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CURVES WITH ONLY TRIPLE RAMIFICATION

by Stefan SCHRÖER

Introduction.

Let C be a smooth proper curve over an algebraically closed ground field k of characteristic $p \geqslant 0$. Assuming $p \neq 2$, Fulton [4] showed that there are generically étale finite maps $C \to \mathbb{P}^1$ such that all ramification points have index $e = 2$. In this paper, I pose the following question: *Does there exist a generically étale finite map $C \to \mathbb{P}^1$ whose ramification points all have index $e = 3$?*

Fried, Klassen, and Kopeliovich [3] took a first step into this direction. They proved that all but finitely many complex elliptic curves admit such a map. In fact, their proof reveals that for any given genus $g \geqslant 1$, the set of Riemann surfaces of genus g admitting such maps is at least 1-dimensional. The arguments, however, are purely topological and involve homeomorphism spaces, Dehn twists, and Teichmüller theory.

The main result of this paper is an improved lower bound on the dimension via purely algebraic methods. We shall prove that the set of points in the moduli space M_g whose corresponding curve admits rational functions with only triple ramification has dimension $\geqslant \max(2g-3, g)$. Our arguments work in all characteristics $p \neq 3$ and rely on deformation theory and the moduli space of stable curves. The basic idea is to deform a covering $X_0 \to \mathbb{P}^1$ where X_0 is a curve with *cuspidal singularities*, so that each cuspidal ramification point breaks up into two regular ramification

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points. The stable reduction process involved in this neatly explains why we miss g dimensions from the $(3g-3)$-dimensional moduli space M_g. One might speculate whether we found the best lower bound.

Actually, Fulton's result on the existence of maps with ramification indices $e=2$ gives some additional information: He proved in [4], Proposition 8.1, that for a given curve C of genus g in characteristic $p \neq 2$ and $n > g$, there are generically étale maps $C \to \mathbb{P}^1$ of degree n such that all ramification points have index $e=2$ and that each fiber contains at most one ramification point. The methods of this paper, however, do not give much information in this direction.

My initial motivation to study this problem was *Belyi's Theorem* [1]. It states that a compact Riemann surface is defined over a number field if and only if it admits a finite map to the Riemann sphere with at most three branch points. Saïdi [14] generalized this to odd characteristics as follows: An algebraic curve C in characteristic $p \geqslant 3$ is defined over a finite field if and only if it admits a *tamely* ramified morphism $C \to \mathbb{P}^1$ with at most three branch points. In characteristic $p=2$, the if part holds true, but the only-if part remains mysterious. However, a curve C over $\overline{\mathbb{F}}_2$ admits a tame function with at most three branch points if it admits a tame function at all. In some sense, the result of this paper tells us that the Belyi-Saïdi Theorem is valid in characteristic $p=2$ at least for a $(2g-3)$-dimensional set.

The question whether a finite morphism $X \to \mathbb{P}^1$ whose ramification points have index $e=3$ exists is also interesting for nonclosed ground fields. There, however, I showed in [15] that the generic curve C_η of genus $g \geqslant 3$ in characteristic $p=2$ does not admit such a map. This relies on Franchetta's Conjecture, which states that $\operatorname{Pic}(C_\eta) = \mathbb{Z}K_{C_\eta}$. Here the ground field is the function field $\kappa(\eta)$ of the moduli space M_g. Of course, it still might be true that the desired map exists over some field extension $\kappa(\eta) \subset L$.

Here is a plan for the paper. In Section 1 we study collisions of triple ramification points in terms of Weierstrass equations for elliptic curves. To globalize this, we collect in Section 2 some general results about deformations of coverings whose fibers are complete intersections. We use this to construct effective formal deformation in Section 3, and explain the resulting increase of transcendence degree in moduli fields. Section 4 contains a construction of maps $\mathbb{P}^1 \to \mathbb{P}^1$ with only triple coverings so that $\mathbb{P}^1$ marked with the ramification points has a large moduli field. We use this to prove our main result in Section 5. The last section contains some applications regarding Belyi's Theorem in positive characteristics.

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1. Collision of triple ramification points.

Fix a ground field k of characteristic $p \neq 3$, and let $h : C \to D$ be a finite generically étale morphism of proper smooth curves. For a rational point $c \in C$ the length $e \geqslant 1$ of the Artin ring $\mathcal{O}_{C,c}/\mathfrak{m}_{D,h(c)}\mathcal{O}_{C,c}$ is called the *ramification index*. We say that $h : C \to D$ has *only triple ramification* if all ramification points are rational have ramification index $e = 3$. The key idea of this paper is to collide triple ramification points in flat families. We now explain this by using elliptic curves, where explicit computations are possible.

Let $A = k[[t]]$ be the formal power series ring in one indeterminate t, and x, y, z be homogeneous coordinates for $\mathbb{P}^2_A$. Consider the closed subscheme $X \subset \mathbb{P}^2_A$ defined by the Weierstrass equation $x^3 = y^2 - ty$, or more precisely by the homogenous equation $x^3 = y^2z - tyz^2$. The generic fiber X_η is an elliptic curve with j-invariant $j = 0$ over the field of formal Laurent series $K = k((t))$. The closed fiber X_0 is a rational curve with a cusp located at $[0, 0, 1] \in \mathbb{P}^2_k$.

The diagonal group scheme $G = \mu_{3,A}$ of third roots of unity acts on $\mathbb{P}^2_A$ via the $\mathbb{Z}/3\mathbb{Z}$-grading on $A[x, y, z]$ given by $\deg(x) = 1$ and $\deg(y) = \deg(z) = 0$, as explained in [9], Exposé I, Proposition 4.7.3. The quotient $\mathbb{P}^2_A/G$ is the homogeneous spectrum of $A[x^3, y, z]$, which is isomorphic to a weighted projective space with weights $(3, 1, 1)$. Clearly, the equation $x^3 = y^2z - tyz^2$ is homogenous with respect to the $\mathbb{Z}/3\mathbb{Z}$-grading, so the closed subscheme $X \subset \mathbb{P}^2_A$ is G-invariant. The corresponding quotient scheme $X/G \subset \mathbb{P}^2_A/G$ is the homogenous spectrum of $A[x^3, y, z]/(y^2z - tyz^2 - x^3) = A[y, z]$. In particular, we have an identification $X/G = \mathbb{P}^1_A$.

