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UNIQUENESS OF CREPANT RESOLUTIONS AND SYMPLECTIC SINGULARITIES

by Baohua FU & Yoshinori NAMIKAWA

1. Introduction.

In this paper, we work over the field $\mathbb{C}$ of complex numbers. Let W be an algebraic variety, smooth in codimension 1, such that K_W is a Cartier divisor. Recall that a resolution of singularities $\pi : X \to W$ is called *crepant* if $\pi^* K_W = K_X$. In this note, we will only consider projective crepant resolutions, i.e. π is projective. Let $\pi^+ : X^+ \to W$ be another (projective) crepant resolution of W.

DEFINITION 1. — (i) *π and π^+ are said isomorphic if the natural birational map $\pi^{-1} \circ \pi^+ : X^+ - - \to X$ is an isomorphism;*

(ii) *π and π^+ are said equivalent if there exists an automorphism ψ of W such that $\psi \circ \pi$ and π^+ are isomorphic.*

As easily seen, any two crepant resolutions of A-D-E singularities are isomorphic. The purpose of this note is to study projective crepant resolutions (mostly for symplectic singularities) up to isomorphisms and up to equivalences.

A special case of crepant resolutions is symplectic resolutions for symplectic singularities. Following [Bea], a variety W, smooth in codimension 1, is said to have *symplectic singularities* if there exists a holomorphic

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symplectic 2-form ω on W_{reg} such that for any resolution of singularities $\pi : X \to W$, the 2-form $\pi^*\omega$ defined a priori on $\pi^{-1}(W_{\text{reg}})$ can be extended to a holomorphic 2-form on X. If furthermore the 2-form $\pi^*\omega$ extends to a holomorphic symplectic 2-form on the whole of X for some resolution of W, then we say that W admits a *symplectic resolution*, and the resolution π is called *symplectic.*

For a symplectic singularity, a resolution is symplectic if and only if it is crepant (see for example Proposition 1.1 [Fu1]). In recent years, there appeared many studies on symplectic resolutions for symplectic singularities (see [CMS], [Fu1], [Fuj], [Ka1], [Ka3], [Na1] and [Wi1] etc.).

Our first theorem on uniqueness of crepant resolutions is the following:

THEOREM (2.2). — *Let $W_i, i = 1, \cdots, k$ be normal locally $\mathbb{Q}$-factorial singular varieties which admit a crepant resolution $\pi_i : X_i \to W_i$ such that $E_i := \mathrm{Exc}(\pi_i)$ is an irreducible divisor. Suppose that $W := W_1 \times \cdots \times W_k$ is locally $\mathbb{Q}$-factorial. Then any crepant resolution of W is isomorphic to the product*

$$\pi := \pi_1 \times \cdots \times \pi_k : X := X_1 \times \cdots \times X_k \to W_1 \times \cdots \times W_k.$$

It applies to many varieties with quotient singularities. For example it shows that for any smooth surface S, its nth symmetric product $S^{(n)}$ admits a unique crepant resolution, which is given by the Hilbert-Chow resolution: $S^{[n]} \to S^{(n)}$. As to the nilpotent orbit closures, we have:

THEOREMS (3.1). — *Let $\mathcal{O}$ be a nilpotent orbit in a complex semi-simple Lie algebra $\mathfrak{g}$. Then $\overline{\mathcal{O}}$ admits at most finitely many non-isomorphic symplectic resolutions.*

This result is an easy corollary of our previous work in [Fu1]. Some other partial results are also presented in Section 3. The above theorem motivates the following:

Conjecture (1). — Any normal symplectic singularity admits at most finitely many non-isomorphic symplectic resolutions.

In Section 4, we prove this conjecture in the 4-dimensional case. As to the relation between two symplectic resolutions, we have the following:

Conjecture (2). — Let W be a normal symplectic singularity. Then for any two symplectic resolutions $f_i : X_i \to W$, $i = 1, 2$, there are deformations $\mathcal{X}_i \xrightarrow{F_i} \mathcal{W}$ of f_i such that, for $s \in S \setminus 0$, $F_{i,s} : \mathcal{X}_{i,s} \to \mathcal{W}_s$ are isomorphisms. In particular, X_1 and X_2 are deformation equivalent.

By constructing explicitly the deformations, we prove this conjecture for symplectic resolutions of nilpotent orbit closures in $\mathfrak{sl}(n)$ in Section 4.

Finally in Section 5, we construct an example of symplectic singularity of dimension 4 which admits two non-equivalent symplectic resolutions.

The following proposition gives some applications of results presented in this note.

PROPOSITION 1.1. — *Let W be an algebraic variety, smooth in codimension 1. If up to isomorphisms, W admits a unique crepant resolution $\pi : X \to W$, then any automorphism of W lifts to X.*

Proof. — Let $\psi : W \to W$ be an automorphism. Then $\psi \circ \pi : X \to W$ is again a crepant resolution, which is isomorphic to π by hypothesis, thus there exists an automorphism $\widetilde{\psi}$ of X lifting ψ. □

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2. Quotient singularities.

LEMMA 2.1. — *Let W be a normal locally $\mathbb{Q}$-factorial variety and $\pi : X \to W$ a projective resolution. Then $\mathrm{Exc}(\pi)$ is of pure codimension 1 and if $\mathrm{Exc}(\pi) = \cup_{i=1}^n E_i$ is the decomposition into irreducible components, then $\mathcal{O}_X(-\sum_i a_i E_i)$ is π-ample for some $a_i > 0$.*

Proof. — The first claim is well-known (see 1.40 [Deb]), which follows from the normality and $\mathbb{Q}$-factority of W. For the second claim, by 1.42 [Deb], $\mathcal{O}_X(-\sum_i a_i E_i)$ is π-very ample for some $a_i \geqslant 0$. Suppose that $a_{i_0} = 0$, then take a point $x \in E_{i_0} - \cup_{i \neq i_0} E_i$ and a π-exceptional curve C passing x. Note that C is not contained in $E_i, i \neq i_0$, thus $C \cdot E_i \geqslant 0, i \neq i_0$. This gives $(-\sum_i a_i E_i) \cdot C \leqslant 0$, which is absurd since $\mathcal{O}_X(-\sum_i a_i E_i)$ is π-very ample. □

We are indebted to M. Brion for pointing out the reference [Deb].

