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THE HUA SYSTEM ON IRREDUCIBLE HERMITIAN SYMMETRIC SPACES OF NONTUBE TYPE

by Dariusz BURACZEWSKI*

1. Introduction.

Let G/K be an irreducible Hermitian symmetric space of noncompact type and let $\{E_j\}$ be an orthonormal basis of \mathfrak{p}^+ . The Hua system, as defined in [JK], is

$$(1.1) \quad \mathcal{H}(F) = \sum_{j,k} E_j E_k^* F \otimes [E_j, E_k^*].$$

After a number of partial results, the earliest going back to Hua, the fundamental theorem concerning the Hua system, proved by K. Johnson and A. Korányi is:

THEOREM 1.1 (K. Johnson, A. Korányi, 1980). — *A function F on a Hermitian symmetric space of tube type satisfies $\mathcal{H}(F) = 0$ if and only if it is the Poisson-Szegő integral of a hyperfunction on the Shilov boundary of G/K .*

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Soon after the result of K. Johnson and A. Korányi appeared, N. Berline and M. Vergne [BV] proved that (1.1) does not annihilate the Poisson-Szegő kernel on type two Hermitian symmetric spaces and described a third order system that characterizes Poisson-Szegő integrals for those spaces, and they raised the question about the nature of the common solutions of the Hua system for the type two Hermitian symmetric spaces, a question that has remained opened for twenty years.

The aim of this paper is to prove that the solutions of the Hua system on type two Hermitian symmetric spaces are the pluriharmonic functions (see Section 4):

THEOREM 1.2. — *Let F be a real valued function on a non-tube irreducible Hermitian symmetric space. Then $\mathcal{H}(F) = 0$ if and only if F is pluriharmonic.*

The system (1.1) annihilates holomorphic and antiholomorphic functions. The origin of (1.1) goes back to L. H. Hua [Hua], who in 1958 wrote a system that annihilates the Poisson-Szegő kernel on some classical domains. His formula was not exactly the one above, but for classical tube domains the zeros of both systems are the same. Then A. Korányi, E. Stein and J. Wolf obtained the formula for general tube domains and in an unpublished paper showed that the Poisson-Szegő kernel is harmonic with respect to the system (see e.g. [JK]). The first results showing that differential equations actually characterize the class of Poisson-Szegő integrals were obtained in special cases [KM], [J1], [J2]. Finally in 1980 K. Johnson and A. Korányi proved Theorem 1.1.

For a particular case of functions having L^2 boundary values, the above theorem was proved in [BBDHPT]. The methods of [BBDHPT] make the utmost use of the strong growth restrictions and are not applicable here.

To treat zeros of \mathcal{H} in full generality we combine two essential ingredients: the method of M. Lassalle [L] and the approach to pluriharmonic functions on symmetric Siegel domains developed in [DHMP], [BDH]. While the first one is based on the semi-simple group G , for the second one the use of the solvable Lie group S acting simply transitively on the corresponding Siegel domain seems indispensable. The reason is that the G -invariant operators don't see pluriharmonicity while the S invariant do⁽¹⁾. The inter-

(1) Bounded pluriharmonic functions are Poisson integrals i.e., they are annihilated by all the G -invariant operators, but there are many G -harmonic functions that are not pluriharmonic.

play between S -picture and G -picture is crucial for our story: the analysis is done on the group S and the special structure of S being the Iwasawa group is essential. That is why we describe so thoroughly both pictures: G/K and S and we pass from one to the other (Sections 2-3).

M. Lassalle [L] reproved Theorem 1.1 introducing new methods and, at the same time, cutting down the number of equations. We adopt his method to reduce the problem to bounded functions. Namely, we prove that a Hua harmonic function is G -harmonic (Section 5)⁽²⁾. To do so we use only a part of the system, the “strongly diagonal operators” (see Section 4). These r equations (r being the rank) correspond to the system \mathcal{H}_h of M. Lassalle. Next we use Harish-Chandra theorem [HC] in order to expand f in terms of its projections on spaces of K -finite functions f_δ . Each of these functions is Hua harmonic, hence G -harmonic. A K -finite and G -harmonic function can be written as a Poisson integral of a continuous bounded function defined on the maximal Furstenberg boundary, therefore all functions f_δ are bounded.

After restricting to bounded functions we transfer our problem to Siegel domains. For this we pass to realization of G/K as a Siegel domain ${}^c\mathcal{D}$ in \mathbb{C}^m , which is described in Section 3.3 following the Korányi-Wolf theory [KW]. However, for our purposes, we have to transform it further on in order to write strongly diagonal operators on the solvable Lie group S , the one that acts simply transitively on ${}^c\mathcal{D}$ (Section 6). This gives an extra advantage: the whole system may be replaced by strongly diagonal operators. In fact, we prove (see Section 4):

THEOREM 1.3. — *Let F be a bounded real valued function on a non-tube Hermitian irreducible symmetric space. If F is annihilated by the Laplace-Beltrami operator and the strongly diagonal Hua operators then F is pluriharmonic.*

Hence only $r + 1$ operators are needed which is considerably less than in either Johnson-Korányi’s or Lassalle’s proof.

The rest of the proof uses S (Section 6). First we show that a bounded function annihilated by the strongly diagonal Hua operators is a Poisson-Szegő integral (see [BBDHPT], [DHP]). Then, we notice that the Laplace-

⁽²⁾ The fact that for type two domains Hua harmonicity implies G -harmonicity was mentioned without proof in [JK].

Beltrami operator Δ_T for the corresponding tube domain T_Ω is a linear combination of the above $r + 1$ operators. Combining these two facts with Theorem 1.1 applied to T_Ω we obtain some more equations (see Section 6).

Now the strategy is to single out operators whose common zeros have to be pluriharmonic. These operators, however, cannot be obtained directly as linear combinations of the ones studied so far. The method we apply is the induction on the rank of the domain. The crucial observation is that S is a semi-direct product

$$S = S^{r-1}S_r,$$

where S^{r-1} is the group acting simply transitively on the Siegel domain of rank $r - 1$ and S_r the group acting simply transitively on the Siegel half plane \mathcal{D}_r (biholomorphic to the complex ball). Since a part of the equations that we have at our disposal are on S_r , we restrict the function F to S_r and apply some Fourier analysis methods on the Heisenberg group (Section 7). The induction produces equations to which we can apply the results of [BDH] and conclude that F is pluriharmonic. Since this method requires that we deal with bounded functions not L^2 , the analytic part here is somewhat more delicate than in [BBDHPT].

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2. Preliminaries on Hermitian symmetric spaces.

Let G/K be an irreducible Hermitian symmetric space of noncompact type and G the connected component of its isometry group. G is a centerless semisimple Lie group, and K is its maximal compact subgroup. We need some standard notation concerning semisimple Lie groups and algebras. For more details we refer to [H1], [Kn] or [K].

2.1. Root space decomposition.

Let \mathfrak{g} and \mathfrak{k} be the Lie algebras of respectively G and K , and let $\theta : \mathfrak{g} \mapsto \mathfrak{g}$ be the Cartan involution on \mathfrak{g} which is identity on \mathfrak{k} . If \mathfrak{p} is the -1 eigenspace of θ , we get the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $[\mathfrak{k}, \mathfrak{p}] = \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. Let $\mathfrak{g}^{\mathbb{C}}$ ($\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{p}^{\mathbb{C}}$ resp.) denote the complexification of \mathfrak{g} (\mathfrak{k} , \mathfrak{p} resp.). We extend θ to be a complex linear involution on $\mathfrak{g}^{\mathbb{C}}$. $G^{\mathbb{C}}$ is the adjoint group of $\mathfrak{g}^{\mathbb{C}}$ with $K^{\mathbb{C}}$ the analytic subgroup corresponding to $\mathfrak{k}^{\mathbb{C}}$.

$\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}^{\mathbb{C}}$. Denote by τ the conjugation operator on $\mathfrak{g}^{\mathbb{C}}$ with respect to \mathfrak{u} . If B is the Killing form on $\mathfrak{g}^{\mathbb{C}}$, then the bilinear form defined by $B_{\tau}(X, Y) = -B(X, \tau Y)$ is positive definite. $\sigma = \tau\theta = \theta\tau$ is the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to \mathfrak{g} . (Usually we shall write \bar{E} instead of σE .)

Choose a Cartan subalgebra \mathfrak{h} in \mathfrak{k} . Then $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Define Δ to be the system of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. Any root space \mathfrak{g}^{α} is contained either in $\mathfrak{k}^{\mathbb{C}}$ or in $\mathfrak{p}^{\mathbb{C}}$. In the first case α is called compact ($\alpha \in C$) and in the second case noncompact ($\alpha \in Q$). Clearly Δ is a disjoint union of C and Q . For a fixed $\alpha \in \Delta$ there are $\tilde{H}_{\alpha} \in i\mathfrak{h}$, $\tilde{E}_{\alpha} \in \mathfrak{g}^{\alpha}$, $\tilde{E}_{-\alpha} \in \mathfrak{g}^{-\alpha}$ such that

$$\begin{aligned}
 \alpha(H) &= B(H, \tilde{H}_{\alpha}), \quad \text{for every } H \in \mathfrak{h}^{\mathbb{C}}, \\
 [\tilde{E}_{\alpha}, \tilde{E}_{-\alpha}] &= \tilde{H}_{\alpha}, \\
 \tau \tilde{E}_{\alpha} &= -\tilde{E}_{-\alpha}, \\
 (2.1) \quad B(\tilde{E}_{\alpha}, \tilde{E}_{-\alpha}) &= 1,
 \end{aligned}$$

([H1], page 220). In particular, \tilde{H}_{α} , \tilde{E}_{α} and $\tilde{E}_{-\alpha}$ span a subalgebra of $\mathfrak{g}^{\mathbb{C}}$, isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

We define a Hermitian product on $(\mathfrak{h}^{\mathbb{C}})^*$ by

$$(2.2) \quad \langle \alpha, \beta \rangle = B(\tilde{H}_{\alpha}, \tilde{H}_{\beta}) = \alpha(\tilde{H}_{\beta}) = \beta(\tilde{H}_{\alpha}),$$

for $\alpha, \beta \in (\mathfrak{h}^{\mathbb{C}})^*$.

Let

$$(2.3) \quad c_{\alpha} = \sqrt{\frac{2}{\alpha(\tilde{H}_{\alpha})}} = \sqrt{\frac{2}{\langle \alpha, \alpha \rangle}}.$$

c_α is well defined because $\alpha(\tilde{H}_\alpha) = B(\tilde{H}_\alpha, \tilde{H}_\alpha) > 0$. Clearly $c_\alpha = c_{-\alpha}$. Using c_α 's we introduce a second normalization:

$$(2.4) \quad \begin{aligned} E_\alpha &= c_\alpha \tilde{E}_\alpha, \\ H_\alpha &= c_\alpha^2 \tilde{H}_\alpha. \end{aligned}$$

By (2.1) these vectors satisfy the following relations:

$$(2.5) \quad \begin{aligned} [E_\alpha, E_{-\alpha}] &= H_\alpha, \\ \tau E_\alpha &= -E_{-\alpha}, \\ B(E_\alpha, E_{-\alpha}) &= \frac{2}{\alpha(\tilde{H}_\alpha)} = \frac{2}{\langle \alpha, \alpha \rangle}, \\ \alpha(H_\alpha) &= 2. \end{aligned}$$

If \mathfrak{c} is the center of \mathfrak{k} , then there exists an element $Z \in \mathfrak{c}$ such that $(\text{ad}Z)^2 = -1$ on $\mathfrak{p}^{\mathbb{C}}$ ([Kn], Theorem 7.117). Let \mathfrak{p}^+ be the (i) -eigenspace of $\text{ad}Z$ and \mathfrak{p}^- be the $(-i)$ -eigenspace of $\text{ad}Z$. Then \mathfrak{p}^+ and \mathfrak{p}^- are Abelian Lie subalgebras invariant under the action of $\mathfrak{k}^{\mathbb{C}}$, and $[\mathfrak{p}^+, \mathfrak{p}^-] \subset \mathfrak{k}^{\mathbb{C}}$ ([H1], page 313). Moreover, there is an ordering of Δ decomposing Q so that $Q = Q^+ \cup Q^-$, and

$$(2.6) \quad \begin{aligned} \mathfrak{p}^+ &= \sum_{\alpha \in Q^+} \mathfrak{g}^\alpha = \sum_{\alpha \in Q^+} \mathbb{C}E_\alpha, \\ \mathfrak{p}^- &= \sum_{\alpha \in Q^+} \mathfrak{g}^{-\alpha} = \sum_{\alpha \in Q^+} \mathbb{C}E_{-\alpha}. \end{aligned}$$

Indeed, we can select an ordering on Δ as follows: for two roots α, β we say that α is bigger than β if and only if $-i(\alpha - \beta)(Z) > 0$. Q^+ is referred as the set of positive noncompact roots, while Q^- is called the set of negative noncompact roots.

For $\alpha \in Q^+$, let

$$(2.7) \quad \begin{aligned} X_\alpha &= E_\alpha + E_{-\alpha}, \\ Y_\alpha &= i(E_\alpha - E_{-\alpha}). \end{aligned}$$

Then the set $\{X_\alpha, Y_\alpha\}_{\alpha \in Q^+}$ spans \mathfrak{p} .

The restriction of $\text{ad}Z$ to \mathfrak{p} gives the complex structure on \mathfrak{p} , which will be denoted by \mathcal{J} . Thus we have

$$\begin{aligned}
 \mathcal{J}X_\alpha &= Y_\alpha, \\
 \mathcal{J}Y_\alpha &= -X_\alpha, \\
 (2.8) \quad E_\alpha &= \frac{1}{2}(X_\alpha - iY_\alpha), \\
 E_{-\alpha} &= \frac{1}{2}(X_\alpha + iY_\alpha).
 \end{aligned}$$

Similarly to (2.7), for every positive compact root α , take $X_\alpha, Y_\alpha \in \mathfrak{k}$ to be

$$\begin{aligned}
 (2.9) \quad X_\alpha &= E_\alpha - E_{-\alpha}, \\
 Y_\alpha &= i(E_\alpha + E_{-\alpha}).
 \end{aligned}$$

2.2. Some algebraic preliminaries.

In this subsection we are going to introduce some further algebraic properties of $\mathfrak{g}^{\mathbb{C}}$, which will be needed later.

First our goal is to describe the restricted root system for G/K . Two roots $\alpha, \beta \in \Delta$ are called strongly orthogonal if neither $\alpha + \beta$ nor $\alpha - \beta$ are roots. One can easily check that strong orthogonality implies orthogonality with respect to the form (2.2). Let

$$(2.10) \quad \Gamma = \{\gamma_1, \dots, \gamma_r\} \subset Q^+$$

($r = \text{rank } G/K$) be a maximal set of strongly orthogonal positive noncompact roots. Then

$$(2.11) \quad \mathfrak{a} = \sum_{\gamma \in \Gamma} \mathbb{R}X_\gamma,$$

is a maximal Abelian subalgebra of \mathfrak{p} . (For a construction of Γ we refer to [H1] pages 385-387.)

