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Peter PFLUG & Włodzimierz ZWONEK

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THE SERRE PROBLEM WITH REINHARDT FIBERS

by Peter PFLUG & Włodzimierz ZWONEK

1. Introduction and the main result.

Our aim is to discuss the Serre problem, i.e., the problem whether a holomorphic fiber bundle $\pi : E \mapsto B$ with a Stein base B and a Stein fiber F is Stein. For a comprehensive list of positive partial results to this problem see e.g. [Siu].

In our paper we consider this problem under the additional assumption that the fiber F is a pseudoconvex hyperbolic Reinhardt domain in \mathbb{C}^2 . Note that the first example showing that the answer to the Serre problem is in general negative were constructed for Reinhardt fibers (see [Sko], [Dem], and [Loeb]). Also first counterexamples with bounded domains as fibers were found in the class of pseudoconvex Reinhardt domains (see [Coe-Loeb]).

We are interested in the problem, which bounded pseudoconvex Reinhardt domains as fibers guarantee that the holomorphic fiber bundle with Stein basis is Stein, in other words for which bounded pseudoconvex Reinhardt domains the answer to the Serre problem is positive.

Since in the class of pseudoconvex Reinhardt domains hyperbolicity (in the sense of Carathéodory, Kobayashi or Brody) is equivalent to the

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boundedness of domains (see [Zwo1]), it is natural that instead of bounded we study the class of pseudoconvex hyperbolic Reinhardt domains.

Let us denote the class of Stein domains D for which the answer to the Serre problem (with the fiber equal to D) is positive by \mathfrak{S} .

Now we may formulate our main theorem, which gives the characterization of hyperbolic pseudoconvex Reinhardt domains in \mathbb{C}^2 belonging to \mathfrak{S} .

THEOREM 1. — *Let D be a pseudoconvex hyperbolic Reinhardt domain in \mathbb{C}^2 . Then $D \notin \mathfrak{S}$ if and only if D is algebraically equivalent to a Reinhardt domain $\tilde{D} \subset \mathbb{C}_*^2$ for which there is a matrix $A \in \mathbb{Z}^{2 \times 2}$ with the eigenvalues λ and $\frac{1}{\lambda}$, where $\lambda > 1$, such that*

$\log \tilde{D} = \{tv + sw : s > \varphi(t), t > 0\}$ (or $\log \tilde{D} = \{tv + sw : s > \varphi(t), t < 0\}$), where $v, w \in \mathbb{R}^2$ are eigenvectors corresponding to the eigenvalues λ and $\frac{1}{\lambda}$ and $\varphi : (0, \infty) \mapsto [0, \infty)$ (respectively, $\varphi : (-\infty, 0) \mapsto [0, \infty)$) is a convex function satisfying the equality $\varphi(t\lambda) = \frac{1}{\lambda}\varphi(t)$, $t \in (0, \infty)$ (respectively, $t \in (-\infty, 0)$).

Note that $D \subset \mathbb{C}_*^2$.

Recall that the first known example of a bounded domain not belonging to \mathfrak{S} was a domain from the class considered in Theorem 1. More precisely, it was a domain associated to $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\varphi \equiv 0$ (defined on $(0, \infty)$) – see [Coe-Loeb]. Later, D. Zaffran in [Zaf] delivered other domains not from \mathfrak{S} of the same type. Namely, he considered domains associated to so-called ‘even Dloussky matrices’ i.e. $A = \begin{bmatrix} 0 & 1 \\ k_1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ k_{2s} & 1 \end{bmatrix}$, $s, k_j \in \mathbb{N} \setminus \{0\}$, and with $\varphi \equiv 0$. Those are closely related to surfaces of Kodaira of class VII_0 .

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2. The structure of the proof of Theorem 1.

In our considerations the key role in the proofs of positive results (i.e. the facts that domains are from the class \mathfrak{S}) will be played by the criterion of Stehlé, which we formulate in the form that we shall use in our paper.

THEOREM 2 (see [Ste] and [Mok]). — *Let D be a domain in \mathbb{C}^n . If there exists a real-valued plurisubharmonic exhaustion function u on D*

such that for any $F \in \text{Aut } D$ the function $u \circ F - u$ is bounded from above on D , then $D \in \mathfrak{S}$.

$\text{Aut } D$ denotes the group of holomorphic automorphisms of D .

Observe that Theorem 2 gives $D \in \mathfrak{S}$ if D is pseudoconvex and $\text{Aut } D$ is compact (see also [Kön] and [Sib]). Therefore, in the proof below we shall be interested only in the case when the group $\text{Aut } D$ is not compact.

The proof of Theorem 1 will be divided into three different cases, depending on the number of axis of \mathbb{C}^2 which intersect the domain D .

Formally, for a pseudoconvex Reinhardt domain $D \subset \mathbb{C}^n$ we define

$$t := t(D) := \#\{j \in \{1, \dots, n\} : D \cap V_j \neq \emptyset\},$$

where $V_j := \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_j = 0\}$, $j = 1, \dots, n$.

The three different cases we shall deal with in the proof of Theorem 1 correspond to the three possible values of t (recall that $n = 2$):

If $t = 2$ (equivalently, $0 \in D$) the result will simply follow from the well-known sufficiency results for a domain to belong to \mathfrak{S} . In fact this case has already been done in [Kön].

In the cases $t = 1$ (these domains will always belong to \mathfrak{S}) and $t = 0$ we shall concentrate on the structure of $\text{Aut } D$. In the case when $t = 1$ there will only be three classes of model domains for which the group is not compact (the result will follow from [Shi]). Two of the classes will be relatively simple to deal with and the third class will consist of one special domain for which we shall use the Stehlé criterion together with Theorem 6.

In the case $t = 0$ we shall use the result of [Shi] to see that $\text{Aut } D$ coincides with the group of algebraic automorphisms, $\text{Aut}_{\text{alg}} D$. Studying the geometric structure of D we shall see that there are two classes of domains admitting non-compact automorphism groups. Because of the geometry of the logarithmic image we call these two classes ‘parabolic’ and ‘hyperbolic’. In the hyperbolic case, which will deliver us a negative answer to the Serre problem, we shall construct a counterexample proceeding as in [Co-Loeb]. On the other hand the parabolic case will be done similarly as the special case of the domain in the case $t = 1$.

3. Results needed for the proof of Theorem 1.

Before we go on to the proof of Theorem 1 let us recall the description of pseudoconvex hyperbolic Reinhardt domains, some results and notions

related to this class of domains, and the structure of automorphism groups of such domains.

Recall that for a pseudoconvex Reinhardt domain in \mathbb{C}^n the logarithmic image of D :

$$\log D := \{(x_1, \dots, x_n) \in \mathbb{R}^n : (e^{x_1}, \dots, e^{x_n}) \in D\}$$

is convex.

