ANNALES

DE

L'INSTITUT FOURIER

Shoetsu OGATA

On projective tori varieties whose defining ideals have minimal generators of the highest degree

Tome 53, nº 7 (2003), p. 2243-2255.

<http://aif.cedram.org/item?id=AIF_2003__53_7_2243_0>

Ann. Inst. Fourier, Grenoble
53, 7 (2003), 2243–2255

ON PROJECTIVE TORIC VARIETIES WHOSE DEFINING IDEALS HAVE MINIMAL GENERATORS OF THE HIGHEST DEGREE

by Shoetsu OGATA

Introduction.

Sturmfels asked in [S2] whether a nonsingular projective toric variety should be defined by only quadrics if it is embedded by global sections of an ample line bundle. An evidence has been obtained by Koelman [K3] before Sturmfels asked the question. Koelman showed that projective toric surfaces are defined by binomials (differences of two monomials) of degree at most three ([K1] and [K2]) and obtained a criterion when the surface needs defining equations of degree three ([K3]). He used combinatorics of plane polygons.

Sturmfels showed in [S1] that for projectively normal toric varieties of dimension n, the defining ideals have minimal generators consisting of elements of degree at most $n+1$ (Theorem 13.14 in [S1]). There are examples showing that this bound is optimal. In this paper we give a generalization of [K3] to higher dimensions, that is, we give a criterion for the ideals defining projectively normal toric varieties of dimension n to be generated by elements of degree less than $n+1$. Bruns, Gubeladze and Trung [BGT] also give a generalization of the results of [K3].

Keywords: Toric varieties – Convex polytopes – Generators of ideals.
Math. classification: 14M25 – 14J40 – 52B20.

A toric variety is a normal algebraic variety with an algebraic action of an algebraic torus of the same dimension of the variety and a dense orbit. Let X be a projective toric variety of dimension n and $T \cong (\mathbb{C}^*)^n$ the algebraic torus acting on X. Let $M = \mathrm{Hom}_{\mathrm{gr}}(T, \mathbb{C}^*)$ be the group of characters, which is isomorphic to $\mathbb{Z}^n$. For $m \in M$, we denote $\mathbf{e}(m)$ the corresponding character of T. Let L be an ample line bundle on X. Then there exist an integral convex polytope P in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and an isomorphism

$$H^0(X, L) \cong \bigoplus_{m \in P \cap M} \mathbb{C}\mathbf{e}(m),$$

where an integral convex polytope is the convex hull of a finite number of elements of M. Let $R(X, L) := \oplus_{l \geqslant 0} H^0(X, L^{\otimes l})$ be the homogeneous coordinate ring of X. Then we have an isomorphism

$$R(X, L) \cong \bigoplus_{l \geqslant 0} \Bigg(\bigoplus_{m \in (lP \cap M)} \mathbb{C}\mathbf{e}(m) \Bigg).$$

This is a normal *polytopal semigroup ring* in the sense of [BGT]. If L is normally generated in the sense of Mumford [M], that is, L satisfies the conditions that it is very ample and that the image of X in $\mathbb{P}(H^0(X, L)^*)$ is projectively normal, then $R = R(X, L)$ is generated by its degree one elements. In this case, R is a quotient ring of the polynomial ring $S = \mathrm{Sym}\, H^0(X, L)$. Let I be the ideal of S with $R \cong S/I$. We call I the defining ideal of (X, L), or of the polytopal semigroup ring of P.

In general an ample line bundle L on a projective toric variety of dimension n is not very ample for $n > 2$. On the other hand, $L^{\otimes i}$ is normally generated for $i \geqslant n-1$ ([EW]), and the defining ideal of $(X, L^{\otimes i})$ is generated by quadrics for $i \geqslant n$ ([BGT], [NO]), or for $i = n-1$ and $n \geqslant 3$ ([Og]). The normal generation of L is equivalent to the condition for the corresponding integral convex polytope P that for all positive integers l, each element x in $(lP) \cap M$ can be expressed as a sum $x = m_1 + \cdots + m_l$ of l elements of $P \cap M$. We call P is normally generated if P satisfies this condition. When $n = 2$, all ample line bundles on projective toric surfaces are normally generated. This is one of difficulties arising in generalization of Koelman's result [K3] to higher dimensions by using combinatorics of polytopes.

We employ a method of algebraic geometry. Specifically, we consider the case of curves which are complete intersections of hyperplane sections and use regular ladders of Fujita [Fj].

THEOREM 1. — *Let P be an integral convex polytope of dimension n ($\geqslant 2$). Assume that P is normally generated. Then the defining ideal of the polytopal semigroup ring of P has generators of degree $n+1$ if and only if P is an n-simplex with standard facets and containing lattice points in its interior.*

We may restate Theorem 1 in terms of algebraic geometry. It is convenient for the readers because we shall prove a part of Theorem by using algebraic geometry.