The fixed scheme X^G for the G-action on X is the homogenous spectrum of

$$A[x, y, z]/(y^2z - tyz^2 - x^3, x) = A[y, z]/(yz(y - tz)).$$

Hence the generic fiber X^G_η comprises the three rational points $[0, 1, 0]$, $[0, 0, 1]$, and $[0, t, 1]$. In contrast, the closed fiber X^G_0 consists of the rational point $[0,1,0]$ together with $\operatorname{Proj} k[y,z]/(y^2)$, which is an Artin scheme of length two around $[0,0,1]$. Intuitively, the two generic fixed points $[0,0,1]$ and $[0,t,1]$ collide in the flat family upon specialization.

Let $f : X \to \mathbb{P}^1_A$ be the quotient map. Its generic fiber $f_\eta : X_\eta \to \mathbb{P}^1_\eta$ is a generically étale finite map with only triple ramification. It has three ramification points, two of which collide in the family. The closed fiber $f_0 : X_0 \to \mathbb{P}^1_k$ is generically étale as well, but the domain X_0 has a cusp resulting from the collision of triple points.

For later use we take a closer look at the complete local rings near the cusp $a \in X_0$ and its image $b \in \mathbb{P}^1_k$. Clearly,

$$\mathcal{O}^\wedge_{\mathbb{P}^1_k,b} = k\Big[\Big[\frac{y}{z}\Big]\Big] \quad \text{and} \quad \mathcal{O}^\wedge_{X_0,a} = k\Big[\Big[\frac{x}{z}, \frac{y}{z}\Big]\Big]\Big/\Big(\Big(\frac{x}{z}\Big)^3 - \Big(\frac{y}{z}\Big)^2\Big).$$

Let $\tilde{X}_0 \to X_0$ be the normalization map and $\tilde{a} \in \tilde{X}_0$ be the preimage of $a \in X_0$. Then there is a uniformizer $s \in \mathcal{O}^\wedge_{\tilde{X}_0,\tilde{a}}$ such that the inclusion $\mathcal{O}^\wedge_{X_0,a} \subset \mathcal{O}^\wedge_{\tilde{X}_0,\tilde{a}}$ is nothing but $x/z \mapsto s^2$ and $y/z \mapsto s^3$. Summing up:

Proposition 1.1. — *Inside $\mathcal{O}^\wedge_{\tilde{X}_0,\tilde{a}} = k[[s]]$, the complete local subalgebra $\mathcal{O}^\wedge_{X_0,a}$ is generated by $s^2 = x/z$ and $s^3 = y/z$, whereas $\mathcal{O}^\wedge_{\mathbb{P}^1_k,b}$ is generated by $s^3 = y/z$.*

The next task is to find regular models for the elliptic curve X_η over $K = k((T))$ whose closed fiber is a reduced divisor with simple normal crossings. This is indeed possible after replacing $A = k[[t]]$ by the finite ring extension $A' = k[[t^{1/3}]]$. Let X'_η be the induced elliptic curve over $K' = k((t^{1/3}))$. The coordinate change $x = t^{2/3}\bar{x}$ and $y = t\bar{y}$ shows that $\bar{x}^3 = \bar{y}^2 - \bar{y}$ is a Weierstrass equation for X'_η, so the corresponding constant elliptic curve over A' is a regular model.

For later applications, however, we prefer a regular model $X' \to \operatorname{Spec}(A')$ such that the projection $X'_\eta \to X_\eta$ extends to a morphism $X' \to X$. Recall that $X \subset \mathbb{P}^2_A$ is defined by the homogeneous equation $x^3 = y^2z - tyz^2$. It follows that $\operatorname{Sing}(X)$ consists of a single point located at $[0,0,1]$, and this singularity is a rational double point of type A_2. Consider the induced surface $X \otimes A'$.

Proposition 1.2. — *The surface $X \otimes A'$ is normal and* $\operatorname{Sing}(X \otimes A')$ *consists of a single point mapping to the singular point on X. The exceptional divisor $E' \subset X'$ for the minimal resolution $X' \to X \otimes A'$ is an elliptic curve with j-invariant $j = 0$ and selfintersection number*

-1. *The strict transform $R' \subset X'$ of X_0 is a smooth rational curve with selfintersection number -1. The closed fiber $X'_0 = E' + R'$ for the projection $X' \to \mathrm{Spec}(A')$ is a simple normal crossing divisor with $E' \cdot R' = 1$.*

Proof. — First note that $X \otimes A'$ satisfies Serre's condition (S_2) by [6], Proposition 6.4.1. It also satisfies the regularity condition (R_1), because $X \to \mathrm{Spec}(A)$ is smooth outside $\mathrm{Sing}(X)$. It follows that $X \otimes A'$ is normal and that $\mathrm{Sing}(X \otimes A')$ consists of at most one point, which must map to $\mathrm{Sing}(X)$.

Let $W \to X$ be the blowing up of the unique singular point $[0, 0, 1] \in X$. A local computation shows that the surface W is regular, and that the closed fiber W_0 is a *degeneration of type IV* of the elliptic curve $W_\eta = X_\eta$. In other words $W_0 = D_1 + D_2 + D_3$, where the D_i are smooth rational curves with selfintersection numbers $D_i^2 = -2$ meeting in a single point. Let $V \to W$ be the blowing up of this intersection point. Then $V_0 = 3F + C_1 + C_2 + C_3$, where F denotes the reduced exceptional divisor and the C_i are the strict transforms of the D_i. Note that $F^2 = -1$ and $C_i^2 = -3$.

To get rid of the multiplicity of $F \subset V_0$ we make a stable reduction process as discussed in [10], Section 3C. Consider the base change $V \otimes A' \to V$ along the A-algebra $A' = k[[t^{1/3}]]$. This is a Kummer covering of degree three with branch locus the closed fiber $V_0 \subset V$. We have $\mathcal{O}_{V \otimes A'} = \mathcal{O}_V \oplus \mathcal{O}_V t^{1/3} \oplus \mathcal{O}_V t^{2/3}$, where the multiplication law is given by the canonical inclusion $\mathcal{O}_V t \subset \mathcal{O}_V$. According to [2], Proposition 4.3, the normalization $V' \to V \otimes A'$ is given by the $\mathcal{O}_V$-algebra

$$\mathcal{O}_{V'} = \mathcal{O}_V \oplus \mathcal{O}_V(-C) \oplus \mathcal{O}_V(-2C),$$

where $C = C_1 + C_2 + C_3$, and the multiplication law is induced by the composite mapping $\mathcal{O}_V(-3C) \subset \mathcal{O}_V t \subset \mathcal{O}_V$. The projection $h : V' \to V$ is a Kummer covering of degree three with branch locus $C \subset V$. The resulting surface V' is regular because the branch locus is regular.