We shall use the notion of an *algebraic map*, i.e. a mapping of the following form:

$$\Phi(z) = (b_1 z^{A^1}, \dots, b_n z^{A^n}),$$

where $A = \begin{bmatrix} A^1 \\ \vdots \\ A^n \end{bmatrix} \in \mathbb{Z}^{n \times n}$ and $b_1, \dots, b_n \in \mathbb{C}_*$ – here for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^\times$ we denote $z^\alpha := z_1^{\alpha_1} \dots z_n^{\alpha_n}$ for $z \in \mathbb{C}^n$ such that if $\alpha_j < 0$ then $z_j \neq 0$.

THEOREM 3 (see [Zwo1]). — *Let D be a pseudoconvex Reinhardt domain in \mathbb{C}^n . Then the following conditions are equivalent:*

- D is (Kobayashi, Carathéodory or Brody) hyperbolic,
- D is algebraically equivalent to a bounded domain,
- $\log D$ contains no straight lines and $D \cap V_j$ is empty or hyperbolic (as a domain in \mathbb{C}^{n-1}), $j = 1, \dots, n$.

Remark. — Observe that the number t remains fixed under algebraic biholomorphism.

In view of Theorem 3 we see that for a pseudoconvex Reinhardt domain $D \subset \mathbb{C}_*^n$:

- (1) D is hyperbolic if and only if $\log D$ contains no straight lines.

If D is a pseudoconvex Reinhardt domain in \mathbb{C}^n , then any element $\Phi \in \text{Aut}_{\text{alg}} D$ must be of the following form:

$$\Phi(z) = (b_1 z^{A^1}, \dots, b_n z^{A^n}), \quad z \in D,$$

where $A \in \mathbb{Z}^{n \times n}$, $|\det A| = 1$, and $b_1, \dots, b_n \in \mathbb{C}_*$ (see [Zwo1]).

Consequently, the mapping $\tilde{\Phi}(x) := Ax + \tilde{b}$, $x \in \log D$, $\tilde{b} := (\log |b_1|, \dots, \log |b_n|)$, is an affine isomorphism of $\log D$.

We may easily verify (from Theorem 3 and Cartan's Theorem on limits of sequences of automorphisms, see e.g. [Nar], Chapter 5, Theorem 4) that for a pseudoconvex hyperbolic Reinhardt domain D the group $\text{Aut}_{\text{alg}} D$ is not compact if and only if

- (2) there is a sequence $(\tilde{\Phi}_n)$ ($\tilde{\Phi}_n$ corresponds to $\Phi_n \in \text{Aut}_{\text{alg}} D$) such that for some (equivalently, any) $x \in \log D$ we have $\|\tilde{\Phi}_n(x)\| \rightarrow \infty$ as $n \rightarrow \infty$.

Following the notation in [Zwo2], for a pseudoconvex Reinhardt domain $D \subset \mathbb{C}^n$ and $a \in \log D$ (chosen arbitrarily) we denote

$$\mathfrak{C}(D) := \{v \in \mathbb{R}^n : a + \mathbb{R}_+ v \subset \log D\}.$$

Recall that $\mathfrak{C}(D)$ is a closed convex cone with the origin at 0, independent of a . It is easy to verify that for any $\Phi \in \text{Aut}_{\text{alg}} D$, where D is a pseudoconvex Reinhardt domain in \mathbb{C}^n ,

$$(3) \quad A(\mathfrak{C}(D)) = \mathfrak{C}(D),$$

where A denotes the matrix associated to Φ .

Remark. — Consider a pseudoconvex hyperbolic Reinhardt domain D in \mathbb{C}^2 with $t = 1$. We claim that in this case $\text{Aut}_{\text{alg}} D$ is compact. Actually, take any $\Phi \in \text{Aut}_{\text{alg}} D$. Without loss of generality $D \cap V_1 = \emptyset$, $D \cap V_2 \neq \emptyset$. Then one may easily verify from the description of $\text{Aut}_{\text{alg}} D$ that $\Phi(z) = (b_1 z_1^{\pm 1}, b_2 z_1^{\alpha_1} z_2)$, $z \in D$, for some $\alpha_1 \in \mathbb{Z}$ and $\Phi(\cdot, 0)$ is a biholomorphism of $D \cap V_2$ (as a subdomain of \mathbb{C}), from which we conclude that $\text{Aut}_{\text{alg}} D$ is compact.

The problem of characterization of automorphism groups of Reinhardt domains was studied in [Shi] and in [Kru], see also [Car]; for Reinhardt domains with smooth boundary see also [Isa-Kra] and papers quoted there. The results obtained there together with the above remarks lead us to the following description of pseudoconvex hyperbolic Reinhardt domains in \mathbb{C}^2 with $t = 1$ and non-compact automorphism groups.

THEOREM 4 (see [Shi], Theorem 5, and [Kru]). — *Let D be a pseudoconvex hyperbolic Reinhardt domain in \mathbb{C}^2 with $t = 1$. Then $\text{Aut } D$ is not compact if and only if D is algebraically equivalent to one of the domains:*

$$(4) \quad \{z_1 \in \mathbb{C} : |z_1| < 1\} \times \{z_2 \in \mathbb{C} : r < |z_2| < 1\} =: \Delta \times P(r, 1), \text{ where } 0 \leq r < 1,$$

$$(5) \quad \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, 0 < |z_2| < (1 - |z_1|^2)^{p/2}\} \text{ for some } p > 0,$$

$$(6) \quad \{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_2| < \exp(-|z_1|^2)\}.$$

Moreover, when D is as in (4), then the group of automorphisms consists of the mappings of the form

$$D \ni (z_1, z_2) \mapsto (a(z_1), b(z_2)) \in D,$$

where a is an automorphism of Δ and b is the automorphism of $P(r, 1)$.

When D is as in (5), then the automorphism group consists of the mappings of the form

$$D \ni (z_1, z_2) \mapsto \left(\alpha \frac{z_1 - \beta}{1 - \bar{\beta}z_1}, \gamma \frac{(1 - |\beta|^2)^{\frac{p}{2}}}{(1 - \beta z_1)^p} z_2 \right) \in D,$$

where $|\alpha| = |\gamma| = 1$, $|\beta| < 1$.

When D is as in (6), then the automorphism group consists of the mappings of the form

$$D \ni (z_1, z_2) \mapsto (\alpha z_1 + \beta, \gamma \exp(-2\alpha\bar{\beta}z_1 - |\beta|^2)z_2) \in D,$$

where $|\alpha| = |\gamma| = 1$, $\beta \in \mathbb{C}$.