THEOREM 1′. — *Let X be a projective toric variety of dimension n ($\geqslant 2$) and let L a very ample line bundle on X which defines an embedding of X as a projectively normal variety. Let P be the integral convex polytope of dimension n determined by the global sections of L. The defining ideal of X needs elements of degree $n+1$ as generators if and only if P is an n-simplex with standard facets and containing lattice points in its interior.*

One half of Theorem is given by Proposition 1.3, which says that if P has only $n+1$ lattice points in the boundary and if it contains at least one lattice point in the interior then the defining ideal needs elements of degree $n+1$ as generators. We can easily see that if P contains only $n+1$ lattice points then $(X, L) \cong (\mathbb{P}^n, \mathcal{O}(1))$. Thus another half of Theorem is that if P contains more than $n+1$ lattice points in the boundary then the defining ideal has generators of degree at most n, which is given by Theorem 2.6.

We know that if X is nonsingular, then P is simplicial and for each vertex v_0 there are n edges $\mathbb{R}_{\geqslant 0} v_i$ ($i = 1, \ldots, n$) meeting at v_0 such that $\{v_1 - v_0, \ldots, v_n - v_0\}$ is a basis of the lattice $\mathbb{Z}^n$. If, in addition, the boundary of P contains only $n+1$ lattice points, then P contains no lattice point in its interior, that is, P is a standard n-simplex. Hence it does not satisfy the condition of Theorem. Thus we have a weak answer to Sturmfels' question.

COROLLARY 1. — *For a nonsingular projectively normal toric variety of dimension n ($\geqslant 2$), its defining ideal embedded by global sections of an ample line bundle has generators of degree at most n.*

Next consider the case that P is an integral n-simplex, that is, $P = \operatorname{Conv}\{u_0, u_1, \ldots, u_n\}$ with $u_0, u_1, \ldots, u_n \in \mathbb{Z}^n =: M$. Let M' be the sublattice of M generated by $u_1 - u_0, \ldots, u_n - u_0$. Then P is a standard n-simplex with respect to M'. Hence (P, M') defines the projective n-space $(\mathbb{P}^n, \mathcal{O}(1))$. From this consideration we see that the toric variety X defined by P is a quotient of the projective n-space by a finite abelian group M/M'. A weighted projective space $\mathbb{P}(q_0, q_1, \ldots, q_n)$ has the same form $\mathbb{P}^n/((\mathbb{Z}/q_0) \times \cdots \times (\mathbb{Z}/q_n))$. If all facets of P are standard $(n-1)$-simplices, then all n elements of $\{q_0, q_1, \ldots, q_n\}$ coincide, hence $\mathbb{P}(q_0, q_1, \ldots, q_n) \cong \mathbb{P}^n$. Thus it does not satisfy the condition of Theorem.

COROLLARY 2. — *The defining ideals of projectively normal weighted projective n-spaces have generators of degree at most n.*

The author should acknowledge the hospitality of Mathematisches Institut Universität Erlangen-Nürnberg, where he could complete this paper. He is also grateful to Michel Brion for his kindly advice on the paper in manuscript.

1. Polarized toric varieties.

First we mention the facts about toric varieties needed in this paper following Oda's book [Od], or Fulton's book [Fl].

Let N be a free $\mathbb{Z}$-module of rank n, M its dual and $<\,,\,>: M \times N \to \mathbb{Z}$ the canonical pairing. By scalar extension to the field $\mathbb{R}$ of real numbers, we have real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$ be the algebraic n-torus over the field $\mathbb{C}$ of complex numbers, where $\mathbb{C}^*$ is the multiplicative group of $\mathbb{C}$. Then $M = \mathrm{Hom}_{\mathrm{gr}}(T_N, \mathbb{C}^*)$ is the character group of T_N. For $m \in M$ we denote $\mathbf{e}(m)$ the corresponding character of T_N. Let Δ be a complete finite fan of N consisting of strongly convex rational polyhedral cones σ, that is, there exist a finite number of elements $v_1, v_2, \ldots, v_s$ in N such that

$$\sigma = \mathbb{R}_{\geqslant 0} v_1 + \cdots + \mathbb{R}_{\geqslant 0} v_s,$$

and $\sigma \cap \{-\sigma\} = \{0\}$. Then we have a complete toric variety $X = T_N \operatorname{emb}(\Delta) := \cup_{\sigma \in \Delta} U_\sigma$ of dimension n (see Section 1.2 [Od], or Section 1.4 [Fl]). Here $U_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ and $\sigma^\vee$ is the dual cone of σ with respect to the pairing $<\,,\,>$. For the origin $\{0\}$, the affine open set $U_{\{0\}} = \operatorname{Spec} \mathbb{C}[M]$ is the unique dense T_N-orbit. We note that a toric variety is always normal.