Let $F', C'_i \subset V'$ be the reduced preimages of $F, C_i \subset V$, respectively. We have $h_*(F') = 3F$ and $h_*(C'_i) = C_i$ and infer that $V'_0 = F' + C'_1 + C'_2 + C'_3$ is reduced. By construction, $F' \to F$ is a Kummer covering of degree three whose branch locus consists of three reduced points, hence F' is an elliptic curve. Since V'_0 has arithmetic genus one, the other components C'_i are smooth rational curves and the closed fiber V'_0 is a divisor with simple normal crossings. Using $h^*(F) = F'$ and $h^*(C_i) = 3C'_i$, we conclude ${F'}^2 = -3$ and ${C'_i}^2 = -1$ with the projection formula. After contracting the (-1)-curves $C'_i \subset V'$, we obtain a relative elliptic curve with generic fiber

$V'_\eta = X'_\eta$, which must be isomorphic to the constant family with Weierstrass equation $x^3 = y^2 - y$. In particular, the elliptic curve F' has j-invariant $j = 0$.

Let $C'_1 \subset V'$ be the reduced strict transform of $X_0 \subset X$, and $V' \to X'$ be the contraction of the remaining two disjoint (-1)-curves $C'_2 \cup C'_3$. Then we have an induced map $X' \to X \otimes A' \to X$. The image $E' \subset X'$ of $F' \subset V'$ is an elliptic curve with j-invariant $j = 0$, which is the exceptional divisor for $X' \to X \otimes A'$. It follows that $X \otimes A'$ actually has a singularity and that $X' \to X \otimes A'$ is the minimal resolution of singularities. The image $R' \subset X'$ of C'_1 is a smooth rational curve, which is also the strict transform of X_0. By construction, $X'_0 = E' + R'$ is a simple normal crossing divisor with intersection numbers ${E'}^2 = -1$, ${R'}^2 = -1$, and $E' \cdot R' = 1$. $\square$

Remarks 1.3. — Laufer showed in [12], Theorem 4.1 that the formal isomorphism class of the singularity on $X \otimes A'$ is uniquely determined by the j-invariant $j(E) = 0$ and the selfintersection number $E^2 = -1$. Wagreich observed in [18], page 425 that the minimal resolution $X' \to X \otimes A'$ is not realized by blowing-up the reduce singular locus and normalizing. More generally, Tomari showed in [17], Theorem 7.4 that an elliptic Gorenstein surface singularity is resolved by a succession of blowing ups with reduced centers and normalizations if and only if the minimal elliptic cycle on the exceptional divisor has selfintersection $\leqslant -2$.

2. Deformations for coverings of complete intersection.

Fix a ground field k of arbitrary characteristic $p \geqslant 0$. Let C and D be two curves without embedded components, and $h : C \to D$ a flat finite morphism that is generically étale. Then $\Omega^1_{C/D}$ is a coherent skyscraper sheaf supported by the ramification points $x \in C$. We shall study infinitesimal deformations of $h : C \to D$. Let R be a local Artin k-algebra with residue field k. A *deformation* of h over A consists of a curve $\mathfrak{C}$ flat and of finite type over A, a morphism $f : \mathfrak{C} \to D \otimes A$, and an isomorphism $f \otimes_A k \simeq h$.

Suppose $I \subset A$ is an ideal with $I^2 = 0$, and $f : \mathfrak{C} \to D \otimes A/I$ is a deformation of g over A/I. According to [11], Proposition 2.1.2.3, the obstruction for extending it to a deformation $f' : \mathfrak{C}' \to D \otimes A$ over A lies in the vector space of hyperextensions

$$\mathbb{E}\mathrm{xt}^2(L^\bullet_{C/D}, \mathcal{O}_C \otimes_k I) = \mathbb{E}\mathrm{xt}^2(L^\bullet_{C/D}, \mathcal{O}_C) \otimes_k I.$$

Here $L^\bullet_{C/D}$ is the *cotangent complex* for h. These obstructions vanish under suitable assumptions. Recall that $h : C \to D$ is a *morphism locally of complete intersection* if for all $c \in C$, the Artin local ring $\mathcal{O}_{C,c}/\mathfrak{m}_{h(c)}\mathcal{O}_{C,c}$ is the quotient of some power series algebra $\kappa(c)[[t_1, \ldots, t_n]]$ by a regular sequence ([8], Definition 19.3.6).

PROPOSITION 2.1. — *If the morphism of curves $h : C \to D$ is locally of complete intersection, then the group $\mathbb{E}\mathrm{xt}^2(L^\bullet_{C/D}, \mathcal{O}_C)$ vanishes.*

Proof. — First note that the cotangent complex $L^\bullet_{C/D}$ has a very explicit form in our situation. Consider the coherent $\mathcal{O}_D$-module $\mathcal{A} = h_*(\mathcal{O}_C)$ and the projective D-scheme $P = \mathbb{P}(\mathcal{A})$. Since $h : C \to D$ is affine, the invertible $\mathcal{O}_C$-module $\mathcal{O}_C$ is very ample with respect to D, so there is a closed embedding $C \subset P$. As $h : C \to D$ is flat and finite, the $\mathcal{O}_D$-module $\mathcal{A}$ is locally free of finite rank, hence the projection $P \to D$ is smooth. It follows from [8], Corollary 19.3.5 that the closed embedding $C \subset P$ is a regular embedding, because $h : C \to D$ is a morphism locally of complete intersection. Hence the conormal sheaf $\mathcal{N}_{C/P} = \mathcal{I}/\mathcal{I}^2$ is locally free. By [11], Proposition 3.3.6, the cotangent complex $L^\bullet_{C/D}$ is quasiisomorphic to the complex concentrated in degrees $[-1, 0]$ given by the canonical map $\mathcal{N}_{C/P} \to \Omega^1_{P/D}|_C$ from the exact sequence

$$\mathcal{N}_{C/P} \longrightarrow \Omega^1_{P/D}|_C \longrightarrow \Omega^1_{C/D} \longrightarrow 0.$$

The map on the left is injective, because it is generically injective and $\mathcal{N}_{C/P}$ is torsion free. It follows that $\mathcal{H}^s(L^\bullet_{C/D}) = 0$ for $s \neq 0$ and $\mathcal{H}^0(L^\bullet_{C/D}) = \Omega^1_{C/D}$.

