_{G^H}(P) - \mathcal{A}_H(P) + \int_0^1 \int_{-\infty}^{+\infty} \dot\beta(s)(H - G^H)(t,u(s,t))\,ds\,dt$$

which implies that $E(u) \leqslant 0$. Thus, $\mathcal{M}^s(P,P)$ consists of just the constant map, $u(s,t) \mapsto P$, and

$$\sigma_{G^H}(P) = P + \sum_{i=1}^{k} m_{y_i} y_i = P + w.$$

Here, $\{P, y_1, \ldots, y_k\}$ are the 1-periodic orbits of H with $\mu_H = 2n$.

To finish the proof we must show that any y_j which appears in w with nonzero coefficient, must satisfy $\mathcal{A}_H(y_j) < \mathcal{A}_H(P)$. For each element $u \in \mathcal{M}^s(P, y_j)$, equation (4) yields

$$0 < \mathcal{A}_{G^H}(P) - \mathcal{A}_H(y_j) + \int_0^1 \int_{-\infty}^{\infty} \dot{\beta}(s)(H - G^H)(t, u(s,t))\, ds\, dt.$$

Hence, $\mathcal{A}_H(y_j) < \mathcal{A}_{G^H}(P) = \mathcal{A}_H(P)$. □

Finally, we use the function K to prove that P satisfies the second condition to be homologically essential for the generator of $HF_{2n}(H)$ with respect to $\mathcal{A}_H$.

PROPOSITION 5.3. — *If $\partial^H a = P + z$ for some $a \in CF_{2n+1}(H)$ and $z \in CF_{2n}(H)$, then $\mathcal{A}_H(z) \geqslant \mathcal{A}_H(P)$.*

Proof. — Let $\partial^H a = P + z$ be given. Consider the linear homotopy from H to K. The induced homotopy chain map σ satisfies

$$\sigma(\partial^H a) = \partial^K \sigma(a).$$

Thus, we have

$$\sigma(P) + \sigma(z) = \partial^K \sigma(a).$$

Since K has no periodic orbits with action larger than $\mathcal{A}_K(P)$, P occurs with coefficient zero in $\partial^K \sigma(a)$. It is also straightforward to show that P occurs in $\sigma(P)$ with coefficient 1 (*i.e.*, $\mathcal{M}^s(P,P)$ contains only the constant map). This means that P must occur in $\sigma(z)$ with coefficient 1. In other words, if $z = \sum_i b_i z_i$, then for some z_i there exists $u \in \mathcal{M}^s(z_i, P)$. Again by (4), this implies that

$$0 < \mathcal{A}_H(z_i) - \mathcal{A}_K(P) + \int_{-\infty}^{+\infty} \int_0^1 \dot{\beta}(s)(K - H)(t, u(s,t))\, dt\, ds$$

and so

$$\mathcal{A}_H(z) \geqslant \mathcal{A}_K(P) = \mathcal{A}_H(P).$$

□

6. Proofs of Theorem 1.5 and Theorem 1.10.

6.1. Proof of Theorem 1.5.

Consider a Hamiltonian $H \in \mathcal{H}$ with a fixed under-twisted global minimum at Q and let

$$\bar{H}(t,x) = -H(t, \phi_H^t(x)).$$

This Hamiltonian generates the path $(\phi_H^t)^{-1}$ and has Q as an under-twisted fixed global maximum. Note that the positive Hofer length of $(\phi_H^t)^{-1}$ is equal to the negative Hofer length of ϕ_H^t. Hence, applying Theorem 1.7 to $\bar{H}$ yields:

COROLLARY 6.1. — *If $\bar{H}$ dominates some Hamiltonian L at Q such that Q is a generically under-twisted fixed global maximum of L and the flow of L has no contractible 1-periodic orbits with action greater than $\mathcal{A}_L(Q) = -\mathcal{A}_H(Q)$, then the path generated by H minimizes the negative Hofer length.*

Theorem 1.5 now follows immediately: simply take H for K in Theorem 1.7 and $\bar{H}$ for L in Corollary 6.1.