Let L be an ample T_N-equivariant invertible sheaf on X. Then the polarized variety (X, L) corresponds to an integral convex polytope. We call the convex hull $\operatorname{Conv}\{u_0, u_1, \ldots, u_r\}$ in $M_{\mathbb{R}}$ of a finite subset

$\{u_0, u_1, \ldots, u_r\} \subset M$ an *integral convex polytope* in $M_{\mathbb{R}}$. The correspondence is given by the isomorphism

$$(1.1) \qquad H^0(X, L) \cong \bigoplus_{m \in P \cap M} \mathbb{C}\mathbf{e}(m),$$

where $\mathbf{e}(m)$ are considered as rational functions on X because they are functions on the open dense subset T_N of X (see Section 2.2 [Od], or Section 3.5 [Fl]).

Let P_1 and P_2 be integral convex polytopes in $M_{\mathbb{R}}$. Then we can consider the Minkowski sum $P_1 + P_2 := \{x_1 + x_2 \in M_{\mathbb{R}}; x_i \in P_i \ (i = 1, 2)\}$ and the multiplication by scalars $rP_1 := \{rx \in M_{\mathbb{R}}; x \in P_1\}$ for a positive real number r. If l is a natural number, then lP_1 coincides with the l times sum of P_1, i.e., $lP_1 = \{x_1 + \cdots + x_l \in M_{\mathbb{R}}; x_1, \ldots, x_l \in P_1\}$. The l-th tensor power $L^{\otimes l}$ corresponds to the convex polytope $lP := \{lx \in M_{\mathbb{R}}; x \in P\}$. Moreover the multiplication map

$$(1.2) \qquad H^0(X, L^{\otimes l}) \otimes H^0(X, L) \to H^0(X, L^{\otimes (l+1)})$$

transforms $\mathbf{e}(u_1) \otimes \mathbf{e}(u_2)$ for $u_1 \in lP \cap M$ and $u_2 \in P \cap M$ to $\mathbf{e}(u_1 + u_2)$ through the isomorphism (1.1). Therefore the equality $(lP \cap M) + (P \cap M) = (l+1)P \cap M$ is equivalent to the surjectivity of (1.2).

In this article we assume that L is *normally generated*, that is, the multiplication map (1.2) is surjective for all $l \geqslant 1$, hence, it is very ample. In terms of polytopes, the normal generation of L means that the equality

$$(1.3) \qquad (lP \cap M) + (P \cap M) = (l+1)P \cap M$$

holds for all positive integers l. It is also equivalent to the condition that for all $l \geqslant 1$, and for any $v \in lP \cap M$, there exist l elements $u_1, \ldots, u_l$ of $P \cap M$ with $v = u_1 + \cdots + u_l$. From this reason we may call P to be *normally generated* if it satisfies (1.3) for all positive integers l.

Let $P \cap M = \{u_0, u_1, \ldots, u_r\}$. By the assumptions we have the embedding by global sections of L;

$$\Phi : X \to \mathbb{P}(H^0(X, L)^*) \cong \mathbb{P}^r.$$

Let $Z_0, Z_1, \ldots, Z_r$ be the homogeneous coordinates of $\mathbb{P}^r$. Then Φ is defined by $Z_i = \mathbf{e}(u_i)$ for $i = 0, 1, \ldots, r$. Set $R := \oplus_{l \geqslant 0} R_l = \oplus_{l \geqslant 0} H^0(X, L^{\otimes l})$ and $S := \oplus_{l \geqslant 0} S_l = \mathbb{C}[Z_0, Z_1, \ldots, Z_r]$. Then we define a surjective ring homomorphism $\varphi : S \to R$ by $\varphi(\prod_i Z_i^{a_i}) = \mathbf{e}(\sum_i a_i u_i)$. Let I be the kernel of φ. Then we see that $I_0 = I_1 = \{0\}$ for the graded ideal $I = \oplus_{l \geqslant 0} I_l$. We call I the *defining ideal* of X in $\mathbb{P}(H^0(X, L)^*)$.

LEMMA 1.1 (Eisenbud-Sturmfels [ES]). — *The defining ideal I is generated by binomials, that is, the differences of two monomials.*

For a proof see Proposition 2.3 in [ES].

PROPOSITION 1.2 (Sturmfels [S1]). — *Let L be a normally generated ample line bundle on a projective toric variety X of dimension n. Then every minimal generator of the ideal defining X in $\mathbb{P}(H^0(X,L)^*)$ has degree at most $n+1$.*

For a proof see Theorem 13.14 in [S1].