Consider the spectral sequence

$$\mathrm{Ext}^r(\mathcal{H}^s(L^\bullet_{C/D}), \mathcal{O}_C) \Rightarrow \mathbb{E}\mathrm{xt}^{r-s}(L^\bullet_{C/D}, \mathcal{O}_C).$$

The sheaves $\mathcal{H}^s(L^\bullet_{C/D})$ are skyscraper sheaves supported by the ramification points $c \in C$. We have $\mathrm{Ext}^0(\mathcal{H}^s(L^\bullet_{C/D}), \mathcal{O}_C) = 0$ because $\mathcal{O}_C$ is torsion free. It follows that the edge map $\mathrm{Ext}^2(\Omega^1_{C/D}, \mathcal{O}_C) \to \mathbb{E}\mathrm{xt}^2(L^\bullet_{C/D}, \mathcal{O}_C)$ from the spectral sequence is surjective. Next, consider the spectral sequence

$$H^r(C, \mathcal{E}xt^s(\Omega^1_{C/D}, \mathcal{O}_C)) \Longrightarrow \mathrm{Ext}^{r+s}(\Omega^1_{C/D}, \mathcal{O}_C).$$

The sheaf $\mathcal{E}xt^0(\Omega^1_{C/D}, \mathcal{O}_C)$ vanishes, because $\Omega^1_{C/D}$ is torsion and $\mathcal{O}_C$ is torsion free. The group $H^1(C, \mathcal{E}xt^1(\Omega^1_{C/D}, \mathcal{O}_C))$ vanishes because $\mathcal{E}xt^1(\Omega^1_{C/D}, \mathcal{O}_C)$ has 0-dimensional support. The sheaf $\mathcal{E}xt^2(\Omega^1_{C/D}, \mathcal{O}_D)$ vanishes, because the stalks of $\Omega^1_{C/D}$ have projective dimension $\leqslant 1$. We conclude that $\mathrm{Ext}^2(\Omega^1_{C/D}, \mathcal{O}_C)$ and hence $\mathbb{E}\mathrm{xt}^2(L^\bullet_{C/D}, \mathcal{O}_C)$ vanishes. □

Suppose again that $I \subset A$ has square zero, and that $f : \mathfrak{C} \to D \otimes A/I$ is a deformation over A/I. According to [11], Proposition 2.1.2.3, the set of isomorphism classes of deformation $f' : \mathfrak{C}' \to D \otimes A$ endowed with an isomorphism $f' \otimes A/I \simeq f$ is an affine space for the vector space of hyperextensions

$$\mathbb{E}\mathrm{xt}^1(L^\bullet_{C/D}, \mathcal{O}_C \otimes_k I) = \mathbb{E}\mathrm{xt}^1(L^\bullet_{C/D}, \mathcal{O}_C) \otimes_k I.$$

This group splits up into local parts:

PROPOSITION 2.2. — *Let $a_i \in C$ be the ramification points and $b_i = h(a_i)$ the corresponding branch points. Then $\mathbb{E}\mathrm{xt}^1(L^\bullet_{C/D}, \mathcal{O}_C) \simeq \bigoplus_i \mathrm{Ext}^1(\Omega^1_{\mathcal{O}^\wedge_{C,a_i}/\mathcal{O}^\wedge_{D,b_i}}, \mathcal{O}^\wedge_{C,a_i})$.*

Proof. — The spectral sequence

$$\mathrm{Ext}^r(\mathcal{H}^s(L^\bullet_{C/D}), \mathcal{O}_C) \Rightarrow \mathbb{E}\mathrm{xt}^{r-s}(L^\bullet_{C/D}, \mathcal{O}_C)$$

gives an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\Omega^1_{C/D}, \mathcal{O}_C) \longrightarrow \mathbb{E}\mathrm{xt}^1(L^\bullet_{C/D}, \mathcal{O}_C) \longrightarrow \mathrm{Hom}(\mathcal{H}^{-1}(L^\bullet_{C/D}), \mathcal{O}_C).$$

The term on the right vanishes because $\mathcal{H}^{-1}(L^\bullet_{C/D})$ is torsion and $\mathcal{O}_C$ is torsion free. The coherent $\mathcal{O}_C$-module $\Omega^1_{C/D}$ is a skyscraper sheaf supported by the $a_i \in C$, so $\Omega^1_{C/D} = \bigoplus_i \Omega^1_{\mathcal{O}^\wedge_{C,a_i}/\mathcal{O}^\wedge_{D,b_i}}$ and the result follows. $\square$

3. Construction of effective formal deformations.

We now apply the results of the preceding section in the following situation. Fix an algebraically closed ground field k of characteristic $p \neq 3$, and let $\tilde{C}$ be a proper smooth curve of genus $g \geqslant 0$ over k. Suppose we have a finite generically étale morphism $\tilde{h} : \tilde{C} \to \mathbb{P}^1_k$ with only triple ramification points.

Let $\tilde{a} \in C$ be such a ramification point. Applying an automorphism of $\mathbb{P}^1_k$, we may assume that $\tilde{a}$ maps to $0 \in \mathbb{P}^1_k$. Using that k is algebraically closed and that $p \neq 3$, one easily sees that there is a uniformizer $s \in \mathcal{O}^\wedge_{\tilde{C},\tilde{a}}$ such that s^3 is a uniformizer for the subring $\mathcal{O}^\wedge_{\mathbb{P}^1,0} \subset \mathcal{O}^\wedge_{\tilde{C},\tilde{a}}$.