Take \mathfrak{h}^- to be the real span of the elements iH_γ , and \mathfrak{h}^+ to be the orthogonal complement of \mathfrak{h}^- in \mathfrak{h} via the Killing form B :

$$\begin{aligned}
 (2.12) \quad \mathfrak{h}^- &= \sum_{\gamma \in \Gamma} \mathbb{R}iH_\gamma, \\
 \mathfrak{h} &= \mathfrak{h}^- \oplus_B \mathfrak{h}^+.
 \end{aligned}$$

Let $\alpha, \beta \in \Delta$. Denote $\alpha \sim \beta$ if and only if $\alpha|_{\mathfrak{h}^-} = \beta|_{\mathfrak{h}^-}$. Define:

$$(2.13) \quad \begin{aligned} C_0 &= \{\alpha \in C : \alpha \sim 0\}, \\ C_i &= \left\{ \alpha \in C : \alpha \sim -\frac{1}{2}\gamma_i \right\} && \text{for } i = 1, \dots, r, \\ C_{ij} &= \left\{ \alpha \in C : \alpha \sim \frac{1}{2}(\gamma_j - \gamma_i) \right\} && \text{for } 1 \leq i < j \leq r, \\ Q_i &= \left\{ \alpha \in Q : \alpha \sim \frac{1}{2}\gamma_i \right\} && \text{for } i = 1, \dots, r, \\ Q_{ij} &= \left\{ \alpha \in Q : \alpha \sim \frac{1}{2}(\gamma_i + \gamma_j) \right\} && \text{for } 1 \leq i < j \leq r. \end{aligned}$$

Then the map $\alpha \mapsto \gamma_i + \alpha$ is a bijection of C_i onto Q_i and C_{ij} onto Q_{ij} . It is also known that Δ^+ is the disjoint union of the sets $C_0, C_i, C_{ij}, \Gamma, Q_i, Q_{ij}$, and Q^+ is the disjoint union of the sets Γ, Q_i, Q_{ij} ([H3], pages 457-460).

We shall call G/K a tube type space if all sets Q_i are empty. Otherwise G/K is a nontube type space.

We introduce numbers $M_{\alpha, \beta}$, which will be helpful in next sections in computing some brackets relations. For $\alpha, \beta \in \Delta$ define $M_{\alpha, \beta}$ by

$$(2.14) \quad \begin{aligned} [E_\alpha, E_\beta] &= M_{\alpha, \beta} E_{\alpha + \beta} && \text{if } \alpha + \beta \in \Delta, \\ M_{\alpha, \beta} &= 0 && \text{if } \alpha + \beta \notin \Delta; \end{aligned}$$

(some properties of these numbers are described in [H1], pages 146-152).

We may assume ([H1], Theorem V.5.5) that

$$(2.15) \quad M_{\alpha, \beta} = -M_{-\alpha, -\beta}.$$

PROPOSITION 2.1. — Fix k between 1 and r .

- a) If $\alpha = \gamma_j \in \Gamma$ then $\alpha(H_{\gamma_k}) = 2\delta_{jk}$.
- b) If $\alpha \in Q_k$ or $\alpha \in Q_{jk}$, and $\beta = \alpha - \gamma_k$, then $\alpha(H_{\gamma_k}) = 1$ and $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = \langle \gamma_k, \gamma_k \rangle$.
- c) For the rest of positive roots α : $\alpha(H_{\gamma_k}) = 0$.

Proof. — a) and c) are obvious. We prove b) for $\alpha \in Q_{jk}$. The second case is similar.

Notice that by (2.2) and (2.4)

$$(2.16) \quad \alpha(H_{\gamma_k}) = \frac{\langle \alpha, \alpha \rangle}{\langle \gamma_k, \gamma_k \rangle} \gamma_k(H_\alpha),$$

therefore it is enough to compute $\alpha(H_{\gamma_k})$ and $\gamma_k(H_\alpha)$. For this purpose we shall look at corresponding roots series. The γ_k -series containing α is $\{\alpha - \gamma_k, \alpha\}$, because $\alpha - 2\gamma_k \sim \frac{\gamma_j - 3\gamma_k}{2}$, which by the above remark implies that $\alpha - 2\gamma_k$ is not a root. If $\alpha - \gamma_k$ is a root, then $\alpha + \gamma_k$ cannot belong to Δ ([H3], Lemma V.4.4), as well. Hence $\alpha(H_{\gamma_k}) = 1$, which is immediate consequence of Theorem V.5.3 in [H3]. Similarly we get $\gamma_k(H_\alpha) = 1$. Therefore $\langle \alpha, \alpha \rangle = \langle \gamma_k, \gamma_k \rangle$. \square

COROLLARY 2.2. — *If $\alpha \in Q_k$ or $\alpha \in Q_{jk}$, $\beta = \alpha - \gamma_k$ then*

$$M_{\gamma_k, \beta} = \pm 1.$$

Proof. — We have just proved that $c_\alpha = c_\beta$ and then the corollary follows from Lemma 5.V.2 in [H1]. \square

3. Irreducible symmetric Siegel domains.

In this chapter we introduce symmetric Siegel domains and following A. Korányi and J. Wolf [KW], we describe realization of a Hermitian symmetric space as a Siegel domain.

3.1. Preliminaries on irreducible symmetric cones.

Let Ω be an irreducible symmetric cone in an Euclidean space. Our aim is to describe a solvable group S_0 acting simply transitively on Ω . We are going to use heavily the language of Jordan algebras so we recall briefly some basic facts which will be needed later. The reader is referred to the book of J. Faraut and A. Koranyi [FK] for more details.

A finite dimensional algebra V with a scalar product $\langle \cdot, \cdot \rangle$ is an Euclidean Jordan algebra, if for all elements x, y and z in V :

$$\begin{aligned} xy &= yx, \\ x(x^2y) &= x^2(xy), \\ \langle xy, z \rangle &= \langle y, xz \rangle. \end{aligned}$$

We denote by $L(x)$ the self-adjoint endomorphism of V given by the multiplication by x , i.e. $L(x)y = xy$. For an irreducible symmetric cone Ω contained in a linear space V of the same dimension, the space V can be made a simple real Euclidean Jordan algebra with unit element e , so that

$$\Omega = \text{int} \{x^2 : x \in V\}.$$

Let G_0 be the connected component of the group of all transformations in $GL(V)$ which leave Ω invariant, and let \mathfrak{g}_0 be its Lie algebra. Then \mathfrak{g}_0 is a subspace of the space of endomorphisms of V which contains all $L(x)$ for all $x \in V$, as well as all $x \square y$ for $x, y \in V$, where

$$(3.1) \quad x \square y = L(xy) + [L(x), L(y)]$$

(see [FK] for these properties).

We fix a Jordan frame $\{c_1, \dots, c_r\}$ in V , that is, a complete system of orthogonal primitive idempotents:

$$\begin{aligned} c_i^2 &= c_i, \\ c_i c_j &= 0 \quad \text{if } i \neq j, \\ c_1 + \dots + c_r &= e \end{aligned}$$

and none of the c_1, \dots, c_r is a sum of two non-zero idempotents. Let us recall that the length r is independent of the choice of the Jordan frame. It is called the rank of V . To have an example in mind, one may think of the space V of the symmetric $r \times r$ matrices endowed with the symmetrized product of matrices $\frac{1}{2}(xy + yx)$. Then the corresponding cone is the set of symmetric positive definite $r \times r$ matrices, the set of diagonal matrices with all entries equal to 0 except for one equal to 1 being a Jordan frame.

The Peirce decomposition of V related to the Jordan frame $\{c_1, \dots, c_r\}$ ([FK], Theorem IV.2.1) may be written as

$$(3.2) \quad V = \bigoplus_{1 \leq i < j \leq r} V_{ij}.$$

It is given by the common diagonalization of the self-adjoint endomorphism $L(c_j)$ with respect to their only eigenvalues $0, \frac{1}{2}, 1$. In particular $V_{jj} = \mathbb{R}c_j$ is the eigenspace of $L(c_j)$ related to 1, and, for $i < j$, V_{ij} is the intersection of the eigenspaces of $L(c_i)$ and $L(c_j)$ related to $\frac{1}{2}$. All V_{ij} , for $i < j$, have the same dimension d .

For each $i < j$, we fix once for all an orthonormal basis of V_{ij} , which we note $\{e_{ij}^\alpha\}$, with $1 \leq \alpha \leq d$. To simplify the notation, we write $e_{ii}^\alpha = c_i$ (α taking only the value 1). Then the system $\{e_{ij}^\alpha\}$, for $i \leq j$ and $1 \leq \alpha \leq \dim V_{ij}$, is an orthonormal basis of V .

Let us denote by \mathfrak{a}_0 the Abelian subalgebra of \mathfrak{g}_0 consisting of elements $H = L(a)$, where

$$a = \sum_{j=1}^r a_j c_j \in \bigoplus_i V_{ii}.$$

We set λ_j the linear form on \mathfrak{a}_0 given by $\lambda_j(H) = a_j$. It is clear that the Peirce decomposition gives also a simultaneous diagonalization of all $H \in \mathfrak{a}_0$, namely

$$(3.3) \quad Hx = L(a)x = \frac{\lambda_i(H) + \lambda_j(H)}{2} x, \quad x \in V_{ij}.$$

Let $A_0 = \exp \mathfrak{a}_0$. Then A_0 is an Abelian group, and this is the Abelian group in the Iwasawa decomposition of G_0 . We now describe the nilpotent part N_0 . Its Lie algebra \mathfrak{n}_0 is the space of elements $X \in \mathfrak{g}_0$ such that, for all $i \leq j$,

$$XV_{ij} \subset \bigoplus_{\substack{k \geq l \\ (k,l) > (i,j)}} V_{kl},$$

where the pairs are ordered lexicographically.

The choice of the Jordan frame determines a solvable Lie group $S_0 = N_0A$, being the semidirect product of N_0 and A . Then the group S_0 acts simply transitively on Ω . This may be found in [FK] Chapter VI, as well as the precise description of \mathfrak{n}_0 which will be needed later. One has

$$(3.4) \quad \mathfrak{n}_0 = \bigoplus_{i < j \leq r} \mathfrak{n}_{ij},$$

where

$$(3.5) \quad \mathfrak{n}_{ij} = \{z \square c_i : z \in V_{ij}\}.$$

This decomposition corresponds to a diagonalization of the adjoint action of \mathfrak{a}_0 since

$$(3.6) \quad [H, X] = \frac{\lambda_j(H) - \lambda_i(H)}{2} X, \quad X \in \mathfrak{n}_{ij}.$$

Finally, let $V^{\mathbb{C}} = V + iV$ be the complexification of V . We extend the action of G to $V^{\mathbb{C}}$ in the obvious way.

3.2. Irreducible symmetric Siegel domains.

Suppose that we are given a complex vector space \mathcal{Z} and a Hermitian symmetric bilinear mapping

$$\Phi : \mathcal{Z} \times \mathcal{Z} \mapsto V^{\mathbb{C}}.$$

We assume that

$$\begin{aligned} \Phi(\zeta, \zeta) &\in \overline{\Omega}, \quad \zeta \in \mathcal{Z}, \\ \text{and } \Phi(\zeta, \zeta) = 0 &\text{ implies } \zeta = 0. \end{aligned}$$

The Siegel domain associated with these data is defined as

$$(3.7) \quad {}^c\mathcal{D} = \{(\zeta, z) \in \mathcal{Z} \times V^{\mathbb{C}} : \Im z - \Phi(\zeta, \zeta) \in \Omega\}^{(3)}.$$

It is called of tube type, if \mathcal{Z} is reduced to $\{0\}$. Otherwise, it is called of type II.

There is a representation $\sigma : S_0 \ni s \rightarrow \sigma(s) \in GL(\mathcal{Z})$ such that

$$(3.8) \quad s\Phi(\zeta, w) = \Phi(\sigma(s)\zeta, \sigma(s)w),$$

and such that all automorphisms $\sigma(s)$, for $s \in A_0$, admit a joint diagonalization (see [KW]). To reduce notations, we shall as well denote by σ the corresponding representation of the algebra \mathfrak{s}_0 . For $X \in \mathfrak{s}_0$, (3.8) implies that

$$(3.9) \quad X\Phi(\zeta, w) = \Phi(\sigma(X)\zeta, w) + \Phi(\zeta, \sigma(X)w).$$

As an easy consequence, one can prove that the only possible eigenvalues for $\sigma(H)$, with $H \in \mathfrak{a}_0$ are $\lambda_j(H)/2$, for $j = 1, \dots, r$. So we may write

$$(3.10) \quad \mathcal{Z} = \bigoplus_{j=1}^r \mathcal{Z}_j$$

⁽³⁾ We denote a Siegel domain by ${}^c\mathcal{D}$ to be consistent with A. Korányi and J. Wolf notation needed in the next subsections.

with the property that

$$(3.11) \quad \sigma(H)\zeta = \frac{\lambda_j(H)}{2}\zeta, \quad \zeta \in \mathcal{Z}_j.$$

Moreover, all the spaces \mathcal{Z}_j have the same dimension. A proof of these two facts may be found in [DHMP]. We call χ the dimension of \mathcal{Z}_j for $j = 1, \dots, r$. Let us remark, using (3.9) and (3.11), that for $\zeta, w \in \mathcal{Z}_j$, we have $L(c_j)\Phi(\zeta, w) = \Phi(\zeta, w)$. Therefore, $\Phi(\zeta, w) = Q_j(\zeta, w)c_j$, for $\zeta, w \in \mathcal{Z}_j$. Moreover, $\langle c_j, \Phi(\zeta, \zeta) \rangle > 0$ for $\zeta \in \mathcal{Z}_j$ and so the Hermitian form Q_j is positive definite on \mathcal{Z}_j .

