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ON REDUCTION OF HILBERT-BLUMENTHAL VARIETIES

by Chia-Fu YU

Introduction.

Let \mathbf{F} be a totally real number field of degree g and $O_{\mathbf{F}}$ be its ring of integers. A Hilbert-Blumenthal variety parameterizes the isomorphism classes of abelian $O_{\mathbf{F}}$ -varieties of dimension g with a certain condition (and with certain additional structure). The purpose of the condition is to exclude some bad points in characteristic p such that the integral model becomes flat. In [R], Rapoport used the condition that the Lie algebra of the abelian $O_{\mathbf{F}}$ -scheme over a base scheme S is a locally free $O_{\mathbf{F}} \otimes S$ -module. We will call it the Rapoport condition and the Rapoport locus for the defined moduli space. The condition was modified later by Deligne and Pappas [DP] in order to confirm the properness of the compactification constructed in [R] in the case of bad reduction. The moduli spaces defined by Deligne and Pappas are usually referred as the Deligne-Pappas spaces. The irreducibility and singularities of the Deligne-Pappas spaces are determined in [DP]. In the present paper we study the reduction of these moduli spaces modulo a fixed rational prime p . More precisely, we consider the moduli spaces of abelian $O_{\mathbf{F}}$ -varieties of dimension g equipped with a compatible prime-to- p polarization. The geometry of reduction of these moduli spaces have been studied by E. Goren and F. Oort in [GO] when p is unramified in $O_{\mathbf{F}}$.

Keywords: Hilbert-Blumenthal varieties – Dieudonné modules – Stratifications – Deformations.

Math. classification: 14G35 – 14L05.

Let \mathcal{M}^{DP} denote the moduli stack over $\text{Spec } \mathbb{Z}_{(p)}$ of separably polarized abelian $O_{\mathbf{F}}$ -varieties of dimension g . This is a separated Deligne-Mumford algebraic stack locally of finite type and one can identify the moduli space defined in [DP] as a connected component of \mathcal{M}^{DP} , see [DP, 2.1]. We will call it the Deligne-Pappas space. Let \mathcal{M}^R denote the Rapoport locus of \mathcal{M}^{DP} , which parameterizes the objects in \mathcal{M}^{DP} satisfying the Rapoport condition. Let $\mathcal{M}_p^{\text{DP}} := \mathcal{M}^{\text{DP}} \otimes \overline{\mathbb{F}}_p$ and $\mathcal{M} := \mathcal{M}^R \otimes \overline{\mathbb{F}}_p$ be the reduction of \mathcal{M}^{DP} and \mathcal{M}^R modulo p respectively. Let $\mathcal{O} := O_{\mathbf{F}} \otimes \mathbb{Z}_p$ and k be an algebraically closed field of characteristic p .

To each abelian $O_{\mathbf{F}}$ -variety A over k , we define two natural invariants called the Lie type and a -type. Lie types are the invariants that classify the Lie algebras $\text{Lie}(A)$ of A as $O_{\mathbf{F}} \otimes k$ -modules, and a -types are those classifying the α -groups of A [LO]. One purpose of this paper is to understand these invariants (including the Newton polygons) using Dieudonné modules. Some results on the relation among these invariants and related conditions are obtained in Sections 2–3.

A natural problem is whether these invariants arising from the Dieudonné modules in question can be realized by abelian varieties with the additional structure (cf. 1.5). This is the integral analogue of a problem of Manin, see [O2]. The following theorem (7.4) answers it affirmatively.

THEOREM 1. — *Any quasi-polarized p -divisible \mathcal{O} -group (H, λ, ι) (1.4) over k is isomorphic to the p -divisible attached to a polarized abelian $O_{\mathbf{F}}$ -variety.*

This result implies that the strata defined by these invariants are all non-empty. As an application of Theorem 1, we construct an explicit example of a point s in the complement of the Rapoport locus, which is both the specialization of a point t_1 in characteristic 0 and also the specialization of an ordinary point t_2 . Notice that both t_1 and t_2 are in the Rapoport locus. This example directly shows that the construction $\overline{\mathcal{M}}^R$ of Rapoport is not proper over $\text{Spec } \mathbb{Z}_{(p)}$ and a modification of the moduli space is needed. Furthermore, it was pointed out in [DP] that the construction of Rapoport compactifies the Deligne-Pappas space.

The proof of the algebraization theorem goes as follows. We first show that the formal isogeny classes are determined by the Newton polygons using a result of Rapoport-Zink [RZ] and of Rapoport-Richartz [RR]. Then we prove the weak Grothendieck conjecture (see 1.13). It follows that any possible Newton polygon can be realized by an abelian variety in question.

By the result on formal isogenies and a theorem of Tate, all the p -divisible groups with additional structures can be realized by abelian $O_{\mathbf{F}}$ -varieties in question.

The main part of this paper is studying the strata induced from these three invariants. The stratification by Lie types coincides with the stratification studied by Deligne and Pappas [DP]. The largest stratum is the Rapoport locus. We define a scheme-theoretic stratification by a -types on the Rapoport locus. The relation between the alpha stratification and the slope stratification on the Rapoport locus is given. As the Rapoport locus is the whole space in the case of good reduction, we recover the main results of E. Goren and F. Oort [GO] on the stratifications.

We now state the results. Write $\mathcal{O} := O_{\mathbf{F}} \otimes \mathbb{Z}_p = \bigoplus_{v|p} \mathcal{O}_v$ and let e_v and f_v be the ramification index and residue degree of v respectively. Let A be an abelian $O_{\mathbf{F}}$ -variety over k and let $A[p^\infty] = \bigoplus_{v|p} H_v$ be the decomposition of the associated p -divisible group (with respect to the \mathcal{O} -action). We define in (1.9) the a -type $\underline{a}(H_v)$ for each component H_v and put $\underline{a}(A) := (\underline{a}(H_v))_v$. When A satisfies the Rapoport condition, the a -type $\underline{a}(H_v)$ of each component is of the form $(a_v^i)_{i \in \mathbb{Z}/f_v\mathbb{Z}}$, where $0 \leq a_v^i \leq e_v$ for all $i \in \mathbb{Z}/f_v\mathbb{Z}$. There is a natural partial order on the set of these a -types. The *reduced a -number* of H_v is defined to be $\dim_k(\alpha_p, H_v[\pi_v])$ (1.9), where π_v is a uniformizer of \mathcal{O}_v .

THEOREM 2. — *Let \underline{a} be an a -type which occurs in \mathcal{M} . The closed subscheme $\mathcal{M}_{\geq \underline{a}}$ of \mathcal{M} that consists of objects with a -type $\geq \underline{a}$ is smooth over $\text{Spec} \bar{\mathbb{F}}_p$ of pure dimension $g - |\underline{a}|$ (Theorem 5.4).*

Let $\underline{a} = (\underline{a}_v)_v$ be an a -type which occurs in \mathcal{M} . We call $\underline{a}_v = (a_v^i)_i$ is *spaced* if $a_v^i a_v^{i+1} = 0$ for all $i \in \mathbb{Z}/f_v\mathbb{Z}$ and \underline{a} is *spaced* if \underline{a}_v is spaced for all $v|p$. We put $\lambda(\underline{a}_v) := \max\{|\underline{b}_v|; \underline{b}_v \leq \underline{a}_v, \underline{b}_v \text{ is spaced}\}$ (cf. [GO, p. 112]). We refer to (1.10) for the definition of the function s_v , which sends certain rational numbers to possible slope sequences at v .

THEOREM 3. — (1) *If $\underline{a} = (\underline{a}_v)_v$ is spaced, then the subset of $\mathcal{M}_{\geq \underline{a}}$ consisting of points whose slope sequence is $(s_v(|\underline{a}_v|))_v$ is dense in $\mathcal{M}_{\geq \underline{a}}$ (Theorem 6.8).*

(2) *The generic point of each irreducible component of $\mathcal{M}_{\geq \underline{a}}$ has slope sequence $\geq (s_v(\lambda(\underline{a}_v)))_v$ (Corollary 6.9).*

THEOREM 4. — (1) *Let U be the subset of \mathcal{M} consisting of points with reduced a -number at most one for each component $v|p$. Then the strong Grothendieck conjecture holds for U (Theorem 6.13).*

(2) *The weak Grothendieck conjecture for \mathcal{M} holds (Corollary 6.16).*

(3) *The strong Grothendieck conjecture for $\mathcal{M}_p^{\text{DP}}$ holds when all the residue fields of $O_{\mathbf{F}}$ are \mathbb{F}_p (Theorem 6.20).*

For the statement of the Grothendieck conjectures, we refer to [GO, 5.1] or (1.13).

The methods are based on the previous works of F. Oort [O2], E. Goren and F. Oort [GO], and the author [Y1]. Using the explicit deformation method developed by Norman [N], Norman-Oort [NO], and T. Zink [Z2], we construct the universal deformation of any Dieudonné module in the Rapoport locus. Then we study systematically the alpha stratification and the slope stratification on the formal neighborhood around the point. The results above are extracted from the formula of iterating the Frobenius map. It is possible to extract finer information beyond the reduced a -number one from our formula. We leave this possible generalization in the future when the finer information becomes useful. We follow the approach of [O2] and [Y1], which is different from that of Goren and Oort [GO]. Therefore, we do not repeat the computation done in loc. cite. F. Andreatta and E. Goren earlier obtained similar results in the case when p is totally ramified. Our work is independent from their results.

In [C], C.-L. Chai studied the combinatorial properties of Newton points for connected reductive quasi-split algebraic groups, inspired by earlier works of R. Kottwitz [K1], K.-Z. Li and F. Oort [LO], M. Rapoport and M. Richartz [RR]. He gave a group-theoretic conjectural description of the dimensions of Newton strata of *good* reduction of Shimura varieties. The main motivation of this work is to examine whether his group-theoretic description of Newton strata for quasi-split groups applies for the simplest case of bad reduction. Our results support his description even when the reductive group G in question is no longer unramified. One may expect that Chai's description applies for the bad reduction case as well, under the assumption of the existence of good integral models of Shimura varieties for certain special subgroups $K_p \subset G(\mathbb{Q}_p)$. We answer a question of Chai affirmatively [C, Question 7.6, p. 984] on the dimensions of Newton strata in the Hilbert-Blumenthal cases when p is totally ramified.

The results of this work reveal an important feature: the stratifications on the smooth locus of bad reduction of Hilbert-Blumenthal varieties behave very similarly as those on the good reduction. The reader can compare the results stated above and those of [GO] for the good reduction case. A fundamental question is whether this feature holds for more general

PEL-type Shimura varieties. We have no idea but expect that the cases of type C still hold.

The following is the structure of this paper. In Section 2 we describe the structure of the Dieudonné module of general polarized abelian $O_{\mathbf{F}}$ -varieties of dimension g . In Section 3 we classify the formal isogeny classes explicitly. In Section 4 we provide the normal forms of the Dieudonné modules in the Rapoport locus. In Section 5 we give the natural generalization the alpha stratification on the Rapoport locus and study its properties. In Section 6 we study systematically the alpha stratification and the slope stratification on formal neighborhoods in the Rapoport locus, by the methods of [N], [NO], [Z2], and [O2] as explained before. In Section 7 we establish a theorem of algebraization concerning the p -divisible groups in question. In Section 8 we give the explicit example as explained before. In the last section we perform a computation of the Hecke correspondence. Using this result, we describe the singularities of the supersingular locus near the superspecial point constructed in Section 8.

We should note that there is no assumption of p in this paper. As most of the time we are dealing with conditions and properties of the associated p -divisible group and local properties of the moduli spaces. It is enough to treat each component of the p -divisible groups and that of the local moduli spaces by the Serre-Tate Theorem. Without loss of generality we may assume that there is one prime over p . It is clear how to state the results in this paper without this assumption of p . We feel not necessary to repeat this.

We use the convenient language of algebraic stacks. The reader is free to replace the word “algebraic stack(s)” by “scheme(s)” in this paper by adding an auxiliary level structure. All schemes here are implicitly assumed to be locally noetherian.

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1. Notations, terminologies and definitions.

1.1. Fix a rational prime p . Let \mathbf{F} be a totally real field of degree g and $O_{\mathbf{F}}$ be the ring of integers. Let v be a prime of $O_{\mathbf{F}}$ over p . Let \mathbf{F}_v denote the completion of \mathbf{F} at v and \mathcal{O}_v denote the ring of integers. Denote by e_v and f_v the ramification index and residue degree of v respectively. Denote by π_v a uniformizer of the ring of integers \mathcal{O}_v and write $g_v := [\mathbf{F}_v : \mathbb{Q}_p]$.

Let v_1, \dots, v_s be the primes of $O_{\mathbf{F}}$ over p , $\mathbf{F}_p := \mathbf{F} \otimes \mathbb{Q}_p = \mathbf{F}_{v_1} \oplus \dots \oplus \mathbf{F}_{v_s}$, and $\mathcal{O} := O_{\mathbf{F}} \otimes \mathbb{Z}_p = \mathcal{O}_{v_1} \oplus \dots \oplus \mathcal{O}_{v_s}$. Write $g_i := g_{v_i} = [\mathbf{F}_{v_i} : \mathbb{Q}_p]$ and $\underline{g} := (g_v)_{v|p}$.

1.2. Let k be a perfect field of characteristic p . Denote by $W := W(k)$ the ring of Witt vectors and $B(k)$ its field of fractions. Let σ be the Frobenius map on W .

1.3. Let B be a finite dimensional semi-simple algebra over \mathbb{Q} with a positive involution $*$. Let O_B be an order of B stable under the involution $*$. Recall that a *polarized abelian O_B -variety* [Z3] is a triple (A, λ, ι) where A is an abelian variety, $\iota : O_B \rightarrow \text{End}(A)$ is a ring monomorphism and $\lambda : A \rightarrow A^t$ is a polarization satisfying the compatible condition $\lambda \iota(b^*) = \iota(b)^t \lambda$ for all $b \in O_B$.

Let A be an abelian variety up to isogeny with $\iota : B \rightarrow \text{End}(A)$. Then the dual abelian variety A^t admits a natural B -action by $\iota^t(b) := \iota(b^*)^t$. The compatible condition above is saying that the polarization $\lambda : A \rightarrow A^t$ is O_B -linear.

1.4. Let B_p be a finite dimensional semi-simple algebra over \mathbb{Q}_p with an involution $*$. Let O_p be an order of B_p stable under the involution $*$. A *quasi-polarized p -divisible O_p -group* is a triple (H, λ, ι) where H is a p -divisible group, $\iota : O_p \rightarrow \text{End}(H)$ is a ring monomorphism and $\lambda : H \rightarrow H^t$ is a quasi-polarization (i.e. $\lambda^t = -\lambda$) such that $\lambda \iota(b^*) = \iota(b)^t \lambda$ for all $b \in O_p$.

For convenience, we also introduce the term “quasi-polarized Dieudonné O_p -modules” for the associated Dieudonné module to a quasi-polarized p -divisible O_p -group over k . It is a Dieudonné module M over k , equipped with a W -valued non-degenerate alternating pairing $\langle \cdot, \cdot \rangle$ and a W -linear action by O_p , that satisfies the usual condition $\langle ax, y \rangle = \langle x, a^*y \rangle$ and $\langle Fx, y \rangle = \langle x, Vy \rangle^\sigma$ for all $a \in O_p$ and $x, y \in M$.

1.5. When $B_p = B \otimes \mathbb{Q}_p$ and $O_p = O_B \otimes \mathbb{Z}_p$. We call a quasi-polarized p -divisible O_p -group (H, λ, ι) over k *algebraizable* if there is a polarized abelian O_B -variety (A, λ_A, ι_A) over k such that the associated p -divisible group $(A(p), \lambda_A(p), \iota_A(p))$ is isomorphic to (H, λ, ι) with the additional structure over k .

1.6. Let S be a base scheme and A be an abelian $O_{\mathbf{F}}$ -scheme of relative dimension g over S . Recall that the abelian $O_{\mathbf{F}}$ -scheme A satisfies *the Rapoport condition* [R] if the Lie algebra $\text{Lie}(A)$ is a locally free $O_{\mathbf{F}} \otimes_{\mathbb{Z}} O_S$ -module. Clearly this condition is local and open on S .

We denote by $\mathcal{P}(A) := \text{Hom}_{O_{\mathbf{F}}}(A, A^t)^{\text{sym}}$ the module of $O_{\mathbf{F}}$ -linear symmetric homomorphisms from A to its dual A^t over S . Notice that $\mathcal{P}(A)$ is the module of global sections of the polarization sheaf. If S is connected and $\mathcal{P}(A)$ is non-zero, then $\mathcal{P}(A)$ is a rank one projective $O_{\mathbf{F}}$ -module together with a notion of positivity. It is shown [R, Prop. 1.12] that $\mathcal{P}(A)$ is non-zero when S is the spectrum of a field or an artinian ring.

We say the abelian $O_{\mathbf{F}}$ -scheme A satisfies *the Deligne-Pappas condition* if for any connected component S' of S , the module $\mathcal{P}(A_{S'})$ is non-zero and the induced morphism

$$\mathcal{P}(A_{S'}) \otimes_{O_{\mathbf{F}}} A_{S'} \rightarrow A_{S'}^t,$$

is isomorphic.

When S is a $\mathbb{Z}_{(p)}$ -scheme, the following simpler condition plays a similar role: the abelian $O_{\mathbf{F}}$ -scheme A admits an $O_{\mathbf{F}}$ -linear prime-to- p degree polarization.

1.7. For the reader's convenience, we recall *the Kottwitz determinant condition* [K2, Sect. 5]. Let V be a one-dimensional \mathbf{F} -vector space. Let $\{e_i\}$ be a \mathbb{Z} -basis of $O_{\mathbf{F}}$ and $\underline{X} = (X_i)$ be some indeterminants. We define $f(\underline{X}) := \det(\sum e_i X_i; V)$. The polynomial f is in $\mathbb{Z}[\underline{X}]$, loc. cit. We say that the Lie algebra $\text{Lie}(A)$ (or the abelian $O_{\mathbf{F}}$ -scheme A) satisfies *the Kottwitz determinant condition* if

$$\det\left(\sum e_i X_i; \text{Lie}(A)\right) = f(\underline{X})$$

in $\mathcal{O}_S[X]$. This condition does not depend on the choice of the basis, and it is a closed condition in a family of abelian $O_{\mathbf{F}}$ -varieties. If $[\text{Lie}(A)] = [O_{\mathbf{F}} \otimes O_S]$ in the Grothendieck group of $O_{\mathbf{F}} \otimes O_S$ -modules of finite type, then A satisfies the Kottwitz determinant condition.