Following Serre's discussion in [16], Chapter VI, Section 1.3, we now construct a singular curve C of arithmetic genus $g+1$ with $\tilde{C}$ as normalization. The underlying topological space of C is the same as $\tilde{C}$, but we write $a \in C$ for the point corresponding to $\tilde{a} \in \tilde{C}$. The structure sheaf is the sheaf of subalgebras $\mathcal{O}_C \subset \mathcal{O}_{\tilde{C}}$ such that $\mathcal{O}_{C,c} = \mathcal{O}_{\tilde{C},c}$ for all $c \neq a$. In contrast, we set $\mathcal{O}_{C,a} = k + \mathfrak{m}^2_{\tilde{C},\tilde{a}}$. Intuitively, C is obtained from $\tilde{C}$ by pinching the first order infinitesimal neighborhood of $\tilde{a} \in \tilde{C}$. The exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{\tilde{C}} \oplus \kappa(a) \longrightarrow \mathcal{O}_{\tilde{C},\tilde{a}}/\mathfrak{m}^2_{\tilde{C},\tilde{a}} \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow k \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^1(\tilde{C}, \mathcal{O}_{\tilde{C}}) \longrightarrow 0,$$

and we infer that C has arithmetic genus $h^1(\mathcal{O}_C) = g+1$. By construction, the canonical bijective morphism $\tilde{C} \to C$ is the normalization, and the image $a \in C$ of $\tilde{a} \in \tilde{C}$ is a cuspidal singularity with $\mathcal{O}^\wedge_{C,a} = k[[s^2, s^3]]$. It follows that the morphism $\tilde{h} : \tilde{C} \to \mathbb{P}^1_k$ induces a finite generically étale morphism $h : C \to \mathbb{P}^1_k$ with $h(a) = 0$. Obviously, the flat morphism $h : C \to \mathbb{P}^1_k$ is locally of complete intersection.

PROPOSITION 3.1. — *Set $A = k[[t]]$. Then there is a flat family $X \to \operatorname{Spec}(A)$ and a finite A-morphism $f : X \to \mathbb{P}^1_A$ such that the following holds:*

(i) *The map on closed fibers $f_0 : X_0 \to \mathbb{P}^1_k$ is isomorphic to $h : C \to \mathbb{P}^1_k$.*

(ii) *The formal completion $\mathcal{O}^\wedge_{X,a}$ is isomorphic to $k[[x,y,t]]/(y^2 - ty - x^3)$ as algebra over $\mathcal{O}^\wedge_{\mathbb{P}^1_A,0} = k[[y,t]]$.*

(iii) *For every closed point $c \in X_0$ with $c \neq a$, the formal completion $\mathcal{O}^\wedge_{X,c}$ is isomorphic to $\mathcal{O}^\wedge_{X_0,c}[[t]]$ as algebra over $\mathcal{O}^\wedge_{\mathbb{P}^1_A,f(c)} = \mathcal{O}^\wedge_{\mathbb{P}^1_k,h(c)}[[t]]$.*

(iv) *The generic fiber X_η is a geometrically connected smooth curve of genus $g+1$.*

Proof. — First, we shall construct a formal flat morphism $\mathfrak{X} \to \operatorname{Spf}(A)$ and a finite formal morphism $\mathfrak{X} \to \mathbb{P}^1_k \times \operatorname{Spf}(A)$ with properties corresponding to (i)–(iii). Set $A_n = A/(t^{n+1})$. Suppose we already have constructed a flat A_{n-1}-scheme X_{n-1} and a morphism $f_{n-1} : X_{n-1} \to \mathbb{P}^1_{A_{n-1}}$ with properties as in (i)–(iii). According to Proposition 2.1, there is a flat A_n-scheme X'_n and a morphism $f'_n : X_n \to \mathbb{P}^1_{A_n}$ whose restriction to A_{n-1} admits an isomorphism $\varphi'_n : f'_n \otimes A_{n-1} \simeq f_{n-1}$. The set of isomorphism classes of such extensions is a torsor under

$$\bigoplus_i \operatorname{Ext}^1\left(\Omega^1_{\mathcal{O}^\wedge_{C,a_i}/\mathcal{O}^\wedge_{\mathbb{P}^1_k,b_i}}, \mathcal{O}^\wedge_{C,a_i}\right)$$

by Proposition 2.2, where $a_i \in C$ are the ramification points and $b_i \in \mathbb{P}^1_k$ are the corresponding branch points. Hence we may choose, for each ramification point $a_i \in C$, an element $\alpha_i \in \operatorname{Ext}^1\left(\Omega^1_{\mathcal{O}^\wedge_{C,a_i}/\mathcal{O}^\wedge_{\mathbb{P}^1_k,b_i}}, \mathcal{O}^\wedge_{C,a_i}\right)$ so that (α_i) applied to the isomorphism class of (X'_n, f'_n, φ'_n) gives another flat A_n-scheme X_n together with a morphism $f_n : X_n \to \mathbb{P}^1_{A_n}$ satisfying our conditions (i)–(iii). By induction we construct for all integers $n \geqslant 0$ morphisms $f_n : X_n \to \mathbb{P}^1_{A_n}$ together with identifications $\varphi_n : f_n \otimes A_{n-1} \simeq f_{n-1}$. Such a system is nothing but the desired morphism of formal schemes.

Being 1-dimensional, the proper scheme X_0 admits an ample invertible sheaf. The obstruction for extending an invertible sheaf from X_{n-1} to X_n lies in the group $H^2(X_0, \mathcal{O}_{X_0}) = 0$. It follows that there is a formal invertible sheaf on $\mathfrak{X}$ that is ample on X_0. By Grothendieck's Algebraization Theorem ([5], Theorem 5.4.5), the formal scheme $\mathfrak{X}$ is the formal completion of a projective A-scheme X. Moreover ([5], Theorem 5.4.1), the morphism $\mathfrak{X} \to \mathbb{P}^1 \times \operatorname{Spf}(A)$ of formal schemes comes from a morphism of schemes $f : X \to \mathbb{P}^1_A$. Properties (i)–(iii) hold because they depend only on the underlying formal schemes.