The representation σ allows to consider S_0 as a group of holomorphic automorphisms of ${}^c\mathcal{D}$. More generally, the elements $\zeta \in \mathcal{Z}$, $x \in V$ and $s \in S_0$ act on ${}^c\mathcal{D}$ in the following way:

$$(3.12) \quad \begin{aligned} \zeta \cdot (w, z) &= (\zeta + w, z + 2i\Phi(w, \zeta) + i\Phi(\zeta, \zeta)), \\ x \cdot (w, z) &= (w, z + x), \\ s \cdot (w, z) &= (\sigma(s)w, sz). \end{aligned}$$

We call $N(\Phi)$ the group corresponding to the first two actions, that is $N(\Phi) = \mathcal{Z} \times V$ with the product

$$(3.13) \quad (\zeta, x)(\zeta', x') = (\zeta + \zeta', x + x' + 2\Im\Phi(\zeta, \zeta')).$$

All three actions generate a solvable Lie group

$$(3.14) \quad S = N(\Phi)S_0 = N(\Phi)N_0A_0 = NA_0,$$

which identifies with a group of holomorphic automorphisms acting simply transitively on ${}^c\mathcal{D}$. The group $N(\Phi)$, that is two-step nilpotent, is a normal subgroup of S . The Lie algebra \mathfrak{s} of S admits the decomposition

$$(3.15) \quad \mathfrak{s} = \mathfrak{n}(\Phi) \oplus \mathfrak{s}_0 = \left(\bigoplus_{1 \leq j \leq r} \mathcal{Z}_j \right) \oplus \left(\bigoplus_{1 \leq i < j \leq r} V_{ij} \right) \oplus \left(\bigoplus_{1 \leq i < j \leq r} \mathfrak{n}_{ij} \right) \oplus \mathfrak{a}_0.$$

Moreover, by (3.3), (3.6) and (3.11), one knows the adjoint action of elements $H \in \mathfrak{a}_0$:

$$(3.16) \quad \begin{aligned} [H, X] &= \frac{\lambda_j(H)}{2} X, & \text{for } X \in \mathcal{Z}_j, \\ [H, X] &= \frac{\lambda_i(H) + \lambda_j(H)}{2} X, & \text{for } X \in V_{ij}, \\ [H, X] &= \frac{\lambda_j(H) - \lambda_i(H)}{2} X, & \text{for } X \in \mathfrak{n}_{ij}. \end{aligned}$$

Since S acts simply transitively on the domain ${}^c\mathcal{D}$, we may identify S and ${}^c\mathcal{D}$. More precisely, we define

$$(3.17) \quad \theta : S \ni s \rightarrow \theta(s) = s \cdot \mathbf{e} \in {}^c\mathcal{D},$$

where \mathbf{e} is the point $(0, ie)$ in ${}^c\mathcal{D}$. The Lie algebra \mathfrak{s} is then identified with the tangent space of ${}^c\mathcal{D}$ at \mathbf{e} using the differential $d\theta_e$. We identify e with the unit element of S . We then transport both the Bergman metric g and the complex structure \mathcal{J} from ${}^c\mathcal{D}$ to S , where they become left-invariant tensor fields on S . We still write \mathcal{J} for the complex structure on S .

3.3. Realization of Hermitian symmetric space as a Siegel domain.

The goal of this subsection is to describe connections between Hermitian symmetric spaces and Siegel domains. For a space G/K , which is supposed to be in the Harish-Chandra realization, we find a biholomorphically equivalent domain, equipped with the structure of Siegel domains. We follow closely [KW] and [K], but at the end we shall need a little more, namely we are interesting in full description of a basis of \mathfrak{s} in terms of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$.

First we recall the Harish-Chandra realization. Let us denote the analytic subgroups of $G^{\mathbb{C}}$ corresponding to subalgebras \mathfrak{p}^+ , \mathfrak{p}^- by P^+ and P^- , respectively. They are Abelian. The exponential map from \mathfrak{p}^{\pm} to P^{\pm} is biholomorphic and P^{\pm} is biholomorphically equivalent with \mathbb{C}^n for some n .

The mapping $(p_1, k, p_2) \mapsto p_1 k p_2$ is a diffeomorphism of $P^+ \times K^{\mathbb{C}} \times P^-$ onto an open submanifold of $G^{\mathbb{C}}$ containing G . For $g \in G$ let $p_+(g)$ denote the unique element in \mathfrak{p}^+ such that $g \in \exp(p_+(g))K^{\mathbb{C}}P^-$. One can show that $p_+(g) = p_+(gk)$ and p_+ is a diffeomorphism of G/K onto a domain $\mathcal{D} \subset \mathfrak{p}^+ = \mathbb{C}^n$. G acts biholomorphically on \mathcal{D} by $g \cdot p_+(\tilde{g}) = p_+(g\tilde{g})$. Let $o = p_+(e)$, then \mathcal{D} is the G -orbit of o and the group K is the stabilizer of the point o . This is the Harish-Chandra embedding and in fact realizes G/K as a bounded symmetric domain (we refer for more details to [H1] or [Kn]).

From now on we shall assume that \mathcal{D} is the above realization. Put $X_{\Gamma} = \sum_{\gamma \in \Gamma} X_{\gamma}$, $E_{\Gamma} = \sum_{\gamma \in \Gamma} E_{\gamma}$ and define an element of $G^{\mathbb{C}}$ called the Cayley transform:

$$(3.18) \quad c = \exp \frac{\pi}{4} i X_{\Gamma},$$

Let

$$\begin{aligned} {}^cG &= \text{Ad}(c)G, \\ {}^cK &= \text{Ad}(c)K, \\ {}^c\mathfrak{g} &= \text{Ad}(c)\mathfrak{g}, \\ {}^c\mathfrak{k} &= \text{Ad}(c)\mathfrak{k}. \end{aligned}$$

For $g \in G$, $c \exp(p_+(g)) \in P^+K^cP^-$ and so the mapping $p_+(g) \mapsto p_+(c \exp p_+(g))$ defines a biholomorphism of \mathcal{D} onto a domain ${}^c\mathcal{D} \subset \mathfrak{p}^+$ ([KW]). Clearly, ${}^c\mathcal{D}$ is the orbit of the point $c \cdot o = iE_\gamma$ under the action of the group cG , and cK is the isotropy group of iE_γ .

A simple computation proves the following lemma

LEMMA 3.1 [[K], Lemma IV.1.1]. — *Let $\gamma \in \Gamma$. Then*

$$\begin{aligned} \text{Ad}(c) \cdot X_\gamma &= X_\gamma, \\ \text{Ad}(c) \cdot Y_\gamma &= H_\gamma, \\ \text{Ad}(c) \cdot H_\gamma &= -Y_\gamma. \end{aligned}$$

Furthermore $\text{Ad}(c)$ acts trivially on \mathfrak{h}^+ .

It can be shown that $c^8 = I$ and $\text{Ad}(c^4)$ preserves \mathfrak{k} and \mathfrak{p} . We decompose both Lie algebras:

- $\mathfrak{p}_T, \mathfrak{q}_2$ is the (± 1) -eigenspace of $\text{Ad}(c^4)$ in \mathfrak{p} ,
- $\mathfrak{k}_T, \mathfrak{q}_2$ is the (± 1) -eigenspace of $\text{Ad}(c^4)$ in \mathfrak{k} ,
- $\mathfrak{g}_T = \mathfrak{k}_T \oplus \mathfrak{p}_T$.

If $\mathfrak{g}_T = \mathfrak{g}$, then the space G/K is of tube type, otherwise it is of nontube type. In the obvious way we introduce Lie algebras $\mathfrak{p}_T^{\mathbb{C}}, \mathfrak{p}_T^+$, etc., and analytic subgroups of $G^{\mathbb{C}}$: G_T, K_T , etc. We denote by $\mathfrak{p}^j, \mathfrak{p}^{jk}, \mathfrak{p}^{\frac{j}{2}}$ subspaces corresponding to restricted roots $\gamma_j, \frac{\gamma_j + \gamma_k}{2}, \frac{\gamma_j}{2}$. Similarly define subspaces of $\mathfrak{k}^{\mathbb{C}}$: $\mathfrak{k}^{kj}, \mathfrak{k}^{\frac{j}{2}}, \mathfrak{m}^{\mathbb{C}}$ which are restricted root spaces of $\frac{\gamma_j - \gamma_k}{2}, \frac{\gamma_j}{2}$ and 0, respectively. Notice that

$$\begin{aligned} \mathfrak{p}_T^+ &= \left(\bigoplus \mathfrak{p}^j \right) \oplus \left(\bigoplus_{j < k} \mathfrak{p}^{jk} \right), \\ \mathfrak{k}_T^{\mathbb{C}} &= \mathfrak{m}^{\mathbb{C}} \oplus (\mathfrak{h}^-)^{\mathbb{C}} \oplus \left(\bigoplus_{j \neq k} \mathfrak{k}^{jk} \right). \end{aligned}$$

Furthermore,

$$\begin{aligned}\mathfrak{p}_2^+ &= \bigoplus \mathfrak{p}^{\frac{j}{2}}, \\ \mathfrak{q}_2^+ &= \bigoplus \mathfrak{k}^{\frac{j}{2}}.\end{aligned}$$

Next we introduce

$$\begin{aligned}\mathfrak{n}^j &= {}^c\mathfrak{g} \cap \mathfrak{p}^j, \\ \mathfrak{n}^{jk} &= {}^c\mathfrak{g} \cap \mathfrak{p}^{jk} \quad \text{for } j < k, \\ \mathfrak{n}_K^{kj} &= {}^c\mathfrak{g} \cap \mathfrak{k}^{kj} \quad \text{for } j > k, \\ \mathfrak{n}^{\frac{j}{2}} &= {}^c\mathfrak{g} \cap (\mathfrak{p}^{\frac{j}{2}} \oplus \mathfrak{k}^{\frac{j}{2}}).\end{aligned}$$

Moreover, let

$$(3.19) \quad \begin{aligned}\mathfrak{n}_T^+ &= \left(\bigoplus \mathfrak{n}^j \right) \oplus \left(\bigoplus_{j < k} \mathfrak{n}^{jk} \right), \\ \mathfrak{n}_2^+ &= \bigoplus_j \mathfrak{n}^{\frac{j}{2}}, \\ \mathfrak{n}_K &= \bigoplus_{j > k} \mathfrak{n}_K^{kj}.\end{aligned}$$

One can easily see that $\mathfrak{n}_T^+ = {}^c\mathfrak{g} \cap \mathfrak{p}_T^+$ and is a real form of \mathfrak{p}_T^+ . Similarly \mathfrak{n}_2^+ is a real form of $\mathfrak{q}_2^+ \oplus \mathfrak{p}_2^+$. One can prove that $\text{Ad}(c^2)$ preserves \mathfrak{k}_T and \mathfrak{p}_T . We define

\mathfrak{l} , \mathfrak{q}_T is the (± 1) -eigenspace of $\text{Ad}(c)$ in \mathfrak{k}_T .

Take $\mathfrak{k}_T^* = \mathfrak{l} \oplus i\mathfrak{q}_T$, then $\mathfrak{k}_T^* = {}^c\mathfrak{g} \cap \mathfrak{k}_T^{\mathbb{C}}$ ([K], Lemma IV.2.6). By K_T^* and L we denote corresponding Lie groups.

Now we are ready to describe the domain ${}^c\mathcal{D}$. The image of the point iE_Γ under the action of K_T^* is a self dual cone in $i\mathfrak{n}_T^+$, with the group L as a stabilizer of iE_Γ ([K], Theorem IV.2.10). We shall denote this cone by Ω . Define a function $\Phi : \mathfrak{p}_2^+ \times \mathfrak{p}_2^+ \mapsto \mathfrak{p}_T^+$ by

$$\Phi(X, Y) = \frac{1}{2} \text{ad}X(\text{ad}Y)^* E_\Gamma,$$

where $(\text{ad}Y)^*$ is the adjoint operator of $\text{ad}Y$ with respect to B_τ . It can be shown ([KW], Lemma 6.4) that Φ satisfies all assumptions listed in the previous subsection and

(3.20)

$${}^c\mathcal{D} = {}^cG \cdot iE_\Gamma = \{X + iY + Z : X, Y \in \mathfrak{n}_T^+, Z \in \mathfrak{p}_2^+, Y - \Phi(Z, Z) \in \Omega\}.$$

([KW], Theorem 6.8). Therefore ${}^c\mathcal{D}$ is a Siegel domain of type I or II.

Now we want to recognize the group S in this picture. The mapping $i\mathfrak{q}_T \mapsto \mathfrak{n}_T^+$ given by

$$i\mathfrak{q}_T \ni X \mapsto [X, E_\Gamma]$$

is a bijection ([KV], Lemma 2.5). Let L be the inverse map, then we define multiplication in \mathfrak{n}_T^+ by

$$XY = \text{ad}L(X)\text{ad}L(Y)E_\Gamma.$$

One can easily prove that the above definition coincides with the one given in [FK], page 49, and multiplication so defined gives the structure of Jordan algebra in \mathfrak{n}_T^+ .

Take $c_i = E_{\gamma_i}$, then $L(c_i) = \frac{1}{2}H_{\gamma_i}$ and the set $\{c_1, \dots, c_r\}$ is a Jordan frame. The Peirce decomposition (3.2) with respect to this frame is given by the decomposition (3.19) of \mathfrak{n}_T^+ with $V_j = \mathfrak{n}^j$ and $V_{jk} = \mathfrak{n}^{jk}$. Then \mathfrak{a}_0 is spanned by vectors H_{γ_i} and the Gauss decomposition (3.4) of $\mathfrak{n}_0 = \mathfrak{n}_K$ coincides with (3.19) for $\mathfrak{n}_{ij} = \mathfrak{n}_K^{ij}$. Therefore, the solvable part of Iwasawa decomposition of ${}^c\mathfrak{g}$ with respect to \mathfrak{a}_0 is

$$\mathfrak{s} = \mathfrak{n}_2^+ \oplus \mathfrak{n}_T^+ \oplus \mathfrak{n}_K \oplus \mathfrak{a}_0,$$

and this is exactly the same decomposition as (3.15). The group S corresponding to \mathfrak{s} acts simply transitively on the domain ${}^c\mathcal{D}$.

3.4. An orthonormal basis for a Siegel domain of type II.

Now we describe an orthonormal basis of \mathfrak{s} for the nontube case corresponding to the decomposition (3.15). This will be the same basis as in [DHMP], [DHP], [BBDHPT]. We begin with finding a basis of \mathfrak{n}^{ij} . Take $\alpha \in Q_{ij}$ and put

$$(3.21) \quad \tilde{\alpha} = \gamma_i + \gamma_j - \alpha,$$

then by [L] (page 141) $\tilde{\alpha} \in Q_{ij}$. The classification theorem for Hermitian symmetric spaces says that in the nontube case the dimension of each space Q_{ij} is even. Since $\alpha \neq \frac{\gamma_i + \gamma_j}{2}$ if and only if $\tilde{\alpha} \neq \frac{\gamma_i + \gamma_j}{2}$, it follows that $\alpha \neq \tilde{\alpha}$. By \bar{Q}_{ij} we shall denote the subset of Q_{ij} such that from each pair of roots α and $\tilde{\alpha}$ exactly one is contained in \bar{Q}_{ij} .