PROPOSITION 1.3. — *Let $P = \mathrm{Conv}\{u_0, u_1, \dots, u_n\}$ be an integral n-simplex such that the equality (1.3) holds for all positive integers l. We assume that the boundary of P contains only $n+1$ lattice points, and that P contains at least one lattice point in its interior. Then the defining ideal I needs an element of degree $n+1$ as a generator.*

Proof. — By a suitable affine translation of M we may assume $u_0 = 0$. Let $\{e_1, \dots, e_n\}$ be a $\mathbb{Z}$-basis of M. The very ampleness of L says that the set of all lattice points in the cone $\sigma^\vee = \mathbb{R}_{\geqslant 0}u_1 + \cdots + \mathbb{R}_{\geqslant 0}u_n$ is generated by $P \cap M$ as a semigroup. In other words, every lattice point in $\sigma^\vee \cap M$ can be written as a sum of elements in $P \cap M$ with positive integer coefficients. Since the lattice points of the face cone $\tau_n^\vee := \mathbb{R}_{\geqslant 0}u_1 + \cdots + \mathbb{R}_{\geqslant 0}u_{n-1}$ of $\sigma^\vee$ are also generated by $\mathrm{Conv}\{u_0, u_1, \dots, u_{n-1}\} \cap M = \{u_0, u_1, \dots, u_{n-1}\}$ as a semigroup, we may set $u_1 = e_1, \dots, u_{n-1} = e_{n-1}$. This shows that every facet of P is a standard $(n-1)$-simplex. Set $u_n = \sum_{i=1}^n a_i e_i$ with integer coefficients. By a change of bases we may set all $a_i \geqslant 0$. Since $\dim P = n$, we have $a_n > 0$. Moreover we may assume that $u_{n+1} := \sum_{i=1}^n e_i$ is in the interior of P. Then we have

$$a_i < a_n \quad \text{for } i = 1, \dots, n-1, \tag{1.4}$$

and

$$(n-2)a_n < a_1 + \cdots + a_{n-1} - 1. \tag{1.5}$$

By componentwise description with respect to the basis of M, we have

$$\begin{aligned} u_1 + \cdots + u_n &= (a_1 + 1, \dots, a_{n-1} + 1, a_n) \\ &= u_{n+1} + (a_1, \dots, a_{n-1}, a_n - 1). \end{aligned}$$

Since $(a_1, \dots, a_{n-1}, a_n - 1)$ is contained in nP from (1.4) and (1.5), there exist $v_{n+2}, \dots, v_{2n+1}$ in $P \cap M$ such that

$$(a_1, \dots, a_{n-1}, a_n - 1) = v_{n+2} + \cdots + v_{2n+1}. \tag{1.6}$$

Corresponding to the relation $u_0 + u_1 + \cdots + u_n = v_{n+2} + \cdots + v_{2n+1}$, we obtain a binomial $B := Z_0 Z_1 \cdots Z_{n+1} - Y_{n+2} \cdots Y_{2n+1}$, where $Y_j = \mathbf{e}(v_j) \in \{Z_0, \ldots, Z_r\}$. Since $(a_1, \ldots, a_{n-1}, a_n - 1)$ is not contained in $(n-1)P$ from (1.5), none of $v_{n+2}, \ldots, v_{2n+1}$ coincides with u_0. If we assume $Y_{n+2} = Z_1$, that is, $v_{n+2} = u_1$, then from (1.4) we have $(a_1 - 1, a_2, \ldots, a_{n-1}, a_n - 1) \notin (n-1)P$, which contradicts (1.6). Hence we see that the binomial B is irreducible.

Next we assume $B = X_1 B_1 + \cdots + X_s B_s$ with binomials $B_i \in I_n$ of degree n and $X_i \in \{Z_0, \ldots, Z_r\}$. If we write binomials B_i as the difference of two monomials $B_i = M_1^{(i)} - M_2^{(i)}$, then we have $X_1 M_1^{(1)} = Y_{n+1} \cdots Y_{2n+1}$ and $X_1 M_2^{(1)} = X_2 M_1^{(2)}, \ldots, X_s M_2^{(s)} = Z_0 Z_1 \cdots Z_n$. We note that for a binomial $B_i = M_1^{(i)} - M_2(i)$ we have $\varphi(M_1^{(i)}) = \varphi(M_2^{(i)}) \in nP \cap M$. If we assume $X_s = Z_0$, then we have $M_2^{(s)} = Z_1 \cdots Z_n$ and

$$\begin{aligned}\varphi(M_1^{(s)}) = \varphi(M_2^{(s)}) &= (a_1 + 1, \ldots, a_{n-1} + 1, a_n) \\ &= u_1 + \cdots + u_n \in \partial(nP).\end{aligned}$$