1.8. Suppose that the ground field k contains the residue fields of $\mathcal{O}_{\mathbf{F}}$ over p . We fix a place v of $\mathcal{O}_{\mathbf{F}}$ dividing p . Let H be a p -divisible \mathcal{O}_v -group of height $2g_v$ over k and M be its associated covariant Dieudonné module. Note that M is a free $W \otimes_{\mathbb{Z}_p} \mathcal{O}_v$ -module of rank two. This follows from that the Frobenius operator F induces a bijection on $M \otimes B(k)$. We identify the set of embeddings $\text{Hom}(\mathcal{O}_v^{\text{ur}}, W)$ with $\mathbb{Z}/f_v\mathbb{Z}$ in a way that $\sigma : i \mapsto i + 1$, where $\mathcal{O}_v^{\text{ur}}$ is the maximal étale extension of \mathbb{Z}_p in \mathcal{O}_v . We write $\mathcal{O}_v \otimes_{\mathbb{Z}_p} k = \bigoplus_{i \in \mathbb{Z}/f_v\mathbb{Z}} k[\pi_v]/(\pi_v^{e_i})$.

The Lie type of H is defined to be

$$\underline{e}(H) := (\{e_1^i, e_2^i\})_{i \in \mathbb{Z}/f_v\mathbb{Z}}$$

if

$$\text{Lie}(H) \simeq \bigoplus_{i \in \mathbb{Z}/f_v\mathbb{Z}} \left(k[\pi_v]/(\pi_v^{e_1^i}) \oplus k[\pi_v]/(\pi_v^{e_2^i}) \right)$$

as $\mathcal{O}_v \otimes k$ -modules for some integers e_1^i, e_2^i .

1.9. Let H and M be as above. The a -type of H is defined to be

$$\underline{a}(H) := (\{a_1^i, a_2^i\})_{i \in \mathbb{Z}/f_v\mathbb{Z}}$$

if

$$M/(F, V)M \simeq \bigoplus_{i \in \mathbb{Z}/f_v\mathbb{Z}} \left(k[\pi_v]/(\pi_v^{a_1^i}) \oplus k[\pi_v]/(\pi_v^{a_2^i}) \right)$$

as $\mathcal{O}_v \otimes k$ -modules for some integers a_1^i, a_2^i . The usual a -number is denoted by $|\underline{a}(H)|$, the dimensional of the k -vector space $M/(F, V)M$.

If H satisfies the Rapoport condition, that is, $\text{Lie}(H)$ is a free $\mathcal{O}_v \otimes_{\mathbb{Z}_p} k$ -module, then the a -type $\underline{a}(H)$ is of the form $(\{0, a^i\})_i$ and we write $\underline{a}(H) = (a^i)_i$ instead. In this case, we define the partial order: $(a^i) \leq (b^i)$ if $a^i \leq b^i$ for all $i \in \mathbb{Z}/f_v\mathbb{Z}$, and define $t(H) := \dim_k M/((F, V)M + \pi_v M)$, called the *reduced (usual) a -number* of H .

1.10. The slope sequence (the Newton polygon) of H we denote by $\text{slope}(H)$. It is either $\{\frac{i}{g_v}, \dots, \frac{i}{g_v}, \frac{g_v-i}{g_v}, \dots, \frac{g_v-i}{g_v}\}$ for some integer $0 \leq i \leq g_v/2$, or $\{\frac{1}{2}, \dots, \frac{1}{2}\}$ (Lemma 3.1).

We often identify a Newton polygon with its slope sequence. Let $S(g_v)$ denote the subset of \mathbb{Q} :

$$S(g_v) := \left\{ i \in \mathbb{Z}; 0 \leq i \leq \frac{g_v}{2} \right\} \cup \left\{ \frac{g_v}{2} \right\}.$$

For each $i \in S(g_v)$, we denote by $s_v(i)$ the slope sequence $\{\frac{i}{g_v}, \dots, \frac{i}{g_v}, \frac{g_v-i}{g_v}, \dots, \frac{g_v-i}{g_v}\}$. The map s_v identifies the set $S(g_v)$ with that of possible slope sequences arising from p -divisible \mathcal{O}_v -groups. The order on $S(g_v)$ induced from \mathbb{Q} is compatible with the Grothendieck specialization theorem.

There are two possible definitions for the slopes of a Dieudonné module. One uses the slopes of the p -divisible group (that is, the F -slopes of the *contravariant* Dieudonné module, or the V -slopes of the covariant one). The other just uses its F -slopes, no matter which Dieudonné theory (covariant or contravariant) one chooses. We adopt the latter. As Dieudonné modules considered in this paper are symmetric, the choice will not effect the results.

1.11. Let A be an abelian $\mathcal{O}_{\mathbf{F}}$ -variety k . The associated p -divisible group $A(p) := A[p^\infty]$ has the decomposition

$$A(p) = \bigoplus_{v|p} H_v.$$

We define the Lie type and a -type of A by

$$\underline{e}(A) := (\underline{e}(H_v))_{v|p}, \quad \underline{a}(A) := (\underline{a}(H_v))_{v|p}.$$

Set $S(\underline{g}) := \prod_{v|p} S(g_v)$, and for each $\underline{i} = (i_v)_v \in S(\underline{g})$, we write $s(\underline{i}) := (s_v(i_v))_v$. The map s identifies the set $S(\underline{g})$ with that of possible Newton polygons arising from abelian $\mathcal{O}_{\mathbf{F}}$ -varieties of dimension g . The slope sequence of A is denoted by

$$\text{slope}(A) := (\text{slope}(H_v))_v.$$

1.12. Let \mathcal{M}^{DP} denote the moduli stack over $\text{Spec} \mathbb{Z}_{(p)}$ of separably polarized abelian $\mathcal{O}_{\mathbf{F}}$ -varieties of dimension g . Let \mathcal{M}^R denote the Rapoport locus of \mathcal{M}^{DP} , which parameterizes the objects in \mathcal{M}^{DP} satisfying the Rapoport condition. Let $k(p)$ be the smallest finite field containing all the residue fields $k(v_i)$. Denote by $\mathcal{M}_p^{\text{DP}}$ the reduction $\mathcal{M}^{\text{DP}} \otimes_{\mathbb{Z}} k(p)$ of \mathcal{M}^{DP} and \mathcal{M} the reduction $\mathcal{M}^R \otimes_{\mathbb{Z}} k(p)$ of the Rapoport locus \mathcal{M}^R .

Let β be an admissible Newton polygon, that is $\beta \in S(\underline{g})$. We denote by \mathcal{M}^β (resp. $\mathcal{M}^{\geq \beta}$) the reduced algebraic substack of \mathcal{M} that consists of points with Newton polygon β (resp. that lies over or equals β).

Let \underline{a} be an a -type on \mathcal{M} . Let $\mathcal{M}_{\underline{a}}$ denote the reduced algebraic substack of \mathcal{M} that consists of points with a -type \underline{a} . In Section 5, we define a closed substack, denoted by $\mathcal{M}_{\geq \underline{a}}$, of \mathcal{M} so that $x \in \mathcal{M}_{\geq \underline{a}}(\bar{k})$ if and only if $\underline{a}(x) \geq \underline{a}$.

1.13. We recall the statement of the Grothendieck conjectures [GO, 5.1] and [O2, Sect. 6]. Let U be an open subset of $\mathcal{M}_p^{\text{DP}}$. We say *the (strong) Grothendieck conjecture holds for U* if for any $x \in U$ and any admissible Newton polygon $\beta < \text{slope}(x)$, then for any neighborhood V of x , there is a point in V whose Newton polygon is β . We say *the weak Grothendieck conjecture holds for U* if given any chain of admissible Newton polygons $\gamma_1 < \gamma_2 < \dots < \gamma_s$, then there exists a chain of irreducible subschemes of U : $V_1 \supset V_2 \supset \dots \supset V_s$ such that $\text{slope}(A_{\eta_i}) = \gamma_i$, where η_i is the generic point of V_i and A_{η_i} is the corresponding abelian variety.

2. Dieudonné modules.

2.1. Let \mathbf{F} be a totally real number field of degree g and $O_{\mathbf{F}}$ be its ring of integers. To simplify notations, we will work on the case that there is one prime of $O_{\mathbf{F}}$ over p in Sections 2–6. The other cases can be reduced to this case if the problem is local, as stated before. Let e be the ramification index and f be the residue degree of this prime v , thus $g = f$. Denote by \mathbf{F}_v the completion of \mathbf{F} at v and \mathcal{O} the ring of integers in \mathbf{F}_v . Let \mathcal{O}^{ur} denote the maximal étale extension of \mathbb{Z}_p in \mathcal{O} . The ring \mathcal{O}^{ur} is isomorphic to $W(\mathbb{F}_{p^f})$. Let π be a uniformizer of \mathcal{O} . The element π can be chosen from $O_{\mathbf{F}}$ and to be totally positive, by the weak approximation. Let $P(T)$ be the monic irreducible polynomial of π over \mathcal{O}^{ur} .

2.2. Let (A, λ, ι) be a polarized abelian $O_{\mathbf{F}}$ -variety of dimension g over a perfect field k containing \mathbb{F}_{p^f} , and let M be its covariant Dieudonné module. The Dieudonné module M is a free $\mathcal{O} \otimes_{\mathbb{Z}_p} W(k)$ -module of rank two equipped with a non-degenerate alternating pairing

$$M \times M \rightarrow W(k)$$

such that on which the Frobenius F and Verschiebung V commute with the action of \mathcal{O} , and that $\langle ax, y \rangle = \langle x, ay \rangle$ and $\langle Fx, y \rangle = \langle x, Vy \rangle^\sigma$ for

all $x, y \in M$ and $a \in \mathcal{O}$. We call it briefly a *quasi-polarized Dieudonné \mathcal{O} -module*.

The ring $\mathcal{O} \otimes_{\mathbb{Z}_p} W(k)$ is isomorphic to

$$\bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} W(k)[T]/(\sigma_i(P(T))),$$

where $\sigma_i, i \in \mathbb{Z}/f\mathbb{Z}$, are embeddings of \mathcal{O}^{ur} into $W(k)$, arranged in a way that $\sigma\sigma_i = \sigma_{i+1}$. Set $W^i := W(k)[T]/(\sigma_i(P(T)))$ and denote again by π the image of T in W^i . The action of the Frobenius map σ on $\mathcal{O} \otimes_{\mathbb{Z}_p} W(k)$ through the right factor gives a map $\sigma : W^i \rightarrow W^{i+1}$ which sends $a \mapsto a^\sigma$ for $a \in W(k)$ and $\sigma(\pi) = \pi$. We also have $W^i \otimes_W k = k[\pi]/(\pi^e)$ and $\mathcal{O} \otimes_{\mathbb{Z}_p} k = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} k[\pi]/(\pi^e)$. Let

$$M^i := \{x \in M \mid ax = \sigma_i(a)x, \forall a \in \mathcal{O}^{\text{ur}}\}$$

be the σ_i -component of M , which is a free W^i -module of rank two. We have the decomposition

$$M = M^0 \oplus M^1 \oplus \dots \oplus M^{f-1}$$

in which $F : M^i \rightarrow M^{i+1}, V : M^{i+1} \rightarrow M^i$. The summands M^i, M^j are orthogonal with respect to the pairing $\langle \cdot, \cdot \rangle$ for $i \neq j$. Conversely, a Dieudonné module together with such a decomposition and these properties is a quasi-polarized Dieudonné \mathcal{O} -module.

2.3 Let $\underline{e}(A) = (\{e_1^i, e_2^i\})_i$ be the Lie type of A defined in (1.8). The invariant $\underline{e}(A)$ has the property that $0 \leq e_j^i \leq e$ for $i \in \mathbb{Z}/f\mathbb{Z}$ and $j = 1, 2, \sum_i e_1^i + e_2^i = g$, and that the invariant $e_1^i + e_2^i$ is a locally constant function in a family. The last one follows from the fact that the σ_i -component $\text{Lie}(A)^i$ of $\text{Lie}(A)$ is a locally free sheaf.

In [DP], Deligne and Pappas showed that the stratum of each Lie type $(\{e_1^i, e_2^i\})_i$, in the Deligne-Pappas space, is a smooth locally closed subscheme, and has dimension $g - 2 \sum_i \min\{e_1^i, e_2^i\}$ provided the stratum is non-empty. We will see that indeed these strata are non-empty (7.4).

2.4. Let (H, λ, ι) be the p -divisible group attached to (A, λ, ι) . If λ_1 is another \mathcal{O} -linear quasi-polarization on H , then $\lambda_1 = \lambda a$ for some $a \in E_0 := \text{End}_{\mathcal{O}}(H) \otimes \mathbb{Q}_p$ with $a^* = a$, where $*$ is the involution induced by λ . We will show that $a \in \mathbf{F}_v$. Let λ_0 be an \mathcal{O} -linear quasi-polarization

of minimal degree. Then any \mathcal{O} -linear quasi-polarization is of the form $\lambda_0 a$ for some $a \in \mathcal{O}$.

The algebra $\text{End}(H) \otimes \mathbb{Q}_p$ has rank $\leq 4g^2$ over \mathbb{Q}_p . Therefore, $[E_0 : \mathbf{F}_v] \leq 4$. If $[E_0 : \mathbf{F}_v] = 4$, then H is supersingular and E_0 is a quaternion algebra over \mathbf{F}_v . In this case, the involution is canonical. If $[E_0 : \mathbf{F}_v] = 2$, then the involution on E_0 is non-trivial. This follows from the non-degeneracy of the alternating pairing. In either case we show that the fixed elements by $*$ lie in \mathbf{F}_v .

Similarly, we can show that given an abelian $O_{\mathbf{F}}$ -variety and let λ_0 be an $O_{\mathbf{F}}$ -linear polarization of minimal degree at p , then any $O_{\mathbf{F}}$ -linear polarization has the form $\lambda_0 a$ for some totally positive element a in $O_{\mathbf{F}} \otimes \mathbb{Z}_{(p)}$.

2.5. Let $\mathcal{D}^{-1} = (\pi^{-d})$ be the inverse of the different of \mathcal{O} over \mathbb{Z}_p . There is a unique $W \otimes \mathcal{O}$ -bilinear pairing $(,) : M \times M \rightarrow W \otimes \mathcal{D}^{-1}$ such that $(x, y) = \text{Tr}_{W \otimes \mathcal{O}/W}(x, y)$. From the uniqueness, we have $(Fx, y) = (x, Vy)^\sigma$ for $x, y \in M$. For each W^i -basis x_1^i, x_2^i of M^i , the π -adic valuation $\text{ord}_\pi(x_1^i, x_2^i)$ is independent of the choice of basis and the degree of the quasi-polarization is p^D [S, Chap. 1, Prop. 12], where

$$D = 2 \sum_{i \in \mathbb{Z}/f\mathbb{Z}} \text{ord}_p \text{Norm}_{W^i/W}(\pi^d(x_1^i, x_2^i)).$$

We can choose two W^i -bases $\{x_1^i, x_2^i\}, \{y_1^i, y_2^i\}$ of M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that

$$Vy_1^{i+1} = \pi^{e_1^i} x_1^i, \quad Vy_2^{i+1} = \pi^{e_2^i} x_2^i.$$

It follows from $(Vx, Vy) = p(x, y)^{\sigma^{-1}}$ that we get

$$\text{ord}_\pi(y_1^{i+1}, y_2^{i+1}) = \text{ord}_\pi(x_1^i, x_2^i) + (e_1^i + e_2^i - e).$$

If $\min_{i \in \mathbb{Z}/f\mathbb{Z}} \text{ord}_\pi(x_1^i, x_2^i) = -d$ (the exponent of the inverse different), say that $i = 0$ achieves the minimum, then

$$\text{ord}_\pi(x_1^i, x_2^i) = -d + \sum_{k=0}^{i-1} (e_1^k + e_2^k - e)$$

and

$$(2.5.1) \quad D = 2 \sum_{i=1}^{f-1} \sum_{k=0}^{i-1} (e_1^k + e_2^k - e).$$

If $\min_{i \in \mathbb{Z}/f\mathbb{Z}} \text{ord}_\pi(x_1^i, x_2^i) > -d$, then we can divide the pairing $(,)$ by a power of π such that $\min_{i \in \mathbb{Z}/f\mathbb{Z}} \text{ord}_\pi(x_1^i, x_2^i) = -d$.

LEMMA 2.6. — (1) *There exists a number N , depending only on g , with the following property: for any abelian $O_{\mathbf{F}}$ -variety (A, ι) of dimension g , there is an $O_{\mathbf{F}}$ -linear polarization λ such that $\text{ord}_p(\deg \lambda) \leq N$.*

(2) *An abelian $O_{\mathbf{F}}$ -variety (A, ι) over k admits an separable $O_{\mathbf{F}}$ -linear polarization if and only if $\dim(A)^i$ are the same for $i \in \mathbb{Z}/f\mathbb{Z}$.*

Proof. — (1) The statement holds as well without the assumption (2.1), so we prove the general case instead. Let (A, ι) be an abelian $O_{\mathbf{F}}$ -variety. By the weak approximation, we can choose an $O_{\mathbf{F}}$ -linear polarization λ such that on each component H_v of $A(p)$ the quasi-polarization λ_v has the minimal degree. Then the exponent of the local degree is given in (2.5.1), and let N' be the sum of these local exponents. The number N' only depends on the Lie type but not on the abelian variety. As there are finitely many possible Lie types with a fixed dimension g , we take N to be the maximal one among all N' .

(2) This follows immediately from (2.5.1). □

LEMMA 2.7. — *Let S be the spectrum of an artinian ring R with residue field of characteristic p . Let (A, ι) be an abelian $O_{\mathbf{F}}$ -scheme over S . Then for any prime $\ell \neq p$, there exists an prime-to- ℓ $O_{\mathbf{F}}$ -linear polarization on A .*

Proof. — We first reduce to the case that R is a field k . Let R be a small extension of R_0 . Suppose there is a prime-to- ℓ polarization λ on $A \otimes_R R_0$, then $p\lambda$ extends over R . This follows from that the obstruction class lies in $H^2(A_k, \mathcal{O}_{A_k})$, which is annihilated by p .

As the map $\text{Hom}(A_k, B_k) \rightarrow \text{Hom}(A_{\bar{k}}, B_{\bar{k}})$ is co-torsion free, the map $\mathcal{P}(A_k) \rightarrow \mathcal{P}(A_{\bar{k}})$ is co-torsion free. It follows that $\mathcal{P}(A_k) \simeq \mathcal{P}(A_{\bar{k}})$. Therefore, we need to verify the case that k is algebraically closed.