Define

$$(3.22) \quad \begin{aligned} \beta &= \alpha - \gamma_i = \gamma_j - \tilde{\alpha}, \\ \theta &= \alpha - \gamma_j = \gamma_i - \tilde{\alpha}, \end{aligned}$$

Then β and θ are positive compact roots. Applying Lemma V.5.1 in [H1] and Corollary 2.2 we obtain:

$$(3.23) \quad \begin{aligned} \varepsilon &= M_{\gamma_i, \beta} = M_{\beta, -\alpha} = M_{-\alpha, \gamma_i} = -M_{-\beta, \alpha} = -M_{\alpha, -\gamma_i} = -M_{-\gamma_i, -\beta}, \\ \delta &= M_{\gamma_j, \theta} = M_{\theta, -\alpha} = M_{-\alpha, \gamma_j} = -M_{-\theta, \alpha} = -M_{\alpha, -\gamma_j} = -M_{-\gamma_j, -\theta}, \\ \rho &= M_{-\gamma_j, \beta} = M_{\beta, \tilde{\alpha}} = M_{\tilde{\alpha}, -\gamma_j} = -M_{-\beta, -\tilde{\alpha}} = -M_{-\tilde{\alpha}, \gamma_j} = -M_{\gamma_j, -\beta}, \\ \sigma &= M_{-\gamma_i, \theta} = M_{\theta, \tilde{\alpha}} = M_{\tilde{\alpha}, -\gamma_i} = -M_{-\theta, -\tilde{\alpha}} = -M_{-\tilde{\alpha}, \gamma_i} = -M_{\gamma_i, -\theta}, \end{aligned}$$

for some $\varepsilon, \sigma, \rho, \delta \in \{-1, 1\}$. We have $\alpha + \tilde{\alpha} - \gamma_i - \gamma_j = 0$, thus it follows from Lemma V.5.3 [H1] that $\varepsilon\rho = \delta\sigma$, which implies

$$(3.24) \quad \varepsilon\delta = (\delta\sigma\rho^{-1})\delta = \delta^2\sigma\rho^{-1} = \sigma\rho.$$

From (3.18) and (3.23) we obtain

$$\begin{aligned} \text{Ad}(c^{-1})E_\alpha &= \frac{1}{2}E_\alpha - \frac{1}{2}\varepsilon\rho E_{-\tilde{\alpha}} - \frac{1}{2}i(\varepsilon E_\beta + \delta E_\theta), \\ \text{Ad}(c^{-1})E_{\tilde{\alpha}} &= \frac{1}{2}E_{\tilde{\alpha}} - \frac{1}{2}\varepsilon\rho E_{-\alpha} + \frac{1}{2}i(\sigma E_{-\theta} + \rho E_{-\beta}), \end{aligned}$$

therefore $\text{Ad}(c^{-1})(E_\alpha - \varepsilon\rho E_{\tilde{\alpha}})$ and $\text{Ad}(c^{-1}) \cdot i(E_\alpha + \varepsilon\rho E_{\tilde{\alpha}})$ are elements of \mathfrak{g} , hence $A_\alpha = E_\alpha - \varepsilon\rho E_{\tilde{\alpha}}$, $B_\alpha = i(E_\alpha + \varepsilon\rho E_{\tilde{\alpha}})$ belong to \mathfrak{n}^{ij} . Calculating dimensions we see that the vectors having the above form are a basis of \mathfrak{n}^{ij} . To write a basis of \mathfrak{n}_{ij} we shall use the formula (3.5) and compute $C_\alpha = 2A_\alpha \square c_i$ and $D_\alpha = 2B_\alpha \square c_i$ (see (3.1)). Take

$$X = \frac{1}{2}(-\delta(E_\theta + E_{-\theta}) - \varepsilon(E_\beta + E_{-\beta})),$$

then $[X, E_\Gamma] = A_\alpha$, therefore $L(A_\alpha) = X$, and

$$\begin{aligned} C_\alpha &= 2([L(A_\alpha), L(c_i)] + L(A_\alpha c_i)) \\ &= [X, H_{\gamma_i}] + X \\ &= \frac{1}{2}(\delta E_\theta - \delta E_{-\theta} - \varepsilon E_\beta + \varepsilon E_{-\beta}) + \frac{1}{2}(-\delta(E_\theta + E_{-\theta}) - \varepsilon(E_\beta + E_{-\beta})) \\ &= -\delta E_{-\theta} - \varepsilon E_\beta. \end{aligned}$$

Similarly we can compute:

$$D_\alpha = i(\delta E_{-\theta} - \varepsilon E_\beta).$$

Our last step it to write a basis of \mathfrak{n}_2^+ . By [KW], Lemma 6.5, the map $\psi = I + \text{Ad}(c^2)\tau$ is a real linear isomorphism of \mathfrak{p}_2^+ onto \mathfrak{n}_2^+ . The dimension argument proves that vectors of the form

$$\psi(E_\alpha), \psi(iE_\alpha)$$

for $\alpha \in Q_i$, form a basis of \mathfrak{n}_2^+ .

The vectors

$$\{E_{\gamma_i}\}, \left\{ \frac{1}{\sqrt{2}}(E_\alpha - \varepsilon\rho E_{\bar{\alpha}}) \right\}, \left\{ \frac{i}{\sqrt{2}}(E_\alpha + \varepsilon\rho E_{\bar{\alpha}}) \right\}, \left\{ \frac{1}{\sqrt{2}}\psi(E_\alpha) \right\}$$

$$\left\{ \frac{1}{2}H_{\gamma_j} \right\}, \left\{ \frac{1}{\sqrt{2}}(-\delta E_{-\theta} - \varepsilon E_\beta) \right\}, \left\{ \frac{i}{\sqrt{2}}(\delta E_{-\theta} - \varepsilon E_\beta) \right\}, \left\{ \frac{1}{\sqrt{2}}\psi(iE_\alpha) \right\}$$

form an orthonormal basis of \mathfrak{s} with respect to the Hermitian product B_τ . We denote the corresponding left-invariant vector fields on S respectively by:

$$X_i, X_\alpha^1, X_\alpha^2, \mathcal{X}_\alpha, H_j, Y_\alpha^1, Y_\alpha^2, \mathcal{Y}_\alpha,$$

and we introduce in \mathfrak{p}^+ coordinates corresponding to the basis $X_i, X_\alpha^1, X_\alpha^2, \mathcal{X}_\alpha$:

$$(3.25) \quad w = \sum_j w_j X_j(e) + \sum_{i,j} \sum_{\alpha \in \bar{Q}_{i,j}} (w_\alpha^1 X_\alpha^1(e) + w_\alpha^2 X_\alpha^2(e)) + \sum_i \sum_{\alpha \in Q_i} w_\alpha \psi^{-1}(\mathcal{X}_\alpha(e)).$$

Given a function f on ${}^c\mathcal{D}$ let

$$\tilde{f}(s) = f(s \cdot (c \cdot o)),$$

then for a left-invariant vector field W on S we have

$$W\tilde{f}(s) = \frac{\partial}{\partial t} f(s \exp tW \cdot (c \cdot o))|_{t=0}$$

and so

$$\begin{aligned}
 (3.26) \quad X_j \tilde{f}(e) &= \partial_{u_j} f(c \cdot o), \\
 X_\alpha^k \tilde{f}(e) &= \partial_{u_\alpha^k} f(c \cdot o), \\
 \mathcal{X}_\alpha \tilde{f}(e) &= \partial_{u_\alpha} f(c \cdot o), \\
 H_j \tilde{f}(e) &= \partial_{v_j} f(c \cdot o), \\
 Y_\alpha^1 \tilde{f}(e) &= \partial_{v_\alpha^k} f(c \cdot o), \\
 \mathcal{Y}_\alpha \tilde{f}(e) &= \partial_{v_\alpha} f(c \cdot o),
 \end{aligned}$$

where $w = u + iv$. Therefore the complex structure \mathcal{J} on \mathfrak{s} , transported from ${}^c\mathcal{D}$ is:

$$\begin{aligned}
 (3.27) \quad \mathcal{J}(X_j) &= H_j, \\
 \mathcal{J}(H_j) &= -X_j, \\
 \mathcal{J}(X_\alpha^k) &= Y_\alpha^k, \\
 \mathcal{J}(Y_\alpha^k) &= -X_\alpha^k, \\
 \mathcal{J}(\mathcal{X}_\alpha) &= \mathcal{Y}_\alpha, \\
 \mathcal{J}(\mathcal{Y}_\alpha) &= -\mathcal{X}_\alpha.
 \end{aligned}$$

Finally,

$$\begin{aligned}
 (3.28) \quad Z_j &= X_j - iH_j, \\
 Z_\alpha^k &= X_\alpha^k - iY_\alpha^k, \\
 \mathcal{Z}_\alpha &= \mathcal{X}_\alpha - i\mathcal{Y}_\alpha
 \end{aligned}$$

are holomorphic vector fields.

Let Z be one of the vectors fields $Z_j, Z_\alpha^k, \mathcal{Z}_\alpha$, w the corresponding coordinate w_j, w_α^k or w_α and let Δ_Z be the unique left-invariant differential operator with the property

$$(3.29) \quad \Delta_Z \tilde{f}(e) = \partial_w \partial_{\bar{w}} f(c \cdot o).$$

Δ_Z is real, second order, elliptic degenerate and annihilates holomorphic (consequently pluriharmonic) functions and any left-invariant operator with the above properties is a linear combination of such. Therefore Δ_Z 's are *building blocks* for admissible operators. Δ_Z can be explicitly computed on the whole group S :

$$(3.30) \quad \Delta_Z = Z\bar{Z} - \nabla_Z \bar{Z} = X^2 + (\mathcal{J}X)^2 - \nabla_X X - \nabla_{\mathcal{J}X} \mathcal{J}X.$$

where ∇ denotes the Riemannian connection on S (see [DHP], [DHMP]).

4. The Hua system of second order operators and the Main Theorem.

Let $\{E_j\}$ be any orthonormal basis of \mathfrak{p}^+ and $\{E_j^*\}$ be a dual basis of \mathfrak{p}^- with respect to the Killing form B of $\mathfrak{g}^{\mathbb{C}}$ (for example $\{\tilde{E}_\alpha\}_{\alpha \in Q^+}$ and $\{\tilde{E}_{-\alpha}\}_{\alpha \in Q^+}$ are such bases). Then the Hua system is

$$(4.1) \quad \mathcal{H} = \sum_{j,k} E_j E_k^* \otimes [E_k, E_j^*].$$

The above definition was given by K. Johnson and A. Koranyi in [JK]. It is clearly an element of $\mathcal{U}^{\mathbb{C}} \otimes \mathfrak{k}^{\mathbb{C}}$, where $\mathcal{U}^{\mathbb{C}}$ is the complexification of the enveloping algebra of \mathfrak{g} . One can easily check that \mathcal{H} does not depend on the chosen basis. For this reason we shall write always the operator \mathcal{H} in terms of the base vectors $\{\tilde{E}_\alpha\}_{\alpha \in Q^+}$. We say that a function f defined on \mathcal{D} is *Hua-harmonic*, if the corresponding function \tilde{f} on G ($\tilde{f}(g) = f(g \cdot o)$) is annihilated by the Hua system. Analogously f is annihilated by a left-invariant operator U on G if $U\tilde{f} = 0$.

Now we are ready to formulate the main result of this paper:

THE MAIN THEOREM. — *Let $\mathcal{D} = G/K$ be an irreducible Hermitian symmetric domain of nontube type and let f be a real function on \mathcal{D} . Then f is Hua-harmonic if and only if f is pluriharmonic.*

Let us recall that f defined on $\mathcal{D} \subset \mathbb{C}^n$ is pluriharmonic if it is the real part of a holomorphic function. Pluriharmonicity is equivalent to being annihilated by all operators $\partial z_j \partial \bar{z}_k$ ($1 \leq j, k \leq n$). Since $E_j E_k^* \tilde{f}(e) = \partial z_j \partial \bar{z}_k f(0)$ (see [JK], formula 3.18), the Hua system annihilates pluriharmonic functions. For this reason we have only to prove that any Hua-harmonic function is pluriharmonic.

In fact we shall not use the whole Hua system, but only a part of it. More precisely for any basis $\{v_k\}$ of $\mathfrak{k}^{\mathbb{C}}$ define elements U_{v_k} of $\mathcal{U}^{\mathbb{C}}$ by

$$(4.2) \quad U_{v_k} = \sum_{\alpha \in Q^+} [v_k, \tilde{E}_\alpha] \tilde{E}_\alpha^*.$$

Then we have a simple proposition

PROPOSITION 4.1 ([JK], [L]). — *Let f be a function on G . Then $\mathcal{H}f = 0$ if and only if for every k : $U_{v_k} f = 0$.*

By [JK] and [L] the Laplace-Beltrami operator is a linear combination of operators U_{v_k} . Next we define second order differential operators U_k , which will be called *strongly diagonal Hua operators*:

$$(4.3) \quad U_k = \sum_{\alpha \in Q^+} [\tilde{H}_{\gamma_k}, \tilde{E}_\alpha] \tilde{E}_\alpha^*.$$

Observe, that in view of Proposition 2.1:

$$(4.4) \quad \begin{aligned} U_k &= \sum_{\alpha \in Q^+} \alpha(\tilde{H}_{\gamma_k}) \tilde{E}_\alpha \tilde{E}_\alpha^* \\ &= \sum_{\alpha \in Q^+} \frac{\langle \alpha, \alpha \rangle}{2} \alpha(\tilde{H}_{\gamma_k}) E_\alpha E_{-\alpha} \\ &= \sum_{\alpha \in Q^+} \frac{\langle \alpha, \alpha \rangle}{\langle \gamma_k, \gamma_k \rangle} \alpha(H_{\gamma_k}) E_\alpha E_{-\alpha} \\ &= 2E_{\gamma_k} E_{-\gamma_k} + \sum_{\alpha \in Q_k} E_\alpha E_{-\alpha} + \sum_j \sum_{\alpha \in Q_{j_k}} E_\alpha E_{-\alpha}. \end{aligned}$$

To prove pluriharmonicity of any Hua-harmonic function, we shall use only the Laplace - Beltrami operator and strongly diagonal Hua operators.

Now we explain the strategy of the proof. Using classical results [HC] we may expand f in terms of its projections on the spaces of K -finite vectors of type π :

$$f = \sum_{\pi \in \hat{K}} \chi_\pi *_{K} f,$$

where \hat{K} is the set of equivalence classes of irreducible unitary representations of K and χ_π is the character of π . Now $f_\pi = \chi_\pi *_{K} f$ are K -finite functions and are clearly Hua-harmonic. In the next section we prove (Theorem 5.1) that every Hua-harmonic function is G -harmonic (i.e. annihilated by all G invariant operators on G/K without constant term), therefore f_π are G -harmonic. Each K -finite, G -harmonic function is Poisson integral of a continuous bounded function defined on the maximal Furstenberg boundary ([H3], Theorem V.6.1), therefore each f_π is bounded, and it is enough to prove the Main Theorem for bounded functions.

Having the boundedness assumption (which is invariant on biholomorphic mappings), we transfer our problem to the group S acting simply transitively on the Siegel domain ${}^c\mathcal{D}$, and using techniques of [BDH], we obtain the result. The details are contained in Sections 6 and 7.