THEOREM 2.2. — *Let $W_i, i = 1, \cdots, k$ be normal locally $\mathbb{Q}$-factorial singular varieties which admit a crepant resolution $\pi_i : X_i \to W_i$ such that $E_i := \mathrm{Exc}(\pi_i)$ is an irreducible divisor. Suppose that $W := W_1 \times \cdots \times W_k$ is locally $\mathbb{Q}$-factorial. Then any crepant resolution of W is isomorphic to the product*

$$\pi := \pi_1 \times \cdots \times \pi_k : X := X_1 \times \cdots \times X_k \to W_1 \times \cdots \times W_k.$$

Proof. — The π-exceptional locus consists of k irreducible divisors $F_i := X_1 \times \cdots \times E_i \times \cdots \times X_k$. We first prove that $-F_i$ is π-nef for all i. Let C be a curve in X such that $\pi(C)$ is a point. Consider the following composite:

$$X = X_1 \times \cdots \times X_k \xrightarrow{p_i} X_i \xrightarrow{\pi_i} W_i.$$

Note that $F_i = p_i^*(E_i)$. If $p_i(C)$ is a point Q, then $(C, F_i) = 0$. If $p_i(C)$ is a curve, then

$$(C, F_i) = [C : p_i(C)](p_i(C), E_i).$$

Applying Lemma 2.1 to the resolution $\pi_i : X_i \to W_i$, we see that $-aE_i$ is π_i-ample for some $a > 0$, thus $(p_i(C), E_i) < 0$, since $\pi_i(p_i(C))$ is a point. Therefore, $-F_i$ is π-nef.

Assume now that there is another crepant resolution $\pi^+ : X^+ \to W$. Then X and X^+ are isomorphic in codimension 1 because π and π^+ are both crepant resolutions. In particular, $\mathrm{Exc}(\pi^+)$ contains exactly k irreducible divisors, say $F_i^+, 1 \leqslant i \leqslant k$. Now apply Lemma 2.1, $L^+ := \mathcal{O}_{X^+}(-\sum_i a_i F_i^+)$ is π^+-ample for some $a_i > 0$. Its proper transform by the birational map $X^+ - - \to X$ coincides with $L := \mathcal{O}_X(-\sum_i a_i F_i)$, which is π-nef.

Since L is π-nef, π-big and π is crepant, the Base Point Free theorem implies that $L^{\otimes m}$ is π-free for a sufficiently large m. So there is a birational morphism $X \to \mathrm{Proj}_W(\oplus_k \pi_* L^{\otimes mk})$. On the other hand, since X and X^+ are isomorphic in codimension 1, there is an isomorphism $\pi_* L^{\otimes mk} \simeq \pi_*^+ L^{+\otimes mk}$. Therefore we have a birational morphism $X \to X^+$ over W. Since X and X^+ are both crepant resolutions of W, this birational morphism should be an isomorphism over W. Hence π and π^+ are isomorphic. □

For a smooth surface S, we denote by $S^{(n)}$ its symmetric n-th products (the Barlet space parametrizing 0 cycles on S of length n), and we denote by $S^{[n]}$ the Hilbert scheme parametrizing 0-dimensional subspaces of S with length n.

COROLLARY 2.3. — *Let $S_i, i = 1, \cdots k$ be a smooth surface. Then any crepant resolution of $S_1^{(n_1)} \times \cdots \times S_k^{(n_k)}$ is isomorphic to the Hilbert-Chow resolution*

$$S_1^{[n_1]} \times \cdots \times S_k^{[n_k]} \to S_1^{(n_1)} \times \cdots \times S_k^{(n_k)}.$$

COROLLARY 2.4. — *Let V be a symplectic vector space and G a finite subgroup of* $\mathrm{Sp}(V)$. *Suppose that the symplectic reflections of G (i.e. $g \in G$ such that* $\mathrm{Fix}(g)$ *is of codimension 2) form a single conjugacy class. Then any two crepant resolutions of V/G are isomorphic.*

Proof. — Let $\pi : X \to V/G$ be a crepant resolution. By McKay correspondence proved by D. Kaledin ([Ka2]), there is a one-to-one correspondence between the conjugacy classes of symplectic reflections in G and closed irreducible sub-varieties E of codimension 1 in X such that $\mathrm{codim}(\pi(E)) = 2$. Notice that such E is exactly irreducible components of $\mathrm{Exc}(\pi)$. By the hypothesis, there is only one such conjugacy class, thus $\mathrm{Exc}(\pi)$ is irreducible. □

Combining this corollary with Proposition 1.1, we immediately have the following corollary:

COROLLARY 2.5. — *Let V be a symplectic vector space and G a finite subgroup of* $\mathrm{Sp}(V)$. *Suppose that the symplectic reflections of G form a single conjugacy class and $\pi : X \to V/G$ is a crepant resolution. Then any action of an algebraic group H on V/G lifts to an H-action on X.*

Remark 2.6. — Corollary 2.4 gives a generalization of a result proved by D. Kaledin (Theorem 1.9 [Ka1]) and Corollary 2.5 strengthens Theorem 1.3 *loc. cit.*.

Example 2.7. — Here is an example to show the condition in Corollary 2.4 that symplectic reflections of G form a single conjugacy class is necessary. This example has also been considered by A. Fujiki ([Fuj]).

Let (x, y, z, w) be the coordinates of $\mathbb{C}^4$. Let G be the subgroup of

$\mathrm{Aut}(\mathbb{C}^4)$ generated by the three elements

$$\sigma_1 : (x, y, z, w) \to (x, y, -z, -w),$$
$$\sigma_2 : (x, y, z, w) \to (-x, -y, z, w),$$
$$\tau : (x, y, z, w) \to (z, w, x, y).$$

Then G is the dihedral group of order 8. Since all the elements of G preserve the two form $dx \wedge dy + dz \wedge dw$, the quotient $W := \mathbb{C}^4/G$ is a symplectic singularity.