6.2. Proof of Theorem 1.10.

The flow of $H(t,x)$ is length minimizing for $t \in [0,\epsilon]$ if and only if the flow of $\epsilon H(\epsilon t, x)$ is length minimizing for $t \in [0,1]$. Let P and Q be the fixed global maximum and minimum of H, respectively. For all sufficiently small $\epsilon > 0$ these will be under-twisted fixed global extrema of $\epsilon H(\epsilon t, x)$.

Let $f_P\colon M \to \mathbb{R}$ be a Morse function for which P is the only critical point with Morse index $2n$ and $f_P(P) = 0$. We also assume that f_P is sufficiently C^2-small so that the only 1-periodic orbits of X_{f_P} are the critical points of f_P.

Set

$$K = \epsilon H(\epsilon t, P) + f_P.$$

Clearly, P is a generically under-twisted fixed global maximum of K, and $\epsilon H(\epsilon t, x)$ dominates K at P for sufficiently small ϵ. Also, the only 1-periodic orbits for the flow of K are the critical points of f_P, so there are also no 1-periodic orbits with action greater than $\mathcal{A}_K(P)$. Theorem 1.7 then implies that the path generated by $\epsilon H(\epsilon t, x)$ minimizes the positive Hofer length.

A similar argument applied to $\epsilon \bar{H}(\epsilon t, x)$ implies that the path generated by $\epsilon H(\epsilon t, x)$ also minimizes the negative Hofer length.

7. Coisotropic manifolds and Hamiltonians generating length minimizing paths for all times.

Let N be a closed manifold endowed with a closed 2-form ω_N of constant rank. Its kernel is then an integrable distribution that gives rise to a foliation $\mathcal{F}$ on N. Up to local diffeomorphisms, there is a unique way of considering (N, ω_N) as a coisotropic submanifold. Indeed, let $\rho: E \to N$ be the vector bundle whose fiber at each point $q \in N$ is the dual K_q^* of the kernel K_q of ω_N at q. Choose a distribution $\mathcal{H}$ on N transversal to $\mathcal{F}$, and define a form on each $T_{Z(q)}E, q \in N$, by

$$(d\rho)^*(\omega_N) + \Pi^*(\omega_{\text{can}})$$

where $Z: N \to E$ is the zero section and the map $\Pi = (\Pi_1, \Pi_2): T_{Z(p)}E \to K \oplus K^*$ is the projection induced by $\mathcal{H}$ (*i.e.*, Π_2 is the projection on the fiber of E and Π_1 is the composition of the $d\rho$ with the projection induced by $\mathcal{H}$). This is a fibrewise symplectic form on the vector bundle $TE|_N$ whose restriction to TN is ω_N. By the Moser-Weinstein theorem, there is an extension of that form to a symplectic form ω on some neighborhood $\mathcal{U}$ of N in E, which is unique up to diffeomorphisms on smaller neighborhoods whose 1-jet act as the identity on $TE|_N$. The submanifold N is then coisotropic in $\mathcal{U}$.

Let us now consider a special case of this construction. Let W be a closed manifold that admits a metric g of non-positive curvature, and let

$$W \hookrightarrow N \xrightarrow{\pi} B$$

be a smooth fibration with structure group $\mathrm{Diff}_g(W)$, the group of g-isometries of W. Assume also that the transition maps $\phi_{i,j}: V_i \cap V_j \to \mathrm{Diff}_g(W)$ are locally constant. Thus each fiber W_b is equipped with a metric g_b of non-positive curvature. As before, $\omega_N \in \Omega^2(N)$ is any closed 2-form whose kernel at each point is the tangent space to the W-fiber at that point. By differentiating the transition functions of the bundle $W \hookrightarrow N \to B$, it can be extended to a T^*W-bundle

$$T^*W \hookrightarrow N' = E \xrightarrow{\pi'} B.$$

There is a symplectic form $d\lambda_b$ on each fiber $T^*(W_b)$ that extends to a closed form τ on E. Since the transition functions are locally constant, one can choose τ such that it coincides with $d\lambda$ in each local chart $V_i \times T^*W$ (where V_i is an open subset of B). The form $\rho^*(\omega_N)+\tau$, is then symplectic on E, where ρ is now the projection $N' \to N$. This form also restricts to ω_N on N.