5. G -harmonic functions.

The aim of this section is to prove the following theorem

THEOREM 5.1. — *If f is a Hua-harmonic function, then f is G -harmonic.*

The above theorem was proved in the tube case by [JK] and [L]. Furthermore the authors of the first cited paper remarked that this result holds also for nontube domains, but they didn't give a proof. Our proof follows closely the argument of [L] and finally we get a system of equations that differs only by constants from the one considered by Lassalle. The main step is to prove the following theorem

THEOREM 5.2. — *If Φ is K -biinvariant, Hua-harmonic function on G , then Φ is constant.*

Using this result we can easily prove Theorem 5.1:

Proof of Theorem 5.1. — As we noticed in the previous section, among the Hua operators there is the Laplace-Beltrami operator. Hence f is an analytic function. Take dk to be unimodular normalized Haar measure on K . For fixed $g \in G$ define a function on G :

$$\Phi(h) = \int_K f(gkhK)dk.$$

Then Φ is K -biinvariant and Hua-harmonic, therefore by Theorem 5.2 Φ is constant. Hence

$$f(gK) = \int f(gK)dk = \int f(gkK)dk = \Phi(e) = \Phi(h) = \int f(gkhK)dk$$

and using the Godement theorem ([H2], page 403) we deduce that f is G -harmonic. \square

We are interested in studying how the Laplace - Beltrami operator and strongly diagonal Hua operators act on a K -biinvariant function f , defined on G . From KAK decomposition of G follows that f depends only on A . Hence its enough to compute radial parts of U_k , which we shall denote by $\Delta(U_k) \in \mathcal{U}(\mathfrak{a}^{\mathbb{C}})$. For this we determine $\Delta(E_{\alpha}E_{-\alpha})$ for all positive

noncompact roots α , considering three cases when α belongs to Γ , Q_j or Q_{ij} . Arguing as in [L] we obtain:

PROPOSITION 5.3. — *The radial parts of operators $E_\alpha E_{-\alpha}$ are:*

- for $\gamma_k \in \Gamma$

$$\Delta(E_{\gamma_k} E_{-\gamma_k}) = \frac{1}{4}(X_{\gamma_k}^2 + 2 \coth 2t_k X_{\gamma_k}),$$

- for $\alpha \in Q_{ij}$

$$\begin{aligned} \Delta(E_\alpha E_{-\alpha} + E_{\tilde{\alpha}} E_{-\tilde{\alpha}}) \\ = \frac{1}{2}(\coth(t_i + t_j)(X_{\gamma_i} + X_{\gamma_j}) + \coth(t_i - t_j)(X_{\gamma_i} - X_{\gamma_j})), \end{aligned}$$

- for $\alpha \in Q_j$

$$\Delta(E_\alpha E_{-\alpha}) = \frac{1}{2} \coth t_i X_{\gamma_i}.$$

Using the above result we may compute radial parts of strongly diagonal Hua operators:

$$\begin{aligned} 2\Delta(U_k) = X_{\gamma_k}^2 + 2 \coth 2t_k X_{\gamma_k} + \beta \coth t_k X_{\gamma_k} \\ + \frac{\lambda}{2} \sum_{j \neq k} (\coth(t_j + t_k)(X_{\gamma_j} + X_{\gamma_k}) + \coth(t_j - t_k)(X_{\gamma_j} - X_{\gamma_k})), \end{aligned}$$

where

$$\beta = \#\{\alpha : \alpha \in Q_k\},$$

$$\lambda = \#\{\alpha : \alpha \in Q_{jk} \text{ for some } j \neq k\},$$

β and λ are independent on j and k .

Take f to be any K -biinvariant analytic function on G . Every element g of G can be written (using KAK decomposition) as

$$g = k_1 \left(\exp \sum_{j=1}^r t_j X_{\gamma_j} \right) k_2, \quad k_1, k_2 \in K, \quad t_j \in \mathbb{R}.$$

Therefore f depends only on t_j and we can think of it as a function defined on \mathbb{R}^r . Furthermore f is W -invariant, where W denotes the classical Weyl group. W acts on \mathfrak{a} by signed permutations, so

$$f(t_1, \dots, t_r) = f(\pm t_{\sigma(1)}, \dots, \pm t_{\sigma(r)})$$

for any permutation σ of r elements. Therefore we can write a new version of Theorem 5.2 as follows:

THEOREM 5.4. — *Let λ and β be strictly positive real numbers and let f be an analytic function on \mathbb{R}^r , invariant under signed permutations, being a solution of differential equations*

$$\left(\partial_{t_k}^2 + 2 \coth 2t_k \partial_{t_k} + \beta \coth t_k \partial_{t_k} + \frac{\lambda}{2} \sum_{j \neq k} (\coth(t_j + t_k)(\partial_{t_j} + \partial_{t_k}) + \coth(t_j - t_k)(\partial_{t_j} - \partial_{t_k})) \right) f = 0,$$

for $k = 1, \dots, l$. Then f is constant.

The theorem for similar system of equations was proved by M. Lassalle ([L], p. 150, Theorem 5). However, the same proof works in our case, therefore we omit it and refer the reader to [L].

6. Hua operators on Siegel domains.

Given a function f on \mathcal{D} we define

$$\begin{aligned} \tilde{f}(g) &= f(g \cdot o) && \text{— a function on } G, \\ {}^c f(x) &= f(c^{-1}x) && \text{— a function on } {}^c\mathcal{D}, \\ {}^c f(g) &= {}^c f(g \cdot (c \cdot o)) && \text{— a function on } {}^cG. \end{aligned}$$

To prove that f is pluriharmonic it is enough to show pluriharmonicity of ${}^c f$, because c is biholomorphic.

One can easily check that ${}^c \tilde{f}(\text{Ad}(c)g) = \tilde{f}(g)$ which implies that for any $X \in \mathcal{U}(G)$:

$$X \tilde{f}(g) = (\text{Ad}(c)X) {}^c \tilde{f}(\text{Ad}(c)g)$$

where $\text{Ad}(c)X \in \mathcal{U}({}^cG)$. Therefore the function ${}^c \tilde{f}$ is annihilated by the left invariant operators $\text{Ad}(c)U_k$. Our aim is now to compute these operators on the domain ${}^c\mathcal{D}$ and on the group S . As a result we shall get Hua operators on Siegel domain written in terms of admissible operators Δ_Z 's, Z being as in (3.28).

THEOREM 6.1. — *Let f be a Hua-harmonic function on \mathcal{D} , then the corresponding function ${}^c f$ on ${}^c\mathcal{D}$ is annihilated by the Laplace-Beltrami*

operator Δ on ${}^c\mathcal{D}$ and \mathcal{H}_k ($1 \leq k \leq r$):

$$(6.1) \quad \mathcal{H}_k = 2\Delta_{Z_k} + \sum_{\alpha \in Q_k} \Delta_{Z_\alpha} + \sum_j \sum_{\alpha \in \overline{Q}_{kj}} (\Delta_{Z_\alpha^1} + \Delta_{Z_\alpha^2}),$$

which we shall call strongly diagonal Hua operators as well. The Laplace-Beltrami operator is given by

$$(6.2) \quad \Delta = \sum_i \Delta_{Z_i} + \sum_i \sum_{\alpha \in Q_i} \Delta_{Z_\alpha} + \sum_{j,k} \sum_{\alpha \in \overline{Q}_{kj}} (\Delta_{Z_\alpha^1} + \Delta_{Z_\alpha^2}).$$

Proof. — The idea of the proof is following. First we compute U_k 's at the point o , denoting the result by V_k :

$$U_k \tilde{f}(e) = V_k f(o).$$

Next using the Cayley transform we find operators cV_k such that

$${}^cV_k {}^c f(c \cdot o) = V_k f(o),$$

and finally we extend them S -invariant on the domain ${}^c\mathcal{D}$ and the group S .

First observe that by [JK] (3.18)

$$E_\alpha \overline{E}_\alpha \tilde{f}(e) = \partial_{z_\alpha} \partial_{\bar{z}_\alpha} f(o),$$

where $\{z_\alpha\}$ are coordinates in \mathfrak{p}^+ with respect to the basis $\{E_\alpha\}_{\alpha \in Q^+}$. Therefore

$$U_k \tilde{f}(e) = V_k f(o) = \left(2\partial_{z_k} \partial_{\bar{z}_k} + \sum_{\alpha \in Q_k} \partial_{z_\alpha} \partial_{\bar{z}_\alpha} + \sum_j \sum_{\alpha \in \overline{Q}_{jk}} \partial_{z_\alpha} \partial_{\bar{z}_\alpha} \right) f(o).$$

Now we have to compute the differential of c . We shall use the formula given in [S], Lemma II.5.3, which says that the Jacobian of the mapping $z \mapsto c \cdot z$ at the point o is given by

$$\text{Jac}(o \mapsto c \cdot o) = \text{Ad } c_K|_{\mathfrak{p}^+},$$

where c_K denotes the component $K^{\mathbb{C}}$ of c in the decomposition $P^+ K^{\mathbb{C}} P^-$. By [KW], Lemma 3.5:

$$c_K = \exp \left(- \sum_{1 \leq i \leq r} \log \cosh \left(\frac{\pi}{4} i \right) \cdot H_{\gamma_i} \right) = \prod_{1 \leq i \leq r} \exp \left(- \log \frac{\sqrt{2}}{2} \cdot H_{\gamma_i} \right).$$

Hence the vectors E_α are eigenvectors of $\text{Ad}(c_K)$:

$$\text{Ad}_K(E_\alpha) = \begin{cases} 2E_\alpha, & \text{for } \alpha \in \Gamma, \\ 2E_\alpha, & \text{for } \alpha \in Q_{ij}, \\ \sqrt{2}E_\alpha, & \text{for } \alpha \in Q_j. \end{cases}$$

Therefore,

$$\begin{aligned} V_k f(o) &= \left(2\partial_{z_k} \partial_{\bar{z}_k} + \sum_{\alpha \in Q_k} \partial_{z_\alpha} \partial_{\bar{z}_\alpha} + \sum_j \sum_{\alpha \in Q_{jk}} \partial_{z_\alpha} \partial_{\bar{z}_\alpha} \right) f(o) \\ &= \left(8\partial_{z_k} \partial_{\bar{z}_k} + 2 \sum_{\alpha \in Q_k} \partial_{z_\alpha} \partial_{\bar{z}_\alpha} + 4 \sum_j \sum_{\alpha \in Q_{jk}} \partial_{z_\alpha} \partial_{\bar{z}_\alpha} \right) {}^c f(c \cdot o) \\ &= {}^c V_k {}^c f(c \cdot o). \end{aligned}$$

Let us observe that for $\alpha \in Q_{ij}$ and $\tilde{\alpha} = \gamma_i + \gamma_j - \alpha$:

$$\begin{aligned} (\partial_{z_\alpha} \partial_{\bar{z}_\alpha} + \partial_{z_{\tilde{\alpha}}} \partial_{\bar{z}_{\tilde{\alpha}}}) {}^c f(c \cdot o) &= (\partial_{x_\alpha}^2 + \partial_{y_\alpha}^2 + \partial_{x_{\tilde{\alpha}}}^2 + \partial_{y_{\tilde{\alpha}}}^2) {}^c f(c \cdot o) \\ &= \left(\frac{1}{2} (\partial_{x_\alpha} - \varepsilon \rho \partial_{x_{\tilde{\alpha}}})^2 + \frac{1}{2} (\partial_{y_\alpha} - \varepsilon \rho \partial_{y_{\tilde{\alpha}}})^2 \right. \\ &\quad \left. + \frac{1}{2} (\partial_{x_\alpha} + \varepsilon \rho \partial_{x_{\tilde{\alpha}}})^2 + \frac{1}{2} (\partial_{y_\alpha} + \varepsilon \rho \partial_{y_{\tilde{\alpha}}})^2 \right) {}^c f(c \cdot o) \\ &= (\partial_{w_\alpha^1} \partial_{\bar{w}_\alpha^1} + \partial_{w_\alpha^2} \partial_{\bar{w}_\alpha^2}) {}^c f(c \cdot o) \end{aligned}$$

for ε, ρ defined by (3.23) and w as in (3.25). Then applying (3.26) and the definition of building blocks we obtain the S -invariant extension of the above operator which is $\Delta_{Z_\alpha^1} + \Delta_{Z_\alpha^2}$. In the same way we get the corresponding operators for $\partial_{z_i} \partial_{\bar{z}_i}$ and $\partial_{z_\alpha} \partial_{\bar{z}_\alpha}$ (for $\alpha \in Q_i$): $\Delta_{Z_i}, 2\mathcal{L}_{Z_\alpha}$. Therefore

$$0 = 8\Delta_{Z_k} + 4 \sum_{\alpha \in Q_k} \Delta_{Z_\alpha} + 4 \sum_j \sum_{\alpha \in Q_{kj}} (\Delta_{Z_\alpha^1} + \Delta_{Z_\alpha^2}) {}^c f(s) = 4\mathcal{H}_k {}^c f(s).$$

Observe that using the same method we can transfer the Laplace-Beltrami operator:

$$\sum_{\alpha \in Q^+} E_\alpha \overline{E_\alpha}$$

and we get formula (6.2). □

Note that to obtain the main result it is enough to prove the following:

THEOREM 6.2. — *Let f be a real, bounded function on S , annihilated by the operators:*

$$\Delta, \mathcal{H}_1, \dots, \mathcal{H}_r.$$

Then f is pluriharmonic.