One sees easily that $W = \mathrm{Sym}^2(\bar{S})$, where $\bar{S} = \mathbb{C}^2/\pm 1$. Let $S \to \bar{S}$ be the minimal resolution. Let C be its exceptional curve. $C \cong \mathbb{P}^1$ and $(C^2)_S = -2$. Now we have a sequence of birational maps

$$X := \mathrm{Hilb}^2(S) \xrightarrow{f_1} \mathrm{Sym}^2(S) \xrightarrow{f_2} \mathrm{Sym}^2(\bar{S}) = W.$$

Let $f : X \to W$ be the composite of the maps, which is a symplectic resolution of W. Note that $f_2^{-1}(0) = \mathrm{Sym}^2(C)(\cong \mathbb{P}^2)$. Let $\Delta_C \subset \mathrm{Sym}^2(C)$ be the diagonal. Put $F := f_1^{-1}(\Delta_C)$. Then F is a $\mathbb{P}^1$ bundle over $\Delta_C(\cong \mathbb{P}^1)$. It can be checked that F is isomorphic to the Hirzebruch surface Σ_4. As a consequence, we have

$$f^{-1}(0) = \mathbb{P}^2 \cup F,$$

where $\mathbb{P}^2$ is the proper transform of $\mathrm{Sym}^2(C)$ by f_1. The intersection $\mathbb{P}^2 \cap F$ is a conic of $\mathbb{P}^2$ and, at the same time, is a negative section of $F \cong \Sigma_4$.

$\mathrm{Sing}(W)$ has two components T_1 (diagonal of $\bar{S} \times \bar{S}$) and T_2. Let $E_i = f^{-1}(T_i)$, $i = 1, 2$. Then $f^{-1}(0) \subset E_1, f^{-1}(0) \cap E_2 = F$. In particular, we see that the resolution f is not symmetric with respect to T_1 and T_2.

Consider the map $u : \mathbb{C}^4 \to \mathbb{C}^4$ defined by $u(x, y, z, w) = (x - z, y - w, x + z, y + w)$. One verifies that

$$u \circ \sigma_1 = \tau \circ u; \quad u \circ \tau = \sigma_2 \circ u; \quad u \circ \sigma_2 = \sigma_1 \circ \sigma_2 \circ \tau \circ u.$$

Thus u gives an automorphism $\bar{u}$ on $W = \mathbb{C}^4/G$, which interchanges T_1 and T_2. So the two crepant resolutions f and $f' := \bar{u} \circ f$ are not isomorphic, though they are equivalent.

In fact one can show that the birational map:

$$(f')^{-1} \circ f : \mathrm{Hilb}^2(S) \dashrightarrow \mathrm{Hilb}^2(S)$$

is exactly the Mukai flop along the subvariety $\mathbb{P}^2$ of $\mathrm{Hilb}^2(S)$.

3. Nilpotent orbits.

Let $\mathfrak{g}$ be a semi-simple complex Lie algebra and $\mathcal{O}$ a nilpotent orbit in $\mathfrak{g}$. Then $\overline{\mathcal{O}}$ is singular and smooth in codimension 1. Let $\widetilde{\mathcal{O}}$ be its normalization, which is a normal variety with symplectic singularities ([Bea]). It is proved in [Fu1] that any projective symplectic resolution of $\widetilde{\mathcal{O}}$ is isomorphic to the collapsing of the zero section of $T^*(G/P)$ for some parabolic subgroup P of G, where G is the adjoint group of $\mathfrak{g}$. Notice that G has only finitely many conjugacy classes of parabolic subgroups, thus we get

THEOREM 3.1. — *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\mathcal{O}$ a nilpotent orbit in $\mathfrak{g}$. Then $\widetilde{\mathcal{O}}$ admits at most finitely many symplectic resolutions, up to isomorphisms.*

Notice that any two Borel subgroups in a semi-simple Lie group are conjugate, thus we have

COROLLARY 3.2. — *Let $\mathcal{N}$ be the nilpotent cone of a semi-simple complex Lie algebra $\mathfrak{g}$. Then any symplectic resolution of $\mathcal{N}$ is isomorphic to the Springer resolution $T^*(G/B) \to \mathcal{N}$, where B is a Borel subgroup of G.*

As to the uniqueness up to isomorphisms of symplectic resolutions for a nilpotent orbit closure, we have following partial results.

PROPOSITION 3.3. — *Let $\mathfrak{g}$ be a simple complex Lie algebra not of type A and $\mathcal{O}$ a nilpotent orbit in $\mathfrak{g}$. Suppose that $\overline{\mathcal{O}} - \mathcal{O} = \overline{C}$ for some nilpotent orbit C of codimension 2 in $\mathcal{O}$. If the singularity $(\overline{\mathcal{O}}, C)$ is of type A_1, then any two symplectic resolutions for $\overline{\mathcal{O}}$ are isomorphic.*

Proof. — Let $\pi : X \to \overline{\mathcal{O}}$ be a symplectic resolution, then over $U := C \cup \mathcal{O}$, π is isomorphic to the blowup of U at C, since $(\overline{\mathcal{O}}, C)$ is of type A_1. By the semi-smallness of symplectic resolutions (Proposition 1.4 [Na1] or Proposition 1.2 [Ka2]), $\mathrm{codim}(\pi^{-1}(\overline{\mathcal{O}} - U)) \geqslant 2$, thus $\mathrm{Exc}(\pi)$ consists of one irreducible divisor. Since $\mathfrak{g}$ is not of type A_k, $\overline{\mathcal{O}}$ is $\mathbb{Q}$-factorial ([Fu1]). Moreover, the π-exceptional fiber over C is isomorphic to $\mathbb{P}^1$, thus connected, so $\overline{\mathcal{O}}$ is normal (Theorem 1 [KP]). Now the proposition follows from Theorem 2.2. □

Then one can use results of H. Kraft and C. Procesi in [KP] to determine all the nilpotent orbits which satisfy the hypothesis of the above proposition. For example, in $\mathfrak{so}(5)$, we find $\mathcal{O}_{[3,1,1]}$. In $\mathfrak{sp}(4)$, we have $\mathcal{O}_{[2,2]}$. In $\mathfrak{so}(8)$, we get $\mathcal{O}_{[3,3,1,1]}$, $\mathcal{O}_{[3,1^5]}$ and $\mathcal{O}_{[2,2,2,2]}$ etc.