Let λ be an $O_{\mathbf{F}}$ -linear polarization on A and \langle, \rangle the induced pairing on the Tate module $T_\ell(A)$. Write $T_\ell(A) = \bigoplus_{w|\ell} T_w$ into the decomposition for $O_{\mathbf{F}} \otimes \mathbb{Z}_\ell = \bigoplus_{w|\ell} \mathcal{O}_w$. Each factor T_w is a free rank two \mathcal{O}_w -module and the pairing \langle, \rangle induces a non-degenerated pairing on T_w . Write $\langle x, y \rangle = \text{Tr}_{\mathcal{O}_w/\mathbb{Z}_\ell}(\pi_w^{-d_w}(x, y))$ for a unique lifting $(,)$: $T_w \times T_w \rightarrow \mathcal{O}_w$ and let $c_w := \text{ord}_w(e_1, e_2)$, where $-d_w$ is the exponent of the inverse different \mathcal{D}_w^{-1} of \mathcal{O}_w over \mathbb{Z}_ℓ and $\{e_1, e_2\}$ is a \mathcal{O}_w -basis for T_w . By the weak approximation, we can choose a totally positive element a in $O_{\mathbf{F}}[\frac{1}{\ell}]$

such that $\text{ord}_w(a) = -c_w$ for all $w|\ell$. Then λa is an $O_{\mathbf{F}}$ -linear polarization of degree prime-to- ℓ . □

PROPOSITION 2.8. — *Let (A, ι) be an abelian $O_{\mathbf{F}}$ -variety over k . Then the following conditions are equivalent:*

- (1) *A satisfies the Deligne-Pappas condition.*
- (2) *A admits a separable $O_{\mathbf{F}}$ -linear polarization.*
- (3) *$[\text{Lie}(A)] = [O_{\mathbf{F}} \otimes k]$ in the Grothendieck group of $O_{\mathbf{F}} \otimes k$ -modules of finite type.*
- (4) *A satisfies the Kottwitz determinant condition.*
- (5) *$\dim_k \text{Lie}(A)^i$ are the same for all $i \in \mathbb{Z}/f\mathbb{Z}$.*

Proof. — We first remark that (1) \implies (3) is given in [DP, Prop. 2.7]. The following does not depend on this result.

Let $\lambda \in \mathcal{P}(A)$ and let (λ) denote the submodule generated by λ . Then the composition $(\lambda) \otimes A \rightarrow \mathcal{P}(A) \otimes A \rightarrow A^t$ is λ . It follows that the degree of the isogeny $\mathcal{P}(A) \otimes A \rightarrow A^t$ divides that of λ . It follows from Lemma 2.7 that the isogeny $\mathcal{P}(A) \otimes A \rightarrow A^t$ has degree a power of p . This shows that (1) \iff (2).

The assertion (2) \iff (5) is Lemma 2.6 (2). It is clear that (3) \implies (4), as the determinant function factors through the Grothendieck group.

The semi-simplification of $\text{Lie}(A)$, as an $O_{\mathbf{F}} \otimes k$ -module, is $\bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} k^{d_i}$, where $d_i = \dim_k \text{Lie}(A)^i$. It follows that (3) \iff (5).

If A satisfies the Kottwitz determinant condition. Then $\text{Lie}(A)$ is a free $\mathcal{O}^{\text{ur}} \otimes k$ -module. Then (5) follows. This completes the proof. □

2.9. Let S be a \mathbb{Z}_p -scheme and let (A, ι) be an abelian $O_{\mathbf{F}}$ -scheme over S . We consider the similar conditions (1')–(5') for (A, ι) over S , where (1'), (2'), and (4') are the same as (1), (2), and (4) in (2.8) and

(3') Locally for the Zariski topology, $\text{Lie}(A)$ and $O_{\mathbf{F}} \otimes \mathcal{O}_S$ are the same in the Grothendieck group of $O_{\mathbf{F}} \otimes \mathcal{O}_S$ -modules of finite type;

(5') $\text{Lie}(A)$ is a locally free $\mathcal{O}^{\text{ur}} \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ -module.

It clear that (5') is an open condition and we have the following implications: (1') \implies (2') and (3') \implies (4') \implies (5').

LEMMA 2.10. — *If $S = \text{Spec } R$, where R is a noetherian local ring over $\mathbb{Z}_{(p)}$, then the condition (2') implies the condition (3').*

Proof. — If A satisfies the condition (2'), then we have, by [DP, Prop. 2.7, Remark 2.8], that $2[\text{Lie}(A)] = 2[O_{\mathbf{F}} \otimes R]$ in the Grothendieck group. We may assume that R is complete, as \hat{R} is faithfully flat over R . Then it suffices to check that $[\text{Lie}(A)_{R_n}] = [O_{\mathbf{F}} \otimes R_n]$ for all $R_n = R/\mathfrak{m}^n$. This follows immediately from the consequence of the Jordan-Hölder Theorem that the Grothendieck group of R' -modules (for any artinian ring R') of finite length is torsion-free. □

LEMMA 2.11. — *Let R be a noetherian local ring and let k be the residue field. Let A and B be abelian schemes over R . The restriction map identifies $\text{Hom}(A, B)$ as a subgroup of $\text{Hom}(A_k, B_k)$. Then for any prime $\ell \neq \text{char}(k)$, the quotient abelian group $\text{Hom}(A_k, B_k)/\text{Hom}(A, B)$ has no ℓ -torsions.*

Proof. — This is a slightly modification of [O4, Lemma 2.1]. We refer to loc. cit. for the proof. □

THEOREM 2.12. — *Let S be an $\mathbb{Z}_{(p)}$ -scheme and (A, ι) be an abelian $O_{\mathbf{F}}$ -scheme over S . If A admits a separable $O_{\mathbf{F}}$ -linear polarization, then A satisfies the Deligne-Pappas condition.*

Proof. — Write \mathcal{P}' for the group scheme over S that represents the functor $T \mapsto \mathcal{P}(A_T)$. We first show that if A satisfies the Deligne-Pappas condition, then \mathcal{P}' is a locally constant group scheme over S . We may assume that S is connected and it suffices to show that for any connected open subset U of S , the restriction map $r : \mathcal{P}'(S) \rightarrow \mathcal{P}'(U)$ is an isomorphism. It is clear that r is injective. As the Deligne-Pappas condition is satisfied, the composition $\mathcal{P}'(S) \otimes_{A_U} \rightarrow \mathcal{P}'(U) \otimes_{A_U} \rightarrow A_U^t$ is isomorphic. This shows that $\mathcal{P}'(U) \otimes_{A_U} \simeq A_U^t$ and $\mathcal{P}'(S) \simeq \mathcal{P}'(U)$.

We now show the statement when $S = \text{Spec } R$, where R is a noetherian local $\mathbb{Z}_{(p)}$ -algebra. Let k be the residue field of R and we identify $\mathcal{P}'(R)$ as a subgroup of $\mathcal{P}'(k)$. It follows from Lemma 2.11 that if a prime $\ell \neq \text{char}(k)$, then $\mathcal{P}'(k)/\mathcal{P}'(R)$ is ℓ -torsion-free. As A admits an separable $O_{\mathbf{F}}$ -polarization, $\mathcal{P}'(k)/\mathcal{P}'(R)$ is torsion free. It follows that $\mathcal{P}'(R) = \mathcal{P}'(k)$, hence that A satisfies the Deligne-Pappas condition.

We now show the statement. We may first assume that S is connected. Let s be a point of S . Then there is a Zariski-open connected neighborhood

U_s of s such that $\mathcal{P}'(U_s) \simeq \mathcal{P}'(\text{Spec}O_{S,s})$, as the latter is generated as a $O_{\mathbf{F}}$ -module by finitely many sections. It follows that A_{U_s} satisfies the Deligne-Pappas condition, hence that \mathcal{P}'_{U_s} is a constant group scheme. This shows that \mathcal{P}' is constant and $\mathcal{P}'(S) = \mathcal{P}'(U_s)$ for any s . Therefore, A satisfies the Deligne-Pappas condition. \square

2.13. Let $\text{Def}[A, \iota]$ denote the equi-characteristic deformation functor of the abelian $O_{\mathbf{F}}$ -variety (A, ι) over k . It follows from crystalline theory that $\text{Def}[A, \iota](k[\epsilon]) = \text{Hom}_{k \otimes O_{\mathbf{F}}}(VM/pM, M/pM)$. From $\dim_k \text{Hom}_{k[\pi]/(\pi^e)}(k[\pi]/(\pi^a), k[\pi]/(\pi^b)) = \min\{a, b\}$, we compute that

$$(2.13.1) \quad \dim_k \text{Def}[A, \iota](k[\epsilon]) = \sum_{i \in \mathbb{Z}/f\mathbb{Z}} \sum_{1 \leq j, k \leq 2} \min\{e_j^i, e - e_k^i\}.$$

If (A, ι) satisfies the Deligne-Pappas condition, then

$$(2.13.2) \quad \dim_k \text{Def}[A, \iota](k[\epsilon]) = ef + 2 \sum_{i \in \mathbb{Z}/f\mathbb{Z}} \min\{e_1^i, e_2^i\}.$$

2.14. Assume that (A, λ, ι) is a *separably* polarized abelian $O_{\mathbf{F}}$ -variety, that is, it lies in the Deligne-Pappas space. Let $\text{Def}[A, \lambda, \iota]$ denote the equi-characteristic deformation functor of (A, λ, ι) . We can choose a $k[\pi]/(\pi^e)$ -basis $\{x_1^i, x_2^i\}$ of $(M/pM)^i$ for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that $(VM/pM)^i$ is generated by $\{\pi^{e_1^i} x_1^i, \pi^{e_2^i} x_2^i\}$ and that $\langle \pi^{e-1} x_1^i, x_2^i \rangle = 1$ and $\langle \pi^k x_1^i, x_2^i \rangle = 0$ for all $i \in \mathbb{Z}/f\mathbb{Z}$ and $0 \leq k < e - 1$. The first order universal deformation (over the deformation ring R) of the abelian $O_{\mathbf{F}}$ -variety (A, ι) is given by the following data:

$$\widetilde{\text{Fil}} \subset H_1^{\text{cris}}(A/R), \quad \widetilde{\text{Fil}} = \bigoplus \widetilde{\text{Fil}}^i, \quad \widetilde{\text{Fil}}^i = \langle X_1^i, X_2^i \rangle_{R[\pi]/(\pi^e)},$$

where

$$X_1^i = \pi^{e_1^i} x_1^i + \sum_{j=0}^{e_1^i-1} a_j^i \pi^j x_1^i + \sum_{j=0}^{e_2^i-1} b_j^i \pi^j x_2^i,$$

$$X_2^i = \pi^{e_2^i} x_2^i + \sum_{j=0}^{e_1^i-1} c_j^i \pi^j x_1^i + \sum_{j=0}^{e_1^i-1} d_j^i \pi^{j+e_2^i-e_1^i} x_2^i.$$

Here we assume that $e_1^i \leq e_2^i$ for simplicity. The condition $\widetilde{\text{Fil}}^i$ being isotropic is given by $\langle X_1^i, \pi^k X_2^i \rangle = 0$ for $0 \leq k \leq e - 1$. When $k \geq e_1^i$, the condition $\langle X_1^i, \pi^k X_2^i \rangle = 0$ is automatic. For $0 \leq k \leq e_1^i - 1$, the condition

$\langle X_1^i, \pi^k X_2^i \rangle = 0$ gives the equation $d_{e_1^i - k - 1}^i + a_{e_1^i - k - 1}^i = 0$. From this we conclude that

$$(2.14.1) \quad \dim_k \text{Def}[A, \lambda, \iota](k[\epsilon]) = ef + \sum_{i \in \mathbb{Z}/f\mathbb{Z}} \min\{e_1^i, e_2^i\}.$$

This shows that $\dim_k \text{Def}[A, \lambda, \iota](k[\epsilon]) = g$ if and only if (A, λ, ι) satisfies the Rapoport condition.

PROPOSITION 2.15. — *The Rapoport locus is the smooth locus in the Deligne-Pappas space.*

Proof. — This follows from the dimension statement of [DP, Thm. 2.2] and (2.14.1). □

We will compare the Kottwitz determinant condition with the Deligne-Pappas condition in an infinitesimal neighborhood. The following lemma will be used for Proposition 2.17.

LEMMA 2.16. — (1) *Let N be an $n \times n$ matrix with entries in a ring R such that the product of any two entries is zero. Let $U = U_n$, where U_n denotes the following lower triangular matrix:*

$$\begin{pmatrix} Y_1 & 0 & \cdots & 0 \\ Y_2 & Y_1 & \cdots & 0 \\ \vdots & & & 0 \\ Y_n & Y_{n-1} & \cdots & Y_1 \end{pmatrix},$$

for some indeterminants Y_i . Then

$$(2.16.1) \quad \det(U + N) = Y_1^n + \sum_{k=1}^n (\text{Tr}_{k-1} N) U_{1,k}$$

where $\text{Tr}_k N := n_{1,1+k} + \cdots + n_{n-k,n}$ ($\text{Tr}_k N := 0$ if $k \geq n$) and $U_{1,k}$ denotes the $(1, k)^{\text{th}}$ -cofactor of U .

(2) *If $U' = \begin{pmatrix} U_{m_1} & 0 \\ 0 & U_{m_2} \end{pmatrix}$ and $N' = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$ (of same block partition), where N' has the same property as N above and $n = m_1 + m_2$, then $\det(U' + N') = Y_1^n + \sum_{k=1}^n (\text{Tr}_{k-1} N_{11} + \text{Tr}_{k-1} N_{22}) U'_{1,k}$.*

Proof. — (1) Write $U = (u_1, \dots, u_n)$ and $N = (n_1, \dots, n_n)$. Then

$$(2.16.2) \quad \begin{aligned} \det(U+N) &= (u_1+n_1) \wedge \dots \wedge (u_n+n_n) \\ &= u_1 \wedge \dots \wedge u_n + \sum_{i=1}^n u_1 \wedge \dots \wedge u_{i-1} \wedge n_i \wedge u_{i+1} \wedge \dots \wedge u_n. \end{aligned}$$

It follows from (2.16.2) and the column expansions that $\det(U+N) = Y_1^n + \sum_{i \leq j} n_{ij} U_{ij}$. From $U_{i,j} = U_{i+1,j+1}$ ($i \leq j$), we get (2.16.1).

(2) It follows from (2.16.2) and (2.16.1) that

$$(2.16.3) \quad \begin{aligned} \det(U' + N') &= \det(U_{m_1} + N_{11}) \det(U_{m_2} + N_{22}) \\ &= \left(Y_1^{m_1} + \sum_{k=1}^{m_1} \text{Tr}_{k-1} N_{11} (U_{m_1})_{1,k} \right) \left(Y_1^{m_2} + \sum_{k=1}^{m_2} \text{Tr}_{k-1} N_{22} (U_{m_2})_{1,k} \right) \\ &= Y_1^n + \sum_{k=1}^n (\text{Tr}_{k-1} N_{11} \cdot (U_{m_1})_{1,k} \cdot Y_1^{m_2} + \text{Tr}_{k-1} N_{22} \cdot (U_{m_2})_{1,k} \cdot Y_1^{m_1}). \end{aligned}$$

Then the statement follows from $(U_{m_2})_{1,k} \cdot Y_1^{m_1} = (U_{m_1})_{1,k} \cdot Y_1^{m_2} = U'_{1,k}$. □

PROPOSITION 2.17. — *Let (A, λ, ι) be a separably polarized abelian $O_{\mathbf{F}}$ -variety. Let $\text{Def}[A, \iota]^K$ denote the subfunctor of $\text{Def}[A, \iota]$ that classifies the objects satisfying the Kottwitz condition. Then $\text{Def}[A, \iota]^K(k[\epsilon]) = \text{Def}[A, \lambda, \iota](k[\epsilon])$.*

Proof. — Let notations be as in (2.14). Let $(\tilde{A}, \tilde{\iota})$ be the universal object over R , which is written as $\otimes_{i \in \mathbb{Z}/f\mathbb{Z}} R^i$. We want to compute the equations defined by the condition

$$\det \left(\sum_{j=1}^e \pi^{j-1} Y_j; \text{Lie}(\tilde{A})^i \right) = \det \left(\sum_{j=1}^e \pi^{j-1} Y_j; (O_{\mathbf{F}} \otimes k)^i \right)$$

in $\mathbb{F}_p[\underline{Y}]$, for $i \in \mathbb{Z}/f\mathbb{Z}$. As $[\widetilde{\text{Fil}}^i] = [\text{Lie}(\tilde{A})^i]$ and the right hand side is Y_1^e , it reduces to the condition

$$\det \left(\sum_{j=1}^e \pi^{j-1} Y_j; \widetilde{\text{Fil}}^i \right) = Y_1^e.$$

It suffices to compute the defining equations on each factor R^i . To ease notations, we suppress the index i .