Since $M_1^{(s)}$ is a monomial of degree n, it is defined by the finite set $\{w_1, \ldots, w_n\} \subset P \cap M$ with $w_1 + \cdots + w_n = u_1 + \cdots + u_n$. From the assumption of very ampleness, $\{u_2 - u_1, \ldots, u_n - u_1\}$ is a basis of the sublattice of M contained in the affine subspace spanned by $\{u_1, \ldots, u_n\}$. Since the expression $(w_1 - u_1) + \cdots + (w_n - u_1) = (u_2 - u_1) + \cdots + (u_n - u_1)$ is unique, we have $\{w_1, \ldots, w_n\} = \{u_1, \ldots, u_n\}$, that is, $M_1^{(s)} = M_2^{(s)}$. This implies $B_s = 0$. If we assume $X_s = Z_i$ for some $i = 1, \ldots, n$, then we can easily see that $M_1^{(s)} = M_2^{(s)}$, hence $B_s = 0$ from the same reason.

This implies that $B \notin S_1 I_n$.

Remark. — If $P = \mathrm{Conv}\{u_0, u_1, \ldots, u_n\}$ does not contain any lattice point in the interior and if P satisfies the equality (1.3) for all positive integers l, then from the proof of Proposition 1.3 we may set $u_0 = 0$, $u_i = e_i$ for $i = 1, \ldots, n-1$ and $u_n = \sum_{i=1}^{n} a_i u_i$ with $a_i \geqslant 0$ and $a_n > 0$ after a suitable affine transformation of M. Since $P \cap M = \{u_0, \ldots, u_n\}$ generates the set of all lattice points in the cone $\mathbb{R}_{\geqslant 0} P$ with the apex $u_0 = 0$, we see that $a_n = 1$. By a change of basis of M, we may set $u_n = e_n$. Thus $(X, L) \cong (\mathbb{P}^n, \mathcal{O}(1))$.

Abe [A] constructs infinitely many examples of integral 3-simplices whose defining ideals need elements of degree 4 as generators. Here we give a part of them.

Example 1.4. — Let l be a positive integer and set $M = \mathbb{Z}^3$.

Let $u_0 = 0, u_1 = (1,0,0), u_2 = (0,1,0)$ and let $u_3 = (1,1,1), u_4 = (3,3,4), \ldots, u_{l+3} = (2l+1, 2l+1, 3l+1)$. Set $P_l = \mathrm{Conv}\{u_0, u_1, u_2, u_{l+3}\}$, a 3-simplex. Then P_l contains the lattice points $u_3, \ldots, u_{l+2}$ as its interior points. The volume of P_l is $(3l+1)/3!$. P_1 is the union of four 3-simplices with the common vertex u_3. Since P_i is the union of P_{i-1} and three 3-simplices with the common vertex u_{i+2}, we see that P_l is devided into the union of $3l+1$ integral 3-simplices, which means that every 3-simplex appearing in the decomposition has volume $1/3!$, hence the polytope P_l has a unimodular triangulation. From Proposition 1.2.2 in [BGT], P_l is normally generated. From Proposition 1.3 we see that P_l defines a projectively normal toric variety of dimension 3 whose defining ideal needs elements of degree 4 as generators.

2. Characterization.

We consider an integral curve C defined by the intersection of general hyperplane sections $Y_1, \ldots, Y_{n-1}$ of the linear system $|L|$, i.e., $C := \cap_{i=1}^{n-1} Y_i$. Set $L_C = L|C$, the restriction of L to the curve C. From easy calculation, we see that

(2.1) $h^0(C, L_C) = h^0(X, L) - n + 1 = {}^\sharp P \cap M - n + 1,$

(2.2) $h^1(C, L_C^{\otimes n-2}) = h^n(X, L^{-1}) = h^0(X, \omega_X \otimes L) = {}^\sharp \operatorname{Int} P \cap M,$

(2.3) $h^1(C, L_C^{\otimes i}) = 0 \quad$ for all $i \geqslant n-1$.

Hence we have $h^0(L_C) - h^1(L_C^{\otimes n-2}) = {}^\sharp \partial P \cap M - n + 1 \geqslant 2$.

LEMMA 2.1 (Iitaka [I]). — *Let D be a Cartier divisor on an integral complete curve C with the properties that the invertible sheaf $\mathcal{O}_C(D)$ is generated by global sections and that the morphism Φ_D associated to D is birational. Assume that $h^0(C, \mathcal{O}_C(D)) = l + 1 \geqslant 4$. Then we have an effective divisor G satisfying*

(1) $\deg G = l - 1$,

(2) $h^0(C, \mathcal{O}_C(D - G)) = 2$,

(3) *the line bundle $\mathcal{O}_C(D - G)$ is generated by global sections and $h^1(C, \mathcal{O}_C(D - G)) = h^1(C, \mathcal{O}_C(D))$.*

For a proof we may see Lemma 3.16 in [I]. Unfortunately it is written in Japanese. Hence we give an outline of a proof.