PROPOSITION 3.4. — *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{sl}(n+1,\mathbb{C})$. Let* $\mathbf{d}=[d_1,\cdots,d_s]$ *be its Jordan decomposition type. If $d_1=\cdots=d_s$, then up to isomorphisms, $\overline{\mathcal{O}}$ admits a unique symplectic resolution.*

Proof. — It is well-known that the closure of any nilpotent orbit in $\mathfrak{sl}(n+1,\mathbb{C})$ is normal and admitting a symplectic resolution. If $d_1=\cdots=d_s$, then all polarizations of $\mathcal{O}$ (i.e. parabolics P such that $T^*(G/P)$ is birational to $\overline{\mathcal{O}}$) form a single conjugacy class (see for example Theorem 3.3 (b) [Hes]), thus $\overline{\mathcal{O}}$ admits a unique symplectic resolution, up to isomorphisms. □

PROPOSITION 3.5. — *Let $\mathcal{O}$ be a nilpotent orbit in a complex simple Lie algebra of type $B-C-D$, with Jordan decomposition type* $\mathbf{d}=[d_1,\cdots,d_s]$. *Suppose that*

(i) *either there exists some integer $k\geqslant 1$ such that $d_1=\cdots=d_s=2k$;*

(ii) *or there exist some integers $q\geqslant 1, k\geqslant 1$ such that $d_1=\cdots=d_q=2k+1$ and $d_{q+1}=\cdots=d_s=2k$.*

Then $\overline{\mathcal{O}}$ admits a unique symplectic resolution, up to isomorphisms.

Proof. — By Proposition 3.21 and Proposition 3.22 [Fu1], such a nilpotent orbit $\overline{\mathcal{O}}$ admits a symplectic resolution. Furthermore by the proofs there (see also [Fu2]), two polarizations of $\mathcal{O}$ have conjugate Levi factors. Thus the number of conjugacy classes of polarizations is given by N_0 of Theorem 7.1 (d) [Hes], which equals 1 in our case. Thus $\overline{\mathcal{O}}$ admits a unique symplectic resolution, up to isomorphisms. □

Now we study symplectic resolutions up to equivalences for a nilpotent orbit $\mathcal{O}:=\mathcal{O}_{[d_1,\cdots,d_k]}$ contained in $\mathfrak{sl}(n)$, where $[d_1,\cdots,d_k]$ is the Jordan decomposition type of $\mathcal{O}$. Let $[s_1,\cdots,s_m]$ be the dual partition of $[d_1,\cdots,d_k]$. We denote by $P_{i_1,\ldots,i_k}\subset SL(n)$ a parabolic subgroup of flag type $(i_1,\ldots,i_k)$; that is, $P_{i_1,\ldots,i_k}$ stabilizes a flag $0=V_0\subset V_1\subset\ldots\subset V_k=\mathbb{C}^n$ such that $\dim V_j/V_{j-1}=i_j$ for $1\leqslant j\leqslant k$. This is equivalent to saying that $SL(n)/P_{i_1,\ldots,i_k}$ is the flag manifold $F(n,n-i_k,n-i_k-i_{k-1},\ldots,i_1)$.

It is well-known that all the polarizations of $\mathcal{O}$ are of the form

$P_{s_{\sigma(1)},\cdots,s_{\sigma(m)}}$ for some permutation $\sigma \in \Sigma_m$ (see for example Theorem 3.3 [Hes]).

PROPOSITION 3.6. — *The two symplectic resolutions*

$$T^*(SL(n)/P_{s_{\sigma(1)},\cdots,s_{\sigma(m)}}) \to \overline{\mathcal{O}}, \quad T^*(SL(n)/P_{s_{\sigma(m)},\cdots,s_{\sigma(1)}}) \to \overline{\mathcal{O}}$$

are equivalent.

Proof. — Take the dual flags, we get an isomorphism between $SL(n)/P_{s_{\sigma(1)},\cdots,s_{\sigma(m)}}$ and $SL(n)/P_{s_{\sigma(m)},\cdots,s_{\sigma(1)}}$. Furthermore $\overline{\mathcal{O}}$ is normal. Now the proposition follows from the following lemma. □

LEMMA 3.7. — *Let W be an affine normal variety and $\pi_i : X_i \to W$, $i = 1, 2$, two crepant resolutions. Then π_1 is equivalent to π_2 if and only if X_1 is isomorphic to X_2.*

Proof. — The isomorphism $X_1 \cong X_2$ induces an isomorphism of $\mathbb{C}$-algebras $\Gamma(X_1, \mathcal{O}_{X_1}) \cong \Gamma(X_2, \mathcal{O}_{X_2})$, thus an isomorphism of algebraic varieties

$$\mathrm{Spec}(\Gamma(X_1, \mathcal{O}_{X_1})) \cong \mathrm{Spec}(\Gamma(X_2, \mathcal{O}_{X_2})).$$

The morphism π_i gives an injective morphism from $\Gamma(W, \mathcal{O}_W) \to \Gamma(X_i, \mathcal{O}_{X_i})$, which is an isomorphism since W is normal. So $W \cong \mathrm{Spec}(\Gamma(X_i, \mathcal{O}_{X_i}))$, $i = 1, 2$. Therefore, the two resolutions π_1 and π_2 are equivalent. □

COROLLARY 3.8. — *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{sl}(n)$ with Jordan decomposition type $[d_1, \cdots, d_k]$. Suppose that $d_1 = 2$, then any two symplectic resolutions for $\overline{\mathcal{O}}$ are equivalent.*

Proof. — Since $d_1 = 2$, the dual partition of $[d_1, \cdots, d_k]$ consists of two parts $[n-t, t]$, where $t = \#\{i | d_i = 2\}$. $\overline{\mathcal{O}}$ has two symplectic resolutions, which are given by cotangent spaces of Grassmanians: $T^* \mathrm{Gr}(n, t) \to \overline{\mathcal{O}}$ and $T^* \mathrm{Gr}(n, n-t) \to \overline{\mathcal{O}}$, thus they are equivalent. □

Some interesting questions related to derived categories for the two symplectic resolutions $T^* \mathrm{Gr}(n, t) \to \overline{\mathcal{O}}$ and $T^* \mathrm{Gr}(n, n-t) \to \overline{\mathcal{O}}$ are discussed in [Na2].