We have

$$R = k[a_i, b_j, c_k, d_\ell ; 0 \leq i, k, \ell < e_1, 0 \leq j < e_2] / (a_i, b_j, c_k, d_\ell)^2,$$

$$\begin{aligned} \widetilde{\text{Fil}} &= \langle X_1, X_2 \rangle_{R[\pi]/(\pi^e)} \\ &= \langle X_1, \pi X_1, \dots, \pi^{e_2-1} X_1, X_2, \pi X_2, \dots, \pi^{e_1-1} X_2 \rangle_R, \end{aligned}$$

as a free R -module and write \mathcal{B} for this R -basis. We compute from (2.14) that

$$(2.17.1) \quad \begin{aligned} \text{for } e_2 \leq k < e, \quad \pi^k X_1 &= \sum_{j=k-e_1}^{e_2-1} a_{j-k+e_1} \pi^j X_1 + \sum_{j=k-e_2}^{e_1-1} b_{j-k+e_2} \pi^j X_2 \\ \text{for } e_1 \leq k < e, \quad \pi^k X_2 &= \sum_{j=k-e_1}^{e_2-1} c_{j-k+e_1} \pi^j X_1 + \sum_{j=k-e_1}^{e_1-1} d_{j-k+e_1} \pi^j X_2. \end{aligned}$$

For $k \geq 1$, let v_k (resp. v'_k) be the column vector for $\pi^{k-1} X_1$ (resp. $\pi^{k-1} X_2$) with respect to the basis \mathcal{B} . The vectors v_k (resp. v'_k) have coordinates in \mathfrak{m}_R except for $k \leq e_2$ (resp. for $k \leq e_1$). In the exceptional case, $v_k = E_k$ and $v'_k = E_{e_2+k}$, where $\{E_i\}$ is the standard basis. The representative matrix of the endomorphism π^{k-1} on the R -module $\widetilde{\text{Fil}}$ with respect to the basis \mathcal{B} is

$$[\pi^{k-1}] = (v_k, \dots, v_{k+e_2-1}, v'_k, \dots, v'_{k+e_1-1}).$$

We have

$$\sum_{j=1}^e [\pi^{j-1}] Y_j = (f_1, \dots, f_{e_2}, f'_1, \dots, f'_{e_1}),$$

where $f_k = \sum_{j=1}^e Y_j v_{j+k-1}$ and $f'_k = \sum_{j=1}^e Y_j v'_{j+k-1}$. Write

$$\begin{aligned} f_k &= u_k + n_k, & u_k &= \sum_{j=1}^{e_2-k+1} Y_j v_{j+k-1}, & n_k &= \sum_{j=e_2-k+2}^e Y_j v_{j+k-1}; \\ f'_k &= u'_k + n'_k, & u'_k &= \sum_{j=1}^{e_1-k+1} Y_j v'_{j+k-1} & n'_k &= \sum_{j=e_1-k+2}^e Y_j v'_{j+k-1} \end{aligned}$$

and put

$$U = (u_1, \dots, u_{e_2}, u'_1, \dots, u'_{e_1}),$$

$$N = (n_1, \dots, n_{e_2}, n'_1, \dots, n'_{e_1}) = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}.$$

Then by Lemma 2.16 (2), we have

$$(2.17.2) \quad \det \left(\sum_{j=1}^e [\pi^{j-1}] Y_j \right) = Y_1^e + \sum_{k=1}^{e_2} (\text{Tr}_{k-1} N_{11} + \text{Tr}_{k-1} N_{22}) U_{1,k}.$$

One directly computes that for $1 \leq k \leq e_1$,

$$\text{Tr}_{k-1} N_{11} + \text{Tr}_{k-1} N_{22} = \sum_{j=1}^{e_1-k+1} j(a_{e_1-k+1-j} + d_{e_1-k+1-j}) Y_j.$$

As the defining equations of $\text{Def}[A, \lambda, \iota](k[\epsilon])$ are $a_i + d_i = 0$ for $0 \leq i \leq e_1 - 1$, one has $\text{Def}[A, \lambda, \iota](k[\epsilon]) \subset \text{Def}[A, \iota]^K(k[\epsilon])$. Conversely, let $Y_i = 0$ for $i \geq 3$, we have

$$U_{1,k} = (-1)^{k+1} Y_1^{e-k} Y_2^{k-1},$$

$$\text{Tr}_{k-1} N_{11} + \text{Tr}_{k-1} N_{22} = (a_{e_1-k} + d_{e_1-k}) Y_1 + 2(a_{e_1-k-1} + d_{e_1-k-1}) Y_2.$$

By comparing the coefficients of (2.17.2) with Y_1^e , we obtain the equations $a_i + d_i = 0$ for $0 \leq i \leq e_1 - 1$, thus $\text{Def}[A, \lambda, \iota](k[\epsilon]) \supset \text{Def}[A, \iota]^K(k[\epsilon])$. This completes the proof. □

Remark 2.18. — (1) It is known [R, Prop. 1.9] that the forgetful morphism $\text{Def}[A, \lambda, \iota] \rightarrow \text{Def}[A, \iota]$ is formally étale if (A, λ, ι) satisfies the Rapoport condition, and that the Rapoport locus is smooth. In [DP, Thm 2.2], Deligne and Pappas concluded that the singular locus had codimension two. However, they actually showed that the complement of the Rapoport locus had codimension two. Proposition 2.15 fills the harmless gap of their assertion on the dimension of the singular locus.

(2) From (2.13.2) and (2.14.1), one can see that the forgetful morphism $\text{Def}[A, \lambda, \iota] \rightarrow \text{Def}[A, \iota]$ is not formally étale anymore when (A, λ, ι) does not satisfy the Rapoport condition. In this case, the first order universal deformation $(\tilde{A}, \tilde{\iota})$ of (A, ι) does not satisfy the Deligne-Pappas condition nor the Kottwitz determinant condition, but the condition (5') in (2.9) still holds for $(\tilde{A}, \tilde{\iota})$.

(3) It seems that the conditions (1')–(4') in (2.9) are equivalent when S is the spectrum of a complete local Noetherian ring. Proposition 2.17 shows some evidence. One can verify this by comparing the defining

equations from condition (2') and (4') in local charts, in the sense of Rapoport and Zink. In fact, it is not hard to verify the equivalence when $e = 2$. However, it is quite complicated in general using this method and we do not attempt to provide the proof here.

2.19. Let $\mathfrak{a}(A)$ denote the a -module of A , which is defined to be the cokernel of the Frobenius map F on $\text{Lie}(A) = M/VM$:

$$M/VM \xrightarrow{F} M/VM \rightarrow \mathfrak{a}(A) \rightarrow 0.$$

Note that in the covariant theory, the Frobenius map F is induced from the Verschiebung morphism $V : A^{(p)} \rightarrow A$ via the covariant functor.

Let each σ_i -component of $\mathfrak{a}(A)$ be

$$\mathfrak{a}(A)^i \simeq k[\pi]/(\pi^{a_1^i}) \oplus k[\pi]/(\pi^{a_2^i})$$

for some integers $\{a_1^i, a_2^i\}$. We define the a -type $\underline{a}(A)$ of A to be the invariant $\underline{a}(A) = (\{a_1^i, a_2^i\})_i$, (1.9). Let $|\underline{a}(A)|$ denote the total a -number of A , which is the dimension of the k -vector space $\text{Hom}_k(\alpha_p, A^t)$. If the abelian $\mathcal{O}_{\mathbf{F}}$ -variety (A, ι) satisfies the Rapoport condition, then $\underline{a}(A)$ is of the form $(\{0, a^i\})_i$ and we write $\underline{a}(A) = (a^i)_i$ instead.

LEMMA 2.20. — *The a -module $\mathfrak{a}(A)$ is canonically isomorphic to the k -linear dual $\text{Hom}_k(\alpha_p, A^t)^*$ of $\text{Hom}_k(\alpha_p, A^t)$ as $k \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbf{F}}$ -modules.*

Proof. — Let M^* denote the *contravariant* Dieudonné functor. Then we have

$$\begin{aligned} \text{Hom}_k(\alpha_p, A^t) &= \text{Hom}_{W[F,V]}(M^*(A^t), k) \simeq \text{Hom}_{W[F,V]}(M(A), k) \\ &= \text{Hom}_k(M(A)/(F, V)M(A), k) = \mathfrak{a}(A)^*. \end{aligned}$$

□

2.21. Let $\underline{a}(A) = (\{a_1^i, a_2^i\})_i$, with $a_1^i \leq a_2^i$, be the a -type of A and $\underline{e}(A) = (\{e_1^i, e_2^i\})_i$ be the Lie type. It follows from the elementary divisor lemma that there are two W^i -bases $\{x_1^i, x_2^i\}, \{y_1^i, y_2^i\}$ of M^i such that

$$VM^{i+1} = \langle \pi^{e_1^i} x_1^i, \pi^{e_2^i} x_2^i \rangle \quad \text{and} \quad FM^{i-1} = \langle \pi^{e-e_1^{i-1}} y_1^i, \pi^{e-e_2^{i-1}} y_2^i \rangle .$$

We may assume that $e_1^i \leq e_2^i$ hence $e - e_2^{i-1} \leq e - e_1^{i-1}$ for each i .

If $e_1^i \leq e - e_2^{i-1}$, write $y_1^i = ax_1^i + bx_2^i, y_2^i = cx_1^i + dx_2^i$, then

$$FM^{i-1} + VM^{i+1} = \langle \pi^{e_1^i} x_1^i, \pi^{e_2^i} x_2^i, \pi^{e-e_1^{i-1}} bx_2^i, \pi^{e-e_2^{i-1}} dx_2^i \rangle.$$

Note that one of b and d is a unit. From this we obtain a bound for a -types: $a_1^i = e_1^i$ and $\min\{e_2^i, e - e_2^{i-1}\} \leq a_2^i \leq \min\{e_2^i, e - e_1^{i-1}\}$ when $e_1^i \leq e - e_2^{i-1}$. Conversely if $e - e_2^{i-1} \leq e_1^i$ then we have $a_1^i = e - e_2^{i-1}$ and $\min\{e - e_1^{i-1}, e_1^i\} \leq a_2^i \leq \min\{e - e_1^{i-1}, e_2^i\}$.

2.22. If A is superspecial, that is $|\underline{a}(A)| = g = ef$. We know that $FM = VM$, in other words that $FM^{i-1} = VM^{i+1}$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. This gives $\{e_1^i, e_2^i\} = \{e - e_1^{i-1}, e - e_2^{i-1}\}$ and $\{a_1^i, a_2^i\} = \{e_1^i, e_2^i\}$ for all i . There are two possibilities:

1. If f is odd, then $\underline{a}(A) = \underline{e}(A) = (\{e_1, e_2\})_i$ for some nonnegative integers e_1 and e_2 with $e_1 + e_2 = e$.

2. If f is even, then there are two nonnegative integers e_1, e_2 , with $0 \leq e_1, e_2 \leq e$, so that $\underline{a}(A) = \underline{e}(A) = (\{e_1, e_2\}, \{e - e_1, e - e_2\}, \{e_1, e_2\}, \dots, \{e - e_1, e - e_2\})$.

LEMMA 2.23. — Let $\underline{a}(A) = (\{a_1^i, a_2^i\})_i$ and $\underline{e}(A) = (\{e_1^i, e_2^i\})_i$. Then $\underline{e}(A^t) = (\{e - e_1^i, e - e_2^i\})_i$ and the a -type $(\{b_1^i, b_2^i\})_i$ of the dual abelian variety A^t is given as follows: $b_1^i = \min\{e - e_1^i, e - e_2^i, e_1^{i-1}, e_2^{i-1}\}$ and $b_1^i + b_2^i = (a_1^i + a_2^i) + (e_1^{i-1} + e_2^{i-1}) - (e_1^i + e_2^i)$.

Proof. — Write $\overline{M} = M/pM$ and $\overline{M}^t = M^t/pM^t$, where M^t is the Dieudonné module of A^t . From the definition of $\underline{e}(A) = (\{e_1^i, e_2^i\})_i$, one concludes that VM^{i+1} is isomorphic to $k[\pi]/(\pi^{e-e_1^i}) \oplus k[\pi]/(\pi^{e-e_2^i})$. The perfect pairing $\overline{M} \times \overline{M}^t \rightarrow k$ induces a perfect pairing

$$V\overline{M}^{i+1} \times \overline{M}^{t,i}/V\overline{M}^{t,i+1} \rightarrow k,$$

thus $\underline{e}(A^t) = (\{e - e_1^i, e - e_2^i\})_i$.

Let $T^i := F\overline{M}^{i-1} \cap V\overline{M}^{i+1}$ in \overline{M}^i . We have $\dim_k T^i = \dim_k F\overline{M}^{i-1} + \dim_k V\overline{M}^{i+1} - \dim_k (F\overline{M}^{i-1} + V\overline{M}^{i+1}) = (a_1^i + a_2^i) + (e_1^{i-1} + e_2^{i-1}) - (e_1^i + e_2^i)$. The perfect pairing $\overline{M} \times \overline{M}^t \rightarrow k$ induces a perfect pairing

$$T^i \times (M^t/(F, V)M^t)^i \rightarrow k,$$

thus $b_1^i + b_2^i = (a_1^i + a_2^i) + (e_1^{i-1} + e_2^{i-1}) - (e_1^i + e_2^i)$. The assertion $b_1^i = \min\{e - e_1^i, e - e_2^i, e_1^{i-1}, e_2^{i-1}\}$ is obtained from (2.21) and $\underline{e}(A^t) = (\{e - e_1^i, e - e_2^i\})_i$. □

3. Formal isogeny classes.

In the rest of this paper, k denotes an algebraically closed field of characteristic $p > 0$.

LEMMA 3.1. — *Let M be a quasi-polarized Dieudonné \mathcal{O} -module. Then the slope sequence $\text{slope}(M)$ of M is either $\{\frac{i}{g}, \dots, \frac{i}{g}, \frac{g-i}{g}, \dots, \frac{g-i}{g}\}$ for some integer $0 \leq i \leq g/2$, or $\{\frac{1}{2}, \dots, \frac{1}{2}\}$.*

Proof. — Suppose that M is not supersingular. The F-isocrystal $M \otimes B(k)$ contains $M_{a,b}^r \oplus M_{b,a}^r$, where $M_{a,b}$ is the simple F-isocrystal of single slope $a/(a+b)$ for some integers a, b . We want to show that $2(a+b)r = 2g$. We first have an embedding $\mathbf{F}_v \rightarrow \text{End}(M_{a,b}^r) = M_r(D_{a/(a+b)})$ and any maximal commutative subalgebra of the latter has \mathbb{Q}_p -dimension $r(a+b)$, so $g|r(a+b)$. On the other hand, we have $2g \geq 2r(a+b)$, from $M \otimes B(k) \supset M_{a,b}^r \oplus M_{b,a}^r$. Hence $M \otimes B(k) = M_{a,b}^r \oplus M_{b,a}^r$. This completes the proof. □

3.2. Let $S(g)$ denote the subset of \mathbb{Q} which parameterizes possible slope sequences arising from abelian $O_{\mathbf{F}}$ -varieties

$$S(g) := \left\{ i \in \mathbb{Z}; 0 \leq i \leq \frac{g}{2} \right\} \cup \left\{ \frac{g}{2} \right\}.$$

For each $i \in S(g)$, we denote by $s(i)$ the slope sequence $\{\frac{i}{g}, \dots, \frac{i}{g}, \frac{g-i}{g}, \dots, \frac{g-i}{g}\}$. The (linear) order on $S(g)$ induced from \mathbb{Q} is compatible with the Grothendieck specialization theorem.

3.3. *Example.* Let $M = M^0 \oplus \dots \oplus M^{f-1}$ be a Dieudonné \mathcal{O} -module, where M^i is a free W^i -module of rank two generated by x_1^i, x_2^i with $x_2^i \in VM^{i+1}$. Let

$$(3.3.1) \quad \begin{aligned} Fx_1^i &= a_{11}^{i+1}x_1^{i+1} + a_{12}^{i+1}x_2^{i+1} \\ Fx_2^i &= pa_{21}^{i+1}x_1^{i+1} + pa_{22}^{i+1}x_2^{i+1} \end{aligned}$$

be the action of the Frobenius F . Then

$$(3.3.2) \quad \begin{aligned} F^f x_1^0 &= \alpha x_1^0 + \beta x_2^0 \\ F^f x_2^0 &= \gamma x_1^0 + \delta x_2^0, \end{aligned}$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = A_1^{(f-1)} \cdots A_{f-1}^{(1)} A_f, \quad A_i := \begin{pmatrix} a_{11}^i & a_{12}^i \\ pa_{21}^i & pa_{22}^i \end{pmatrix}, \quad A_f = A_0.$$

Here we write $A^{(n)}$ for A^{σ^n} . Note that M satisfies the Rapoport condition.

Let $a, b \in \mathbb{Z}, a + b = g, 0 \leq a \leq b$. Write $a = de + r, 0 \leq r < e$. Take

$$A_i = \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} \text{ for } 1 \leq i \leq 2d,$$

$$A_i = \begin{pmatrix} 1 & 1 \\ -p & 0 \end{pmatrix} \text{ for } 2d < i < f, \text{ and } A_f = \begin{pmatrix} \pi^r & 1 \\ -p & 0 \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (-1)^d p^d \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}, \quad \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \prod_{i=2d+1}^f A_i \equiv \begin{pmatrix} \pi^r & 1 \\ 0 & 0 \end{pmatrix} \pmod{p}.$$

Note that $A_i^\sigma = A_i$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. The characteristic polynomial of F^f is $X^2 - (\alpha + \delta)X + p^f$ and $\text{ord}_p(\alpha + \delta) = \frac{a}{e}$. Therefore $\text{slope}(M) = \{ \frac{a}{g}, \dots, \frac{a}{g}, \frac{b}{g}, \dots, \frac{b}{g} \}$.

Remark 3.4. — The example constructed above is non-trivial, due to the following constraint. Let $M = M_{a,b} \oplus M_{b,a}, a + b = g, a \neq b$ be a Dieudonné \mathcal{O} -module which satisfies the Rapoport condition, where $M_{a,b}$ (resp. $M_{b,a}$) is a Dieudonné submodule of single slope $\frac{b}{a+b}$ (resp. $\frac{a}{a+b}$). Then a is a multiple of e . Therefore, one can not construct a special Dieudonné module, in the sense of Manin, with an \mathcal{O} -action which satisfies the Rapoport condition and has arbitrary possible slope sequences in Lemma 3.1. The proof of this fact is as follows.

Let $M_{a,b} = M^0 \oplus \dots \oplus M^{f-1}$ as in (2.2), where M^i is a free W^i -module of rank one. We can choose a basis x_i for M^i such that $Vx_{i+1} = \pi^{n_i} x_i$ for $1 \leq i \leq f - 1$ and $Vx_1 = u_0 \pi^{n_0} x_0$, for some $u_0 \in (W^0)^\times$. As M satisfies the Rapoport condition, we have $n_i = 0$ or e . As the slope of $M_{a,b}$ is $\frac{b}{a+b}$, we have $V^f x_0 = \pi^a u x_0$ for some unit u hence $a = n_0 + \dots + n_{f-1}$, which is a multiple of e .

3.5. Let V be a 2-dimensional \mathbf{F}_v -vector space with a non-degenerate alternating form ψ on V with values in \mathbb{Q}_p such that $\psi(ax, y) = \psi(x, ay)$ for $a \in \mathbf{F}_v$, $x, y \in V$. Let G be the algebraic group of \mathbf{F}_v -linear similitudes over \mathbb{Q}_p and let G_1 be the derived group of G . The algebraic group G_1 is simply-connected and it is the kernel of the multiplier map $c : G \rightarrow \mathbb{G}_m$.

Let M be a quasi-polarized Dieudonné \mathcal{O} -module. We choose an isomorphism between $M \otimes_W B(k)$ and $V \otimes_{\mathbb{Q}_p} B(k)$ for skew-symmetric \mathbf{F}_v -modules over $B(k)$. Let $b \in G(B(k))$ be the element obtained by the transport of structure of the Frobenius F on M . The σ -conjugacy class $B(G)$ classifies the (F) -isocrystals with G -structure. The fibre of $\nu(b)$ under the Newton map $\nu : B(G) \rightarrow S(g)$ is classified by $H^1(\mathbb{Q}_p, J_b)$ [RR, Prop. 1.17], where J_b be the algebraic group over \mathbb{Q}_p which represents the group functor [RZ, Prop. 1.12]

$$R \mapsto \{g \in G(R \otimes_{\mathbb{Q}_p} B(k)); g(b\sigma) = (b\sigma)g\}.$$

LEMMA 3.6. — $H^1(\mathbb{Q}_p, J_b) = 0$.