Now we are going to change slightly the notation for the one used in [BDH] and [BBDHPT]. Take all vector fields

$$X_\alpha^1, X_\alpha^2, \quad \text{for } \alpha \in \overline{Q}_{ij},$$

and denote them

$$X_{ij}^\beta, \quad 1 \leq \beta \leq d.$$

Corresponding vectors

$$Y_\alpha^1, Y_\alpha^2$$

denote by

$$Y_{ij}^\beta,$$

using the same ordering. Analogously we change

$$\mathcal{X}_\alpha, \mathcal{Y}_\alpha$$

for

$$\mathcal{X}_j^\beta, \mathcal{Y}_j^\beta.$$

Hence we get the left-invariant vector fields:

$$X_j, X_{ij}^\beta, \mathcal{X}_j^\beta, H_j, Y_{ij}^\beta, \mathcal{Y}_j^\beta.$$

Finally, for the buildings blocks we adopt the notation

$$\begin{aligned} \mathcal{L}_j^\beta &= \Delta_{Z_\beta}, \\ \Delta_i &= \Delta_{Z_i}, \\ \Delta_{ij}^\beta &= \Delta_{Z_\beta^k}. \end{aligned}$$

Then formulas (6.1) and (6.2) can be written as:

$$\begin{aligned} \mathcal{H}_k &= \mathcal{L}_k + 2\Delta_k + \sum_{\substack{1 \leq i < k \\ \beta}} \Delta_{ik}^\beta + \sum_{\substack{k < i \leq r \\ \beta}} \Delta_{ki}^\beta, \\ \Delta &= \sum_i \mathcal{L}_i + \sum_i \Delta_i + \sum_{\substack{1 \leq i < j \leq r \\ \beta}} \Delta_{ij}^\beta, \end{aligned}$$

for $\mathcal{L}_k = \sum_{\beta} \mathcal{L}_k^{\beta}$. Furthermore by [DHMP], Theorem 1.23:

$$(6.3) \quad \begin{aligned} \Delta_j &= X_j^2 + H_j^2 - H_j, \\ \mathcal{L}_j^{\alpha} &= (\mathcal{X}_j^{\alpha})^2 + (\mathcal{Y}_j^{\alpha})^2 - H_j, \\ \Delta_{ij}^{\alpha} &= (X_{ij}^{\alpha})^2 + (Y_{ij}^{\alpha})^2 - H_j. \end{aligned}$$

From [DHMP] and [BDH] it follows that pluriharmonicity is equivalent to being annihilated by each building block \mathcal{L}_i^{β} , Δ_{ij}^{β} , Δ_i separately, and that is just what we are going to prove about f :

THEOREM 6.3. — *Let f be a real, bounded function on S , annihilated by Δ and by \mathcal{H}_j , for $j = 1, \dots, r$. Then*

$$\Delta_i f = 0, \quad \mathcal{L}_j^{\beta} f = 0, \quad \Delta_{ij}^{\beta} f = 0,$$

for all i, j, β .

A simple calculation shows that as a linear combination of Δ and \mathcal{H}_j 's we may obtain the Laplace-Beltrami operator Δ_T for the tube domain $T_{\Omega} = V + i\Omega$, which is identified with the subgroup VS_0 of the group S . Indeed,

$$\Delta_T = \sum_{1 \leq j \leq r} \mathcal{H}_j - \Delta = \sum_{1 \leq j \leq r} \Delta_j + \sum_{\substack{1 \leq i < j \leq r \\ \beta}} \Delta_{ij}^{\beta}.$$

Given $\zeta \in \mathcal{Z}$, the function $f_{\zeta}(xya) = f((\zeta, x)ya)$ becomes a function on the tube domain T_{Ω} and all the operators that are linear combinations of the blocks Δ_{ij}^{β} , Δ_i have perfect sense as operators there. In particular using Δ , Δ_T and Poisson integrals both on ${}^c\mathcal{D}$ and on T_{Ω} we are able to prove that for fixed ζ , f_{ζ} is the Poisson-Szegö integral on T_{Ω} and so using Theorem 1.1 we obtain

THEOREM 6.4 ([BBDHPT], Theorem 3.1). — *Let f be a bounded function on S annihilated by Δ and by \mathcal{H}_j , for $j = 1, \dots, r$. Then*

$$\mathcal{H}_j^T f = 0 \quad \text{for } j = 1, \dots, r$$

and

$$\mathcal{L}_j f = \sum_{\beta} \mathcal{L}_j^{\beta} f = 0 \quad \text{for } j = 1, \dots, r,$$

where $\mathcal{H}_j^T = 2\Delta_j + \sum_{k < j} \sum_{\beta} \Delta_{kj}^{\beta}$ are strongly diagonal Hua operators for VS_0 .

Proof of Theorem 6.3. — To prove the theorem we are going to use induction on the rank of the cone. Assuming $r > 1$, we show how to reduce the problem to the rank $r - 1$.

First let us decompose the domain ${}^c\mathcal{D}$. Take Ω^{r-1} to be the rank $r - 1$ cone determined by the frame c_1, \dots, c_{r-1} . The underlying space V^{r-1} for Ω^{r-1} is the subspace

$$V^{r-1} = \bigoplus_{1 \leq i \leq j < r} V_{ij}.$$

Next let us define

$$\mathcal{Z}^{r-1} = \bigoplus_{j=1}^{r-1} \mathcal{Z}_j.$$

Then it is easily seen that $\mathcal{Z}^{r-1} \times \mathcal{Z}^{r-1}$ is mapped by Φ into $(V^{r-1})^{\mathbb{C}}$. Moreover, $\Phi(\zeta, \zeta)$ belongs to Ω^{r-1} when $\zeta \in \mathcal{Z}^{r-1}$. So we may define the corresponding Siegel domain \mathcal{D}^{r-1} as

$$\mathcal{D}^{r-1} = \{(\zeta, z) \in \mathcal{Z}^{r-1} \times (V^{r-1})^{\mathbb{C}} : \Im z - \Phi(\zeta, \zeta) \in \Omega^{r-1}\}.$$

Now let

$$(6.4) \quad \mathfrak{h}_r = \mathcal{Z}_r \oplus \left(\bigoplus_{j < r} V_{jr} \right) \oplus \left(\bigoplus_{j < r} \mathfrak{n}_{jr} \right) \oplus V_{rr}.$$

Looking at the corresponding brackets relations one can prove that \mathfrak{h}_r is isomorphic to the Lie algebra of the Heisenberg group \mathbb{H}^m with $m = \chi + (r - 1)d$. Therefore

$$(6.5) \quad \exp \mathfrak{h}_r = \mathbb{C}^m \times \mathbb{R} = \mathbb{H}^m$$

(V_{rr} is the center of the group).

Put $A_r = \exp \mathbb{R}H_r$ and $S_r = \exp \mathfrak{h}_r \cdot A_r$ then S_r is a solvable group acting simple transitively on the classical Siegel half plane \mathcal{D}_r which is biholomorphic equivalent to the complex ball in \mathbb{C}^{m+1} (see also the next section). We can identify S_r with \mathcal{D}_r .

On the level of the Lie algebra we have the decomposition

$$(6.6) \quad \mathfrak{s} = \mathfrak{s}^{r-1} \oplus \mathfrak{s}_r,$$

where

$$(6.7) \quad \begin{aligned} \mathfrak{s}^{r-1} &= \left(\bigoplus_{j < r} \mathcal{Z}_j \right) \oplus \left(\bigoplus_{1 \leq i \leq j < r} V_{ij} \right) \oplus \left(\bigoplus_{1 \leq i < j < r} \mathfrak{n}_{ij} \right) \oplus \left(\bigoplus_{1 \leq j < r} \mathfrak{a}_{0j} \right), \\ \mathfrak{s}_r &= \mathcal{Z}_r \oplus \left(\bigoplus_{1 \leq j < r} V_{jr} \right) \oplus \left(\bigoplus_{1 \leq j < r} \mathfrak{n}_{jr} \right) \oplus V_{rr} \oplus \mathfrak{a}_{0r}. \end{aligned}$$

\mathfrak{s}^{r-1} is a subalgebra of \mathfrak{s} and \mathfrak{s}_r is an ideal. Therefore we can decompose the group as a semidirect product of a subgroup S^{r-1} and normal subgroup S_r :

$$(6.8) \quad S = S^{r-1} S_r.$$

S^{r-1} acts simply transitively on the Siegel domain \mathcal{D}^{r-1} of the rank $r - 1$.

The complex structure on \mathfrak{s}_r (transferred from \mathcal{D}_r) is the restriction of the one on \mathfrak{s} . Hence

$$(6.9) \quad \begin{aligned} \mathcal{J}(X_r) &= H_r, \\ \mathcal{J}(H_r) &= -X_r, \\ \mathcal{J}(X_{jr}^\alpha) &= Y_{jr}^\alpha, \\ \mathcal{J}(Y_{jr}^\alpha) &= -X_{jr}^\alpha, \\ \mathcal{J}(\mathcal{X}_r^\alpha) &= \mathcal{Y}_r^\alpha, \\ \mathcal{J}(\mathcal{Y}_r^\alpha) &= -\mathcal{X}_r^\alpha. \end{aligned}$$

Furthermore the basis

$$\mathcal{X}_r^\alpha, \mathcal{Y}_r^\alpha, X_{jr}^\alpha, Y_{jr}^\alpha, X_r, H_r$$

of \mathfrak{s}_r is orthonormal with respect to the Riemannian form g_r on S_r . We can compute the Riemannian connection $\tilde{\nabla}$ on \mathfrak{s}_r :

LEMMA 6.5. — *The Riemannian connection $\tilde{\nabla}$ is given by*

$$\begin{aligned}\tilde{\nabla}_{H_r} H_r &= 0, \\ \tilde{\nabla}_{X_{jr}^\alpha} X_{jr}^\alpha &= \frac{1}{2} H_r, \\ \tilde{\nabla}_{X_r} X_r &= H_r, \\ \tilde{\nabla}_{Y_{ir}^\alpha} Y_{ir}^\alpha &= \frac{1}{2} H_r, \\ \tilde{\nabla}_{\mathcal{X}_r^\alpha} \mathcal{X}_r^\alpha &= \frac{1}{2} H_r, \\ \tilde{\nabla}_{\mathcal{Y}_r^\alpha} \mathcal{Y}_r^\alpha &= \frac{1}{2} H_r.\end{aligned}$$

Defining operators $\tilde{\Delta}_r$, $\tilde{\mathcal{L}}_r^\alpha$ and $\tilde{\Delta}_{jr}^\alpha$ as in (3.30) and (6.3), with $\tilde{\nabla}$ in place of ∇ , using formula (3.30) and Lemma 6.5, we get:

$$(6.10) \quad \begin{aligned}\tilde{\Delta}_r &= X_r^2 + H_r^2 - H_r, \\ \tilde{\mathcal{L}}_r^\alpha &= (\mathcal{X}_r^\alpha)^2 + (\mathcal{Y}_r^\alpha)^2 - H_r, \\ \tilde{\Delta}_{ir}^\alpha &= (X_{ir}^\alpha)^2 + (Y_{ir}^\alpha)^2 - H_r.\end{aligned}$$

Since the above operators act from the right, they have perfect sense both on S_r and S , by (6.3) we may write:

$$(6.11) \quad \begin{aligned}\tilde{\Delta}_r(f_{s'}) &= (\Delta_r f)(s'_r), \\ \tilde{\mathcal{L}}_r^\alpha(f_{s'}) &= (\mathcal{L}_r^\alpha f)(s'_r), \\ \tilde{\Delta}_{jr}^\alpha(f_{s'}) &= (\Delta_{jr}^\alpha f)(s'_r).\end{aligned}$$

where $s' \in S^{r-1}$, $s_r \in S_r$ and $f_{s'}(s_r) = f(s'_r)$.

In the next section we are going to restrict operators \mathcal{L}_r and \mathcal{H}_j^T to left cosets of S_r and to prove that left-hand sides of (6.11) are zero. But in view of (6.11) this means that f is annihilated by the strongly diagonal Hua operators and the Laplace-Beltrami operator corresponding to the domain \mathcal{D}^{r-1} , which completes the proof of theorem (6.3). \square

7. Bounded pluriharmonic functions on the Siegel upper half plane.

Now we are going to prove the last step of the Main Theorem. The idea of our proof is very similar to the proof of Theorem 6.7 in [BDH]. But

we shall use somewhat different operators, and for this reason we have to strengthen some of the lemmas. For completeness we give the whole proof. We should mention, that notation used in this section is independent on the notation contained in the rest part of the paper.

We shall consider the classical Siegel domain \mathcal{D} defined by

$$\mathcal{D} = \left\{ (z, z_{n+1}) \in \mathbb{C}^n \times \mathbb{C} : \Im z_{n+1} > \frac{1}{4}|z|^2 \right\}.$$

It is well-known, that \mathcal{D} is biholomorphically equivalent to the unit ball in \mathbb{C}^{n+1} . Let G denote the group of holomorphic isometries of \mathcal{D} . Take the Iwasawa decomposition of G :

$$G = \mathbb{H}^n AU(\mathbb{C}^n).$$

Then the elements $(\zeta, u) \in \mathbb{H}^n$ (Heisenberg group), $a \in A = \mathbb{R}^+$ (dilataions group) and $\rho \in U(\mathbb{C}^n)$ (the group of unitary mappings of \mathbb{C}^n) act on \mathcal{D} in the following way:

$$\begin{aligned} (\zeta, u) \cdot (z, z_{n+1}) &= (\zeta + z, u + z_{n+1} + \frac{1}{2}\Im\langle z, \zeta \rangle + i|\zeta|^2), \\ \delta_a \cdot (z, z_{n+1}) &= (\sqrt{a}z, az_{n+1}), \\ \rho \cdot (z, z_{n+1}) &= (\rho(z), z_{n+1}). \end{aligned}$$

The semidirect product of \mathbb{H}^n and A acts simply transitively on \mathcal{D} . Therefore we can identify the group $S = \mathbb{H}^n A$ with the domain \mathcal{D} :

$$S \ni s = (\zeta, u, a) \sim (\zeta, u, a) \cdot (0, i) \in \mathcal{D}.$$

Let us denote by $\mathcal{X}_j, \mathcal{Y}_j, T$ left-invariant fields on \mathbb{H}^n , which in \mathbf{e} (the identity element of \mathbb{H}^n) agree respectively with $\partial_{x_j}, \partial_{y_j}, \partial_u$ ($\zeta_j = x_j + iy_j$). Then the operators $\tilde{\mathcal{X}}_j, \tilde{\mathcal{Y}}_j, \tilde{T}, H$ given by

$$\begin{aligned} \tilde{\mathcal{X}}_j &= \sqrt{a}\mathcal{X}_j, \\ \tilde{\mathcal{Y}}_j &= \sqrt{a}\mathcal{Y}_j, \\ \tilde{T} &= aT, \\ H &= a\partial_a, \end{aligned}$$

are left-invariant on the group S , and form a basis of the Lie algebra \mathfrak{s} of S .

For fixed k ($1 \leq k \leq n$), we are going to consider the following operators:

$$(7.1) \quad \begin{aligned} D &= \sum_{1 \leq j \leq k} \left((\tilde{\mathcal{X}}_j)^2 + (\tilde{\mathcal{Y}}_j)^2 \right) - kH, \\ L &= \sum_{k+1 \leq j \leq n} \left((\tilde{\mathcal{X}}_j)^2 + (\tilde{\mathcal{Y}}_j)^2 \right) - (n-k)H + 2(\tilde{T}^2 + H^2 - H), \\ L_B &= \sum_{1 \leq j \leq n} \left((\mathcal{X}_j)^2 + (\mathcal{Y}_j)^2 \right). \end{aligned}$$

D and L are admissible operators (compare (6.3)), L_B is called a sublaplacian on the Heisenberg group. All the operators are related by the following equation:

$$(7.2) \quad D + L = aL_B + 2a^2(\partial_u^2 + \partial_a^2) - an\partial_a.$$

Notice that if

$$\{\tilde{\mathcal{X}}_j, \tilde{\mathcal{Y}}_j : j = 1, \dots, k\} = \{\mathcal{X}_r^\alpha, \mathcal{Y}_r^\alpha : \alpha = 1, \dots, \chi\}$$

and

$$\{\tilde{\mathcal{X}}_j, \tilde{\mathcal{Y}}_j : j = k+1, \dots, n\} = \{X_{j_r}^\alpha, Y_{j_r}^\alpha : j = 1, \dots, r-1, \alpha = 1, \dots, d\},$$

where $\mathcal{X}_r^\alpha, \mathcal{Y}_r^\alpha, X_{j_r}^\alpha, Y_{j_r}^\alpha$ are left invariant vector fields on S and S_r defined in Section 6, then $\mathcal{L}_r, \mathcal{H}_r^T$ restricted to S_r are D and L , respectively.