Example 3.9. — Here we give an example where a nilpotent orbit admits two non-isomorphic symplectic resolutions. Let $n \geqslant 2$ be an integer. Consider the symplectic resolution $T^*\mathbb{P}^n \xrightarrow{\pi} \overline{\mathcal{O}}_{\min}$, where $\mathcal{O}_{\min} = \mathcal{O}_{[2,1^n]}$

is the minimal nilpotent orbit in $\mathfrak{sl}(n+1,\mathbb{C})$. Now we perform a Mukai flop along the zero section $P \simeq \mathbb{P}^n$ of $T^*\mathbb{P}^n$, i.e. we first blow up $T^*\mathbb{P}^n$ along P, then blow down along the other direction to get another symplectic resolution $T^*\mathbb{P}^n \xrightarrow{\pi^+} \overline{\mathcal{O}}_{\min}$. Notice that the birational map $(\pi^+)^{-1}\circ\pi : T^*\mathbb{P}^n \dashrightarrow T^*\mathbb{P}^n$ is not defined at the points of P. So π and π^+ are not isomorphic. In fact, the two symplectic resolutions come from non-conjugate parabolic subgroups in G, one is the stabilizer of a line in $\mathbb{C}^{n+1}$ and the other is the stabilizer of a vector subspace of codimension 1 in $\mathbb{C}^{n+1}$.

Example 3.10. — Here we give an example of a nilpotent orbit closure which admits three non-equivalent symplectic resolutions. Let $\mathcal{O}$ be the nilpotent orbit of $\mathfrak{sl}(6)$ with Jordan decomposition type $[3,2,1]$. Then there are six non-conjugate polarizations $P_{(\sigma(1),\sigma(2),\sigma(3))}$ of $\mathcal{O}$, where σ is a permutation of $\{1,2,3\}$. There are six non-isomorphic symplectic resolutions of $\bar{\mathcal{O}}$ corresponding to the six polarizations. Among these, the following pairs are equivalent resolutions by Proposition 3.6:

$$(T^*F(6,3,1), T^*F(6,5,3))$$
$$(T^*F(6,3,2)), T^*F(6,4,3))$$
$$(T^*F(6,5,2)), T^*F(6,4,1)).$$

We now show that there are exactly three non-equivalent resolutions. Assume that two of the three cotangent bundles $T^*F(6,3,1)$, $T^*F(6,3,2)$ and $T^*F(6,5,2)$ are equivalent resolutions of $\bar{\mathcal{O}}$. Let us consider the fibers of each resolution. Since the fibers with $\dim = 1/2 \dim T^*F$ are central fibers, if two resolutions are equivalent, then the corresponding flag manifolds are mutually isomorphic. We shall prove that this is absurd. We observe ample cones of these varieties. Since these varieties have Picard number two, they have at most two different fibrations. $F(6,3,1)$ has two fibrations $F(6,3,1) \to F(6,3)$ and $F(6,3,1) \to F(6,1)$. The first one is a $\mathbb{P}^2$-bundle and the second one is a $\mathrm{Gr}(5,2)$-bundle. $F(6,3,2)$ has two fibrations $F(6,3,2) \to F(6,3)$ and $F(6,3,2) \to F(6,2)$. The first one is a $\mathbb{P}^2$-bundle and the second one is a $\mathbb{P}^3$-bundle. $F(6,5,2)$ has two fibrations $F(6,5,2) \to F(6,5)$ and $F(6,5,2) \to F(6,2)$. The first one is a $\mathrm{Gr}(5,2)$-bundle and the second one is a $\mathbb{P}^3$-bundle. If two of these varieties are isomorphic, they should have three different fibrations, which is absurd.

By Lemma 3.7, we see that neither two of $T^*F(6,3,1), T^*F(6,3,2)$ and $T^*F(6,5,2)$ are isomorphic.

4. Finiteness of symplectic resolutions and deformations.

We propose the following conjecture:

CONJECTURE 1. — *Let W be a normal symplectic singularity. Then W admits at most finitely many non-isomorphic symplectic resolutions.*

Note that for nilpotent orbits, this conjecture is proved in Theorem 3.1. Here we prove the conjecture in the case of $\dim(W) = 4$.

THEOREM 4.1. — *There are only finitely many non-isomorphic symplectic resolutions of a symplectic singularity W of dimension 4.*

Proof. — Fix a symplectic resolution $f : X \to W$. Let $f^+ : X^+ \to W$ be another symplectic resolution. Then, X and X^+ are connected by a finite sequence of Mukai flops over W. In fact, any small birational contraction of a symplectic 4-fold is locally isomorphic to the Mukai flop by [CMS] or [WW], from which the existence of flops follows. The termination of the flop sequence follows from [Mat]. For the existence of flops for general 4-folds, see [Sho]. Now we can apply the argument of [KM] to prove our theorem.□

Example 4.2. — Let A be an abelian surface and $\sigma : A \to A$ the involution $x \mapsto -x$. Then $A_0 := A/ < \sigma >$ has 16 double points. Let $B \to A_0$ be the minimal resolution. Then $\pi : B \times B \to A_0 \times A_0$ is a symplectic resolution. Notice that the 2-dimensional π-exceptional fibers are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, thus no Mukai flop can be performed. Thus π is the unique symplectic resolution for $A_0 \times A_0$, up to isomorphisms.

Example 4.3. — Let $f : \mathrm{Hilb}^2(S) \to \mathrm{Sym}^2(\bar{S})$ be the symplectic resolution considered in Example 2.7. The only 2-dimensional f-exceptional fiber is $f^{-1}(0) = \mathbb{P}^2 \cup F$. We cannot apply a Mukai flop to f more than one time, thus $\mathrm{Sym}^2(\bar{S})$ admits exactly two non-isomorphic symplectic resolutions: f and $\bar{u} \circ f$.

Recall that a deformation of a variety X is a flat morphism $\mathcal{X} \xrightarrow{p} S$ from a variety $\mathcal{X}$ to a pointed smooth connected curve $0 \in S$ such that $p^{-1}(0) \cong X$. Moreover, a deformation of a proper morphism $f : X \to Y$ is a proper S-morphism $F : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X} \to S$ is a deformation of X and $\mathcal{Y} \to S$ is a deformation of Y.