Let us formulate one more auxiliary result. In the proof of Lemma 6 (and later in the proof of Theorem 8) the important role will be played once more by a result of S. Shimizu (which is combined below with Theorem 3).

THEOREM 5 (see [Shi]). — *Let D be a pseudoconvex hyperbolic Reinhardt domain in \mathbb{C}_*^n . Then $\text{Aut } D = \text{Aut}_{\text{alg}} D$.*

LEMMA 6. — *Let \tilde{D} be a pseudoconvex Reinhardt domain in \mathbb{C}^2 such that*

$$(7) \quad \tilde{D} \subset \mathbb{C} \times \Delta, \quad \mathbb{C}_* \times \{0\} \subset \tilde{D} \text{ and } (1, 0) \notin \mathfrak{C}(\tilde{D}).$$

Then for any $\Phi \in \text{Aut } \tilde{D}$:

$$\Phi(\tilde{D} \cap (\mathbb{C} \times \{0\})) = \tilde{D} \cap (\mathbb{C} \times \{0\}).$$

Put $D := \tilde{D} \setminus (\mathbb{C} \times \{0\})$. Assume additionally that

$$(8) \quad (0, 0) \in \tilde{D} \text{ or } \mathfrak{C}(D) = \mathbb{R}_+(0, -1).$$

Then $\text{Aut } D = \text{Aut } \tilde{D}|_D$.

Proof of Lemma 6. — To prove the first part of the lemma it suffices to show only the inclusion ‘ \subset ’. Denote $\Phi := (\Phi_1, \Phi_2)$. Since

$$\mathbb{C}_* \ni z_1 \mapsto \Phi_2(z_1, 0) \in \Delta,$$

$\Phi_2(\tilde{D} \cap (\mathbb{C} \times \{0\})) = \{z_2^0\}$ for some $|z_2^0| < 1$. It remains to show that $z_2^0 = 0$. Suppose the contrary. Then the fact that $(1, 0) \notin \mathfrak{C}(D)$ implies that the well-defined holomorphic function $\mathbb{C}_* \ni z_1 \mapsto \Phi_1(z_1, 0) \in \mathbb{C}$ is bounded, so constant. Therefore, Φ is constant on $\tilde{D} \cap (\mathbb{C} \times \{0\})$ – contradiction.

Assume now additionally (8). It follows from the first part of the lemma that $(\text{Aut } \tilde{D})|_D \subset \text{Aut } D$.

Assume for a while that each $\Phi \in \text{Aut } D$ extends holomorphically onto \tilde{D} . We claim that such an extension maps \tilde{D} to \tilde{D} . In fact, let $\Phi \in \text{Aut } D$ and let $\tilde{\Phi}$ denote the extension of Φ to \tilde{D} , $\tilde{\Phi} : \tilde{D} \mapsto \tilde{D}$. Our aim is to show that $\tilde{\Phi}(\tilde{D}) \subset \tilde{D}$. In the case $(0, 0) \in \tilde{D}$ the existence of plurisubharmonic peak functions for $\partial\tilde{D}$ together with the maximum principle for subharmonic functions easily shows that $\partial\tilde{D} \cap \tilde{\Phi}(\tilde{D}) = \emptyset$, which finishes the proof in this case. So assume that $\mathcal{C}(D) = \mathbb{R}_+(0, -1)$. Then $(0, 0) \in \partial\tilde{D}$. Similarly as in the previous case (use the plurisubharmonic peak functions and the maximum principle for subharmonic functions) we see that $\tilde{\Phi}(\tilde{D}) \cap (\partial\tilde{D} \setminus \{(0, 0)\}) = \emptyset$. Suppose that $(0, 0) \in \tilde{\Phi}(\tilde{D})$. Then certainly $(0, 0) = \tilde{\Phi}(z_1^0, 0)$ for some $z_1^0 \in \mathbb{C}_*$. Let $z_2^0 \in \mathbb{C}_*$ be such that $(z_1^0, z_2^0) \in D$. Since $\Phi \in \text{Aut } D = \text{Aut}_{\text{alg}} D$ (use Theorem 5) and because of the equality $A(\mathcal{C}(D)) = \mathcal{C}(D)$, where A is the matrix corresponding to Φ , we get that $\tilde{\Phi}(z_1^0, |z_2^0|\Delta) = (\tilde{z}_1^0, |\tilde{z}_2^0|\Delta)$ for some $(\tilde{z}_1^0, \tilde{z}_2^0) \in D$, which contradicts the continuity of $\tilde{\Phi}$ and the equality $\tilde{\Phi}(z_1^0, 0) = (0, 0)$.

Therefore, to prove the other inclusion $(\text{Aut } \tilde{D})|_D \supset \text{Aut } D$ we have to show that each $\Phi \in \text{Aut } D$ extends holomorphically onto \tilde{D} . Let $\Phi \in \text{Aut } D$. Note that Φ_2 is bounded, so it extends holomorphically onto \tilde{D} . Therefore, we may expand Φ_2 into the Hartogs-Taylor series in \tilde{D} :

$$\Phi_2(z_1, z_2) = z_2^{j_0} \sum_{j \geq j_0} c_j(z_1) z_2^{j-j_0},$$

where $j_0 \geq 0$ and $c_{j_0} \neq 0$.

Write the Hartogs-Laurent expansion of Φ_1 in D :

$$\Phi_1(z_1, z_2) = \sum_{j \in \mathbb{Z}} d_j(z_1) z_2^j.$$

Since Φ_1 is not constant, there is an $s \in \mathbb{Z}$ such that $d_s \neq 0$. Note that there is a $j \in \mathbb{Z}$ such that $d_k \equiv 0$ for any $k < j$. Actually, otherwise the function $\Phi_1 \cdot \Phi_2$ would be unbounded on D , which would contradict (8). Let j_1 denote the smallest j satisfying this property. To finish the proof it is sufficient to show that $j_1 \geq 0$. Suppose the contrary. Then there is a $k \in \mathbb{N}$ such that $kj_1 + j_0 < 0$. But this implies that the function $\Phi_1^k \cdot \Phi_2$ is unbounded on D , which contradicts (8). □

Let us formulate a result we shall need in the proof of Theorem 1.

THEOREM 7. — *Let \tilde{D} be a pseudoconvex domain in \mathbb{C}^n and let M be a pure one-codimensional analytic subset of \tilde{D} . Put $D := \tilde{D} \setminus M$.*

Assume, additionally, that $\text{Aut } \tilde{D}|_D = \text{Aut } D$. Then the fact that $\tilde{D} \in \mathfrak{S}$ implies that $D \in \mathfrak{S}$.