Proof. — Let $N := M \otimes B(k)$ be the isocrystal with G -structure. The group J_b is $\text{Aut}_G(N)$, viewed as an algebraic group over \mathbb{Q}_p . If M is supersingular, then $\text{Aut}_G(N)$ is the multiplicative group of a quaternion algebra over \mathbf{F}_v with reduced norm in \mathbb{Q}_p . If M is not supersingular, then $\text{Aut}_G(N) = \mathbf{F}_v^\times \times \mathbb{Q}_p^\times$. In both cases, $H^1(\mathbb{Q}_p, J_b) = 0$. □

COROLLARY 3.7. — *Let M_1 and M_2 be two quasi-polarized Dieudonné \mathcal{O} -modules. If $\text{slope}(M_1) = \text{slope}(M_2)$, then $M_1 \otimes_W B(k)$ and $M_2 \otimes_W B(k)$ are isomorphic as quasi-polarized isocrystals with the action by \mathbf{F}_v .*

COROLLARY 3.8. — *Any polarized abelian $O_{\mathbf{F}}$ -variety over k is isogenous to one which satisfies the Rapoport condition.*

Proof. — It follows from (3.3) and Corollary 3.7 that the statement holds for the associated p -divisible group. Then the assertion follows from a theorem of Tate. □

4. Normal forms.

4.1. Let M be a non-ordinary separably quasi-polarized Dieudonné \mathcal{O} -module over k which satisfies the Rapoport condition. Let $\underline{a}(M) = (a^i)_i$

be the a -type of M . Note that the Lie type of M is $(\{0, e\})_i$ and the constraint for the a -type is $0 \leq a^i \leq e$. Denote by $\tau(M)$ the a -index of M , which is defined to be the subset of $\mathbb{Z}/f\mathbb{Z}$:

$$\tau(M) := \{i \in \mathbb{Z}/f\mathbb{Z}; a^i \neq 0\}.$$

Write τ for $\tau(M)$.

Let $\mathcal{D}^{-1} = (\pi^{-d})$ be the inverse of the different of \mathcal{O} over \mathbb{Z}_p . There is a unique $W \otimes \mathcal{O}$ -bilinear pairing $(,) : M \times M \rightarrow W \otimes \mathcal{O}$ such that $\langle x, y \rangle = \text{Tr}_{W \otimes \mathcal{O}/W}(\pi^{-d}(x, y))$. Write $\overline{M} := M/\pi M$. The module \overline{M} is a $2f$ -dimensional vector space over k together with a k -linear action by $\mathcal{O}/\pi = \mathcal{O}^{\text{ur}}/p$ that commutes with the action of F and V . The perfect pairing $(,)$ on M induces an perfect pairing $(,)$ on \overline{M} that satisfies $(Fx, y) = (x, Vy)^p$ and $(ax, y) = (x, ay)$ for all $x, y \in \overline{M}, a \in \mathcal{O}^{\text{ur}}/p$. The decomposition of \overline{M} (2.2) into σ_i -eigenspaces

$$\overline{M} = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} \overline{M}^i$$

respects the perfect pairing as before (2.2).

As M satisfies the Rapoport condition, $V\overline{M}$ and $F\overline{M}$ are free $k \otimes \mathcal{O}^{\text{ur}}$ -module of rank one, and we have that $F\overline{M} = \ker V$ and $V\overline{M} = \ker F$. Consider \overline{M} as a $k[F, V]$ -module, it is isomorphic to N/pN for a separably quasi-polarized Dieudonné \mathcal{O}^{ur} -module N over k of rank $2f$ with the same a -index as M .

PROPOSITION 4.2. — *There exists a k -basis $\{x_i, y_i\}$ of \overline{M}^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that*

- $(x_i, y_i) = 1$ and $y_i \in V\overline{M}$ for all $i \in \mathbb{Z}/f\mathbb{Z}$,

- $F x_i = \begin{cases} x_{i+1} & \text{if } i + 1 \notin \tau \\ -y_{i+1} & \text{if } i + 1 \in \tau \end{cases}$

$$V y_i = \begin{cases} y_{i-1} & \text{if } i \notin \tau \\ 0 & \text{if } i \in \tau \end{cases} \quad V x_i = \begin{cases} y_{i-1} & \text{if } i \in \tau \\ 0 & \text{if } i \notin \tau \end{cases} .$$

Proof. — It follows from [Y1, Prop. 4.1]. □

Note that this proposition gives the classification of the π -torsion subgroup scheme $A[\pi]$ of separably polarized abelian $O_{\mathbf{F}}$ -varieties over k , classified by the a -indices.

The following results generalize [Y1, Prop. 4.1, Prop. 4.2, Lemma 4.3], the proofs are the same and omitted.

PROPOSITION 4.3. — *There exists a W^i -basis $\{X_i, Y_i\}$ of M for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that*

$$\begin{aligned} & \text{(i) } (X_i, Y_i) = 1 \text{ and } Y_i \in (VM)^i, \forall i \in \mathbb{Z}/f\mathbb{Z}, \\ & \text{(ii) } FX_i = \begin{cases} X_{i+1} & \text{if } i+1 \notin \tau \\ -Y_{i+1} + c_{i+1}\pi X_{i+1} & \text{if } i+1 \in \tau, \end{cases} \\ & FY_i = \begin{cases} pY_{i+1} & \text{if } i+1 \notin \tau \\ pX_{i+1} & \text{if } i+1 \in \tau, \end{cases} \\ & \text{for some } c_{i+1} \in W^{i+1}. \end{aligned}$$

PROPOSITION 4.4. — *There exists a W^i -basis $\{X_i, Y_i\}$ of M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that*

$$\begin{aligned} & \text{(i) } (X_i, Y_i) \in (W^i)^\times \text{ and } Y_i \in (VM)^i, \forall i \in \mathbb{Z}/f\mathbb{Z}, \\ & \text{(ii) } FX_i = \begin{cases} X_{i+1} & \text{if } i+1 \notin \tau \\ Y_{i+1} + c_{i+1}\pi X_{i+1} & \text{if } i+1 \in \tau, \end{cases} \\ & FY_i = \begin{cases} \pi^e Y_{i+1} & \text{if } i+1 \notin \tau \\ \pi^e X_{i+1} & \text{if } i+1 \in \tau, \end{cases} \\ & \text{for some } c_{i+1} \in W^{i+1}. \end{aligned}$$

Note that the i -th component a^i of $\underline{a}(M)$ is $\min\{e, \text{ord}_\pi(c_i\pi)\}$ for $i \in \tau$.

LEMMA 4.5. — *If M is superspecial (2.22) (in this case $a^i = e$, $\forall i \in \mathbb{Z}/f\mathbb{Z}$), then*

(1) *There exists a W^i -basis $\{X_i, Y_i\}$ of M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that*

- $Y_i \in (VM)^i$ and $(X_i, Y_i) = 1$,
- $FX_i = -Y_{i+1}$, $FY_i = pX_{i+1}$,

for all $i \in \mathbb{Z}/f\mathbb{Z}$.

(2) *There exists a W^i -basis $\{X_i, Y_i\}$ of M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that*

- $Y_i \in (VM)^i$ and $(X_i, Y_i) = u_i \in (W^i)^\times$ with $u_i^{\sigma^r} = u_i$, where $r = \text{lcm}(2, f)$,
- $FX_i = Y_{i+1}$, $FY_i = pX_{i+1}$,

for all $i \in \mathbb{Z}/f\mathbb{Z}$.

For applications, we need to classify all quasi-polarized superspecial Dieudonné \mathcal{O} -modules, not just separably-polarized ones or those satisfying the Rapoport condition.

LEMMA 4.6. — *Let M be a quasi-polarized superspecial Dieudonné \mathcal{O} -module over k and let e_1, e_2 be as in (2.22).*

(1) *If f is even, then there is a W^i -basis X_i, Y_i for M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that*

$$(i) (X_i, Y_i) = \begin{cases} \pi^n & \text{if } i \text{ is odd,} \\ \pi^{n+e-e_1-e_2} & \text{if } i \text{ is even,} \end{cases} \text{ for all } i \in \mathbb{Z}/f\mathbb{Z} \text{ and}$$

some $n \in \mathbb{Z}$.

$$(ii) FX_i = \begin{cases} -\pi^{e_1}Y_{i+1} & \text{if } i \text{ is odd,} \\ -\pi^{e-e_2}Y_{i+1} & \text{if } i \text{ is even,} \end{cases}$$

$$FY_i = \begin{cases} v\pi^{e_2}X_{i+1} & \text{if } i \text{ is odd,} \\ v\pi^{e-e_1}X_{i+1} & \text{if } i \text{ is even,} \end{cases}$$

where $v\pi^e = p$.

(2) *If f is odd, then there is a W^i -basis X_i, Y_i for M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that*

$$(i) (X_i, Y_i) = \pi^n \text{ for some } n \in \mathbb{Z}.$$

$$(ii) FX_i = -\pi^{e_1}Y_{i+1}, FY_i = v\pi^{e_2}Y_{i+1} \text{ for } i \in \mathbb{Z}/f\mathbb{Z},$$

where $v\pi^e = p$.

Proof. — The proof is similar to that of [Y1, Lemma 4.3], hence is sketched.

(1) Write $f = 2c$ and let $M' := \{x \in M \mid F^c x = (-1)^c V^c x\}$. Since M is superspecial, we have $F^2 M' = pM'$ and $M' \otimes_{W(\mathbb{F}_{p,f})} W(k) \simeq M$. We can choose bases $\{X_0, Y_0\}, \{X_1, Y_1\}$ for $(M')^0, (M')^1$ respectively such that $FX_0 = -\pi^{e-e_2}Y_1, FY_0 = v\pi^{e-e_1}X_1$, and $(X_1, Y_1) = \pi^n$ for some $n \in \mathbb{Z}$.

Define X_i, Y_i recursively for $2 \leq i \leq f$ by (ii). Then it is straight forward to verify that $X_f = X_0, Y_f = Y_0$ and (i).

(2) Write $f = 2c + 1$ and let $M' := \{x \in M \mid F^{2f} x + p^f x = 0\}$. From (ii), Y_0 is required to be $(-1)^{c+1} \pi^{-e_1} p^{-c} F^f X_0$. Let X_0, Z_0 be a basis for $(M')^0$ such that $FX_0 = \pi^{e_1} X'_1, FZ_0 = \pi^{e_2} Y'_1$ for some basis $\{X'_1, Y'_1\}$. Let $Y_0 := (-1)^{c+1} \pi^{-e_1} p^{-c} F^f X_0$. If $e_1 < e_2$, then X_0, Y_0 form a basis. If $e_1 = e_2$, then we can choose again X_0 such that X_0, Y_0 form a basis for $(M')^0$. Define X_i, Y_i recursively for $1 \leq i \leq f$ by (ii) and it is easy to check that $X_f = X_0$ and $Y_f = Y_0$. We found a basis satisfying (ii).

Write $(X_0, Y_0) = \mu\pi^n$ for some unit μ and some integer n . It follows from $(F^f X_0, F^f Y_0) = p^f(X_0, Y_0)^{\sigma^f}$ that $\mu^{\sigma^f} = \mu$. If we replace X_0 by λX_0 , then μ will change to $\lambda\lambda^{\sigma^f}\mu$. Since $B(\mathbb{F}_{p^{2f}})[\pi]/B(\mathbb{F}_{p^f})[\pi]$ is a quadratic unramified extension, we can adjust X_0 by choosing a suitable λ such that $(X_0, Y_0) = \pi^n$, hence (i) is satisfied. \square

4.7. Let M be a non-ordinary separably quasi-polarized Dieudonné \mathcal{O} -module satisfying the Rapoport condition. Let $\underline{a}(M) = (a^i)$ be the a -type of M and $\tau = \tau(M) = \{n_1, n_2, \dots, n_t\}$ be the a -index of M , where $t = |\underline{a}(M/\pi M)|$, the reduced a -number of M . We assume that $0 \leq n_1 < n_2 < \dots < n_t < f$ and let $n_0 := n_t - f$ and $n_{t+1} = n_1$. Denote by $A(e, f)$ the set of possible a -types on the Rapoport locus

$$A(e, f) := \{(a^i); i \in \mathbb{Z}/f\mathbb{Z}, a^i \in \mathbb{Z}, 0 \leq a^i \leq e\},$$

with the partial order that $(a^i) \leq (b^i)$ if $a^i \leq b^i, \forall i \in \mathbb{Z}/f\mathbb{Z}$. An a -type $(a^i) \in A(e, f)$ is called spaced if $a^i a^{i+1} = 0, \forall i \in \mathbb{Z}/f\mathbb{Z}$, cf. [GO, Sect. 1, p.112].

By Proposition 4.4, we can choose a W^i -basis $\{X_i, Y_i\}$ of M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that

$$\begin{pmatrix} FX_{i-1} \\ FY_{i-1} \end{pmatrix} = A_i \begin{pmatrix} X_i \\ Y_i \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} 1 & 0 \\ 0 & \pi^e \end{pmatrix} \text{ if } i \notin \tau; \quad A_i = \begin{pmatrix} \pi^{a^i} c_i & 1 \\ \pi^e & 0 \end{pmatrix} \text{ if } i \in \tau,$$

for some $c_i \in W^i$ for $i \in \tau$ and c_i is a unit if $a^i < e$.

Let $\ell_i := n_i - n_{i-1}$ for $1 \leq i \leq t$. We have

$$\begin{pmatrix} F^{\ell_i} X_{n_{i-1}} \\ F^{\ell_i} Y_{n_{i-1}} \end{pmatrix} = \begin{pmatrix} \pi^{a^i} c_{n_i} & 1 \\ \pi^{e\ell_i} & 0 \end{pmatrix} \begin{pmatrix} X_{n_i} \\ Y_{n_i} \end{pmatrix}, \quad \forall 1 \leq i \leq t.$$

For $1 \leq s \leq t$, suppose that

$$\begin{pmatrix} F^{\ell_s + \ell_{s+1} + \dots + \ell_t} X_{n_{s-1}} \\ F^{\ell_s + \ell_{s+1} + \dots + \ell_t} Y_{n_{s-1}} \end{pmatrix} = \begin{pmatrix} \alpha_s & \beta_s \\ \gamma_s & \delta_s \end{pmatrix} \begin{pmatrix} X_{n_t} \\ Y_{n_t} \end{pmatrix},$$

for some coefficients $\alpha_s, \beta_s, \gamma_s, \delta_s$. It follows from

$$\begin{aligned}
 (4.7.1) \quad & \begin{pmatrix} F^{\ell_s+\ell_{s+1}+\dots+\ell_t} X_{n_{s-2}} \\ F^{\ell_s+\ell_{s+1}+\dots+\ell_t} Y_{n_{s-2}} \end{pmatrix} = \begin{pmatrix} F^{\ell_s+\ell_{s+1}+\dots+\ell_t} (F^{\ell_{s-1}} X_{n_{s-2}}) \\ F^{\ell_s+\ell_{s+1}+\dots+\ell_t} (F^{\ell_{s-1}} Y_{n_{s-2}}) \end{pmatrix} \\
 & = \begin{pmatrix} \pi^{a^{n_{s-1}}} c_{n_{s-1}}^{(\ell_s+\ell_{s+1}+\dots+\ell_t)} & 1 \\ \pi^{e\ell_{s-1}} & 0 \end{pmatrix} \begin{pmatrix} F^{\ell_s+\ell_{s+1}+\dots+\ell_t} X_{n_{s-1}} \\ F^{\ell_s+\ell_{s+1}+\dots+\ell_t} Y_{n_{s-1}} \end{pmatrix}
 \end{aligned}$$

that

$$\begin{aligned}
 (4.7.2) \quad & \begin{pmatrix} \alpha_{s-1} & \beta_{s-1} \\ \gamma_{s-1} & \delta_{s-1} \end{pmatrix} = \begin{pmatrix} u_{s-1} & 1 \\ \pi^{e\ell_{s-1}} & 0 \end{pmatrix} \begin{pmatrix} \alpha_s & \beta_s \\ \gamma_s & \delta_s \end{pmatrix}, \\
 & \begin{pmatrix} \alpha_t & \beta_t \\ \gamma_t & \delta_t \end{pmatrix} = \begin{pmatrix} \pi^{a^{n_t}} c_{n_t} & 1 \\ \pi^{e\ell_t} & 0 \end{pmatrix},
 \end{aligned}$$

where $u_{s-1} = \pi^{a^{n_{s-1}}} c_{n_{s-1}}^{(\ell_s+\ell_{s+1}+\dots+\ell_t)}$. Recall that we write $a^{(n)}$ for a^{σ^n} (3.3). Therefore,

$$(4.7.3) \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \begin{pmatrix} u_1 & 1 \\ \pi^{e\ell_1} & 0 \end{pmatrix} \begin{pmatrix} u_2 & 1 \\ \pi^{e\ell_2} & 0 \end{pmatrix} \cdots \begin{pmatrix} u_t & 1 \\ \pi^{e\ell_t} & 0 \end{pmatrix}.$$

4.8. Consider the case that $c_i = 0$ for all $i \in \tau$. If t is even, write $t = 2d$, then we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \pi^{e(\ell_2+\ell_4+\dots+\ell_{2d})} & 0 \\ 0 & \pi^{e(\ell_1+\ell_3+\dots+\ell_{2d-1})} \end{pmatrix}$$

and $\text{slope}(M) = s(i)$ (3.2), where $i = \min\{e(\sum_{1 \leq j \leq d} \ell_{2j}), e(\sum_{1 \leq j \leq d} \ell_{2j-1})\}$. If t is odd, write $t = 2d + 1$, then we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^2 = \begin{pmatrix} \pi^g & 0 \\ 0 & \pi^g \end{pmatrix}$$

and M is supersingular.

PROPOSITION 4.9. — *Let M be a separably quasi-polarized Dieudonné \mathcal{O} -module over k that satisfies the Rapoport condition. If $\underline{a}(M)$ is spaced, that is $\ell_i \geq 2$ for all $1 \leq i \leq t$, then $\text{slope}(M) \geq s(|\underline{a}(M)|)$ (3.2).*

Proof. — We may assume that M is non-ordinary. It follows from (4.7.3) that

$$\begin{pmatrix} F^f \pi^e X_{n_t} \\ F^f Y_{n_t} \end{pmatrix} = A \begin{pmatrix} \pi^e X_{n_t} \\ Y_{n_t} \end{pmatrix}, \quad A = \prod_{i=1}^t A_i = \pi^{|\underline{a}(M)|} \prod_{i=1}^t (\pi^{-a_{n_i}} A_i),$$

$$A_i = \begin{pmatrix} u_i & \pi^e \\ \pi^{e(\ell_i-1)} & 0 \end{pmatrix}.$$

Let N be the Dieudonné \mathcal{O} -submodule of M generated by $\pi^e X_{n_t}, Y_{n_t}$. As $F^f(N) \subset \pi^{|\underline{a}(M)|} N$, we get $\text{slope}(M) \geq s(|\underline{a}(M)|)$. \square

In general the slope sequence $\text{slope}(M)$ is not determined by $\underline{a}(M)$. This is known even in the unramified case [GO]. We will see in (6.8) that the bound in (4.9) is sharp for spaced a -types, while it is not the case for non-spaced ones, see (6.10).