Concerning the last property (iv), observe that $C = X_0$ is geometrically integral, because $\tilde{C} \to C$ is a birational universal homeomorphism. Then X_η is geometrically integral as well by [7], Theorem 12.2.1. Hence $h^0(\mathcal{O}_{X_0}) = h^0(\mathcal{O}_{X_\eta})$ and in turn $g + 1 = h^1(\mathcal{O}_{X_0}) = h^1(\mathcal{O}_{X_\eta})$ by flatness. □

Starting with a proper smooth k-curve $\tilde{C}$ of genus $g \geqslant 1$ endowed with a finite generically étale morphism $\tilde{C} \to \mathbb{P}^1_k$ with only triple ramification, Proposition 3.1 produces a proper geometrically connected smooth curve X_η over $k((t))$ of genus $g+1$ endowed with a finite generically étale morphism $X_\eta \to \mathbb{P}^1_\eta$ with only triple ramification. The curve X_η defines a morphism $\operatorname{Spec} k((T)) \to M_{g+1}$ into the moduli space of smooth curves of genus $g+1$. Let $x \in M_{g+1}$ be the image point. Its residue field $\kappa(x)$ is called the *moduli field* for the smooth curve X_η. The crucial observation is:

PROPOSITION 3.2. — *The moduli field for the smooth curve X_η has transcendence degree $\geqslant 1$ over k.*

Proof. — Set $A = k[[t]]$ and consider the family $X \to \operatorname{Spec}(A)$ constructed above. The generic fiber X_η defines a rational map $\operatorname{Spec}(A) \dashrightarrow M_{g+1}$. This rational map is not necessarily everywhere defined, because

M_{g+1} is not proper. However, the moduli space $\overline{M}_{g+1}$ of stable curves of genus $g+1$ yields a compactification $M_{g+1} \subset \overline{M}_{g+1}$, and the rational map extends to a morphism $\mathrm{Spec}(A) \to \overline{M}_{g+1}$.

To prove our assertion it suffices to check that the latter map does not factor over a closed point. To see this we make base change with respect to the ring extension $A' = k[[t^{1/3}]]$. In light of Proposition 3.1 (ii), the surface X at its singular point $a \in X$ is formally isomorphic to the surface X studied in Section 1. It then follows from Proposition 1.2 that the induced surface $X \otimes A'$ has a unique singularity, and its minimal resolution $X' \to X \otimes A'$ yields a family of stable curves $X' \to \mathrm{Spec}(A')$. More precisely, the exceptional curve for the minimal resolution is an elliptic curve with j-invariant $j = 0$. In turn, the image of the classifying map $\mathrm{Spec}(A') \to \overline{M}_{g+1}$ hits the boundary divisor $\overline{M}_{g+1} - M_{g+1}$, whereas the generic point maps to the interior M_{g+1}. It follows that this morphism does not factor over a closed point, and therefore the moduli field of X_η is not algebraic. $\square$

4. Moduli fields for pointed rational curves.

In this section we discuss another method to achieve large moduli fields, namely to use pointed rational curves. Throughout we fix an integer $n \geqslant 3$. Recall that an *n-pointed smooth curve of genus zero* over a scheme S is a smooth proper map $f : X \to S$ whose fibers are isomorphic to $\mathbb{P}^1$, together with n disjoint sections $X_i \subset X$. The following is well known:

LEMMA 4.1. — *In the above situation, there is a unique isomorphism $X \to \mathbb{P}^1_S$ sending $X_1, X_2, X_3 \subset X$ to the constant sections $0, 1, \infty \subset \mathbb{P}^1_S$, respectively.*

Proof. — The $\mathcal{O}_S$-module $\mathcal{E} = f_*\mathcal{O}_X(X_1)$ is locally free of rank 2, and $X = \mathbb{P}(\mathcal{E})$. The sections $X_i \subset X$ correspond via $X_i = \mathbb{P}(\mathcal{L}_i)$ to invertible quotients $\mathcal{L}_i = \mathcal{E}/\mathcal{K}_i$, and the condition of disjointness means $\mathcal{K}_i \cap \mathcal{K}_j = 0$ for $i \neq j$. In turn, the canonical maps $\mathcal{E} \to \mathcal{L}_1 \oplus \mathcal{L}_2$ and $\mathcal{L}_1 \leftarrow \mathcal{K}_2 \to \mathcal{L}_3$ are bijective, so we may assume $\mathcal{L}_1 = \mathcal{L}_2$, $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_1$, and that $\mathcal{K}_2 \subset \mathcal{E}$ is the diagonal submodule. Tensoring $\mathcal{E}$ and the $\mathcal{L}_i$ with $\mathcal{L}_1^{-1}$, we obtain the desired isomorphism $X \to \mathbb{P}^1_S$ sending X_1, X_2, X_3 to $0, 1, \infty$, respectively. This isomorphism is unique because any automorphism of $\mathcal{O}_S \oplus \mathcal{O}_S$ fixing the summands and the diagonal submodule is multiplication by a scalar. $\square$

It follows that the functor sending a scheme S to the set of isomorphism classes of n-pointed smooth curves of genus zero over S is representable by the scheme $M_{0,n} = (\mathbb{P}^1 \times \ldots \times \mathbb{P}^1) - D$. The product has $n-3$ factors, and D denotes the closed subset of all points $(x_4, \ldots, x_n)$ with $x_i \in \{0, 1, \infty\}$ for some $4 \leqslant i \leqslant n$ or $x_i = x_j$ for some $4 \leqslant i < j \leqslant n$.

Fix a ground field k. In light of this explicit nature of $M_{0,n}$, it is easy to compute moduli fields. Suppose $(\mathbb{P}^1_K, 0, 1, \infty, x_4, \ldots, x_n)$ is an n-pointed smooth curve of genus zero over a field extension $k \subset K$. Using the identifications $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$ and $\mathbb{A}^1(K) = K$, we see that the rational points $x_4, \ldots, x_n \in \mathbb{P}^1_K$ correspond to certain scalars $t_4, \ldots, t_n \in K - \{0, 1\}$.