Two varieties X_1 and X_2 are said *deformation equivalent* if there is a flat morphism $\mathcal{X} \xrightarrow{p} S$ from a variety $\mathcal{X}$ to a connected (not necessarily irreducible) curve S such that X_1 and X_2 are isomorphic to two fibers of p. As to the relation between two symplectic resolutions, we have the following:

CONJECTURE 2. — *Let W be a normal symplectic singularity. Then for any two symplectic resolutions $f_i : X_i \to W, i = 1, 2$, there are deformations $\mathcal{X}_i \xrightarrow{F_i} \mathcal{W}$ of f_i such that, for $s \in S \setminus 0$, $F_{i,s} : \mathcal{X}_{i,s} \to \mathcal{W}_s$ are isomorphisms. In particular, X_1 and X_2 are deformation equivalent.*

If W is a projective symplectic variety (with singularities), then we have Kuranishi spaces $\mathrm{Def}(W)$ and $\mathrm{Def}(X_i)$ for W and X_i. Since W has only rational singularities, we have the maps $(f_i)_* : \mathrm{Def}(X_i) \to \mathrm{Def}(W)$. By Theorem 2.2, [Na1], the Kuranishi spaces are all non-singular and $(f_i)_*$ are finite coverings. Now take a map $\Delta \to \mathrm{Def}(W)$ from a 1-dimensional disk such that this map factors through both $\mathrm{Def}(X_i)$. By pulling back the semi-universal families by this map, we have three flat families of varieties. If we take the map sufficiently general, then these families give the desired ones in the conjecture. One can say more. D. Huybrechts in [Huy] proved that if two compact hyper-Kähler manifolds X_1 and X_2 are birationally equivalent, then they are deformation equivalent. Here we do not need the intermediate variety W any more.

Let us return to our local case. When W is an isolated singularity, we also have the Kuranishi spaces for W and X_i. Moreover, by [CMS] and [WW], f_i's give a Mukai flop in this case. Then one can show the Conjecture applying the deformation theory as well as the projective case. The problem is when W is not an isolated singularity. We do not have appropriate spaces like the Kuranishi spaces any more. Sometimes, the formal approach could be possible, but its convergence is a difficult problem. D. Kaledin proved this conjecture under some hypothesis in [Ka3]. For the last statement of the conjecture, we proved in [Fu2] that X_1 is deformation equivalent to X_2 when they are symplectic resolutions of nilpotent orbit closures in a classical simple complex Lie algebra. Here we prove Conjecture 2 for nilpotent orbit closures in $\mathfrak{g} = \mathfrak{sl}(n)$. The construction is elementary and it may be of independent interest (see [Na2]).

THEOREM 4.4. — *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{sl}(n)$ with Jordan decomposition type $[d_1, \cdots, d_k]$. Then Conjecture 2 holds for $\overline{\mathcal{O}}$.*

Proof. — Let $[s_1, \cdots, s_m]$ be the dual partition of $[d_1, \cdots, d_k]$. The polarizations of $\mathcal{O}$ are $P_{s_{\sigma(1)},\cdots,s_{\sigma(m)}}$, $\sigma \in \Sigma_m$. Define

$$F_\sigma := SL(n)/P_{s_{\sigma(1)},\cdots,s_{\sigma(m)}}.$$

Let

$$\tau_1 \subset \cdots \subset \tau_{m-1} \subset \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{O}_{F_\sigma}$$

be the universal subbundles on F_σ. A point of T^*F_σ is expressed as a pair (p,ϕ) of $p \in F_\sigma$ and $\phi \in \mathrm{End}(\mathbb{C}^n)$ such that

$$\phi(\mathbb{C}^n) \subset \tau_{m-1}(p), \cdots, \phi(\tau_2(p)) \subset \tau_1(p), \phi(\tau_1(p)) = 0.$$

The Springer resolution

$$s_\sigma : T^*F_\sigma \to \bar{\mathcal{O}}$$

is defined as $s_\sigma((p,\phi)) := \phi$.

First, we shall define a vector bundle $\mathcal{E}_\sigma$ over F_σ and an exact sequence

$$0 \to T^*F_\sigma \to \mathcal{E}_\sigma \xrightarrow{\eta_\sigma} \mathcal{O}_{F_\sigma}^{\oplus m-1} \to 0.$$

Let $T^*F_\sigma(p)$ be the cotangent space of F_σ at $p \in F_\sigma$. Then, for a suitable basis of $\mathbb{C}^n$, $T^*F_\sigma(p)$ consists of the matrices of the following form:

$$\begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $\mathcal{E}_\sigma(p)$ be the vector subspace of $\mathfrak{sl}(n)$ consisting of the matrices A of the following form:

$$\begin{pmatrix} a_{\sigma(1)} & * & \cdots & * \\ 0 & a_{\sigma(2)} & \cdots & * \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & a_{\sigma(m)} \end{pmatrix},$$

where $a_i := a_i I_{s_i}$ and I_{s_i} is the identity matrix of the size $s_i \times s_i$. Since $A \in \mathfrak{sl}(n)$, $\Sigma_i s_i a_i = 0$. We define a map $\eta_\sigma(p) : \mathcal{E}_\sigma(p) \to \mathbb{C}^{\oplus m-1}$ as $\eta_\sigma(p)(A) := (a_1, a_2, \cdots, a_{m-1})$. Then we have an exact sequence of vector spaces

$$0 \to T^*F_\sigma(p) \to \mathcal{E}_\sigma(p) \xrightarrow{\eta_\sigma(p)} \mathbb{C}^{\oplus m-1} \to 0.$$

We put $\mathcal{E}_\sigma := \cup_{p \in F_\sigma} \mathcal{E}_\sigma(p)$. Then $\mathcal{E}_\sigma$ becomes a vector bundle over F_σ, and we get the desired exact sequence. Note that we have a morphism

$$\eta_\sigma : \mathcal{E}_\sigma \to \mathbb{C}^{m-1}.$$

Next, let $\overline{N} \subset \mathfrak{sl}(n)$ be the set of all matrices which is conjugate to a matrice of the following form:

$$\begin{pmatrix} b_1 & * & \cdots & * \\ 0 & b_2 & \cdots & * \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & b_m \end{pmatrix},$$

where $b_i = b_i I_{s_i}$ and I_{s_i} is the identity matrix of order s_i. Furthermore the zero trace condition requires $\sum_i s_i b_i = 0$. For $A \in \overline{N}$, let $\phi_A(x) := \det(xI - A)$ be the characteristic polynomial of A. Let $\phi_i(A)$ be the coefficient of x^{n-i} in $\phi(A)$. We define the characteristic map $ch : \overline{N} \to \mathbb{C}^{n-1}$ by $ch(A) := (\phi_2(A), ..., \phi_n(A))$. Note that $\phi_1(A) = 0$.