Since the structure group preserves the metric, one may define a Hamiltonian

$$H: \mathcal{U} \to \mathbb{R}$$

given by $H(b,p,q) = f(\|p\|^2)$ in local coordinates, where f is the identity map near 0 and becomes constant for values larger than some $\epsilon > 0$. Extend H outside $\mathcal{U}$ by the constant map. Since in local coordinates $V_i \times T^*W$, the symplectic structure is $\rho^*(\omega_N) + d\lambda$, where $\rho^*(\omega_N)$ has each T^*W-fiber as a kernel, the flow of H is the geodesic flow along each leaf of the isotropic foliation of the coisotropic submanifold N. Because the metric has non-positive curvature, there is no non-constant contractible closed orbit. Since this Hamiltonian can be approximated by an autonomous Hamiltonian that is generically under-twisted at its fixed maximum and minimum and has no non-trivial periodic orbit, we then have, as a consequence of Theorem 1.5 and Lemma 8.1 of Section 8:

COROLLARY 7.1. — *Let (M,ω) be an aspherical symplectic manifold that contains a coisotropic submanifold of the above form. If the fundamental group of each leaf injects in the fundamental group of M, the flow of H is length minimizing for all times.*

This result is very likely valid in non-aspherical manifolds as well, though we will not present a general proof here. This corollary admits the following two extreme cases.

Examples.

(1) The submanifold N is a Lagrangian submanifold. In this case, we recover the constructions due to Schwarz in [Sc]. It is interesting to note that this can also be derived in the Lagrangian case from previous results obtained by very different techniques that do not involve Floer's homology. Actually, if $L \subset M$ is a Lagrangian submanifold, the hypotheses that L admits a metric with non-positive curvature and that its fundamental group injects into the fundamental group of M means that the embedding of L lifts to an embedding of the contractible space $\widetilde{L}$ into $\widetilde{M}$. By Theorem 1.4.A

in Lalonde-Polterovich [LP], this implies that any neighborhood of $\widetilde{L}$ has infinite capacity (there exist embedded balls of arbitrarily large capacity) — this was incidentally used in [LP] to prove that no Hamiltonian isotopy can disjoin L from itself. Using the same techniques as in Lalonde-McDuff [LM2], Lemma 5.7, one constructs an autonomous Hamiltonian flow ϕ_t whose lift to the universal cover disjoins balls B_t that have capacity equal to the energy of ϕ_t. This implies that ϕ_t is length minimizing for all times by the energy-capacity inequality. Note that the aspherical condition is not needed here.

(2) The submanifold N is a hypersurface. This is a simple case since any 1-dimensional manifold admits a flat metric. Note that our construction of coisotropic submanifolds implies both the stability of the coisotropic submanifold and the non-existence of closed contractible orbits. This is what we must require here, *i.e.*, the right statement is the following: the existence of a length minimizing autonomous Hamiltonian flow is guaranteed as soon as there exists in (M, ω) a stable hypersurface whose closed orbits are all non-contractible (by stable, we mean here that there is a neighbourhood of N of the form $[-\epsilon, \epsilon) \times N$ in which each leaf $\{s\} \times N$ has a characteristic flow conjugated to the one on $N = \{0\} \times N$). This is the case for instance for fibered symplectic manifolds over a surface of strictly positive genus

$$M \hookrightarrow (S, \Omega) \to \Sigma$$

i.e., fibered manifolds equipped with a symplectic form Ω whose restriction to each M-fiber is non-degenerate. The inverse image $N = \pi^{-1}(\gamma)$ of any non-contractible loop in Σ yields a hypersurface whose characteristic foliation is transverse to the M-fibers. A closed orbit of such a foliation must necessarily project to a non-vanishing multiple of γ, and is therefore non-contractible. Note that the symplectic structure in a neighborhood of the hypersurface has the form $\Omega|_N + d(t\alpha)$ where t is the coordinate in the normal direction to N and $\alpha \in \Omega^1(N)$ is any 1-form that does not vanish on the characteristic directions. Choosing α as the pull-back of $d\theta \in \Omega^1(\gamma)$, the symplectic form becomes $\Omega|_N + dt \wedge d\theta$, for which N is stable as required.