PROPOSITION 4.2. — *The moduli field of $(\mathbb{P}^1_K, 0, 1, \infty, x_4, \ldots, x_n)$ is nothing but the subfield $k(t_4, \ldots, t_n) \subset K$.*

Proof. — Let $L \subset K$ be the moduli field in question, and $(X, x'_1, \ldots, x'_n)$ be a pointed smooth curve of genus zero over L inducing our given pointed curve over K. We may assume $x'_1 = 0, x'_2 = 1, x'_3 = \infty$ by Lemma 4.1. Then the remaining $x'_4, \ldots, x'_n$ correspond to scalars $t'_4, \ldots, t'_n \in L - \{0, 1\}$. The uniqueness in Lemma 4.1 implies $t'_i = t_i$, so we have $k(t_4, \ldots, t_n) \subset L$. The reverse inclusion is obvious. □

We now examine the effect of finite morphisms on moduli fields:

PROPOSITION 4.3. — *Let $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be a finite morphism and $x_1, \ldots, x_n \in \mathbb{P}^1_K$ be rational points such that the images $y_1 = f(x_i), \ldots, y_n = f(x_n)$ are pairwise different. Then the moduli fields for the n-pointed smooth curves $(\mathbb{P}^1_K; x_1, \ldots, x_n)$ and $(\mathbb{P}^1_K; y_1, \ldots, y_n)$ have the same transcendence degree over k.*

Proof. — The finite morphism $f^n : \prod_{i=1}^n \mathbb{P}^1_K \to \prod_{i=1}^n \mathbb{P}^1_K$ induces on the moduli space a rational map $\psi : M_{0,n} \dashrightarrow M_{0,n}$ sending an n-pointed smooth curve of genus zero $(\mathbb{P}^1, x_1, \ldots, x_n)$ to $(\mathbb{P}^1, y_1, \ldots, y_n)$. The domain of definition for ψ comprises those pointed curves for which the image points $y_1, \ldots, y_n$ are pairwise different.

The rational map $\psi : M_{0,n} \dashrightarrow M_{0,n}$ is quasifinite on its domain of definition, because f is a finite map. So if $a \in M_{0,n}$ is a point in the domain of definition, and $b = \psi(a)$ is its image point, then $\kappa(b) \subset \kappa(a)$ is a finite field extension. This immediately implies our assertion. □

We now relate these moduli fields to coverings with only triple ramification. For this we assume that our ground field k has characteristic $p \neq 3$. Choose an algebraically closed extension field $k \subset K$ of transcendence degree $n-3$, and algebraically independent elements $t_4, \ldots, t_n \in K$. Consider the n-pointed smooth curve $(\mathbb{P}^1_K, y_1, \ldots, y_n)$ with $y_1 = 0, y_2 = 1, y_3 = \infty$ and such the remaining marked points $y_4, \ldots, y_n \in \mathbb{P}^1_K$ correspond to the scalars $t_4, \ldots, t_n \in K$.

PROPOSITION 4.4. — *There is a generically étale finite map $h : \mathbb{P}^1_K \to \mathbb{P}^1_K$ with only triple ramification such that the $y_i \in \mathbb{P}^1_K$ occur as branch points. If we choose for each $y_i \in \mathbb{P}^1_K$ a ramification point $x_i \in h^{-1}(y_i)$, then the moduli field for $(\mathbb{P}^1_K, x_1, \ldots, x_n)$ has transcendence degree $n-3$.*

Proof. — Consider the polynomial map $r : \mathbb{P}^1_K \to \mathbb{P}^1_K$ given by $[z_0, z_1] \mapsto [z_0^3, z_1^3]$. Then r is a generically étale finite map of degree three with only triple ramification, whose ramification and branch points are $0, \infty \in \mathbb{P}^1_K$. For suitable $\varphi \in \mathrm{PGL}_2(K)$, the composition φr realizes any given pair of rational points $a, b \in \mathbb{P}^1_K$ as branch locus.

We now construct the desired map h by induction. Suppose we already have a generically étale finite map $h_i : \mathbb{P}^1_K \to \mathbb{P}^1_K$ with only triple ramification such that the y_j occur as branch points for $1 \leqslant j \leqslant i$ and h_i is étale over y_j for $i+1 \leqslant j \leqslant n$. Fix a rational point $x' \in h_i^{-1}(y_{i+1})$. Choose $\varphi \in \mathrm{PGL}_2(K)$ so that x' is a branch point for φr, but that φr is étale over $h_i^{-1}(y_j)$ for all points y_j with $j \neq i+1$. Then $h_{i+1} = h_i \varphi r$ is the desired map.

It remains to check the assertion on moduli fields. According to Proposition 4.2, the moduli field for $(\mathbb{P}^1_K, y_1, \ldots, y_n)$ is $k(t_4, \ldots, t_n) \subset K$, which has transcendence degree $n-3$. It then follows from Proposition 4.3 that the moduli field for $(\mathbb{P}^1_K, x_1, \ldots, x_n)$ has transcendence degree $n-3$ as well. $\square$

5. Curves with only triple ramification.

We come to the main result of this paper:

THEOREM 5.1. — *Let k be a field of characteristic $p \neq 3$. For each integer $g \geqslant 0$, there is finitely generated field extension $k \subset K$ and a smooth proper curve C of genus g over K with the following properties:*

(i) *The moduli field of C has transcendence degree over k at least* $\max(2g-3, g)$.

(ii) *There is a finite generically étale map $C \to \mathbb{P}^1_K$ with only triple ramification.*

Let me reformulate this in terms of moduli spaces over algebraically closed ground fields.

COROLLARY 5.2. — *Suppose k is algebraically closed of characteristic $p \neq 3$ and assume $g \geqslant 2$. Let $S \subset M_g$ be closure for the set of all closed points such that the corresponding curve C admits a finite generically étale map $C \to \mathbb{P}^1_k$ with only triple ramification. Then we have* $\dim(S) \geqslant 2g-3$.

Proof. — Let $C \to \mathbb{P}^1_K$ be as in Theorem 5.1, and choose an integral k-scheme U of finite type whose field of rational functions is $K = \kappa(U)$. Shrinking U, we may extend C to a smooth relative curve $X \to U$, and $C \to \mathbb{P}^1_K$ to a U-morphism $f : X \to \mathbb{P}^1_U$. Shrinking further, we may assume that all fibers $f_u : X_u \to \mathbb{P}^1_u$ are generically étale finite maps with only triple ramification.