Outline of Proof. — We use an induction on l. The image $W = \Phi_D(C)$ is a curve in $\mathbb{P}^l$ and is not contained in any hyperplane. Take general points p, q on W so that the line in $\mathbb{P}^l$ through p and q meets W at only two points. These points are nonsingular points of W and the map Φ_D has an inverse on an open subset containing these points. Set $P_1 = \Phi_D^{-1}(p)$ and $P_2 = \Phi_D^{-1}(q)$. Then $\mathcal{O}_C(D - (P_1 + P_2))$ is generated by global sections. Let $D' := D - P_1$. Then $\mathcal{O}_C(D')$ is generated by global sections and the map $\Phi_{D'}$ is birational.

On the other hand, we have $h^0(\mathcal{O}_C(D')) = h^0(\mathcal{O}_C(D)) - 1 = l$. By the assumption of induction for D' we have a divisor G'. Set $G = G' + P_1$. Then this divisor G satisfies (1), (2) and (3).

When $l = 3$, we set $G = P_1 + P_2$. By Riemann-Roch Theorem we have $h^1(\mathcal{O}_C(D)) = h^1(\mathcal{O}_C(D - G))$.

Remark. — We note that the divisor D given in Lemma 2.1 consists of general $l - 1$ points on the curve C.

A very ample invertible sheaf L on a projective variety X defines an embedding $\Phi_L : X \to \mathbb{P}(H^0(X, L)^*) = \mathbb{P}^l$. Set $M_L := \Phi_L^* \Omega^1_{\mathbb{P}^l}(1)$ so that there exists the following exact sequence of vector bundles:

$$(2.4) \qquad 0 \to M_L \to H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \to L \to 0.$$

Taking wedge product in (2.4) and twisting by $L^{\otimes k-1}$, we obtain an exact sequence

$$(2.5) \quad 0 \to \wedge^2 M_L \otimes L^{\otimes k-1} \to \wedge^2 H^0(X, L) \otimes_{\mathbb{C}} L^{\otimes k-1} \to M_L \otimes L^{\otimes k} \to 0.$$

LEMMA 2.2 (Green-Lazarsfeld [GL]). — *Assume that L is normally generated. Let k_0 be an integer such that the maps induced by (2.5)*

$$(2.6) \qquad \sigma_k : \wedge^2 H^0(L) \otimes H^0(L^{\otimes k-1}) \to H^0(M_L \otimes L^{\otimes k})$$

are surjective for all $k \geqslant k_0$. Then every minimal generator of the homogeneous ideal defining X in $\mathbb{P}^l$ has degree k_0 or less.

In our situation we shall show $k_0 = n$ for $(X, L) = (C, L_C)$.

PROPOSITION 2.3. — *Let L_C be a very ample line bundle on an integral complete curve C and let $n \geqslant 2$ an integer with $H^1(C, L_C^{\otimes i}) = 0$ for $i \geqslant n-1$. Then we have $H^1(C, \wedge^2 M_{L_C} \otimes L_C^{\otimes i}) = 0$ for $i \geqslant n$. Furthermore if we have the inequality $h^0(L_C) - h^1(L_C^{\otimes n-2}) \geqslant 3$ for $n \geqslant 2$, then we have $H^1(C, \wedge^2 M_{L_C} \otimes L_C^{\otimes n-1}) = 0$.*

Proof. — When $l := h^0(L_C) - 1 = 2$, from the condition we have $h^1(L_C^{\otimes n-2}) = 0$. Since $\operatorname{rank} M_{L_C} = 2$, we have $\wedge^2 M_{L_C} \cong L_C^{-1}$ from the sequence (2.4), hence, we have $H^1(C, \wedge^2 M_{L_C} \otimes L_C^{\otimes i}) \cong H^1(C, L_C^{\otimes i-1}) = 0$ for $i \geqslant n-1$.

When $l \geqslant 3$, we can apply Lemma 2.1 to $L_C = \mathcal{O}_C(D)$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{L_C(-G)} & \longrightarrow & H^0(L_C(-G)) \otimes \mathcal{O}_C & \longrightarrow & L_C(-G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{L_C} & \longrightarrow & H^0(L_C) \otimes \mathcal{O}_C & \longrightarrow & L_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma_G & \longrightarrow & H^0(L_C|G) \otimes \mathcal{O}_C & \longrightarrow & L_C|G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here we write as Σ_G the kernel of $H^0(L_C|G) \otimes \mathcal{O}_C \to L_C|G$. Since $h^0(L_C(-G)) = 2$, the vector bundle $M_{L_C(-G)} \cong L_C^{-1}(G)$ is a line bundle. And since G is a general divisor of degree $l-1$, we may write $G = \sum_{i=1}^{l-1} P_i$, hence, we have $\Sigma_G \cong \oplus_{i=1}^{l-1} \mathcal{O}_C(-P_i)$. Thus we have the exact sequence

$$0 \to L_C^{-1}(G) \to M_{L_C} \to \bigoplus_{i=1}^{l-1} \mathcal{O}_C(-P_i) \to 0. \tag{2.7}$$