The goal of this section is to prove the following theorem:

THEOREM 7.1. — *Let F be a real bounded function on \mathcal{D} annihilated by D and L . Then F is a pluriharmonic function, which implies that for every j , $(\tilde{\mathcal{X}}_j^2 + \tilde{\mathcal{Y}}_j^2 - H)F = 0$ and $(\tilde{T}^2 + H^2 - H)F = 0$.*

In terms of the previous section this theorem says that F is annihilated by all building blocks: $\Delta_{j_r}^\alpha$ and \mathcal{L}_r^α .

Being $D + L$ harmonic, F can be written as Poisson integral of its boundary value $f \in L^\infty(\mathbb{H}^n)$:

$$F((\zeta, u)a) = f *_{\mathbb{H}^n} P_a(\zeta, u) =: F_a(\zeta, u)$$

(see [DH], [R]).

PROPOSITION 7.2. — *Let F be any bounded function on \mathcal{D} annihilated by D and L , and let $f \in L^\infty(\mathbb{H}^n)$ be its boundary value. Then f satisfies the following equation:*

$$(7.3) \quad (L_B^2 + n^2 \partial_u^2) f(\zeta, u) = 0.$$

Proof. — This proposition is a straightforward consequence of the fact, that the family of functions F_a tends in the distribution sense to the function f , when a goes to 0. The convergence is sufficient to “transfer” differential operators L_B and ∂_u to f . However our assumptions give us more, namely limits

$$\lim_{a \rightarrow 0} \partial_a^m F_a$$

exist in the distribution sense as well, for all nonnegative integer numbers m . To prove this put

$$\bar{D} = \sum_{1 \leq j \leq k} ((\mathcal{X}_j)^2 + (\mathcal{Y}_j)^2).$$

Then using (7.1) and an observation that ∂_a and \bar{D} commute we have

$$\partial_a^m F_a(\zeta, u) = \frac{1}{k^m} \bar{D}^m F_a(\zeta, u),$$

which implies

$$(7.4) \quad \lim_{a \rightarrow 0} \partial_a^m F_a(\zeta, u) = \lim_{a \rightarrow 0} \frac{1}{k^m} \bar{D}^m F_a(\zeta, u) = \frac{1}{k^m} \bar{D}^m f(\zeta, u).$$

Now we are able to deal with the operator L . Let

$$\bar{L} = \sum_{k+1 \leq j \leq n} ((\mathcal{X}_j)^2 + (\mathcal{Y}_j)^2)$$

then by (7.1)

$$0 = \lim_{a \rightarrow 0} (\bar{L} + 2a(\partial_u^2 + \partial_a^2) - (n - k)\partial_a) F_a(\zeta, u) = (\bar{L} - \frac{n - k}{k} \bar{D}) f(\zeta, u).$$

Hence

$$(7.5) \quad \bar{L} f(\zeta, u) = \frac{n - k}{k} \bar{D} f(\zeta, u).$$

Notice that the operators \bar{L} and \bar{D} commute, so

$$\bar{L}^2 f(\zeta, u) = \bar{L} \frac{n-k}{k} \bar{D} f(\zeta, u) = \frac{n-k}{k} \bar{D} \bar{L} f(\zeta, u) = \left(\frac{n-k}{k} \right)^2 \bar{D}^2 f(\zeta, u).$$

Therefore we easily check that

$$(7.6) \quad L_B^2 f(\zeta, u) = (\bar{L} + \bar{D})^2 f(\zeta, u) = \frac{n^2}{k^2} \bar{D}^2 f(\zeta, u).$$

Finally applying again (7.1), differentiating it with respect to a and using formulas (7.4), (7.5) and (7.6) we conclude that

$$\begin{aligned} 0 &= \lim_{a \rightarrow 0} \partial_a (\bar{L} + 2a(\partial_u^2 + \partial_a^2) - (n-k)\partial_a) F_a(\zeta, u) \\ &= \lim_{a \rightarrow 0} (\partial_a \bar{L} + 2(\partial_u^2 + \partial_a^2) + 2a(\partial_a \partial_u^2 + \partial_a^3) - (n-k)\partial_a^2) F_a(\zeta, u) \\ &= \left(\frac{1}{k} \bar{L} \bar{D} + 2\partial_u^2 + \frac{2}{k^2} \bar{D}^2 - \frac{n-k}{k^2} \bar{D}^2 \right) f(\zeta, u) \\ &= \left(2\partial_u^2 + \left(\frac{n-k}{k^2} + \frac{2}{k^2} - \frac{n-k}{k^2} \right) \bar{D}^2 \right) f(\zeta, u) \\ &= \frac{2}{n^2} (n^2 \partial_u^2 + L_B^2) f(\zeta, u), \end{aligned}$$

which complete the proof. \square

Now the strategy is to reduce the problem to functions F whose boundary value are easier to handle, due to special properties of their Fourier transforms. First, we may clearly assume that

$$(7.7) \quad f = \rho * \tilde{f},$$

for $\tilde{f} \in L^\infty(\mathbb{H}^n)$, $\rho \in \mathcal{S}(\mathbb{H}^n)$. Indeed, F_a may be approximated by functions $\rho_n * F_a$ and on the level of boundary values this means (7.7). Furthermore, we prove the following

PROPOSITION 7.3. — *Let F be as in Theorem 7.1 and let $f = \rho * \tilde{f}$ ($\tilde{f} \in L^\infty(\mathbb{H}^n)$, $\rho \in \mathcal{S}(\mathbb{H}^n)$) be its boundary value. Denote by $f(\zeta, \hat{\lambda})$ the distributional partial Fourier transform of f along \mathbb{R} . Then there exists a sequence of functions $\eta_n \in C_c^\infty(\mathbb{R} \setminus \{0\})$ such that for f_n and F_n defined by*

$$\begin{aligned} f_n(\zeta, u) &= \eta_n *_{\mathbb{R}} f(\zeta, u), \\ F_n(\zeta, u, a) &= (f_n) *_{\mathbb{H}^n} P_a(\zeta, u) = \eta_n *_{\mathbb{R}} F(\zeta, u, a), \end{aligned}$$

the following conditions hold:

- F_n is annihilated by L and D ;
- $\text{supp}f_n(\cdot, \widehat{\cdot}) \subset \mathbb{C}^n \times \{\lambda \in \mathbb{R} : \varepsilon_n < |\lambda| < M_n\}$ and the constants ε_n and M_n are dependent only on η_n ;
- the sequence F_n tends to $F + c$, where c is a constant.

It is clear that for any sequence of functions η_n such that $\widehat{\eta}_n \in C_c^\infty(\mathbb{R} \setminus \{0\})$ the first two conditions are true. The main difficulties is to build a sequence that ensures the last condition.

Let ϕ be a Schwartz function on \mathbb{R} such that

$$(7.8) \quad \widehat{\phi}(\lambda) = \begin{cases} 1 & \text{for } |\lambda| \leq 1, \\ 0 & \text{for } |\lambda| \geq 2, \end{cases}$$

and $\widehat{\phi}(\lambda) = \widehat{\phi}(-\lambda)$.

For a given sequence $\{k_n\}_{n=1,2,\dots}$ of natural numbers tending to infinity, let

$$\psi_n(u) = \frac{1}{k_n} \phi\left(\frac{u}{k_n}\right).$$

Given a bounded function g on \mathbb{R} , there is $\{k_n\}_{n=1,2,\dots}$, $k_n \rightarrow \infty$ such that

$$\lim_{n \rightarrow +\infty} \langle g, \psi_n \rangle \text{ exists.}$$

But by [BDH] (Lemma 4.4) even more is true. We can choose a sequence $\{k_n\}_{n=1,2,\dots}$ such that

$$(7.9) \quad \lim_{n \rightarrow +\infty} \langle f_\zeta, \psi_n \rangle \text{ exists for all } \zeta \in \mathbb{C}^n.$$

Now using (7.3) we prove that the above limit does not depend on ζ .

LEMMA 7.4. — *The function*

$$\mathbb{C}^n \ni \zeta \mapsto H(\zeta) = \lim_{n \rightarrow +\infty} \langle f_\zeta, \psi_n \rangle$$

is constant.

Proof. — Consider the function \widetilde{H} on \mathbb{H}^n defined by

$$\widetilde{H}(\zeta, u) = H(\zeta).$$

Then $\tilde{H} \in L^\infty(\mathbb{H}^n)$ and it is a boundary value of a smooth function $\tilde{H}_a \in L^\infty(S)$, given by

$$\tilde{H}_a(\zeta, u) = \tilde{H} *_{\mathbb{H}^n} P_a(\zeta, u).$$

Observe that for every $u \in \mathbb{R}$,

$$\lim_{n \rightarrow +\infty} \psi_n *_{\mathbb{R}} f_\zeta(u) = \lim_{n \rightarrow +\infty} \langle \psi_n, f_\zeta \rangle.$$

Indeed,

$$\begin{aligned} & \lim_{n \rightarrow +\infty} |\langle \psi_n, f_\zeta \rangle - \psi_n *_{\mathbb{R}} f_\zeta(u)| \\ &= \lim_{n \rightarrow +\infty} \left| \int \frac{1}{k_n} \left(\phi\left(\frac{y}{k_n}\right) - \phi\left(\frac{u-y}{k_n}\right) \right) f_\zeta(y) dy \right| \\ &\leq c \lim_{n \rightarrow +\infty} \int \left| \phi(y) - \phi\left(y - \frac{u}{k_n}\right) \right| dy \\ &= 0. \end{aligned}$$

Hence by the Lebesgue dominated convergence theorem

$$\begin{aligned} \tilde{H}_a(\zeta, u) &= \left(\lim_{n \rightarrow +\infty} \psi_n *_{\mathbb{R}} f(\cdot) \right) *_{\mathbb{H}^n} P_a(\zeta, u) \\ &= \lim_{n \rightarrow +\infty} (\psi_n *_{\mathbb{R}} f *_{\mathbb{H}^n} P_a(\zeta, u)) \\ &= \lim_{n \rightarrow +\infty} \psi_n *_{\mathbb{R}} F_a(\zeta, u). \end{aligned}$$

pointwise hence also in the distributional sense on S . Each of functions $\psi_n *_{\mathbb{R}} F_a(\zeta, u)$ is annihilated by L and D , so \tilde{H}_a is as well. Therefore by Proposition 7.2

$$(L_B^2 + n^2 \partial_u^2) \tilde{H}(\zeta, u) = 0,$$

but \tilde{H} is independent on u , thus

$$\Delta_\zeta^2 \tilde{H}(\zeta, u) = \Delta_\zeta^2 H(\zeta) = 0,$$

where Δ_ζ^2 is the Laplacian on $\mathbb{C}^n = \mathbb{R}^{2n}$.

Taking the Fourier transform of both sides we obtain

$$|\lambda|^4 \hat{H}(\lambda) = 0,$$

which means

$$\text{supp}\widehat{H} \subset \{0\}.$$

Boundedness of H forces it to be constant. □

Proof of Proposition 7.3. — Let ϕ be a function defined by (7.8). We select the sequence $\{k_n\}$ such that the condition (7.9) was satisfied. Take η_n to be

$$\eta_n(x) = k_n\phi(k_nx) - \frac{1}{k_n}\phi\left(\frac{x}{k_n}\right).$$

It is easy to check that this is the required sequence of functions. □

Thus we may assume that for a positive ε, M :

$$\text{supp}f(\zeta, \widehat{\lambda}) \subset \mathbb{C}^n \times \{\lambda : \varepsilon < |\lambda| < M\}.$$

Let $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R})$ be defined as follows:

$$\begin{aligned} \widehat{\phi}_1(\lambda) &= \begin{cases} 1 & \text{for } \lambda \in \langle \varepsilon, M \rangle, \\ 0 & \text{for } \lambda \notin \langle \frac{\varepsilon}{2}, 2M \rangle, \end{cases} \\ \widehat{\phi}_2(\lambda) &= \widehat{\phi}_1(-\lambda). \end{aligned}$$

We may decompose functions f and F into the sum of two functions

$$\begin{aligned} f &= f_1 + f_2, \\ F &= F_1 + F_2, \end{aligned}$$

where

$$\begin{aligned} f_i(\zeta, u) &= \phi_i *_{\mathbb{R}} f_{\zeta}(u), \\ F_i(\zeta, u, a) &= f_i *_{\mathbb{H}^n} P_a(\zeta, u), \end{aligned}$$

and we see that

$$\begin{aligned} \widehat{f}_1 |_{\mathbb{R}^+} &= \widehat{f} |_{\mathbb{R}^+}, \\ \widehat{f}_2 |_{\mathbb{R}^-} &= \widehat{f} |_{\mathbb{R}^-}. \end{aligned}$$

Furthermore

$$\widehat{f}_2(-\lambda) = \overline{\widehat{f}_1(\lambda)},$$

which imply

$$F_2 = \overline{F_1}.$$

So to prove Theorem 7.1 it remains to show

THEOREM 7.5. — F_1 is holomorphic.

To prove this theorem we shall need some elementary theory of unitary representations of the Heisenberg group for which we refer to [T].

Let U^λ be the Schrödinger representation of \mathbb{H}^n , ([T], 1.2.1). In the underlying Hilbert space $\mathcal{H}_\lambda = L^2(\mathbb{R}^n)$ we consider the basis consisting of properly scaled Hermite functions ξ_α^λ (1.4.18 and Section 2.1 of [T]). Let

$$(7.10) \quad \phi_{\alpha,\beta}^\lambda(\zeta, u) = (U_{(\zeta,u)}^\lambda \xi_\alpha^\lambda, \xi_\beta^\lambda).$$

Then

$$(7.11) \quad \phi_{\alpha,\beta}^\lambda(\zeta, u) = (2\pi)^{n/2} e^{i\lambda u} \Phi_{\alpha,\beta}(\sqrt{|\lambda|}\zeta),$$

where $\Phi_{\alpha,\beta}$ are the special Hermite functions, ([T], 1.4.19). $\Phi_{\alpha,\beta}$ functions belong to the Schwartz class on \mathbb{C}^n and are eigenfunctions of the sublaplacian L_B :

$$(7.12) \quad L_B \phi_{\alpha,\beta}^\lambda(\zeta, u) = -(2|\alpha| + n)|\lambda| \phi_{\alpha,\beta}^\lambda(\zeta, u).$$

Let

$$(7.13) \quad e_k^\lambda(\zeta, u) = \sum_{|\alpha|=k} \phi_{\alpha,\alpha}^\lambda(\zeta, u),$$

and for $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$:

$$(7.14) \quad \psi_\phi^k(\zeta, u) = \int_{\mathbb{R}} e_k^\lambda(\zeta, u) \phi(\lambda) d\lambda.$$

Then $\psi_\phi^k \in \mathcal{S}(\mathbb{C}^n \times \mathbb{R})$ and by (7.12)

$$(7.15) \quad L_B \psi_\phi^k = -(2k + n) \psi_{\phi'}^k,$$

where $\phi'(\lambda) = |\lambda| \phi(\lambda)$.