(3) One may construct lots of examples in between, *i.e.*, coisotropic submanifolds of dimensions $n < d < 2n - 1$ that satisfy the conditions of our construction. The following example shows that the existence of length minimizing paths is stable under pull-backs via certain symplectic fibrations. Let

$$M \hookrightarrow (S, \Omega) \xrightarrow{\pi} (B, \sigma)$$

be a fibration with compatible symplectic forms. By this we mean that for each point $p \in S$ and 1-form $\tau \in (T_{d\pi(p)}B)^*$, the projection of the Ω-symplectic gradient of $(d\pi|_p)^*(\tau)$ is equal to the σ-symplectic gradient of τ. This is the case for instance if

$$(M,\omega) \hookrightarrow S \xrightarrow{\pi} B$$

is a fibration with structure group equal to $\mathrm{Diff}^0_\omega(M,\omega)$, the identity component of the symplectic group, such that the ω_b-forms on each fiber admit an extension to a closed 2-form τ on the total space S that has constant rank equal to $\dim M$. Let σ be a symplectic form on B and take on S the symplectic form $\Omega = \pi^*(\sigma) + \tau$ (it is indeed symplectic because the kernel of τ has constant rank and must be transversal to the fibers).

Suppose now that there is a quasi-autonomous Hamiltonian $H_{t\in[0,T]}: B \to \mathbb{R}$ whose flow has no non-constant contractible closed orbit at time T and for which there are fixed maximum and minimum where the linearization of the flow satisfies the conditions of Theorem 1.5. Then the pull-back of $H_{t\in[0,T]}$ induces a length minimizing path on S. This is obvious since, with our definition of Ω, the symplectic gradient of $H_t \circ \pi$ projects to the symplectic gradient of H_t, thus any non-constant closed T-orbit projects to a non-contractible closed orbit in B and must therefore be non-contractible too. Apply to the two fibers that represent the maximum and minimum of $H_t \circ \pi$ the same argument on the linearizations and approximate the Hamiltonian near the two fibers so that it becomes generically under-twisted. Theorem 1.5 then yields the desired conclusion.

8. Appendix: Proof of Claim 5.1.

The following preliminary result implies that "nearby" Hamiltonians may be used when one considers the length minimizing properties of a Hamiltonian path. We refer to Oh [Oh3], Lemma 5.1, for the natural optimal C^0-version stated here.

LEMMA 8.1. — *Let $\{G_i\} \subset \mathcal{H}$ be a sequence of normalized Hamiltonians such that each G_i generates a path that minimizes the positive Hofer length. If there is a normalized function G such that the following convergence conditions are satisfied:*

(1) $\phi^1_{G_i} \to \phi^1_G$ *in the C^0-norm*

(2) $\|G_i - G\| \to 0$,

then G also minimizes the positive Hofer length.

Remark 8.2. — Similar statements apply to the negative and standard Hofer lengths.

Remark 8.3. — The convergence criteria of Lemma 8.1 are satisfied if $G_i \to G$ in the C^2-topology.

Now, let $H, K \in \mathcal{H}$ satisfy the hypotheses of Theorem 1.7. Recall that this means that

(1) H has an under-twisted fixed global maximum at P,

(2) H dominates K at P,

(3) K has a generically under-twisted fixed global maximum at P,

(4) the flow of K has no contractible 1-periodic orbits with action greater than $\mathcal{A}_K(P)$.

To prove Claim 5.1, we must find a pair of functions H' and K' such that: H' and K' satisfy the hypotheses of Theorem 1.7, H' and K' have the additional properties described in Claim 5.1, and H' is arbitrarily close to H in the sense of Lemma 8.1.

To do this, we will perturb H and K several times until the resulting functions H' and K' have the properties described in Claim 5.1. Each perturbation will be small in the sense of Lemma 8.1 and preserve the conditions required by the hypotheses of Theorem 1.7. The only difficulty occurs in the verification that the flow of K' still has no contractible 1-periodic orbits with action greater than $\mathcal{A}_{K'}(P)$.

We begin by replacing H and K with the periodic Hamiltonians

$$H_\alpha = \dot{\alpha}(t)H(\alpha(t), x) \quad \text{and} \quad K_\alpha = \dot{\alpha}(t)K(\alpha(t), x)$$

where $\alpha\colon [0,1] \to [0,1]$ is a smooth function such that $\alpha(t) = 0$ near $t = 0$, $\alpha(t) = 1$ near $t = 1$, and $\alpha(t) - t$ is C^0-small. These periodic Hamiltonians are arbitrarily close to H and K in the sense of Lemma 8.1 and they still satisfy the hypotheses of Theorem 1.7. In fact, the resulting flows are just reparametrizations of the original ones by $t \to \tau(t)$. Thus, for simplicity, we just may assume that H and K are periodic to begin with.