4.10. When $t = |\tau| = 1$, say $\tau = \{0\}$, we have

$$\begin{pmatrix} F^f X_0 \\ F^f Y_0 \end{pmatrix} = \begin{pmatrix} c_0 & 1 \\ \pi^g & 0 \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}$$

for some $c_0 \in \pi^{a_0} W^0$. It is easy to see that M satisfies a Cayley-Hamilton equation $([O2],[Y1]) F^{2f} X_0 - c_0^{(f)} F^f X_0 - \pi^g X_0 = 0$, thus $\text{slope}(M) = s(i)$, where $i = \min\{\frac{g}{2}, \text{ord}_\pi(c_0)\}$.

4.11. When $t = |\tau| = 2$, say $\tau = \{0, \ell_2\}$ and $\ell_2 \leq \ell_1$, we have

$$\begin{pmatrix} F^f X_{\ell_1} \\ F^f Y_{\ell_1} \end{pmatrix} = \begin{pmatrix} u_1 & 1 \\ \pi^{e\ell_1} & 0 \end{pmatrix} \begin{pmatrix} u_2 & 1 \\ \pi^{e\ell_2} & 0 \end{pmatrix} \begin{pmatrix} X_{\ell_1} \\ Y_{\ell_1} \end{pmatrix},$$

for some coefficients u_1, u_2 ($u_1 = c_0^{(\ell_2)}, u_2 = c_{\ell_2}$). Then

$$(4.11.1) \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} u_1 u_2 + \pi^{e\ell_2} & u_1 \\ \pi^{e\ell_2} u_2 & \pi^{e\ell_2} \end{pmatrix}.$$

If $u_1 = 0$, then the matrix is $\begin{pmatrix} \pi^{e\ell_2} & 0 \\ \pi^{e\ell_1} u_2 & \pi^{e\ell_1} \end{pmatrix}$, hence $\text{slope}(M) = s(e\ell_2)$. If $u_1 \neq 0$, then we have the Cayley-Hamilton equation

$$\beta F^{2f} X_{\ell_1} - (\beta^{(f)} \delta + \beta \alpha^{(f)}) F^f X_{\ell_1} + \beta^{(f)} (\alpha \delta - \beta \delta) X_{\ell_1} = 0.$$

As $\text{ord}_\pi(\delta) = e\ell_1 \geq \frac{g}{2}$, we see that $\text{slope}(M) = s(i)$, where $i = \min\{\frac{g}{2}, \text{ord}_\pi(a^{(f)})\}$. In both cases we have that $\text{slope}(M) = s(i)$, where $i = \min\{\frac{g}{2}, \text{ord}_\pi(u_1u_2 + \pi^{e\ell_2})\}$.

5. Alpha stratification.

5.1. Let p be a fixed prime number. Let \mathcal{M}^{DP} denote the moduli stack over $\text{Spec}\mathbb{Z}_{(p)}$ of polarized abelian $O_{\mathbf{F}}$ -varieties (A, λ, ι) of dimension $g = [F : \mathbb{Q}]$ with the polarization λ of prime-to- p degree. It is a separated Deligne-Mumford algebraic stack over $\text{Spec}\mathbb{Z}_{(p)}$ locally of finite type. In [DP], Deligne and Pappas showed that the algebraic stack \mathcal{M}^{DP} is flat and a locally complete intersection over $\text{Spec}\mathbb{Z}_{(p)}$ of relative dimension g , and the closed fibre $\mathcal{M}^{\text{DP}} \otimes_{\mathbb{F}_p}$ is geometrically normal and has singularities of codimension at least two. It follows from Deligne-Pappas' results and the compactification of Rapoport that the irreducible components of geometric special fibre $\mathcal{M}^{\text{DP}} \otimes_{\overline{\mathbb{F}}_p}$ are in bijection correspondence with those of geometric generic fibre $\mathcal{M}^{\text{DP}} \otimes_{\overline{\mathbb{Q}}}$. Those are parameterized by the isomorphism classes of non-degenerate skew-symmetric $O_{\mathbf{F}}$ -modules $H_1(A(\mathbb{C}), \mathbb{Z})$ for all $(A, \lambda, \iota) \in \mathcal{M}^{\text{DP}}(\mathbb{C})$.

Let \mathcal{M}^R denote the Rapoport locus of \mathcal{M}^{DP} . It is the smooth locus of \mathcal{M}^{DP} (2.15). Let \mathcal{M} denote the reduction $\mathcal{M}^R \otimes_{\mathbb{Z}} k(v)$ of \mathcal{M}^R modulo v , where $k(v)$ is the residue field of $O_{\mathbf{F}}$ at v . We will define the stratification on \mathcal{M} by a -types scheme-theoretically, (cf. [Y1, Sect. 3]).

5.2. Let S be a locally noetherian scheme over $\text{Spec}k(v)$ and $\pi_A : (A, \lambda, \iota) \rightarrow S \in \mathcal{M}(S)$ a polarized abelian $O_{\mathbf{F}}$ -scheme over S . The sheaf $R^1(\pi_A)_*(\mathcal{O}_A)$ is a locally free rank one $O_{\mathbf{F}} \otimes O_S$ -module. It admits a decomposition (2.2),

$$R^1(\pi_A)_*(\mathcal{O}_A) = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} R^1(\pi_A)_*(\mathcal{O}_A)^i$$

with respect to the action by $O_{\mathbf{F}}$. Each component $R^1(\pi_A)_*(\mathcal{O}_A)^i$ is a locally free $O_S[\pi]/(\pi^e)$ -module of rank one and it admits a filtration of locally free O_S -modules

$$R^1(\pi_A)_*(\mathcal{O}_A)^i = \mathcal{F}^{i,0} \supset \mathcal{F}^{i,1} \supset \dots \supset \mathcal{F}^{i,e} = 0, \quad \mathcal{F}^{i,j} := \ker \pi^{e-j}.$$

Let $F_{A/S} : A \rightarrow A^{(p)}$ be relative Frobenius morphism over S , where $A^{(p)} := A \times_{S, F_{\text{abs}}} S$ and F_{abs} is the absolute Frobenius morphism on S . It induces an $\mathcal{O}_S \otimes \mathcal{O}_{\mathbf{F}}$ -linear morphism $F_{A/S}^* : R^1(\pi_A)_*(\mathcal{O}_A)^{(p)} \rightarrow R^1(\pi_A)_*(\mathcal{O}_A)$.

Let $\underline{a} = (a^i) \in A(e, f)$ (4.7) be an a -type. Let $\mathcal{M}_{\geq \underline{a}}$ denote the substack of \mathcal{M} whose objects $\pi_A : (A, \lambda, \iota) \rightarrow S$ satisfy the following condition:

$$(5.2.1) \quad F_{A/S}^* \left(R^1(\pi_A)_*(\mathcal{O}_A)^{(p), i} \right) \subset \mathcal{F}^{i, a^i}, \quad \forall i \in \mathbb{Z}/f\mathbb{Z}.$$

It is clear that the condition (5.2.1) is closed and locally for Zariski topology $\mathcal{M}_{\geq \underline{a}}$ is defined by finitely many equations. Therefore $\mathcal{M}_{\geq \underline{a}}$ is a closed algebraic substack of \mathcal{M} .

It is clear that $\mathcal{M}_{\geq \underline{a}}$ is non-empty as it contains $E \otimes \mathcal{O}_{\mathbf{F}}$ with a prime-to- p polarization, where E is a supersingular elliptic curve.

LEMMA 5.3. — *Let $(A, \lambda, \iota) \in \mathcal{M}(k)$ and $\underline{a}(A) = (a^i)$ be the a -type of A (2.19). Then $\text{coker} F_{A/k}^* = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} k[\pi]/(\pi^{a^i})$.*

Proof. — Let M^* denote the contravariant Dieudonné module of A . We identify $M^*/pM^* = H_{DP}^1(A/k)$ and have $H^1(A, \mathcal{O}_A) = M^*/VM^*$ and $\text{coker} F_{A/k}^* = M^*/(F, V)M^*$. On the other hand, the quasi-polarization gives an isomorphism $M \simeq M^*$ of Dieudonné \mathcal{O} -modules, which induces an isomorphism $M/(F, V)M \simeq M^*/(F, V)M^*$ of $\mathcal{O}_{\mathbf{F}} \otimes k$ -modules. □

THEOREM 5.4. — *The algebraic stack $\mathcal{M}_{\geq \underline{a}}$ is smooth over $\text{Spec } k(v)$ of pure dimension $g - |\underline{a}|$.*

Proof. — Let $x : \text{Spec } k \rightarrow \mathcal{M}_{\geq \underline{a}}$ be a geometric point. Then there is an affine open neighborhood U of x and a polarized abelian $\mathcal{O}_{\mathbf{F}}$ -scheme $(A, \lambda, \iota) \in \mathcal{M}(U)$ such that $R^1(\pi_A)_*(\mathcal{O}_A)^{(p)}$ and $R^1(\pi_A)_*(\mathcal{O}_A)$ are free $\mathcal{O}_U \otimes \mathcal{O}_{\mathbf{F}}$ -modules. Let $x_i^{(p)}$ and x_i be $\mathcal{O}_U[\pi]/(\pi^e)$ -bases of $R^1(\pi_A)_*(\mathcal{O}_A)^{(p), i}$ and $R^1(\pi_A)_*(\mathcal{O}_A)^i$ respectively for each $i \in \mathbb{Z}/f\mathbb{Z}$. Let

$$F_{A/S}^*(x_i^{(p)}) = \sum_{0 \leq j < e} f_{i,j} \pi^j x_i, \quad f_{i,j} \in \mathcal{O}_U.$$

Then locally $\mathcal{M}_{\geq \underline{a}}$ is defined by the equations $f_{i,j}$, where $(i, j) \in I = \{(i, j); i \in \mathbb{Z}/f\mathbb{Z}, 0 \leq j < a^i\}$. Therefore, $\dim_x \mathcal{M}_{\geq \underline{a}} \geq g - |I| = g - |\underline{a}|$.

Let $\overline{M} := H^1_{\text{cris}}(A_x/k)$, $\overline{M}^{(p)} := H^1_{\text{cris}}(A_x^{(p)}/k) = \overline{M} \otimes_{k,\sigma} k$. We have the Hodge filtration:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Fil}^{(p)} & \longrightarrow & \overline{M}^{(p)} & \longrightarrow & Q^{(p)} := H^1(A_x, \mathcal{O}_{A_x})^{(p)} \longrightarrow 0 \\ & & & & \downarrow F^* & & \downarrow F^* \\ 0 & \longrightarrow & \text{Fil} & \longrightarrow & \overline{M} & \longrightarrow & Q := H^1(A_x, \mathcal{O}_{A_x}) \longrightarrow 0 \end{array}$$

By Proposition 4.3 and Lemma 5.3, we can choose a $k[\pi]/(\pi^e)$ -basis $\{x_i, y_i\}$ be of $(\overline{M})^i$ for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that $y_i \in \text{Fil}^i$ and $F^*(x_i \otimes 1) = -y_i + c_i x_i$ for some $c_i \in \pi^{a_i} k[\pi]/(\pi^e)$. For any PD-extension R of k , the differential forms x_i, y_i have unique horizontal liftings in $\widetilde{M} := H^1_{\text{cris}}(A/R)$ with respect to the Gauss-Manin connection, which we will denote the liftings by x_i, y_i again.

Let $R := k[[\underline{t}]]/(\underline{t})^2$ be the first order universal deformation ring of $(A_x, \lambda_x, \iota_x)$, where $\underline{t} = (t_{i,j}), i \in \mathbb{Z}/f\mathbb{Z}, 0 \leq j < e$. The first order universal deformation $(\widetilde{A}, \widetilde{\lambda}, \widetilde{\iota})$ gives rise to

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widetilde{\text{Fil}}^{(p)} & \longrightarrow & \widetilde{M}^{(p)} & \longrightarrow & \widetilde{Q}^{(p)} \longrightarrow 0 \\ & & & & \downarrow F^* & & \downarrow F^* \\ 0 & \longrightarrow & \widetilde{\text{Fil}} & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{Q} \longrightarrow 0 \end{array}$$

where $\widetilde{\text{Fil}} = \text{span}\langle y_i + \sum_{0 \leq j < e} t_{i,j} \pi^j x_i \rangle$, $\widetilde{\text{Fil}}^{(p)} = \text{span}\langle (y_i + \sum_{0 \leq j < e} t_{i,j} \pi^j x_i) \otimes 1 \rangle = \text{span}\langle y_i \otimes 1 \rangle$, $\widetilde{Q} = \text{span}\langle [x_i] \rangle$, and $\widetilde{Q}^{(p)} = \text{span}\langle [x_i \otimes 1] \rangle$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. Here the bracket $[\]$ denotes the class modulo $\widetilde{\text{Fil}}$ and $\widetilde{\text{Fil}}^{(p)}$. It follows from

$$F^*([x_i \otimes 1]) = [-y_i] = \sum_{0 \leq j < e} t_{i,j} [\pi^j x_i]$$

that $R/(t_{i,j})_{(i,j) \in I}$ is the first order deformation ring of $\mathcal{M}_{\geq \underline{a}}$ at x and the tangent space has dimension $g - |\underline{a}| \leq \dim_x \mathcal{M}_{\geq \underline{a}}$. The assertion follows. \square

COROLLARY 5.5. — *The ordinary points are dense in $\mathcal{M}^{\text{DP}} \otimes \mathbb{F}_p$.*

Proof. — Note that for ordinary points $a^i = 0, \forall i \in \mathbb{Z}/f\mathbb{Z}$. The statement follows from Theorem 5.4 and the density of the Rapoport locus. \square

5.6. Let $\mathcal{M}_{\underline{a}}$ denote the subset of \mathcal{M} that consists of points with a -type \underline{a} . It is a locally closed subset of \mathcal{M} , hence regarded as a locally closed algebraic substack of \mathcal{M} with the reduced induced structure. Lemma 5.3 says that $(A, \lambda, \iota) \in \mathcal{M}_{\geq \underline{a}}(k)$ if and only if $\underline{a}(A) \geq \underline{a}$. It follows from Theorem 5.4 that $\mathcal{M}_{\underline{a}}$ is a dense open substack of $\mathcal{M}_{\geq \underline{a}}$ and $\mathcal{M}_{\geq \underline{a}}$ is the scheme-theoretic closure of $\mathcal{M}_{\underline{a}}$ in \mathcal{M} . This justifies our notation.

6. Deformations of Dieudonné modules.

6.1. We follow the convenient setting of [N, Sect. 0] and [CN, Sect. 2, p. 1011]. As we will only deal with smooth functors, the deformation theory developed by P. Norman [N] and Norman-Oort [NO] is enough for our purpose. We refer the reader to Zink [Z2] for the generalized theory of displays over more general base ring.

Let R be a commutative ring of characteristic p . Let $W(R)$ denote the ring of Witt vectors over R , equipped with the Verschiebung τ and Frobenius σ :

$$(a_0, a_1, \dots)^\tau = (0, a_0, a_1, \dots)$$

$$(a_0, a_1, \dots)^\sigma = (a_0^p, a_1^p, \dots).$$

Let $\text{Cart}_p(R)$ denote the Cartier ring $W(R)[F][[V]]$ modulo the relations

- $FV = p$ and $VaF = a^\tau$,
- $Fa = a^\sigma F$ and $Va^\tau = aV, \forall a \in W(R)$.

A left $\text{Cart}_p(R)$ -module is *uniform* if it is complete and separated in the V -adic topology. A uniform $\text{Cart}_p(R)$ -module M is *reduced* if V is injective on M and M/VM is a free R -module. A *Dieudonné module over R* is a finitely generated reduced uniform $\text{Cart}_p(R)$ -module.

There is an equivalence of categories between the category of finite dimensional commutative formal group over R and the category of Dieudonné module over R . We denote this functor by \mathbf{D}_* . The tangent space of a formal group G is canonically isomorphic to $\mathbf{D}_*(G)/V\mathbf{D}_*(G)$.

6.2. Let $\underline{a} = (a^i)$ be an a -type and $(A_0, \lambda_0, \iota_0) \in \mathcal{M}_{\geq \underline{a}}(k)$ (1.12) (5.2) be a *non-ordinary* polarized abelian $O_{\mathbf{F}}$ -variety. By Lemma 3.1, the associated p -divisible group $G_0 = A_0[p^\infty]$ is connected, hence it is a smooth formal group. As the forgetful functor $\text{Def}[A_0, \lambda_0, \iota_0] \rightarrow \text{Def}[A_0, \iota_0]$ induces

an equivalence of deformation functors, we will consider deformations of abelian $\mathcal{O}_{\mathbf{F}}$ -varieties and their associated formal groups.

Let I be the set $\{(i, j); i \in \mathbb{Z}/f\mathbb{Z}, a^i \leq j < e - 1\}$ and let $R := k[[t_{i,j}]]_{(i,j) \in I}$. We have

$$\mathcal{O} \otimes_{\mathbb{Z}_p} W(R) = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} W(R)[T]/(\sigma_i(P(T))).$$

Set $W_R^i := W(R)[T]/(\sigma_i(P(T)))$ and denote again by π the image of T in W_R^i .

Let M_0 be the covariant Dieudonné module of A_0 . Let $\underline{b} = (b^i)$ be the a -type of A_0 and τ be the a -index of M_0 . By Proposition 4.4, we can choose a W^i -basis of M_0^i for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that

$$(6.2.1) \quad FX_{i-1} = \begin{cases} X_i & \text{if } i \notin \tau \\ c_i X_i + Y_i & \text{if } i \in \tau \end{cases} \quad FY_{i-1} = \begin{cases} \pi^e Y_i & \text{if } i \notin \tau \\ \pi^e X_i & \text{if } i \in \tau, \end{cases}$$

for some $c_i \in \pi^{b^i} W^i$.