Let us consider the vector $\mathbf{a} = (a_1, a_2, ..., a_m)$ of length n where a_i appear exactly s_i times. Define $\phi_{i,\mathbf{a}}$ to be the $\phi_i(A)$ for the following diagonal matrix A of the size $n \times n$:

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & a_m \end{pmatrix},$$

where $a_i = a_i I_{s_i}$ with $s_i \times s_i$ identity matrix I_{s_i}. For $(a_1, a_2, ..., a_{m-1}) \in \mathbb{C}^{m-1}$ we put $\mathbf{a}' := (a_1, ..., a_{m-1}, -\Sigma_{i=1}^{m-1} s_i a_i / s_m)$, and we define a map

$$\pi : \mathbb{C}^{m-1} \to \mathbb{C}^{n-1}$$

by $\pi(a_1, ..., a_{m-1}) = (\phi_{2,\mathbf{a}'}, ..., \phi_{n,\mathbf{a}'})$. Pulling back $ch : \overline{N} \to \mathbb{C}^{n-1}$ by π, we have

$$ch' : \overline{N}' \to \mathbb{C}^{m-1}.$$

Each point of $\mathcal{E}_\sigma$ is expressed as a pair of a point $p \in F_\sigma$ and $\phi \in \mathrm{End}(\mathbb{C}^n)$. Now we define

$$\bar{s}_\sigma : \mathcal{E}_\sigma \to \overline{N}$$

as $\bar{s}_\sigma(p, \phi) = \phi$. This map is a generically finite morphism. Since $ch \circ \bar{s}_\sigma = \pi \circ \eta_\sigma$, we have a morphism

$$\bar{s}'_\sigma : \mathcal{E}_\sigma \to \overline{N}'.$$

Let $\widetilde{N}$ be the normalization of $\overline{N}'$ and let f be the composite: $\widetilde{N} \to \overline{N}' \xrightarrow{ch'} \mathbb{C}^{m-1}$. Then $\bar{s}'_\sigma$ factors through $\widetilde{N}$ and we have a morphism

$$\tilde{s}_\sigma : \mathcal{E}_\sigma \to \widetilde{N}.$$

Now, $\tilde{s}_\sigma$ becomes a birational morphism. Moreover, for a general point $\mathbf{t} \in \mathbb{C}^{m-1}$, $\tilde{s}_{\sigma,\mathbf{t}}$ is an isomorphism. The flat deformations

$$\mathcal{E}_\sigma \xrightarrow{\tilde{s}_\sigma} \widetilde{N} \xrightarrow{f} \mathbb{C}^{m-1}$$

give desired deformations in the conjecture. □

5. An example.

In this section we construct a symplectic singularity W of dim 4 which has two non-equivalent symplectic resolutions. We already have such examples by the nilpotent orbit construction (cf. Example 3.10). But here we introduce another construction. Our construction is elementary.

A similar example has also been constructed by J. Wierzba (Section 7.2.3 [Wi2]), using a different approach. Finally we note that such an example can be constructed by hyper-Kähler quotients [Got].

5.1. The idea.

Let $f : V \to W$ be a symplectic resolution such that

(i) for some point $0 \in W$, $f^{-1}(0) = \mathbb{P}^2 \cup \Sigma_1$, where $\mathbb{P}^2 \cap \Sigma_1$ is a line on $\mathbb{P}^2$ and, is, at the same time, a negative section of Σ_1;

(ii) the singular locus Σ of W is 2-dimensional. And for $p \in \Sigma$ such that $p \neq 0$, $(W, p) \cong (A_1 -$ surface singularity$) \times (\mathbb{C}^2, 0)$.

Over such a point p, f will become the minimal resolution. Now flop V along $\mathbb{P}^2$; then we get a new symplectic resolution $f^+ : V^+ \to W$ such that ${f^+}^{-1}(0) = \mathbb{P}^2 \cup \mathbb{P}^2$ where two $\mathbb{P}^2$ intersect in one point. Then, it is clear that the two symplectic resolutions are not equivalent. In fact, if they are equivalent, then there should be an isomorphism $V \cong V^+$ which sends $f^{-1}(0)$ isomorphically onto ${f^+}^{-1}(0)$. But this is absurd.

5.2. Construction of the example.

5.2.1. Set-up.

Let $\bar{S}$ be the germ of an A_2-surface singularity and let $\pi : S \to \bar{S}$ be its minimal resolution with exceptional curves C and D. There are natural birational morphisms

$$\mathrm{Hilb}^2(S) \xrightarrow{\nu} \mathrm{Sym}^2(S) \to \mathrm{Sym}^2(\bar{S}).$$

We denote by $g : \mathrm{Hilb}^2(S) \to \mathrm{Sym}^2(\bar{S})$ the composition. $\mathrm{Sym}^2(S)$ contains $\mathrm{Sym}^2(C)$ and $\mathrm{Sym}^2(D)$. Let P_C and P_D be their proper transforms on $\mathrm{Hilb}^2(S)$. Note they are isomorphic to $\mathbb{P}^2$. Let us consider the double cover

$\alpha : S \times S \to \mathrm{Sym}^2(S)$. Then $\alpha(C \times D) = \alpha(D \times C)$ and $\alpha(C \times D) \cong C \times D$. Let Q be the proper transform of $\alpha(C \times D)$ on $\mathrm{Hilb}^2(S)$. Now Q is isomorphic to the one point blowup of $C \times D$. If $p := C \cap D$ in S, then the center of the blowup is $(p, p) \in C \times D$. Let $l_C \subset Q$ (resp. $l_D \subset Q$) be the proper transform of $C \times \{p\}$ (resp. $\{p\} \times D$) by the blowup. Let $e \subset Q$ be the exceptional curve. Then l_C, l_D and e are (-1)-curves of Q with $(l_C, l_D) = 0$, $(l_C, e) = (l_D, e) = 1$. The relationship between P_C, P_D and Q are the following:

(i) P_C and P_D are disjoint.

(ii) Q intersects both P_C and P_D.

(iii) In Q, $Q \cap P_C$ coincides with l_C and $Q \cap P_D$ coincides with l_D.

(iv) In P_C, $Q \cap P_C$ is a line, and, in P_D, $Q \cap P_D$ is a line.