Now, let U_P be a Darboux chart around P. Because P is a generically under-twisted fixed maximum of K, it is an isolated fixed point and we can choose U_P to be small enough so that none of the other 1-periodic orbits of ϕ_K^t enter U_P. We denote the canonical norm in this chart by $|\ |$ and the ball of radius ρ by $B_\rho(P)$.

Consider the small bump function

$$g(|(\phi_K^t)^{-1}(x)|^2),$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is a smooth nonnegative function which strictly decreases on $[0, \rho^2]$ and vanishes on $[\rho^2, \infty]$. We choose ρ to be sufficiently small so that $(\phi_K^t)^{-1}(x) \in U_P$ for all $t \in [0,1]$ and $x \in B_\rho(P)$.

Set

$$H' = H + g(|(\phi_K^t)^{-1}(x)|^2) \quad \text{and} \quad K' = K + g(|(\phi_K^t)^{-1}(x)|^2).$$

When g is sufficiently C^1-small, P is still an under-twisted fixed global extrema for these new Hamiltonians which still clearly satisfy the first three hypotheses of Theorem 1.7 and now have the second property of Claim 5.1.

We must now check that K' has no 1-periodic orbits with action greater than $\mathcal{A}_{K'}(P)$. First we prove that the addition of the bump does not create any new 1-periodic orbits. To do this, we adapt an argument of Siburg's in [Si].

The flow of K' is equal to the composed flow $\phi_K^t \circ \phi_g^t(x)$. The inside flow is $\phi_g^t(x) = R(t, |x|^2)$, where R is the rotation matrix

$$R(t,s) = e^{2g'(s)it}\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}.$$

Note that $\phi_K^t \circ \phi_g^t(x) = \phi_K^t(x)$ whenever $x \notin B_\rho(P)$. Hence, we only need to prove that there are no new periodic orbits starting at $x \in B_\rho(P)$.

Let S_r be a sphere of any radius r. We define on each S_r the normalized distance function, $d_r\colon S_r \times S_r \to \mathbb{R}^+$, which identifies S_r with the unit sphere and measures the distance between points with respect to the usual round metric.

Consider the lower semi-continuous function $\mathcal{D}_K\colon B_\rho(P) \to \mathbb{R}^+$ given by

$$\mathcal{D}_K(x) = \begin{cases} d_{|x|}(x, \phi_K^1(x)) & \text{if } |\phi_K^1(x)| = |x|, \\ 1 & \text{otherwise.} \end{cases}$$

Using the fact that the flow ϕ_K^t is close to its generically under-twisted linearized flow near P , it is straightforward to check that $\mathcal{D}_K$ is bounded away from zero on $B_\rho(P)$, for ρ sufficiently small. The flow ϕ_g^t preserves each sphere S_r for $r \in [0, \rho]$. For a sufficiently C^1-small g, we can conclude that the perturbed function $\mathcal{D}_{K'}$ is also bounded away from zero. Hence, the flow $\phi_{K'}^t = \phi_K^t \circ \phi_g^t(x)$ has no new 1-periodic orbits.

By our construction, the 1-periodic orbits of K' are identical to those of K. Furthermore, only the $\mathcal{A}_{K'}$-value of P is different than its $\mathcal{A}_K$-value and $\mathcal{A}_{K'}(P) > \mathcal{A}_K(P)$. So, not only is $\mathcal{A}_{K'}(P)$ still a maximum in $\mathrm{Crit}(\mathcal{A}_{K'})$, but it is now an *isolated* maximum.

Finally, we recall that the space $\mathcal{H}_{\mathrm{reg}}$ is of second category in $C^\infty(S^1 \times M, \mathbb{R})$. We may then perturb H' and K', *away from* P, so that the resulting functions (which we still call H' and K') are in $\mathcal{H}_{\mathrm{reg}}$. After this final perturbation, $\mathcal{A}_{K'}(P)$ is still an isolated maximum in $\mathrm{Crit}(\mathcal{A}_{K'})$. The other properties also persist.

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