By [NO, Lemma 0.2], we construct a Dieudonné module M_R over R with a $\text{Cart}_p(R)$ -linear action by \mathcal{O} as follows. It is the $\text{Cart}_p(R)$ -module generated by $\{\pi^j X_i, \pi^j Y_i\}_{i \in \mathbb{Z}/f\mathbb{Z}, 0 \leq j < e}$ with the relations

$$(6.2.2) \quad FX_{i-1} = \begin{cases} X_i & \text{if } i \notin \tau \\ c_i X_i + (Y_i + \sum_{a^i \leq j < e} T_{i,j} \pi^j X_i) & \text{if } i \in \tau \end{cases}$$

$$(6.2.3) \quad Y_{i-1} = \begin{cases} Vu(Y_i + \sum_{a^i \leq j < e} T_{i,j} \pi^j X_i) & \text{if } i \notin \tau \\ Vu X_i & \text{if } i \in \tau, \end{cases}$$

where $T_{i,j}$ is the Teichmüller lift of $t_{i,j}$, $\forall (i, j) \in I$, and $u = p^{-1}\pi^e$. We impose the natural W_R^i -module structure on the free $W(R)$ -submodule generated by $\{\pi^j X_i, \pi^j Y_i\}_{0 \leq j < e}$, and impose the trivial W_R^k -module structure on this submodule for $k \neq i$. The action of \mathcal{O} on M_R comes from the natural embedding $\mathcal{O} \hookrightarrow \mathcal{O} \otimes_{\mathbb{Z}_p} W(R)$.

LEMMA 6.3. — *The Dieudonné \mathcal{O} -module M_R over R is isomorphic to the universal deformation of M_0 for $\mathcal{M}_{\geq \underline{a}}$.*

Proof. — It is clear that $\text{Cart}_p(k) \otimes_{\text{Cart}_p(R)} M_R = M_0$. On M_R/VM_R , we have

$$F[X_{i-1}] = c_i[X_i] + \sum_{a^i \leq j < e} t_{i,j} \pi^j [X_i], \quad \forall i \in \tau.$$

By the Serre-Tate theorem, we obtain a morphism $\text{Spf} R \rightarrow \mathcal{M}_{\geq \underline{a}, x}^\wedge$. By [N, Thm. 1], this construction induces an injection of tangent spaces. By Theorem 5.4, $\mathcal{M}_{\geq \underline{a}}$ is smooth and $\dim \mathcal{M}_{\geq \underline{a}} = \dim R$. Hence the morphism is an isomorphism. \square

Remark 6.4. — Another construction of M_R is using tensor products modulo relations. Let P_R is a free $\mathcal{O} \otimes_{\mathbb{Z}_p} W(R)$ -module of rank two, with a W_R^i -basis $\{X_i, Y_i\}$ for each component P_R^i . Then we construct M_R to be the quotient of $\text{Cart}_p(R) \otimes_{W(R)} P_R$ modulo the relations (6.2.2) and (6.2.3).

6.5. Let M_R be as in (6.2). Let

$$(6.5.1) \quad T_i := \begin{cases} \sum_{a^i \leq j < e} T_{i,j} \pi^j & \text{if } i \notin \tau \\ c_i + \sum_{a^i \leq j < e} T_{i,j} \pi^j & \text{if } i \in \tau. \end{cases}$$

Then we have

$$\begin{pmatrix} FX_{i-1} \\ FY_{i-1} \end{pmatrix} = A_i \begin{pmatrix} X_i \\ Y_i \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} 1 & 0 \\ T_i \pi^e & \pi^e \end{pmatrix} \text{ if } i \notin \tau; \quad A_i = \begin{pmatrix} T_i & 1 \\ \pi^e & 0 \end{pmatrix} \text{ if } i \in \tau.$$

LEMMA 6.6. — *The non-ordinary locus of \mathcal{M}_x^\wedge is defined by $\prod_{i \in \tau} t_{i,0} = 0$.*

Proof. — Take $\underline{a} = (0, 0, \dots, 0)$. On $M_R/(VM_R + \pi M_R)$, we have

$$F[X_{i-1}] = [X_i] \text{ if } i \notin \tau; \quad F[X_{i-1}] = t_{i,0}[X_i] \text{ if } i \in \tau.$$

As each slope stratum is reduced, it induces a closed reduced subscheme, if not empty, of \mathcal{M}_x^\wedge . A point $\mathfrak{p} \in \text{Spec } R$ is in the non-ordinary locus if and only if any of $t_{i,0}, i \in \tau$, vanishes on R/\mathfrak{p} . Therefore the defining equation is $\prod_{i \in \tau} t_{i,0} = 0$. \square

6.7. We continue with (6.5). Let $t = |\underline{a}(M_0/\pi M_0)|$ and $\tau = \{n_1, n_2, \dots, n_t\}$. We assume that $0 \leq n_1 < n_2 < \dots < n_t < f$ and let $n_0 := n_t - f$ and $n_{t+1} = n_1$. Let $\ell_i := n_i - n_{i-1}$ for $1 \leq i \leq t$ (4.7).

From (6.5), we have

$$\begin{aligned}
 \begin{pmatrix} F^{\ell_i} X_{n_{i-1}} \\ F^{\ell_i} Y_{n_{i-1}} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ T_{n_{i-1}+1}^{(\ell_i-1)} \pi^e & \pi^e \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ T_{n_{i-1}+\ell_i-1}^{(1)} \pi^e & \pi^e \end{pmatrix} \\
 &\quad \begin{pmatrix} T_{n_i} & 1 \\ \pi^e & 0 \end{pmatrix} \begin{pmatrix} X_{n_i} \\ Y_{n_i} \end{pmatrix} \\
 (6.7.1) \quad &= \begin{pmatrix} 1 & 0 \\ V'_i & \pi^{e(\ell_i-1)} \end{pmatrix} \begin{pmatrix} U'_i & 1 \\ \pi^e & 0 \end{pmatrix} \begin{pmatrix} X_{n_i} \\ Y_{n_i} \end{pmatrix} \\
 &= \begin{pmatrix} U'_i & 1 \\ U'_i V'_i + \pi^{e\ell_i} & V'_i \end{pmatrix} \begin{pmatrix} X_{n_i} \\ Y_{n_i} \end{pmatrix}
 \end{aligned}$$

where

$$U'_i := T_{n_i}, \quad \text{and} \quad V'_i := \sum_{j=1}^{\ell_i-1} T_{n_{i-1}+j}^{(\ell_i-j)} \pi^{ej}.$$

Recall that we write $T^{(n)}$ for T^{σ^n} (3.3) (4.7). For $1 \leq s \leq t$, suppose that

$$\begin{pmatrix} F^{\ell_s+\ell_{s+1}+\dots+\ell_t} X_{n_{s-1}} \\ F^{\ell_s+\ell_{s+1}+\dots+\ell_t} Y_{n_{s-1}} \end{pmatrix} = \begin{pmatrix} \alpha_s & \beta_s \\ \gamma_s & \delta_s \end{pmatrix} \begin{pmatrix} X_{n_t} \\ Y_{n_t} \end{pmatrix},$$

for some coefficients $\alpha_s, \beta_s, \gamma_s, \delta_s$ in $W_R^{n_t}$. It follows from the same computation (4.7.1) of (4.7) that

$$(6.7.2) \quad \begin{pmatrix} \alpha_{s-1} & \beta_{s-1} \\ \gamma_{s-1} & \delta_{s-1} \end{pmatrix} = \begin{pmatrix} U_{s-1} & 1 \\ U_{s-1} V_{s-1} + \pi^{e\ell_{s-1}} & V_{s-1} \end{pmatrix} \begin{pmatrix} \alpha_s & \beta_s \\ \gamma_s & \delta_s \end{pmatrix},$$

$$(6.7.3) \quad \begin{pmatrix} \alpha_t & \beta_t \\ \gamma_t & \delta_t \end{pmatrix} = \begin{pmatrix} U_t & 1 \\ U_t V_t + \pi^{e\ell_t} & V_t \end{pmatrix},$$

where

$$U_s := T_{n_s}^{(\ell_{s+1}+\ell_{s+2}+\dots+\ell_t)} \quad \text{and} \quad V_s := \sum_{i=1}^{\ell_s-1} T_{n_{s-1}+i}^{(\ell_s+\dots+\ell_t-i)} \pi^{ei}.$$

Therefore,

$$(6.7.4) \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \prod_{i=1}^t \begin{pmatrix} U_i & 1 \\ U_i V_i + \pi^{e\ell_i} & V_i \end{pmatrix}.$$

THEOREM 6.8. — *If $\underline{a} = (a^i)$ is spaced, then the points in $\mathcal{M}_{\geq \underline{a}}$ with slope sequence $s(|\underline{a}|)$ are dense in $\mathcal{M}_{\geq \underline{a}}$.*

Proof. — We will show that for each point $x = (A_0, \lambda_0, \iota_0) \in \mathcal{M}_{\underline{a}}(k)$, there is a deformation in $\mathcal{M}_{\geq \underline{a}}$ whose generic point has slope sequence $s(|\underline{a}|)$. Let M_0 be the Dieudonné module of A_0 . Let $R = k[[t_{i,a^i}]]$, $i \in \tau(M_0)$. We construct a deformation M_R by (6.2) with $T_{i,j} = 0$ except for $i \in \tau(M_0)$ and $j = a^i$. Note that $V_i = 0$ in (6.7.4). We have (cf. 4.9)

$$\begin{pmatrix} F^f \pi^e X_{n_t} \\ F^f Y_{n_t} \end{pmatrix} = \pi^{|\underline{a}|} A \begin{pmatrix} \pi^e X_{n_t} \\ Y_{n_t} \end{pmatrix}, \quad A = \prod_{i=1}^t \begin{pmatrix} U_i & \pi^{e-a^{n_i}} \\ \pi^{e(\ell_i-1)-a^{n_i}} & 0 \end{pmatrix},$$

where $U_i = (c_{n_i} + T_{n_i, a^{n_i}})^{(\ell_{i+1} + \ell_{i+2} + \dots + \ell_i)}$ for some c_{n_i} in W^{n_i} . We may assume that $e(\ell_i - 1) > a^{n_i}$ for some i , otherwise $|\underline{a}| = \frac{g}{2}$ and there is nothing to prove. Assume that $e(\ell_1 - 1) > a^{n_1}$, we have

$$(6.8.1) \quad \begin{aligned} A &\equiv \begin{pmatrix} U_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_2 & * \\ * & 0 \end{pmatrix} \dots \begin{pmatrix} U_t & * \\ * & 0 \end{pmatrix} \text{ mod } \pi \\ &\equiv \begin{pmatrix} P & * \\ 0 & 0 \end{pmatrix} \text{ mod } \pi \end{aligned}$$

where P is a nonzero polynomial in U_i 's. Hence P is a unit in $W(K)[\pi]$, where K is the perfection of $\text{Frac}(R)$. Therefore $M_K = \text{Cart}_p(K) \otimes_{\text{Cart}_p(R)} M_R$ has slope sequence $s(|\underline{a}|)$. □

COROLLARY 6.9. — *The generic point of each irreducible component of $\mathcal{M}_{\geq \underline{a}}$ has slope sequence $\geq s(\lambda(\underline{a}))$, where $\lambda(\underline{a}) := \max\{|\underline{b}|; \underline{b} \leq \underline{a}, \underline{b} \text{ is spaced}\}$.*

Proof. — This follows from Proposition 4.9 and Grothendieck's specialization theorem. □

PROPOSITION 6.10. — *Let $\underline{a} = (a^i)$ be an a -type with $a^i \leq [e/2]$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. Let $x = (A_0, \lambda_0, \iota_0) \in \mathcal{M}(k)$ such that A_0 is superspecial.*

Then in every open neighborhood of x in $\mathcal{M}_{\geq \underline{a}}$ there exists a point of slope sequence $s(\underline{a})$.

Proof. — Let $c := [e/2]$ and M_0 be the Dieudonné module of A_0 . By Lemma 4.5 (2), we can choose a basis $\{X_i, Y_i\}$ for M^i such that

$$\begin{pmatrix} FX_{i-1} \\ FY_{i-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \pi^e & 0 \end{pmatrix} \begin{pmatrix} X_i \\ Y_i \end{pmatrix}, \quad \forall i \in \mathbb{Z}/f\mathbb{Z}.$$

Let $R = k[[t_i]]_{i \in \mathbb{Z}/f\mathbb{Z}}$. We construct a deformation M_R of M_0 by (6.2):

$$\begin{pmatrix} FX_{i-1} \\ FY_{i-1} \end{pmatrix} = \begin{pmatrix} \pi^{a^i} T_i & 1 \\ \pi^e & 0 \end{pmatrix} \begin{pmatrix} X_i \\ Y_i \end{pmatrix}, \quad \forall i \in \mathbb{Z}/f\mathbb{Z},$$

where T_i is the Teichmüller lift of t_i . We have (cf. 4.9, 6.10)

$$\begin{pmatrix} F^f \pi^c X_{f-1} \\ F^f Y_{f-1} \end{pmatrix} = \pi^{|\underline{a}|} A \begin{pmatrix} \pi^c X_{f-1} \\ Y_{f-1} \end{pmatrix}, \quad A = \prod_{i=0}^{f-1} \begin{pmatrix} U_i & \pi^{c-a^i} \\ \pi^{e-c-a^i} & 0 \end{pmatrix},$$

where $U_i = T_i^{(f-1-i)}$. We may assume that $a^i < e/2$ for some i , otherwise $|\underline{a}| = \frac{e}{2}$ and there is nothing to prove. Say $a^0 < e/2$, we have

$$\begin{aligned} (6.10.1) \quad A &\equiv \begin{pmatrix} U_0 & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_1 & * \\ * & 0 \end{pmatrix} \cdots \begin{pmatrix} U_{f-1} & * \\ * & 0 \end{pmatrix} \pmod{\pi} \\ &\equiv \begin{pmatrix} P & * \\ 0 & 0 \end{pmatrix} \pmod{\pi} \end{aligned}$$

where P is a nonzero polynomial in U_i 's. Hence P is a unit in $W(K)[\pi]$, where K is the perfection of $\text{Frac}(R)$. Therefore $M_K = \text{Cart}_p(K) \otimes_{\text{Cart}_p(R)} M_R$ has slope sequence $s(\underline{a})$. □

Remark 6.11. — Goren and Oort showed [GO, Thm. 5.4.11] that when p is inert in \mathbf{F} , the inequality \geq in Corollary 6.9 can be strengthened by equality $=$. However, the equality does not hold in general by Proposition 6.10. It will be interesting to have the sharp formula of the slope sequence of the generic points of *any* alpha stratum. When the a -types are spaced, the equality can be achieved as it stands in Theorem 6.8.

6.12. Let $x = (A_0, \lambda_0, \iota_0)$ and M_0 be as in (6.2). Suppose that *the reduced a -number $t := |\tau|$ of x is one*, where τ is the a -index of M_0 . We

assume that $\tau = \{0\}$ for simplicity. By Proposition 4.4, we choose a basis for M_0 as in (6.2.1) of (6.2).

Let $\ell = \text{ord}_\pi(c_0)$. We have $a^0 = \min\{e, \ell\}$ and $\text{slope}(M_0) = s(\min\{\frac{g}{2}, \ell\})$ by (4.10).

Take $\underline{a} = (0, 0, \dots, 0)$ and construct the universal deformation M_R of M_0 for \mathcal{M} by (6.2). It is the Dieudonné module of the formal group attached to the universal formal deformation $(\tilde{A}, \tilde{\lambda}, \tilde{t})$ over $\mathcal{M}_x^\wedge = \text{Spf } R$. For each $m \in S(g)$, $m \leq \ell$, let $\mathcal{M}_x^{\geq s(m)}$ denote the reduced closed subscheme (in \mathcal{M}_x^\wedge) consisting of points with slope sequence $\geq s(m)$ (1.12). We will find the defining equations of the subscheme $\mathcal{M}_x^{\geq s(m)}$ and show that the generic point of $\mathcal{M}_x^{\geq s(m)}$ has slope sequence $s(m)$.

From (6.7.4), we have

$$\begin{pmatrix} F^f X_0 \\ F^f Y_0 \end{pmatrix} = \begin{pmatrix} U_1 & 1 \\ U_1 V_1 + \pi^g & V_1 \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix},$$

where

$$U_1 = T_0 \quad \text{and} \quad V_1 = \sum_{i=1}^{f-1} T_i^{(f-i)} \pi^{ei}.$$

Therefore we have the Cayley-Hamilton equation $F^{2f} X_0 - (U_1^{(f)} + V_1) F^f X_0 - \pi^g = 0$. The subscheme $\mathcal{M}_x^{\geq s(m)}$ is defined by the equations obtained from “ $\text{ord}_\pi(U_1^{(f)} + V_1) \geq m$ ”.

From (6.5.1) of (6.5), write

$$\begin{aligned} (6.12.1) \quad U_1^{(f)} + V_1 &= c_0^{(f)} + \sum_{i=0}^{f-1} \sum_{j=0}^{e-1} T_{i,j}^{(f-i)} \pi^{ei+j} \\ &= c_0^{(f)} + \sum_{i=0}^{g-1} T_k \pi^k \end{aligned}$$

where $T_k := T_{i,j}^{(f-i)}$, $t_k = t_{i,j}$ for $k = ei + j$, $0 \leq j \leq e - 1$. We can see that the defining equations for $\mathcal{M}_x^{\geq s(m)}$ are $t_0 = t_1 = \dots = t_{m-1} = 0$. Let K_m be the perfection of the generic residue field of $\mathcal{M}_x^{\geq s(m)}$. The element T_m is a unit in $W(K_m)$, therefore the generic point of $\mathcal{M}_x^{\geq s(m)}$ has slope sequence $s(m)$.

THEOREM 6.13. — *Let $x : \text{Spec } k \rightarrow \mathcal{M}$ be a geometric point of reduced a -number one. Then each closed subscheme $\mathcal{M}_x^{\geq s(m)}$, for $m \in S(g)$, $s(m) \leq \text{slope}(x)$ (3.2), is formally smooth of codimension $[m]$ and its generic point has slope sequence $s(m)$, where $[m]$ denotes the smallest integer not less than m .*

COROLLARY 6.14. — *Let U be the subset of \mathcal{M} consisting of points with reduced a -number ≤ 1 . Then the strong Grothendieck conjecture holds for U (1.13).*

COROLLARY 6.15. — *The strong Grothendieck conjecture for \mathcal{M} holds when p is totally ramified in \mathbf{F} (1.13).*

COROLLARY 6.16. — *The weak Grothendieck conjecture for \mathcal{M} holds (1.13).*

Proof. — It follows from Lemma 7.2 that there is a supersingular point of reduced a -number one. Then the assertion follows from Theorem 6.13. □

6.17. *In the rest of this section we assume that p is totally ramified in \mathbf{F} . Denote by $\mathcal{M}_v^{\text{DP}}$ the reduction $\mathcal{M}^{\text{DP}} \otimes_{\mathbb{Z}} k(v)$ of \mathcal{M}^{DP} modulo v . Let $x = (A_0, \lambda_0, \iota_0) \in \mathcal{M}_v^{\text{DP}}(k)$ be a geometric point and let $\underline{e}(A_0) = \{e_1, e_2\}$ and $\underline{a}(A_0) = \{a_1, a_2\}$. We assume that $e_1 \leq e_2$ and $a_1 \leq a_2$. Let M_0 be the covariant Dieudonné module of A_0 . We can choose two $W[\pi] = \mathcal{O} \otimes W$ -bases $\{X_1, X_2\}, \{Y_1, Y_2\}$ of M_0 such that*

$$VuY_1 = \pi^{e_2} X_1 \quad \text{and} \quad VuY_2 = \pi^{e_1} X_2,$$

where $pu = \pi^e$. We have

$$FX_1 = \pi^{e_1} Y_1 \quad \text{and} \quad FX_2 = \pi^{e_2} Y_2.$$

Write $Y_1 = \alpha X_1 + \beta X_2$ and $Y_2 = \gamma X_1 + \delta X_2$. Then we have

$$\begin{pmatrix} FX_1 \\ FX_2 \end{pmatrix} = \begin{pmatrix} \pi^{e_1} \alpha & \pi^{e_1} \beta \\ \pi^{e_2} \gamma & \pi^{e_2} \delta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

and

$$(F, V)M = \langle \pi^{e_1} X_2, \pi^{e_2} X_1, \pi^{e_1} \alpha X_1 \rangle_{W[\pi]}.$$

It follows that $a_1 = e_1$ and $a_2 = \min\{e_2, e_1 + \text{ord}_\pi(\alpha)\}$. As $e_2 \geq \frac{e}{2}$, by the argument in (4.11) we have $\text{slope}(A_0) = s(i)$, where $i = \min\{\frac{e}{2}, e_1 +$

$\text{ord}_\pi(\alpha)\}$. Note that the a -type $\underline{a}(A_0)$ determines the invariant $\underline{e}(A_0)$ and slope(A_0): $e_1 = a_1$ and $\text{slope}(A_0) = s(\min\{\frac{e}{2}, a_2\})$.