By Chevalley's Theorem, the image $V = \varphi(U)$ of the classifying map $\varphi : U \to M_g$ is constructible. Shrinking U, we may assume that $V \subset M_g$ is a subscheme. Since the moduli field of C has transcendence degree $\geqslant 2g-3$, the dimension of V is at least $2g-3$. For each rational point $\sigma \in V$, the fiber $\varphi^{-1}(\sigma) \subset U$ contains a rational point because k is algebraically closed, and the result follows. □

We also extend the result on elliptic curves of Fried, Klassen, and Kopeliovich [3] to all characteristics:

COROLLARY 5.3. — *Suppose k is algebraically closed of characteristic $p \neq 3$. Then for all but finitely many j-invariants $j \in k$, the corresponding elliptic curve E admits a finite generically étale map $E \to \mathbb{P}^1_k$ with only triple ramification.*

Proof. — The argument is as for the preceding corollary, except that one uses the moduli space $M_{1,1}$ instead of M_g. □

Proof of Theorem 5.1. — First consider the case $g \geqslant 3$. We shall construct by induction on $n \geqslant 0$ a pointed smooth stable curve

$(C_n, c_1^n, \ldots, c_{g-n}^n)$ of genus n with $g-n$ marked points over some algebraically closed field extension K_n, so that the moduli field has transcendence degree $\geqslant n+g-3$. Moreover, there will be a generically étale finite map $h_n : C_n \to \mathbb{P}^1_{K_n}$ with only triple ramification such that the c_i^n are ramification points. Induction terminates at $n = g$.

According to Proposition 4.4, the desired curve exists for $n = 0$. Suppose we already found by induction $(C_n, c_1^n, \ldots, c_{g-n}^n)$ and $h_n : C_n \to \mathbb{P}^1_{K_n}$ for some $n < g$. The idea now is to trade the last marked point for a genus increase. Conforming with the notation in Section 3, we set $\tilde{C} = C_n$ and $\tilde{a} = c_{g-n}^n$, and let C be the corresponding cuspidal curve of genus $n+1$ with normalization $\tilde{C}$. According to Proposition 3.1, there is an effective deformation $X \to \operatorname{Spec}(A)$ over $A = K_n[[t]]$ with closed fiber isomorphic to C. Moreover, our given map $C \to \mathbb{P}^1_{K_n}$ extends to a family $f : X \to \mathbb{P}^1_A$ whose generic fiber $X_\eta \to \mathbb{P}^1_\eta$ is a generically étale finite map with only triple ramification. Furthermore, the rational ramification points $c_1^n, \ldots, c_{g-n-1}^n \in C$ extend to ramification sections over A, and define rational ramification points $c_1^{n+1}, \ldots, c_{g-(n+1)}^{n+1} \in X_\eta$ in the generic fiber.

Let $X' \to \operatorname{Spec}(A')$ be the stable reduction over the base change $A' = k[[t^{1/3}]]$ for $X \to \operatorname{Spec}(A)$ constructed in the proof for Proposition 3.2. Then we have a classifying morphism $\operatorname{Spec}(A') \to \overline{M}_{n+1,g-(n+1)}$. The image of the closed point $0 \in \operatorname{Spec}(A')$ is a point $\sigma \in \overline{M}_{n+1,g-(n+1)}$ corresponding to the pointed stable curve $(C_n \cup E, c_1^n, \ldots, c_{g-(n+1)}^n)$ of genus $n+1$. Here E is an elliptic curve with $j = 0$ as in the proof for Proposition 3.2, and $C_n \cap E = \{c_{g-n}^n\}$. It follows that the residue field $\kappa(\sigma)$ has transcendence degree $\geqslant n+g-3$. As a consequence, the image of the generic point $\eta \in \operatorname{Spec}(A')$ in $\overline{M}_{n+1,g-(n+1)}$ has residue field of transcendence degree $\geqslant (n+1)+g-3$. We now let K_{n+1} be the algebraic closure of $K_n((T))$, set $C_{n+1} = X_\eta \otimes K_{n+1}$, and choose as marked points $c_1^{n+1}, \ldots, c_{g-(n+1)}^{n+1}$. This completes the induction.

In this way we obtain a smooth proper curve C of genus g satisfying properties (i) and (ii). The field extension $k \subset K$ in the construction is algebraically closed, but it follows from [7], Theorem 8.8.2 that the map $C \to \mathbb{P}^1_K$ is already defined over some finitely generated field extension. This finishes the case $g \geqslant 3$.

It remains to treat the case $g \leqslant 2$. For $g = 0$, we simply take $C = \mathbb{P}^1_k$ and the identity map $C \to \mathbb{P}^1_k$. For $g = 1$ or $g = 2$, we choose $C_0 = \mathbb{P}^1$, $c_1^0 = 0$, $c_2^0 = 1$, and $c_3^0 = \infty$ without caring for the moduli field, and apply the preceding deformation argument once ore twice, respectively. $\square$

6. Connections with Belyi's Theorem.

Belyi's Theorem [1] states that a compact Riemann surface is defined over a number field if and only if it admits a rational function with at most three critical values. Saïdi [14] generalized this to odd characteristics $p \geqslant 3$. Let me rephrase the part of his result that holds true for all characteristics:

PROPOSITION 6.1. — *Let k be an algebraically closed field of characteristic $p > 0$.*

(i) *A smooth proper curve C over k is defined over a finite field if there is a finite map $C \to \mathbb{P}^1_k$ with only tame ramification and at most three branch points.*

(ii) *A smooth proper curve C over $\overline{\mathbb{F}}_p$ admits a finite map $h : C \to \mathbb{P}^1_{\overline{\mathbb{F}}_p}$ with only tame ramification and at most three branch points if there is at least one finite map $g : C \to \mathbb{P}^1_{\overline{\mathbb{F}}_p}$ with only tame ramification.*

Proof. — For convenience, I recall Saïdi's argument: The first statement follows from Grothendieck's theory of the tame fundamental group (compare [13], Theorem 6.1). For the second statement, let $h_n : \mathbb{P}^1_{\overline{\mathbb{F}}_p} \to \mathbb{P}^1_{\overline{\mathbb{F}}_p}$ be the polynomial map $[z_0, z_1] \mapsto [z_0^{p^n-1}, z_1^{p^n-1}]$. Then $h = h_n \circ g$ is the desired map for some n sufficiently large, as explained in [14], Theorem 5.6. □

In characteristic $p \geqslant 3$, tame functions $g : C \to \mathbb{P}^1_{\overline{\mathbb{F}}_p}$ as in (ii) exist by [4], Proposition 8.1. In characteristic $p = 2$, Corollary 5.2 tells us that this holds true for a set of of curves of dimension at least $2g - 3$.

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