Proof of Theorem 7. — Since $\text{Aut } \tilde{D}|_D = \text{Aut } D$, we may replace the typical fiber D of E by \tilde{D} and obtain a new bundle \tilde{E} such that $\tilde{E} \setminus E$ is a hypersurface. □

Now, as preparations for the case $t = 0$ we shall show some auxiliary results. More precisely, we characterize all pseudoconvex hyperbolic Reinhardt domains D in \mathbb{C}_*^2 with non-compact automorphism groups.

Additionally, for our future needs we give some necessary conditions on the form of automorphisms in one of the cases.

THEOREM 8. — *Let D be a hyperbolic pseudoconvex Reinhardt domain in \mathbb{C}_*^2 (i.e. $t = 0$). Then $\text{Aut}(D)$ is not compact if and only if D is algebraically biholomorphic to a Reinhardt domain \tilde{D} in \mathbb{C}_*^2 of one of the following two types:*

(11) *there are a matrix $A \in \mathbb{Z}^{2 \times 2}$ and a number $\beta_2 \neq 0$ such that*

$$\log \tilde{D} = \{tv + sw : t \in \mathbb{R}, s > \psi(t)\} \text{ (if } \beta_2 > 0 \text{) or}$$

$$\log \tilde{D} = \{tv + sw : t \in \mathbb{R}, s < \psi(t)\} \text{ (if } \beta_2 < 0 \text{)}$$

and 1 is the only eigenvalue of A with the eigenvector w (so $Aw = w$), $Av = v + w$ for some $v \in \mathbb{R}^2$, and $\psi : \mathbb{R} \mapsto \mathbb{R}$ is a convex (or concave in the second case) function satisfying the property $\psi(t + \beta_2) = t + \psi(t)$, $t \in \mathbb{R}$;

(12)

there is a matrix $A \in \mathbb{Z}^{2 \times 2}$ with the eigenvalues λ and $\frac{1}{\lambda}$, $\lambda > 1$, such that

$$\log \tilde{D} = \{tv + sw : s > \varphi(t), t > 0\} \text{ or } \log \tilde{D} = \{tv + sw : s > \varphi(t), t < 0\},$$

where $v, w \in \mathbb{R}^2$ are eigenvectors corresponding to the eigenvalues λ and $\frac{1}{\lambda}$ and $\varphi : (0, \infty) \mapsto [0, \infty)$ (or $\varphi : (-\infty, 0) \mapsto [0, \infty)$) is a convex function satisfying the equality $\varphi(t\lambda) = \frac{1}{\lambda}\varphi(t)$, $t \in (0, \infty)$ (or $t \in (-\infty, 0)$).

Moreover, in the case (11) each automorphism $\tilde{\Phi}$ must be such that

$$\tilde{\Phi}(x) = \tilde{A}x + \tilde{b},$$

where $\tilde{A} \in \mathbb{Z}^{2 \times 2}$, $|\det \tilde{A}| = 1$, $\tilde{A}w = w$ and one of three possibilities holds:

– $\tilde{A} = \mathbb{I}_2, \tilde{b} = 0$;

– there is some $\tilde{v} \in \mathbb{R}^2$ such that $\tilde{A}\tilde{v} = w + \tilde{v}$,

– the number -1 is the second eigenvalue of \tilde{A} with the corresponding eigenvector equal to \tilde{v} .

Additionally, in all cases, if we denote $x = tv + sw$ and $\tilde{\Phi}(x) = \tilde{t}v + \tilde{s}w$ then $\tilde{s} - \psi(\tilde{t}) = s - \psi(t)$, $t, \tilde{t}, s, \tilde{s} \in \mathbb{R}$.

Remark 9. — In fact the domains representing two different cases in (11) are actually algebraically equivalent (use the biholomorphism $z^{-\mathbb{I}_2}$, where \mathbb{I}_2 denotes the unit matrix).

The examples of functions φ from (12) are the functions defined as follows $\varphi(t) := \frac{a}{t}$, $t > 0$ (or $t < 0$), where a is some fixed number, $a \geq 0$ (or $a \leq 0$).

The examples of functions ψ from (11) are the functions defined as follows $\psi(t) := \frac{t(t-\beta_2)}{2\beta_2}$, $t \in \mathbb{R}$.

One of the examples of matrices satisfying (12) has already been given in the remarks after Theorem 1. More generally, the examples of matrices satisfying (12) may be of the following form $A = \begin{bmatrix} k & 1 \\ k-1 & 1 \end{bmatrix}$, $k \in \mathbb{N}$, $k \geq 2$ or even Dloussky matrices as defined in [Zaf].

Let us note that as an example of a matrix A satisfying (11) we may take the matrix $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, where $k \in \mathbb{Z} \setminus \{0\}$. Then $w = (0, 1)$, $v = (\frac{1}{k}, 0)$.

Because of the geometry of $\log D$ let us call the domains satisfying (11) of ‘parabolic’ type and those satisfying (12) of ‘hyperbolic’ type.

Now let us go to the proof of Theorem 8.

Proof of Theorem 8. — Assume that $\text{Aut } D$ is not compact. Then in view of (1), (2) and (3) $\mathfrak{C}(D)$ equals

$$(13) \quad \mathbb{R}_+ v', v' \neq 0 \text{ or } \mathbb{R}_+ v' + \mathbb{R}_+ w', \text{ where } v', w' \text{ are linearly independent.}$$

Let $\Phi \in \text{Aut } D = \text{Aut}_{\text{alg}} D$. Denote the corresponding mapping $\tilde{\Phi}(x) = Ax + \tilde{b}$, $x \in \mathbb{R}^2$.

First we claim that A must have a positive eigenvalue. We consider two possibilities as given in (13). Consider the first case $\mathfrak{C}(D) = \mathbb{R}_+ v'$. Then from the invariance $A(\mathfrak{C}(D)) = \mathfrak{C}(D)$ we easily get that v' is an eigenvector with the positive eigenvalue. So assume that $\mathfrak{C}(D) = \mathbb{R}_+ v' + \mathbb{R}_+ w'$. We use once more the equality $A(\mathfrak{C}(D)) = \mathfrak{C}(D)$ to see that two cases have to be discussed, namely:

$$\begin{aligned} A(\mathbb{R}_{>0} v') &= \mathbb{R}_{>0} v', \quad A(\mathbb{R}_{>0} w') = \mathbb{R}_{>0} w' \text{ or} \\ A(\mathbb{R}_{>0} v') &= \mathbb{R}_{>0} w', \quad A(\mathbb{R}_{>0} w') = \mathbb{R}_{>0} v'. \end{aligned}$$

Note that in the first case we are done. In the second one using a continuity argument we easily find the existence of a $u \in \mathbb{R}_{>0}v' + \mathbb{R}_{>0}w'$ such that $A(\mathbb{R}_{>0}u) = \mathbb{R}_{>0}$, which also finishes the proof of our claim.