We shall need more information about $\Phi_{\alpha,\alpha}$. Let L_k be the k -th Laguerre polynomial, i.e.,

$$L_k(t) e^{-t} = \frac{1}{k!} \partial_t^k (e^{-t} t^k).$$

Given a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\zeta \in \mathbb{C}^n$ let

$$(7.16) \quad L_\alpha(\zeta) = L_{\alpha_1} \left(\frac{1}{2} |\zeta_1|^2 \right) \cdot \dots \cdot L_{\alpha_n} \left(\frac{1}{2} |\zeta_n|^2 \right).$$

Then

$$(7.17) \quad \Phi_{\alpha,\alpha}(\zeta) = (2\pi)^{-n/2} L_{\alpha}(\zeta) e^{-\frac{1}{4}|\zeta|^2},$$

(see [T] 1.4.20).

LEMMA 7.6. — For every $k \neq 0$ and $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$:

$$\int_{\mathbb{H}^n} f_1(\eta, t) \overline{\psi_{\phi}^k(\eta, t)} d\eta dt = 0.$$

Proof. — Let $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$ and $\tilde{\phi}(\lambda) = \lambda^{-2}\phi(\lambda)$. Applying (7.15) and Proposition 7.2 we get

$$\begin{aligned} 0 &= \int_{\mathbb{H}^n} (L_B^2 + n^2 \partial_t^2) f_1(\eta, t) \overline{\psi_{\phi}^k(\eta, t)} d\eta dt \\ &= \int_{\mathbb{H}^n} f_1(\eta, t) \overline{(L_B^2 + n^2 \partial_t^2) \psi_{\phi}^k(\eta, t)} d\eta dt \\ &= ((2k + n)^2 - n^2) \int_{\mathbb{H}^n} f_1(\eta, t) \overline{\psi_{\phi}^k(\eta, t)} d\eta dt \\ &= 4k(k + n) \int_{\mathbb{H}^n} f_1(\eta, t) \overline{\psi_{\phi}^k(\eta, t)} d\eta dt \end{aligned}$$

and the lemma follows. □

Furthermore, f_1 translated by any element $(\zeta, u) \in \mathbb{H}^n$ on the left is the boundary value of F_1 translated on the left by (ζ, u) . Therefore, repeating the above argument we obtain:

$$(7.18) \quad f_1 *_{\mathbb{H}^n} \psi_{\phi}^k(\zeta, u) = \int_{\mathbb{H}^n} f_1((\zeta, u)(\eta, t)) \overline{\psi_{\phi}^k(\eta, t)} d\eta dt = 0,$$

for $k \neq 0$ and $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Let us define

$$(7.19) \quad g(\zeta, u) = \phi_1 *_{\mathbb{R}} P_a(\zeta, u).$$

e_k^λ are joint eigenfunctions of L_B and T , and we can expand g in terms of its spectral projections:

$$(7.20) \quad g(\zeta, u) = (2\pi)^{-n-1} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} g *_{\mathbb{H}^n} e_k^\lambda(\zeta, u) |\lambda|^n d\lambda$$

(see Theorem 2.1.1 [T]), where the above series converges in $L^2(\mathbb{H}^n)$ norm. We are going to prove that

PROPOSITION 7.7. — *The series*

$$(7.21) \quad \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} g *_{\mathbb{H}^n} e_k^\lambda(\zeta, u) |\lambda|^n d\lambda$$

converges in $L^1(\mathbb{H}^n)$.

Proof. — To prove this proposition we will use the following property of the Laguerre functions ([BDH]):

For every $l, p \in \mathbb{N}$ there exist $c = c(l, p)$ and $M = M(l, p)$ such that

$$(7.22) \quad \int_0^\infty (1+t)^l |\partial_t^p L_k(t)|^2 e^{-t} dt \leq ck^M.$$

To estimate $L^1(\mathbb{H}^n)$ norm of

$$\int_{-\infty}^{\infty} g *_{\mathbb{H}^n} e_k^\lambda(\zeta, u) |\lambda|^n d\lambda,$$

let us notice first that

$$\begin{aligned} g *_{\mathbb{H}^n} e_k^\lambda(\zeta, u) &= \phi_1 *_{\mathbb{R}} P_a *_{\mathbb{H}^n} e_k^\lambda(\zeta, u) \\ &= P_a *_{\mathbb{H}^n} (\phi_1 *_{\mathbb{R}} e_k^\lambda)(\zeta, u) \\ &= \widehat{\phi}_1(\lambda) P_a *_{\mathbb{H}^n} e_k^\lambda(\zeta, u). \end{aligned}$$

By (7.12) we get

$$\begin{aligned} &\| \int_{-\infty}^{\infty} g *_{\mathbb{H}^n} e_k^\lambda(\zeta, u) |\lambda|^n d\lambda \|_{L^1(\mathbb{H}^n)} \\ &= \| \int_{-\infty}^{\infty} (P_a *_{\mathbb{H}^n} e_k^\lambda)(\zeta, u) \widehat{\phi}_1(\lambda) |\lambda|^n d\lambda \|_{L^1(\mathbb{H}^n)} \\ &= \frac{1}{(2k+n)^N} \| \int_{-\infty}^{\infty} (P_a *_{\mathbb{H}^n} (L_B^N e_k^\lambda))(\zeta, u) \widehat{\phi}_1(\lambda) |\lambda|^{n-N} d\lambda \|_{L^1(\mathbb{H}^n)} \\ &= \frac{1}{(2k+n)^N} \| \int_{-\infty}^{\infty} (L_B^N P_a *_{\mathbb{H}^n} e_k^\lambda)(\zeta, u) \widehat{\phi}_1(\lambda) |\lambda|^{n-N} d\lambda \|_{L^1(\mathbb{H}^n)} \\ &\leq \frac{c}{(2k+n)^N} \| P_a \|_{L^1(\mathbb{H}^n)} \| \int_{-\infty}^{\infty} \widehat{\phi}_1(\lambda) e_k^\lambda(\zeta, u) |\lambda|^{n-N} d\lambda \|_{L^1(\mathbb{H}^n)}. \end{aligned}$$

For the last inequality we estimated $|L_B^N P_a|$ by P_a pointwise due to Harnack's principle. To obtain the thesis it is enough to show that

$$\|\psi_{\eta_N}^k\|_{L^1(\mathbb{H}^n)} \leq c_N k^m$$

for some m , independent on N , where

$$\eta_N(\lambda) = \widehat{\phi}_1(\lambda)|\lambda|^{n-N}.$$

According to (7.11) and (7.13) we can write

$$\psi_{\eta_N}^k(\zeta, u) = \sum_{|\alpha|=k} (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}} e^{i\lambda u} \eta_N(\lambda) \Phi_{\alpha, \alpha}(\sqrt{\lambda} \zeta) d\lambda.$$

Hence the Schwartz inequality, (7.17) and (7.22) yield

$$\begin{aligned} \int_{\mathbb{H}^n} |\psi_{\eta_N}^k(\zeta, u)| du d\zeta &\leq c \int_{\mathbb{H}^n} (1 + u^2) |\psi_{\eta_N}^k(\zeta, u)|^2 du d\zeta \\ &\leq c \sum_{|\alpha|=k} \left(\int_{\mathbb{C}^n} \int_{\mathbb{R}} |\partial_\lambda(\eta_N(\lambda) \Phi_{\alpha, \alpha}(\sqrt{\lambda} \zeta))|^2 d\zeta d\lambda \right. \\ &\qquad \qquad \qquad \left. + \int_{\mathbb{C}^n} \int_{\mathbb{R}} |\eta_N(\lambda) \Phi_{\alpha, \alpha}(\sqrt{\lambda} \zeta)|^2 d\zeta d\lambda \right) \\ &\leq c_N \sum_{|\alpha|=k} \int_{\mathbb{C}^n \times [\varepsilon, M]} \left(|\partial_\lambda(\Phi_{\alpha, \alpha}(\sqrt{\lambda} \zeta))|^2 + |\Phi_{\alpha, \alpha}(\sqrt{\lambda} \zeta)|^2 \right) d\zeta d\lambda \leq c_N k^m. \end{aligned}$$

Now taking N large enough we obtain (7.21). □

LEMMA 7.8. — For every $a \in \mathbb{R}^+$, the function $\mathbb{H}^n \ni (\zeta, u) \mapsto P_a(\zeta, u)$ is radial.

Proof. — For $\rho \in U(\mathbb{C}^n)$ define

$$\widetilde{P}(\zeta, u) = P(\rho\zeta, u),$$

and

$$\widetilde{P}_a(\zeta, u) = a^{-1} \widetilde{P}(\delta_{a^{-1}}(\zeta), u).$$

Then

$$\widetilde{P}_a(\zeta, u) = P_a(\rho\zeta, u).$$

We claim that \tilde{P}_a is annihilated by the operator $L + D$. For this we apply the fact that L_B and ∂_u commute with the action of $U(n)$ on \mathbb{H}^n :

$$\begin{aligned} (aL_B + a^2\partial_u^2)\tilde{P}_a(\zeta, u) &= (aL_B + a^2\partial_u^2)P_a(\rho\zeta, u) \\ &= (an\partial_a - 2a^2\partial_a^2)P_a(\rho\zeta, u) \\ &= (an\partial_a - 2a^2\partial_a^2)\tilde{P}_a(\zeta, u). \end{aligned}$$

Moreover, \tilde{P}_a is an approximate identity. These two properties uniquely define the Poisson kernel ([DH], Corollary 3.13). Hence

$$P_a(\zeta, u) = \tilde{P}_a(\zeta, u). \quad \square$$

Since e_k^λ are radial as well, we get

COROLLARY 7.9. — For every $\lambda \neq 0$, $k \in \mathbb{N}$:

$$e_k^\lambda * P_a(\zeta, u) = P_a * e_k^\lambda(\zeta, u).$$

Now we are ready to expand F_1 . Using (7.20), Proposition 7.7 and the above corollary we have:

$$\begin{aligned} F_1(\zeta, u, a) &= f_1 *_{\mathbb{H}^n} (\phi_1 *_{\mathbb{R}} P_a)(\zeta, u) \\ &= f_1 *_{\mathbb{H}^n} \left((2\pi)^{-n-1} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \phi_1 *_{\mathbb{R}} P_a *_{\mathbb{H}^n} e_k^\lambda |\lambda|^n d\lambda \right) (\zeta, u) \\ &= (2\pi)^{-n-1} \sum_{k=0}^{\infty} \int_{\mathbb{R}} f_1 *_{\mathbb{H}^n} (P_a *_{\mathbb{H}^n} \phi_1 *_{\mathbb{R}} e_k^\lambda) (\zeta, u) |\lambda|^n d\lambda \\ &= (2\pi)^{-n-1} \sum_{k=0}^{\infty} \int_{\mathbb{R}} f_1 *_{\mathbb{H}^n} e_k^\lambda *_{\mathbb{H}^n} P_a(\zeta, u) \hat{\phi}_1(\lambda) |\lambda|^n d\lambda \\ &= (2\pi)^{-n-1} \sum_{k=0}^{\infty} f_1 * \psi_\eta^k * P_a(\zeta, u), \end{aligned}$$

where

$$\eta(\lambda) = \hat{\phi}_1(\lambda) |\lambda|^n.$$

Hence by (7.18)

$$F_1(\zeta, u, a) = (2\pi)^{-n-1} f_1 *_{\mathbb{H}^n} P_a *_{\mathbb{H}^n} \psi_\eta^0(\zeta, u).$$

On the other hand, by [DHMP] (Lemma 2.37),

$$U_{P_a}^\lambda \xi_0^\lambda = e^{-\lambda a} \xi_0^\lambda.$$

Therefore,

$$\begin{aligned} P_a * \psi_\eta^0(\zeta, u) &= \int_{\mathbb{H}^n} P_a(\eta, t) \left(\int_{\mathbb{R}} e_0^\lambda((\eta, t)^{-1}(\zeta, u)) \widehat{\phi}_1(\lambda) |\lambda|^n d\lambda \right) d\eta dt \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{H}^n} P_a(\eta, t) \phi_{0,0}^\lambda((\eta, t)^{-1}(\zeta, u)) d\eta dt \right) \widehat{\phi}_1(\lambda) |\lambda|^n d\lambda \\ &= \int_{\mathbb{R}} \left(U_{(\zeta, u)}^\lambda \xi_0^\lambda, U_{P_a}^\lambda \xi_0^\lambda \right) \widehat{\phi}_1(\lambda) |\lambda|^n d\lambda \\ &= \int_{\mathbb{R}} e^{-\lambda a} \phi_{0,0}^\lambda(\zeta, u) \widehat{\phi}_1(\lambda) |\lambda|^n d\lambda. \end{aligned}$$

So it suffices to prove that the function

$$G(\zeta, u, a) = e^{-\lambda a} \phi_{0,0}^\lambda(\zeta, u)$$

is holomorphic. We have

$$G(\zeta, u, a) = (2\pi)^{-n/2} e^{-\lambda a} e^{i\lambda x} e^{-\frac{1}{4} \lambda |\zeta|^2} = (2\pi)^{-n/2} e^{i\lambda(x+ia+\frac{1}{4}i|\zeta|^2)} = e^{i\lambda z}$$

which proves Theorem 7.5.

8. Appendix.

In [DHP] and [BBDHPT] the authors introduced a geometric definition of the Hua system. The definition was written in terms of the curvature tensor R of the domain, which allows to generalize Hua operators to any Kählerian manifold. On Hermitian symmetric spaces both definitions are equivalent, which is an consequence of Theorem 4.2, page 180 in [H1]. Explicitly, this theorem says that for $X, Y \in \mathfrak{p}$, at the base point o ,

$$R(X, Y) = -\text{ad}[X, Y]|_{\mathfrak{p}}.$$

Therefore, one can apply the adjoint representation of \mathfrak{k} on \mathfrak{p} to the Hua system \mathcal{H} , to express it using the curvature tensor. Finally, if we use non-degeneracy of the adjoint representation and the invariance of the system under the group of biholomorphic transformations of the domain, we get the formula from previous papers.

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