Let $E \subset \mathrm{Hilb}^2(S)$ be the exceptional divisor of the birational morphism $\nu : \mathrm{Hilb}^2(S) \to \mathrm{Sym}^2(S)$. Let $E_C := E \cap \nu^{-1}(\mathrm{Sym}^2(C))$ and $E_D := E \cap \nu^{-1}(\mathrm{Sym}^2(D))$. E_C is a $\mathbb{P}^1$-bundle over the diagonal $\Delta_C \subset \mathrm{Sym}^2(C)$. Let f_C be a fiber of this bundle. Similarly, E_D is a $\mathbb{P}^1$-bundle over $\Delta_D \subset \mathrm{Sym}^2(D)$, and let f_D be its fiber. Note that

$$g^{-1}(0) = Q \cup P_C \cup P_D \cup E_C \cup E_D.$$

5.2.2. Mukai flop.

Flop $\mathrm{Hilb}^2(S)$ along the center P_C to get a new 4-fold V. We denote by $P'_C \subset V$ the center of this flop. There is a birational morphism $g^+ : V \to \mathrm{Sym}^2(\bar{S})$.

Let $P'_D \subset V$ be the proper transform of P_D, and let $Q' \subset V$ be the proper transform of Q. Since P_D is disjoint from P_C, P'_D is naturally isomorphic to P_D; hence $P'_D \cong \mathbb{P}^2$. On the other hand, Q' is isomorphic to the blowdown of Q along l_C. Now Q' becomes the Hirzebruch surface Σ_1. The intersection $P'_D \cap Q'$ is a line of P'_D, and is a negative section of $Q' \cong \Sigma_1$. On the other hand, Q' and P'_C intersect in one point. Let $E'_C \subset V$ (resp. $E'_D \subset V$) be the proper transform of E_C (resp. E_D).

5.2.3. Idea.

We shall construct a birational contraction map $f : V \to W$ over $\mathrm{Sym}^2(\bar{S})$ such that, in $(g^+)^{-1}(0)$, P'_D and Q' are contracted to a point by f, E'_C is contracted along the ruling to a curve, and both E'_D and P'_C are birationally mapped onto their images. We put $f(P'_D \cap Q') = q$

and let W^0 be a sufficiently small open neighborhood of $q \in W$, and let $V^0 := f^{-1}(W^0)$. Then $f^0(:= f|_{V^0}) : V^0 \to W^0$ satisfies the conditions of Section 5.1. Let $(f^0)^+ : (V^0)^+ \to W^0$ be another symplectic resolution obtained by flopping P'_D. Then f^0 and $(f^0)^+$ are not equivalent.

5.2.4. The construction of f.

Let $\mu : B(S \times S) \to S \times S$ be the blowup along the diagonal Δ_S. Let F be the exceptional divisor of the blowup. We have a double cover $\tilde{\alpha} : B(S \times S) \to \mathrm{Hilb}^2(S)$. We can write

$$\tilde{\alpha}_* \mathcal{O}_{B(S\times S)} = \mathcal{O}_{\mathrm{Hilb}^2(S)} \oplus M$$

for some $M \in \mathrm{Pic}(\mathrm{Hilb}^2(S))$. Note that $M^{\otimes 2} = \mathcal{O}(-E)$. Choose $L \in \mathrm{Pic}(S)$ in such a way that $(L.C) = 0$ and $(L.D) = 1$. The line bundle $\mu^*(p_1^* L \otimes p_2^* L)$ on $B(S \times S)$ can be written as the pull-back by $\tilde{\alpha}$ of a line bundle N on $\mathrm{Hilb}^2(S)$. Define

$$\mathcal{L} := N \otimes M.$$

Then we have

$$(\mathcal{L}.e) = 1, (\mathcal{L}.l_C) = -1, (\mathcal{L}.l_D) = 0$$

$$(\mathcal{L}.f_C) = 1, (\mathcal{L}.f_D) = 1.$$

We have the following situation after the flop along P_C:

(i) The proper transform e' of e is a ruling of $Q' \cong \Sigma_1$.

(ii) The proper transform l'_D of l_D is a negative section of $Q' \cong \Sigma_1$, and at the same time, is a line of P'_D.

Let f'_C be the proper transform of f_C, and let f'_D be the proper transform of f_D. Then, for the proper transform $\mathcal{L}' \in \mathrm{Pic}(V)$ of $\mathcal{L}$, we have

$$(\mathcal{L}'.e') = 0, (\mathcal{L}'.l'_D) = 0, (\mathcal{L}'.f'_C) = 0, (\mathcal{L}'.f'_D) = 1.$$

Moreover, for a line l of P'_C, we see that $(\mathcal{L}'.l) = 1$ because $\mathrm{Hilb}^2(S) \dashrightarrow V$ is the flop along P_C and $(\mathcal{L}.l_C) = -1$. These implies that $\mathcal{L}'$ is g^+-nef (and, of course, g^+-big). Since g^+ is a crepant resolution, by the base point free theorem, $\mathcal{L}'^{\otimes n}$ is g^+-free for a sufficiently large n. By this line bundle we define $f : V \to W$. An irreducible curve on V is contracted to a point if and only if it has no intersection number with $\mathcal{L}'$. Since $(\mathcal{L}'.l'_D) = 0$, f contracts P'_D to a point by (ii). Moreover, since $(\mathcal{L}'.e') = 0$, f contracts Q' to the same point. Finally, since $(\mathcal{L}'.f'_C) = 0$, f contracts every ruling of E'_C to ponits. Similarly we can check that P'_C and E'_D are birationally mapped onto their images by f.

5.2.5. Detailed description of f.

Among the irreducible components of ${g^+}^{-1}(0)$, Q', P'_D and E'_C are f-exceptional. The birational morphism f factorizes g^+ as

$$V \xrightarrow{f} W \xrightarrow{h} \mathrm{Sym}^2(\bar{S}).$$

Then $h^{-1}(0)$ consists of two components; one of them is $f(P'_C) \cong \mathbb{P}^2$ and the other one is $f(E'_D)$, which is the blowdown of $E'_D \cong \Sigma_4$ along the negative section. These two components intersect in one point. Note that $f(E'_C)$ is a conic on $f(P'_C)$.

Remark 5.1. — It follows from Lemma 3.7 that for the two symplectic resolutions $V \to W$ and $V^+ \to W$, V is not isomorphic to V^+.

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