Remark that if $A = \mathbb{I}_2$ then, because of (1), $\tilde{b} = 0$.

Note also that if A has a negative eigenvalue different from -1 then taking instead of Φ the automorphism Φ^2 we see that $\text{Aut } D$ has an element with the associated matrix A having two positive eigenvalues, both different from 1. Therefore, we see that if $\text{Aut } D$ consists of more elements than those associated to $A = \mathbb{I}_2$ (and then automatically, $\tilde{b} = 0$) then $\text{Aut } D$ must contain an element of one of the following forms:

(14) A has two eigenvalues; one of them equals 1 and the other -1 ,

(15) A has only one eigenvalue equal to 1 and $A \neq \mathbb{I}_2$,

(16) A has two positive eigenvalues λ and $\frac{1}{\lambda}$, $\lambda > 1$.

Since a subgroup of affine transformations of \mathbb{R}^2 which stabilizes a proper convex cone and is generated by elements whose linear part consist of reflections has at most two elements, we may assume that $\text{Aut}(D)$ contains a transformation whose linear part A satisfies (15) or (16).

Let us make one more remark. If we choose $A \in \mathbb{Z}^{2 \times 2}$ with $|\det A| = 1$ and with all eigenvalues different from 1 then moving the domain, if necessary, we may assume that $\tilde{b} = 0$. In fact, since $\det(A - I) \neq 0$ there is a vector $x_0 \in \mathbb{R}^2$ such that $Ax_0 + \tilde{b} = x_0$. Consequently, for any $x \in \mathbb{R}^2$ the following equalities hold:

$$\Phi(x) = Ax + \tilde{b} = A(x - x_0) + Ax_0 + \tilde{b} = A(x - x_0) + x_0.$$

Therefore, moving the coordinate system, if necessary, we may assume that

$$(17) \quad \tilde{\Phi}(x) = Ax,$$

where A is as above.

In addition to the previous remark note that, when the only eigenvalue is 1 and A is not the identity, then some simplification of the form of $\tilde{\Phi}$ is also possible. Namely, then there are linearly independent vectors v, w such that $Aw = w$ and $Av = v + w$. Then $(A - \mathbb{I}_2)(\mathbb{R}^2) = \mathbb{R}w$. Write any element $x \in \mathbb{R}^2$ in the form $x = \alpha_1w + \alpha_2v$, $\tilde{b} = \beta_1w + \beta_2v$. Then there is an $x_0 \in \mathbb{R}^2$ such that

$$Ax_0 + \tilde{b} = x_0 + \beta_2v.$$

Consequently,

$$\tilde{\Phi}(x) = Ax + \tilde{b} = A(x - x_0) + Ax_0 + \tilde{b} = A(x - x_0) + x_0 + \beta_2 v,$$

which implies that moving the coordinate system, if necessary, we may assume that

$$(18) \quad \tilde{\Phi}(x) = Ax + \beta_2 v.$$

Assume now that there is an automorphism of D such that the associated matrix A satisfies (15). Then we may also assume that $\tilde{\Phi}$ satisfies (18). There are linearly independent vectors v, w such that $Aw = w$ and $Av = w + v$. Note that

$$(19) \quad \tilde{\Phi}^{(k)}(x) = x + k(\alpha_2 w + \beta_2 v) + \frac{k(k-1)}{2} \beta_2 w, \quad x = \alpha_1 w + \alpha_2 v, \quad k \in \mathbb{Z}.$$

Now (1), (19), and the convexity of $\log D$ imply that $\beta_2 \neq 0$ and for any $t \in \mathbb{R}$ there is (exactly one) $s := s(t) \in \mathbb{R}$ such that $sw + tv \in \partial \log D$. Moreover, the convexity of $\log D$ together with (19) implies that if $\beta_2 > 0$ then $sw + tv \in \log D$ for any $s > s(t)$ and if $\beta_2 < 0$ then $sw + tv \in \log D$ for any $s < s(t)$. Denote $\psi(t) := s(t)$. Then because of the equality $\tilde{\Phi}(\partial(\log D)) = \partial(\log D)$, the property (19) (applied for $k = 1$), and the convexity of $\log D$, we get the convexity (or concavity) of the function ψ and the property $\psi(t + \beta_2) = \psi(t) + t$, $t \in \mathbb{R}$, which gives us the form as in (11). Note also that the domain as in (11) has a non-compact automorphism group. In fact, note that $\tilde{\Phi}(\log D) = \log D$ and $\|\tilde{\Phi}^{(k)}(x)\| \rightarrow \infty$ as $k \rightarrow \infty$, $x \in \log D$, where $\tilde{\Phi}(x) = Ax + \beta_2 v$, $x \in \log D$.

Consider now the case when there is an automorphism Φ of D as in (16). Then because of (17) we may assume that $\tilde{b} = 0$.

Consider any point $x = sw + tv \in \log D$, $t, s \in \mathbb{R}$, where $v, w \in \mathbb{R}^2$ are eigenvectors corresponding to eigenvalues $\lambda, \frac{1}{\lambda}$. Then $Ax = \frac{s}{\lambda} w + t \lambda v$ and, consequently, for any $k \in \mathbb{Z}$:

$$(20) \quad A^k(x) = \frac{s}{\lambda^k} w + t \lambda^k v.$$

Taking $-\log D$ instead of $\log D$, if necessary, (which corresponds to the mapping $z^{-\mathbb{I}_2}$), we may assume that there is a vector $x_0 = s_0 w + t_0 v \in \log D$, where $s_0 > 0$, $t_0 \neq 0$. Assume that $t_0 > 0$ (the case $t_0 < 0$ goes along the same lines). Then it easily follows from (20), the convexity of $\log D$, and (1) that $\log D \subset \{sw + tv : t, s > 0\}$. Now one may easily see from (20) and the convexity of $\log D$ that $\{t > 0 : \text{there is an } s > 0 \text{ such that } tv + sw \in \log D\}$ is an open interval $(0, \infty)$. Moreover, for any $t > 0$ there is exactly one $s(t) \geq 0$ such that $sw + tv \in \log D$ for $s > s(t)$ and

$sw + tv \notin \log D$ for any $s < s(t)$. We define $\varphi(t) := s(t)$. The convexity of φ follows from the convexity of $\log D$. The property $\varphi(t\lambda) = \frac{1}{\lambda}\varphi(\lambda)$ easily follows from the property (20).

If we assume that $\log D$ is of the form as in (12) then it follows from the properties of A that $\|A^k(x)\| \rightarrow \infty$ as $k \rightarrow \infty$, $x \in \log D$, which gives non-compactness of $\text{Aut } D$.