Taking wedge product in (2.7) and twisting by L_C^{k-1}, we obtain an exact sequence

$$0 \to \bigoplus_{i=1}^{l-1} L_C^{\otimes k-2}(G - P_i) \to \wedge^2 M_{L_C} \otimes L_C^{\otimes k-1} \to \bigoplus_{i<j} L_C^{\otimes k-1}(-P_i - P_j) \to 0. \tag{2.8}$$

Since $h^1(L_C^{\otimes k-1}) = 0$ for $k \geqslant n$ and since P_i are general, we have that $h^1(L_C^{\otimes k-1}(-P_i - P_j)) = h^1(L_C^{\otimes k-1}) = 0$ for $k \geqslant n$ and that $h^1(L_C^{\otimes k-2}(G - P_i)) = h^1(L_C^{\otimes k-2}) = 0$ for $k \geqslant n+1$. Hence we have $H^1(\wedge^2 M_{L_C} \otimes L_C^{\otimes i}) = 0$ for $i \geqslant n$.

Next set $k = n-1$. If $h^1(L_C^{\otimes n-2}(G - P_i)) = 0$, then the proof of the proposition is completed. Suppose that $h^1(L_C^{\otimes n-2}(G - P_i)) > 0$. Since the divisor $G - P_i$ consists of general $l-2$ points, then we have

$$\begin{aligned} h^1(L_C^{\otimes n-2}(G - P_i)) &= h^1(L_C^{\otimes n-2}) - \deg(G - P_i) = h^1(L_C^{\otimes n-2}) - (l-2) \\ &= h^1(L_C^{\otimes n-2}) - (l+1) + 3 = h^1(L_C^{\otimes n-2}) - h^0(L_C) + 3. \end{aligned}$$

The assumption $h^0(L_C) - h^1(L_C^{\otimes n-2}) \geqslant 3$ implies the inequality $0 \geqslant h^1(L_C^{\otimes n-2}(G-P_i))$, which is a contradiction. Hence we have $h^1(L_C^{\otimes n-2}(G-P_i)) = 0$.

COROLLARY 2.4. — *Let L_C be a normally generated ample line bundle on an integral complete curve C. If $h^1(L_C^{\otimes i}) = 0$ for $i \geqslant n-1$ and if $h^0(L_C) - h^1(L_C^{\otimes n-2}) \geqslant 3$ for $n \geqslant 2$, then the defining ideal of C in $\mathbb{P}(H^0(C, L_C)^*)$ has generators of degree at most n.*

Proof. — From Proposition 2.3, we have the surjectivity of the map σ_n of (2.6). Thus the statement follows from Lemma 2.2.

LEMMA 2.5 (Fujita [Fj]). — *Let Y be an irreducible member of $|L|$ with $H^0(X, L) \to H^0(Y, L_Y)$ surjective. Let $\delta \in H^0(X, L)$ be the class corresponding to Y, and let ξ_α $(\alpha = 1, \ldots, k)$ be homogeneous elements of the graded ring $R(X, L) := \oplus_{t \geqslant 0} H^0(X, L^{\otimes t})$ with $\deg \xi_\alpha = d_\alpha$ and let η_α be the restriction of ξ_α to $R(Y, L_Y) = \oplus_{t\geqslant 0} H^0(Y, L_Y^{\otimes t})$. Suppose that $\{\eta_1, \ldots, \eta_k\}$ generates $R(Y, L_Y)$. Let g_i $(i = 1, \ldots, l)$ be homogeneous polynomials in k variables $Y_1, \ldots, Y_k$ with $\deg Y_i = d_i$.*

Suppose that all relations among $\{\eta_\alpha\}$ in $R(Y, L_Y)$ are derived from $g_1(\eta_1, \ldots, \eta_k) = 0, \ldots, g_l(\eta_1, \ldots, \eta_k) = 0$. Then there exist l homogeneous polynomials $f_1, \ldots, f_l$ in $k+1$ variables $X_0, X_1, \ldots, X_k$ with $\deg X_0 = 1, \deg X_i = d_i$ for $i = 1, \ldots, k$ such that $f_i(0, Y_1, \ldots, Y_k) = g_i(Y_1, \ldots, Y_k)$ for $i = 1, \ldots, k$ and that all relations among $\delta, \xi_1, \ldots, \xi_k$ in $R(X, L)$ are derived from $f_1(\delta, \xi_1, \ldots, \xi_k) = 0, \ldots, f_k(\delta, \xi_1, \ldots, \xi_k) = 0$.