6.18. In [DP], Deligne and Pappas have defined a closed algebraic substack \mathcal{N}_i of $\mathcal{M}_v^{\text{DP}}$ which classifies the objects of Lie type $\{e_1, e_2\}$ with $e_1 \geq i$. They have shown that the complement \mathcal{N}_i^o of \mathcal{N}_{i+1} in \mathcal{N}_i is a smooth algebraic stack of dimension $e - 2i$ if it is non-empty. It follows from their results and Theorem 7.4 that points of Lie type $\{e_1, e_2\}$ are dense in \mathcal{N}_{e_1} . It follows from (6.17) that any point in \mathcal{N}_{e_1} has slope sequence $\geq s(e_1)$. The following lemma confirms the density of points with slope sequence $s(e_1)$ in \mathcal{N}_{e_1} .

LEMMA 6.19. — *If $a_2 > a_1$, then there is a deformation $(\tilde{A}, \tilde{\lambda}, \tilde{\iota})$ over $k[[t]]$ of $(A_0, \lambda_0, \iota_0)$ whose generic point has a -type $(a_1, a_2 - 1)$.*

Proof. — As the forgetful map $\text{Def}[A_0, \lambda_0, \iota_0] \rightarrow \text{Def}[A_0, \iota_0]$ induces an equivalence of deformation functors in \mathcal{N}_{e_1} , we will construct a deformation of abelian $O_{\mathbf{F}}$ -varieties in \mathcal{N}_{e_1} . By the reduction step in [DP, 4.3] and the construction of (6.2), we can construct a Dieudonné \mathcal{O} -module M_R over $R := k[[t]]$ of M_0 such that

$$(6.19.1) \quad \begin{aligned} FX_1 &= \pi^{e_1}\alpha X_1 + \pi^{e_1}\beta(X_2 + T\pi^{a_2-e_1-1}X_1) \\ FX_2 &= \pi^{e_2}\gamma X_1 + \pi^{e_2}\delta(X_2 + T\pi^{a_2-e_1-1}X_1), \end{aligned}$$

where T is the Teichmüller lift of t . Note that β is a unit in $W[\pi]$: if $\beta \equiv 0 \pmod{\pi}$, then α is a unit and $a_2 = a_1$. By base change to $K := k((t))^{\text{perf}}$, we have

$$(F, V)M_K = \langle \pi^{e_1}X_2, \pi^{e_2}X_1, (\pi^{e_1}\alpha + \pi^{a_2-1}\beta T)X_1 \rangle_{W(K)[\pi]},$$

thus $\underline{a}(M_K) = (a_1, a_2 - 1)$. This completes the proof. □

THEOREM 6.20. — *The strong Grothendieck conjecture for $\mathcal{M}_v^{\text{DP}}$ holds when p is totally ramified in \mathbf{F} .*

Proof. — Let $x \in \mathcal{M}_v^{\text{DP}}(k)$ be a geometric point with a -type (a_1, a_2) and slope sequence $s(i)$, where $i = \min\{\frac{e}{2}, a_2\}$. It suffices to deform the point x to a point with slope sequence $s(i - 1)$. If $a_2 > a_1$, then by Lemma 6.19 we can deform to a point with slope sequence $s(i - 1)$. Suppose that $a_2 = a_1 = e_1$, we have slope(x) = $s(e_1)$. By (6.18), we can deform to a point of Lie type $\{e_1 - 1, e_2 + 1\}$. As points with slope sequence $s(e_1 - 1)$

are dense in \mathcal{N}_{e_1-1} , we can further deform to a point with slope sequence $s(e_1 - 1)$. □

COROLLARY 6.21. — *Each Newton stratum of $\mathcal{M}_v^{\text{DP}}$ with slope sequence $s(m)$, $m \in S(g)$ (3.2), has pure dimension $g - [m]$ when p is totally ramified in \mathbf{F} .*

Proof. — This follows from the purity of Newton strata [dJO] and Theorem 6.20. □

Remark 6.22. — In [C] Chai gives a group-theoretic dimension formula for Newton strata arising from quasi-split groups. He expects that it is so for good reduction of PEL-type Shimura varieties [C, Question 7.6, p. 984]. Corollary 6.21 suggests that his description can be applied for a larger class, not necessarily restricted to the good reduction case.

7. An algebraization theorem.

7.1. Let notations be as in (1.1) and (1.12). The set of possible Newton polygons in question is parameterized by $S(g_1) \times \dots \times S(g_s)$ (1.11), where $g_i = [\mathbf{F}_{v_i} : \mathbb{Q}_p]$. For each abelian $O_{\mathbf{F}}$ -variety A , the associated p -divisible group $A(p) := A[p^\infty]$ has a decomposition $A(p) = A(p)_{v_1} \oplus \dots \oplus A(p)_{v_s}$.

Recall (1.5) that a quasi-polarized p -divisible \mathcal{O} -group (H, λ, ι) over k is *algebraizable* if it is attached to a polarized abelian $O_{\mathbf{F}}$ -variety (A, λ_A, ι_A) over k .

LEMMA 7.2. — *Any supersingular quasi-polarized p -divisible \mathcal{O} -group (H, λ, ι) over k is algebraizable.*

Proof. — Let E be a supersingular elliptic curve over k , and $A' := E \otimes O_{\mathbf{F}}$. It is clear that A' satisfies the Rapoport condition. By [R, Prop. 1.10], there exists a separable $O_{\mathbf{F}}$ -linear polarization λ' on A' . Let $(H_1, \lambda_1, \iota_1)$ be the p -divisible group attached to (A', λ', ι') . It follows from Corollary 3.7 that $(H_1, \lambda_1, \iota_1)$ is isogenous to (H, λ, ι) . By a theorem of Tate, there is a polarized abelian $O_{\mathbf{F}}$ -variety (A, λ_A, ι_A) whose p -divisible group is isomorphic to (H, λ, ι) . □

THEOREM 7.3. — (1) *The weak Grothendieck conjecture for \mathcal{M} holds.*

(2) *The strong Grothendieck conjecture for $\mathcal{M}_p^{\text{DP}}$ holds when all residue degrees f_i are one.*

Proof. — (1) It follows from Lemma 7.2 that there is a supersingular polarized abelian $O_{\mathbf{F}}$ -variety A such that each component $A(p)_{v_i}$ of the associated p -divisible group $A(p)$ has reduced a -number one. Then the assertion follows from the theorem of Serre-Tate and Theorem 6.13.

(2) This follows from the theorem of Serre-Tate and Theorem 6.20. \square

THEOREM 7.4. — *Any quasi-polarized p -divisible O -group (H, λ, ι) over k is algebraizable.*

Proof. — It follows from Theorem 7.3 (1) that any slope sequence in $S(g_1) \times \dots \times S(g_s)$ can be realized by a point in \mathcal{M} . Then the theorem follows from Corollary 3.7. \square

8. An example.

In this section we construct a separably polarized abelian $O_{\mathbf{F}}$ -scheme A over a complete DVR R whose close fibre A_k does not satisfy the Rapoport condition.

Let $g = 2$ and $p > 3$ be a ramified prime in the totally quadratic real field \mathbf{F} . We have $O_{\mathbf{F}} \otimes W(k) = W(k)[\pi], \pi^2 = p$. Let M be a free $W(k)[\pi]$ -module generated by e_1, e_2 with the Verschiebung action

$$Ve_1 = \pi e_2, Ve_2 = \pi e_1,$$

and with the alternating form determined by

$$\langle e_1, e'_2 \rangle = \langle e'_1, e_2 \rangle = 1$$

and other pairing are 0 for the W -basis e_1, e'_1, e_2, e'_2 , where $e'_1 := \pi e_1, e'_2 := \pi e_2$. By the algebraization theorem, there is a polarized abelian $O_{\mathbf{F}}$ -variety A_0 over k with the prescribed Dieudonné module M .

We have the Hodge filtration of $M/pM = H^1_{DP}(A_0/k)^*$

$$0 \rightarrow VM/pM = k\bar{e}'_1 \oplus k\bar{e}'_2 \rightarrow M/pM \rightarrow M/VM = k\bar{e}_1 \oplus \bar{e}_2 \rightarrow 0.$$

Clearly, A_0 does not satisfy the Rapoport condition, as $\pi = 0$ on M/VM .

Let $R := W(k)[\sqrt{p}]$. By Grothendieck-Messing’s theory, Serre-Tate’s Theorem and Grothendieck’s Existence Theorem, we can lift the abelian variety A_0 with the additional structure over R by lifting the Hodge filtration with respect to the addition structure, see [Y2, Sect. 4]. Let N be the R -submodule of $M \otimes_W R$ generated by $e'_1 + \sqrt{p}e_1$ and $e'_2 - \sqrt{p}e_2$. It is easy to check that N is stable by $O_{\mathbf{F}}$ -action, $N \otimes_R k = VM/pM$ and $\langle N, N \rangle = 0$. Thus, we get a desired polarized abelian $O_{\mathbf{F}}$ -scheme over R .

9. A computation of the Hecke correspondence.

9.1. Let M_0 be the quasi-polarized Dieudonné \mathcal{O} -module of the polarized abelian variety A_0 in the previous section. Let N be the quasi-polarized Dieudonné module containing M_0 with $VN = M$ and $\langle \cdot, \cdot \rangle_N = \frac{1}{p} \langle \cdot, \cdot \rangle$. We have

$$\begin{aligned} Ve_1 &= \pi e_2 = e'_2 & Fe'_2 &= pe_1 \\ Ve_2 &= \pi e_1 = e'_1 & Fe'_1 &= pe_2 \\ Ve'_1 &= \pi Ve_1 = pe_2 & Fe_2 &= e'_1 \\ Ve'_2 &= \pi Ve_2 = pe_1 & Fe_1 &= e'_2 \end{aligned}$$

and $\langle e_1, e'_2 \rangle = 1, \langle e'_1, e_2 \rangle = 1$. From $N = \frac{1}{p}FM$, we have

$$N = \langle e_1, e_2, \frac{1}{p}e'_1, \frac{1}{p}e'_2 \rangle_W.$$

Write $X_1 = e_1, X_2 = e_2, X'_1 = \frac{1}{p}e'_1, X'_2 = \frac{1}{p}e'_2$, we have

$$M_0 = \langle X_1, X_2, pX'_1, pX'_2 \rangle_W$$

and $\langle X_1, X'_2 \rangle_N = \langle X'_1, X_2 \rangle_N = 1$. Denote by \bar{N} the quotient of N modulo pN and \bar{M}_0 the image of M_0 in \bar{N} . Write x_i, x'_i the image of X_i, X'_i in \bar{N} .

9.2. Let \mathcal{X} be space of π -invariant maximal isotropic “Dieudonné” subspaces of \bar{N} over k :

$$\mathcal{X}(k) = \{0 \subset \bar{M} \subset \bar{N}; \dim \bar{M} = 2, \pi \bar{M} \subset \bar{M}, F\bar{M} \subset \bar{M}, V\bar{M} \subset \bar{M}, \langle \bar{M}, \bar{M} \rangle_N = 0\}.$$

We regard \mathcal{X} as a reduced subscheme over k . It is a closed subscheme of the Lagrangian Grassmanian $LG(2, 4)$, hence a projective variety.

There is a finite morphism $\text{pr} : \mathcal{X} \rightarrow \mathcal{M}_p^{\text{DP}}$ (cf. [Y1, Sect. 6]) sending $\bar{M}_0 \mapsto (A_0, \lambda_0, \iota_0)$ and the morphism pr factors through the supersingular locus \mathcal{S}^{DP} of $\mathcal{M}_p^{\text{DP}}$.

We have $\bar{M}_0 = \langle x_1, x_2 \rangle_k \in \mathcal{X}(k)$. Let $\bar{M}_t \in \mathcal{X}$ be the k -subspace of \bar{N} generated by

$$\tilde{x}_1 = x_1 + t_{11}x'_1 + t_{12}x'_2, \quad \tilde{x}_2 = x_2 + t_{21}x'_1 + t_{22}x'_2.$$

The points \bar{M}_t form a Zariski open neighborhood of \bar{M}_0 in \mathcal{X} , which we denote by \mathcal{U} . We will show that $\mathcal{U} \simeq \text{Spec}k[t_1, t_2, t_3]/(t_1^{p+1} - t_2^{p+1}, t_1^2 + t_2t_3)$.

9.3. From $\langle \bar{M}_t, \bar{M}_t \rangle_N = 0$, we get $t_{11} + t_{22} = 0$. From $X'_i = \frac{1}{p}e'_i = \frac{1}{\pi}e_i = \frac{1}{\pi}X_i$, we have

$$\pi x'_i = x_i, \quad \pi x_i = 0, \quad i = 1, 2.$$

One computes

$$\pi \tilde{x}_1 = t_{11}x_1 + t_{12}x_2 = t_{11}\tilde{x}_1 + t_{12}\tilde{x}_2 - (t_{11}^2 + t_{12}t_{21})x'_1 - (t_{11}t_{12} + t_{12}t_{22})x'_2$$

and concludes from $\pi \tilde{x}_1 \in \bar{M}_t$ that

$$(9.3.1) \quad t_{11}^2 + t_{12}t_{21} = 0, \quad \text{and} \quad t_{12}(t_{11} + t_{22}) = 0.$$

Similarly from $\pi \tilde{x}_2 \in \bar{M}_t$, one gets

$$(9.3.2) \quad t_{22}^2 + t_{12}t_{21} = 0, \quad \text{and} \quad t_{21}(t_{11} + t_{22}) = 0.$$

For the stability of \bar{M}_t by F and V , one computes

$$F\tilde{x}_1 = t_{11}^p x_2 + t_{12}^p x_1 = t_{11}^p \tilde{x}_2 + t_{12}^p \tilde{x}_1 - (t_{11}^p t_{21} + t_{12}^p t_{11})x'_1 - (t_{11}^p t_{22} + t_{12}^{p+1})x'_2$$

and obtains

$$(9.3.3) \quad t_{11}^p t_{21} + t_{12}^p t_{11} = 0$$

$$(9.3.4) \quad t_{11}^p t_{22} + t_{12}^{p+1} = 0.$$

Similarly from $F\tilde{x}_2 \in \bar{M}_t$, one obtains

$$(9.3.5) \quad t_{21}^{p+1} + t_{22}^p t_{11} = 0$$

$$(9.3.6) \quad t_{21}^p t_{22} + t_{22}^p t_{12} = 0.$$

Applying $V\bar{M}_t \subset \bar{M}_t$, one does not get new equations but (9.3.3)–(9.3.6). From (9.3.4), one has $t_{12} = \alpha t_{11}, \alpha^{p+1} = 1$. From (9.3.5), one has $t_{21} = \beta t_{22} = -\beta t_{11}, \beta^{p+1} = 1$. From (9.3.1), one has $1 = \alpha\beta$. These parameters satisfy the equations (9.3.3) and (9.3.6).

We computed that $\mathcal{U} = \{(t_{11}, t_{12}, t_{21}, t_{22}) = (t, \alpha t, -\frac{1}{\alpha}t, -t); \alpha^{p+1} = 1\}$, hence $\mathcal{U} \simeq \text{Spec } k[t_1, t_2, t_3]/(t_1^{p+1} - t_2^{p+1}, t_1^2 + t_2 t_3)$. Compared with a result of [BG, p. 476, 3], the morphism pr maps \mathcal{X} onto the $(p+1)$ irreducible components of \mathcal{S}^{DP} containing $(A_0, \lambda_0, \iota_0)$.

PROPOSITION 9.4. — *Let \mathcal{U} be as above. There is a Zariski open neighborhood \mathcal{V} of 0 in \mathcal{U} such that $\text{pr} : \mathcal{V} \rightarrow \mathcal{S}^{\text{DP}}$ is an étale neighborhood of \mathcal{S}^{DP} at $(A_0, \lambda_0, \iota_0)$.*

Proof. — Choose a finite étale cover $\mathcal{M}_{p,n}^{\text{DP}} \rightarrow \mathcal{M}_p^{\text{DP}}$ by adding a prime-to- p level structure with $n \geq 3$. Choose a lift $(A_0, \lambda_0, \iota_0, \eta_0)$ of $(A_0, \lambda_0, \iota_0)$. Then there is a lift $\tilde{\text{pr}} : \mathcal{X} \rightarrow \mathcal{M}_{p,n}^{\text{DP}}$ which sends \bar{M}_0 to $(A_0, \lambda_0, \iota_0, \eta_0)$. The morphism $\tilde{\text{pr}}$ becomes a closed immersion as the automorphisms of the objects are trivial. It again factors through the supersingular locus $\mathcal{S}_n^{\text{DP}}$. As \mathcal{X} and $\mathcal{S}_n^{\text{DP}}$ are reduced schemes, \mathcal{X} is isomorphic to its image. The image is the union of the $p+1$ irreducible components of \mathcal{S}^{DP} containing $(A_0, \lambda_0, \iota_0, \eta_0)$. Then there is a Zariski open \mathcal{V} of \mathcal{U} such that the \mathcal{V} is an étale neighborhood of \mathcal{S}^{DP} at $(A_0, \lambda_0, \iota_0)$. \square

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