Now let us go to the study of the necessary form of the automorphisms of D in the case (11).

Since the cones $\mathfrak{C}(D)$ in both cases (11) and (12) are not linearly isomorphic we easily conclude from the considerations that led us to the construction of the domain as in (11) that each of the automorphisms must be of one of the forms as in (14) or (15) or its corresponding matrix must be the identity. Therefore, to finish the proof it is sufficient to verify the invariance condition. Note that $\tilde{\Phi}(\partial \log D) = \partial \log D$. Then elementary calculations show that the invariance condition holds for all the possible automorphisms. \square

4. Proof of Theorem 1.

Proof of Theorem 1 for $t = 2$. — In this case $0 \in D$ and D is bounded, so D is Carathéodory complete (see [Pfl]) and, consequently, because of [Hir] $D \in \mathfrak{S}$. As mentioned earlier this case has already been done in [Kön]. \square

Proof of Theorem 1 for $t = 1$. — As earlier announced we consider only cases when $\text{Aut } D$ is not compact. When D is as in (4) then one may easily verify that the following exhausting function:

$$u(z) := \max\{-\log(1 - |z_1|^2), -\log \text{dist}(z_2, \mathbb{C} \setminus P(r, 1))\}$$

satisfies the assumptions of Stehlé's criterion.

When D is as in (5), then one may easily verify that the following exhausting function:

$$\max\left\{\log |z_2| - \frac{p}{2} \log(1 - |z_1|^2), -\log |z_2|\right\}, z \in D$$

satisfies the assumptions of the criterion of Stehlé.

Now assume that D is of the form as in (6).

Denote

$$\tilde{D} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_2| < \exp(-|z_1|^2)\}.$$

Therefore, we get from Lemma 6 (note that $\mathfrak{C}(D) = \mathfrak{C}(\tilde{D}) = \mathbb{R}_+(-1, 0) + \mathbb{R}_+(0, -1)$),

$$(9) \quad \text{Aut } \tilde{D}|_D = \text{Aut } D.$$

Now we prove that $\tilde{D} \in \mathfrak{S}$. Elementary calculations show that for any $\Phi \in \text{Aut } \tilde{D}$:

$$(10) \quad \tilde{u}(\Phi(z)) = \tilde{u}(z), \quad z \in \tilde{D},$$

where $\tilde{u}(z) := \log |z_2| + |z_1|^2, z \in \tilde{D}$.

Define $(\log^+ |\lambda| := \max\{\log |\lambda|, 0\}, \lambda \in \mathbb{C})$,

$$u(z) := \max\{\rho(\tilde{u}(z)), \log^+ |z_1|\}, z \in \tilde{D},$$

where $\rho : [-\infty, 0) \mapsto [0, \infty)$ is a continuous, C^2 -smooth on $(-\infty, 0)$, convex and increasing function such that $\lim_{t \rightarrow 0^-} \rho(t) = \infty$ (e.g. $\rho(t) := \frac{1}{t}, t < 0$). Then it is trivial to see that u is exhausting for \tilde{D} . Calculating the Levi form of $\rho \circ \tilde{u}$ we see that $\rho \circ \tilde{u}$ is plurisubharmonic on D (and consequently, because of the Riemann extension theorem, on \tilde{D}). Moreover, it follows from (10) and the form of $\text{Aut } \tilde{D}$ that for any $\Phi \in \text{Aut } \tilde{D}$ the function $u \circ \Phi - u$ is bounded from above on \tilde{D} , which implies, in view of the criterion of Stehlé, that $\tilde{D} \in \mathfrak{S}$.

Then, because of (9) we may make use of Theorem 7 to see that $D \in \mathfrak{S}$, too. □

Proof of Theorem 1 for $t = 0$. — As noted earlier it suffices to consider only the cases of non-compact $\text{Aut } D$. As proven in Theorem 8 there are two possibilities. We consider the first (hyperbolic) one. We show that if D is such that [12] is satisfied then $D \notin \mathfrak{S}$.

We proceed as in [Coe-Loeb], we even follow the notation from that paper. We define

$$V := \{\zeta_1 v + \zeta_2 w : \zeta_1, \zeta_2 \in \mathbb{C}, \text{Im } \zeta_1 > 0 \text{ (respectively, } \text{Im } \zeta_1 < 0), \\ \text{Im } \zeta_2 > \varphi(\text{Im } \zeta_1)\}.$$

It is obvious that V/\mathbb{Z}^2 is biholomorphic to D . We put $\Omega := \mathbb{C} \times V$.

We define the group of automorphisms $G_{\mathbb{Z}}$ induced by $\mathbb{Z} \times \mathbb{Z}^2$ on Ω as follows. Let $(\zeta_0, b_0) \in \mathbb{Z} \times \mathbb{Z}^2$. Then

$$\Omega \ni (\zeta, b) \mapsto (\zeta + \zeta_0, A^{\zeta_0} b + b_0) = \left(\zeta + \zeta_0, \zeta_1 \lambda^{\zeta_0} v + \zeta_2 \frac{1}{\lambda^{\zeta_0}} w + b_0 \right) \in \Omega,$$

where $(\zeta, b) = (\zeta, \zeta_1 v + \zeta_2 w) \in \Omega$.

Note that the fact that the functions defined above leave the set Ω invariant follows from the properties of φ . Namely,

$$\frac{1}{\lambda^{\zeta_0}} \operatorname{Im} \zeta_2 > \frac{1}{\lambda^{\zeta_0}} \varphi(\operatorname{Im} \zeta_1) = \varphi(\operatorname{Im} \lambda^{\zeta_0}) \text{ if and only if } \operatorname{Im}(\zeta_2) > \varphi(\operatorname{Im} \zeta_1),$$

for any $\zeta_0 \in \mathbb{Z}$, $(\zeta_1, \zeta_2) \in \mathbb{C}^2$ with $\operatorname{Im} \zeta_1 > 0$ (respectively, with $\operatorname{Im} \zeta_1 < 0$).

Now, we define the desired holomorphic fiber bundle $E := \Omega/G_{\mathbb{Z}}$, which has V/\mathbb{Z}^2 as the fiber and \mathbb{C}/\mathbb{Z} as the basis.

Below we show that there is no plurisubharmonic exhaustion function on E . Suppose the contrary. Let u denote a plurisubharmonic exhaustion function on E .