For a proof see Propositions 2.2 and 2.4 in [Fj].

THEOREM 2.6. — *Let P be an integral convex polytope of dimension n satisfying (1.3) for all positive integers l. We assume that the boundary of P contains at least $n+2$ lattice points. Then the defining ideal I has generators of degree at most n.*

Proof. — Let C be an integral curve defined by the intersection of $n-1$ general hyperplane sections of the linear system $|L|$. Then the condition $h^1(L_C^{n-2}) - h^0(L_C) \geqslant 3$ is equivalent to the condition $^\sharp\partial P \cap M \geqslant n+2$ from the equalities (2.1) and (2.2). From Corollary 2.4 we have the statement of the theorem for the integral complete curve C in $\mathbb{P}(H^0(C, L_C)^*)$.

Let D be a general member of the linear system $|L|$. Then D is irreducible and reduced, and the restriction map $H^0(X,L) \to H^0(D,L|_D)$ is surjective from the vanishing of cohomologies: We have $H^i(X,L^{\otimes j}) = 0$ for $0 < i < n$ and all j, and $H^n(X,L^{\otimes j}) = 0$ for all $j \geqslant 0$. Thus we have a sequence $X = D_n \supset D_{n-1} \supset \cdots \supset D_1 = C$ with $\dim D_j = j$, $D_{j-1} \in |L|_{D_j}|$ and the surjective restriction $H^0(D_j, L|_{D_j}) \to H^0(D_{j-1}, L|_{D_{j-1}})$. This sequence is called *regular ladder* in [Fj]. By applying Lemma 2.5 to a regular ladder of (X,L), we have that every minimal generator of the homogeneous ideal defining X in $\mathbb{P}(\Gamma(X,L)^*)$ has degree n or less.

BIBLIOGRAPHY

[A] T. ABE, On the study of integral convex polytopes and toric varieties, Master thesis, Tohoku University (2002) (Japanese).

[BGT] W. BRUNS, J. GUBELADZE and N. V. TRUNG, Normal polytopes, triangulations, and Koszul algebras, J. reine angew. Math., 485 (1997), 123–160.

[ES] D. EISENBUD and B. STURMFELS, Binomial ideals, Duke Math. J., 84 (1996), 1–45.

[EW] G. EWALD and U. WESSELS, On the ampleness of invertible sheaves in complete projective toric varieties, Results in Mathematics, 19 (1991), 275–278.

[Fj] T. FUJITA, Defining Equations for Certain Types of Polarized Varieties, in Complex Analysis and Algebraic Geometry (edited by Bailly and Shioda), Iwanami and Cambridge Univ. Press, 1977, 165–173.

[Fl] W. FULTON, Introduction to Toric Varieties, Ann. of Math. Studies N° 131, Princeton Univ. Press, 1993.

[GL] M. GREEN and R. LAZARSFELD, A simple proof of Petri's Theorem on canonical curves, Geometry of Today, Giornate di Geometria, Roma 1984, vol. 60, Birkhäuser Boston Inc., 1985, 129–142.

[I] S. IITAKA, Commutative rings, Kiso Sugaku Algebra, vol. 4, Iwanami Shoten, Tokyo, 1977 (Japanese).

[K1] R.J. KOELMAN, The number of moduli of families of curves on toric surfaces, thesis, Katholieke Universiteit te Nijmengen, 1991.

[K2] R.J. KOELMAN, Generators for the ideal of a projectively embedded toric surfaces, Tohoku Math. J., 45 (1993), 385–392.

[K3] R.J. KOELMAN, A criterion for the ideal of a projectively embedded toric surfaces to be generated by quadrics, Beiträger zur Algebra und Geometrie, 34 (1993), 57–62.

[M] D. MUMFORD, Varieties defined by quadric equations, In : Questions on Algebraic Varieties, Corso CIME (1969), 30–100.

[NO] K. NAKAGAWA and S. OGATA, On generators of ideals defining projective toric varieties, Manuscripta Math., 108 (2002), 33–42.

[Od] T. ODA, Convex Bodies and Algebraic Geometry, Ergebnisse der Math. 15, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1988.

[Og] S. OGATA, Quadratic generation of ideals defining projective toric varieties, Kodai Math. J., 26 (2003), 137–146.

[S1] B. STURMFELS, Gröbner bases and Convex Polytopes, University Lecture Series, Vol. 8, Providence, RI, American Mathematics Society, 1995.

[S2] B. STURMFELS, Equations defining toric varieties, Algebraic Geometry-Santa Cruz 1995, Proc. Sympos. Pure Math., 62 (1997), 437–449.

Manuscrit reçu le 18 septembre 2002,
accept le 15 juillet 2003.

Shoetsu OGATA,
Tohoku University
Mathematical Institute
Sendai 980 (Japan).
ogata@math.tohoku.ac.jp