First recall that there is a family $(f_R)_{R>1}$ of holomorphic functions $\bar{\Delta} \mapsto \mathbb{C}$ satisfying the following properties:

$$\begin{aligned} 0 < \operatorname{Im} f_R < \pi \text{ on } \Delta, \\ \operatorname{Re} f_R(0) &= 0, \\ \lim_{R \rightarrow 1^+} \int_0^{2\pi} e^{\pm \operatorname{Re} f_R(e^{i\theta})} d\theta &= \infty. \end{aligned}$$

One may define $f_R(\zeta) := \log i \frac{R+\zeta}{R-\zeta}$ – see [Coe-Loeb].

Note that $\varphi(t\lambda^k) = \frac{1}{\lambda^k} \varphi(t)$, $k \in \mathbb{Z}$, $t \in [1, \lambda]$ (respectively, $t \in [-\lambda, -1]$) and φ is continuous. Therefore, there is a constant $a > 0$ (respectively, $a < 0$) such that $\frac{a}{t} > \varphi(t)$, $t > 0$ (respectively, $t < 0$).

Now for any $R > 1$ we find functions g_R and h_R holomorphic on Δ , continuous on $\bar{\Delta}$ and such that

$$\operatorname{Im} g_R(\zeta) = e^{\operatorname{Re} f_R(\zeta)}, \quad \operatorname{Im} h_R(\zeta) = a e^{-\operatorname{Re} f_R(\zeta)}, \quad |\zeta| = 1$$

(respectively,

$$\operatorname{Im} g_R(\zeta) = -e^{\operatorname{Re} f_R(\zeta)}, \quad \operatorname{Im} h_R(\zeta) = -a e^{-\operatorname{Re} f_R(\zeta)}, \quad |\zeta| = 1).$$

Since for any $|\zeta| = 1$, the inequality $\varphi(\operatorname{Im} g_R(\zeta)) - \operatorname{Im} h_R(\zeta) < 0$ holds, the maximum principle for subharmonic functions (note that $\varphi \circ h$, where h is harmonic on Δ , is subharmonic on Δ) implies that for any $\zeta \in \bar{\Delta}$, the inequality $\varphi(\operatorname{Im} g_R(\zeta)) - \operatorname{Im} h_R(\zeta) < 0$ holds (or, equivalently, $g_R(\zeta)v + h_R(\zeta)w \in V$, $\zeta \in \bar{\Delta}$).

Define

$$\Psi_R(\zeta) := u\left(\left[\left(\frac{f_R(\zeta)}{\log \lambda}, g_R(\zeta)v + h_R(\zeta)w\right)\right]_{G_{\mathbb{Z}}}\right), \zeta \in U_R \supset \bar{\Delta}.$$

Certainly, Ψ_R is subharmonic on some neighborhood of $\bar{\Delta}$.

It follows from the definition of E that

$$\Psi_R(\zeta) = u \left(\left[\left(\frac{f_R(\zeta)}{\log \lambda} - \left[\frac{\operatorname{Re} f_R(\zeta)}{\log \lambda} \right], \lambda^{-\left[\frac{\operatorname{Re} f_R(\zeta)}{\log \lambda} \right]} g_R(\zeta) v + \lambda^{\left[\frac{\operatorname{Re} f_R(\zeta)}{\log \lambda} \right]} h_R(\zeta) w \right) \right]_{G_{\mathbb{Z}}} \right),$$

for any $\zeta \in \bar{\Delta}$ ($[x]$ in the inner brackets denotes the largest integer not exceeding x).

Note that the real part of the first component in the formula above is from the interval $[0, 1)$ and its imaginary part is from the interval $(0, \frac{\pi}{\log \lambda})$. Moreover, for $|\zeta| = 1$:

$$\operatorname{Im} \left(\lambda^{-\left[\frac{\operatorname{Re} f_R(\zeta)}{\log \lambda} \right]} g_R(\zeta) \right) = e^{\operatorname{Re} f_R(\zeta) - \log \lambda \left[\frac{\operatorname{Re} f_R(\zeta)}{\log \lambda} \right]} \in [1, \lambda)$$

and, similarly,

$$\operatorname{Im} \left(\lambda^{\left[\frac{\operatorname{Re} f_R(\zeta)}{\log \lambda} \right]} h_R(\zeta) \right) \in \left(\frac{a}{\lambda}, a \right]$$

(respectively,

$$\operatorname{Im} \left(\lambda^{-\left[\frac{\operatorname{Re} f_R(\zeta)}{\log \lambda} \right]} g_R(\zeta) \right) \in (-\lambda, -1],$$

$$\operatorname{Im} \left(\lambda^{\left[\frac{\operatorname{Re} f_R(\zeta)}{\log \lambda} \right]} h_R(\zeta) \right) \in \left(\frac{-a}{\lambda}, -a \right].$$

Consequently, there is some constant $M \in \mathbb{R}$ such that for any $R > 1$:

$$\Psi_R(\zeta) \leq M, \quad |\zeta| = 1.$$

Then the maximum principle for subharmonic functions gives for any $R > 1$:

$$\Psi_R(0) \leq M.$$

Note that (the sign depends on one of two possible cases)

$$\lim_{R \rightarrow 1^+} \operatorname{Im} g_R(0) = \pm \lim_{R \rightarrow 1^+} \frac{1}{2\pi} \int_0^{2\pi} e^{\operatorname{Re} f_R(e^{i\theta})} d\theta = \pm \infty.$$

Similarly,

$$\lim_{R \rightarrow 1^+} \operatorname{Im} h_R(0) = \infty.$$

Therefore, since $\operatorname{Re} f_R(0) = 0$, $\operatorname{Im} f_R(0) \in (0, \pi)$, and u is exhaustive on E , we get $\lim_{R \rightarrow 1^+} \Psi_R(0) = \infty$ – contradiction. This finishes the proof of the hyperbolic case.

Now we are left only with the parabolic case.

Applying the mapping $z^{-\mathbb{I}_2}$ we may reduce ourselves to the case $\beta_2 < 0$ (and then ψ is concave). Note that we may assume that v and w are from \mathbb{Z}^2 and the coordinates of w are relatively prime. Using some algebraic biholomorphism (mapping w to $(-1, 0)$ and being such that the determinant of the corresponding matrix composing of integers is one) we may assume additionally that $w = (0, -1)$. Note that in this case $\mathfrak{C}(D) = \mathbb{R}_+(0, -1)$.

Consider now the domain $\tilde{D} := \text{int } \bar{D}$. Note that $\tilde{D} \cap (\mathbb{C} \times \{0\}) = \mathbb{C}_* \times \{0\}$, $\tilde{D} \subset \mathbb{C} \times R \cdot \Delta$ for some $R > 0$ and $\mathfrak{C}(\tilde{D}) = \mathbb{R}_+(0, -1)$. Then Lemma 6 implies that $\text{Aut } D = \text{Aut } \tilde{D}|_D$. Therefore, as earlier, because of Theorem 7, it is sufficient to show that $\tilde{D} \in \mathfrak{S}$.

Now we define

$$u(z) := \max\{\rho(\tilde{u}(z)), \log |z_1|, -\log |z_1|\}, \quad z \in \tilde{D},$$

where $(\log |z_1|, \log |z_2|) = t(z)v + s(z)w$, $\tilde{u}(z) = s(z) - \psi(t(z))$, $\rho : [-\infty, 0) \mapsto \mathbb{R}$ is a continuous, increasing and a convex function, C^2 -smooth on $(-\infty, 0)$ and $\lim_{t \rightarrow 0^-} \rho(t) = \infty$ (e.g. $\rho(t) = \frac{-1}{t}$, $t < 0$). Note that assuming that ψ is additionally C^2 -smooth we may verify, calculating the Levi form of $\rho \circ \tilde{u}$, that $\rho \circ \tilde{u}$ is plurisubharmonic on D . Then applying the standard approximation of a concave function with the help of the increasing sequence of C^2 -smooth concave functions we get that $\rho \circ \tilde{u}$ is plurisubharmonic on D without the additional assumption on its smoothness, too. Consequently, u is plurisubharmonic on D , and then also on \tilde{D} .

It is clear that u is an exhausting function for \tilde{D} . We claim that for any $\Phi \in \text{Aut } \tilde{D}$, $u \circ \Phi - u$ is bounded from above on \tilde{D} .

Actually, take $\Phi \in \text{Aut } \tilde{D}$. It follows from the description of $\text{Aut } D$ in Theorem 8 that $\rho(\tilde{u}(\Phi(z))) = \rho(\tilde{u}(z))$, $z \in \tilde{D}$. One may also verify that for any $\Phi \in \text{Aut } \tilde{D}$, $\max\{\log |\Phi_1(z)|, -\log |\Phi_1(z)|\} - \max\{\log |z_1|, -\log |z_1|\}$ is bounded from above on \tilde{D} . Then the Stehlé criterion applies and $\tilde{D} \in \mathfrak{S}$. □

Remark 10. — As we saw in the proof of Theorem 1 there were three non-trivial cases. The domain $\{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_2| < \exp(-|z_1|^2)\}$ for $t = 1$ and the domains of parabolic type for $t = 0$ are domains for which the automorphism group is non-compact. The proof that they belong to class \mathfrak{S} relies upon the proof of belonging to the class \mathfrak{S} of some larger

domain. On the other hand the domains of hyperbolic type are always not from \mathfrak{S} and the proof is based upon the construction of Coeuré and Loeb.

It is natural to ask the question what happens in higher dimension. Is there a similar geometric-like description of the class of hyperbolic pseudoconvex Reinhardt domains from \mathfrak{S} ?

BIBLIOGRAPHY

- [Car] H. CARTAN, Sur les transformations analytiques des domaines cerclés et semicerclés bornés, *Math. Ann.*, 106 (1932), 540–573.
- [Coe-Loeb] G. COEURÉ & J.-J. LOEB, A counterexample to the Serre problem with a bounded domain in \mathbb{C}^2 as fiber, *Ann. Math.*, 122 (1985), 329–334.
- [Dem] J.-P. DEMAÏLLY, Un exemple de fibré holomorphe non de Stein á fibré \mathbb{C}^2 ayant pour base le disque ou le plan, *Invent. Math.*, 48 (1978), 293–302.
- [Hir] A. HIRSCHOWITZ, Domains de Stein et fonctions holomorphes bornées, *Math. Ann.*, 213 (1975) 185–193.
- [Isa-Kra] A.V. ISAEV & S.G. KRANTZ, Domains with non-compact automorphism group: a survey, *Adv. Math.*, 146 (1999), 1–38.
- [Kön] K. KÖNIGSBERGER, Über die Holomorphie-Vollständigkeit lokaltrivialer Faserräume, *Math. Ann.*, 189 (1970) 178–184.
- [Kru] N.G. KRUZHILIN, Holomorphic automorphism of hyperbolic Reinhardt domains, *Math. USSR Izv.*, 32 (1998) 158–38.
- [Loeb] J.-J. LOEB, Un nouveau contre-exemple a une conjecture de Serre, *Publ. IRMA Lille*, 6(4) (1984).
- [Mok] N. MOK, The Serre problem on Riemann surfaces, *Math. Ann.*, 258 (1981), 145–168.
- [Nar] R. NARASIMHAN, Several complex variables, *Chicago Lectures in Mathematics*. The University of Chicago Press, Chicago-London, 1971.
- [Pfl] P. PFLUG, About the Carathéodory completeness of all Reinhardt domains, In: *Functional Analysis, Holomorphy and Approximation Theory II*, (1984), North-Holland, Amsterdam, 331–337, G. Zapata ed.
- [Shi] S. SHIMIZU, Automorphisms and equivalence of bounded Reinhardt domains not containing the origin, *Tohoku Math. J.*, 40(1) (1988), 119–152.
- [Sib] N. SIBONY, Fibrés holomorphes et metrique de Carathéodory, *C. R. Acad. Sc. Paris*, 279 (1974), 261–264.
- [Siu] Y.T. SIU, Holomorphic fiber bundles whose fibers are bounded Stein domains with first Betti number, *Math. Ann.*, 219 (1976), 171–192.
- [Sko] H. SKODA, Fibrés holomorphes à base fibre de Stein, *Invent. Math.*, 43 (1977), 97–107.
- [Ste] J.-L. STEHLÉ, Fonctions plurisousharmoniques et convexité holomorphe de certains fibrés analytiques. In *Séminaire Pierre Lelong (Analyse)*, Année 1973/1974 (ed. P. Lelong, P. Doulbeaut, H. Skoda), *Lecture Notes in Mathematics* 474. Springer, Berlin (1975), 155–179.

- [Zaf] D. ZAFFRAN, Serre problem and Inoue-Hirzebruch surfaces, *Math. Ann.*, 319(2) (2001), 395-420.
- [Zwo1] W. ZWONEK, On hyperbolicity of pseudoconvex Reinhardt domains, *Arch. Math.*, 72(4) (1999), 304-314.
- [Zwo2] W. ZWONEK, Completeness, Reinhardt domains and the method of complex geodesics in the theory of invariant functions, *Diss. Math.*, 388 (2000) 1-103.

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Peter PFLUG,
Universität Oldenburg
Fachbereich Mathematik
Postfach 2503
D-26111 Oldenburg (Germany)
pflug@mathematik.uni-oldenburg

Włodzimierz ZWONEK,
Uniwersytet Jagielloński
Instytut Matematyki
Reymonta 4
30-059 Kraków (Poland).
zwonek@im.uj.edu.pl