ANNALES

DE

L'INSTITUT FOURIER

Urban CEGRELL

The general definition of the complex Monge-Ampère operator

<http://aif.cedram.org/item?id=AIF_2004__54_1_159_0>

THE GENERAL DEFINITION OF THE COMPLEX MONGE-AMPÈRE OPERATOR

by Urban CEGRELL

———

1. Introduction.

Denote by $PSH(\Omega)$ the plurisubharmonic functions on Ω and by $PSH^-(\Omega)$ the subclass of negative functions. A set $\Omega \subset \mathbb{C}^n$ is said to be a hyperconvex domain if it is open, bounded, connected and if there exists $\varphi \in PSH^-(\Omega)$ such that $\{z \in \Omega; \varphi(z) < -c\} \subset\subset \Omega$, $\forall c > 0$. Such function is called an exhaustion function for Ω. Throughout this paper, Ω will always denote a hyperconvex domain. In the first part of this paper, we consider global approximation of negative plurisubharmonic functions by decreasing sequences of negative plurisubharmonic functions that are continuous on $\bar{\Omega}$, equal to zero on $\partial\Omega$ and with bounded Monge-Ampère mass. The elements of this class of functions serves as "test functions". Theorem 2.1 below states that global approximation is possible in PSH^-. We use Theorem 2.1 to show that integration by parts is almost always allowed (Corollary 3.4). In the second part of this paper, we discuss a general definition of the complex Monge-Ampère operator. This is done by introducing a class $\mathcal{E}$ of plurisubharmonic functions which consists of all functions that are locally equal to decreasing limits of test functions described above. The Monge-Ampère operator can be extended to $\mathcal{E}$, and this is the most general definition if we require the operator to be continuous under decreasing limits (Theorem 4.5). In the remaining part of the paper, we study the Monge-Ampère operator using this general definition.

———

It is a great pleasure to thank Norman Levenberg, Frank Wikström, Yang Xing and Per Åhag for many fruitful comments.

2. Approximation by continuous plurisubharmonic functions.

In this section, we prove an approximation theorem for negative plurisubharmonic functions, used throughout the paper.

THEOREM 2.1. — *Suppose $u \in PSH^-(\Omega)$. Then there is a decreasing sequence of functions $u_j \in PSH(\Omega) \cap C(\bar{\Omega})$ with $u_j|_{\partial\Omega} = 0$, $\forall j \in \mathbb{N}$, $\lim_{j \to +\infty} u_j(z) = u(z)$, $\forall z \in \Omega$ and $\int_{\Omega} (dd^c u_j)^n < +\infty$.*

Proof. — Denote by h_E the relative extremal function for $E \subset\subset \Omega$. See [17]. Then $\operatorname{supp}(dd^c h_E)^n \subset\subset \Omega$ and if $E = B$ is a ball, it follows from a theorem by Walsh [18] that h_B is continuous on $\bar{\Omega}$. Thus, there is a continuous exhaustion function v for Ω with $\operatorname{supp}(dd^c v)^n \subset\subset \Omega$, in particular $\int_{\Omega} (dd^c v)^n < +\infty$. See [5] and [14] for details.

For each $j \in \mathbb{N}$, choose $\{r_j\}_{j=1}^{\infty}$ a decreasing sequence such that

$$0 < r_j < \operatorname{dist}\left(\left\{z \in \Omega; \; v(z) < -\frac{1}{2j^2}\right\}, \complement\Omega\right).$$

Denote by u_{r_j} the usual regularization of u, defined on Ω_{r_j} where

$$\Omega_m = \{z \in \Omega; \; \operatorname{dist}(z, \complement\Omega) > m\}.$$

Define $u_j = \sup_{j \leqslant m}(u_{r_m} - \frac{1}{m}, mv)$ on Ω_m and mv otherwise on Ω.

Note that if $d(z_0, \complement\Omega) \leqslant r_j$ then $d(z_0, \complement\Omega) < \operatorname{dist}\left(\{v < -\frac{1}{2j^2}\}, \complement\Omega\right)$ so $v(z_0) \geqslant -\frac{1}{2j^2}$ and $jv(z_0) \geqslant \frac{-1}{2j} > \frac{-1}{j}\left(\geqslant u_{r_j}(z_0) - \frac{1}{j}\right)$. Therefore, each $\max\left(u_{r_j} - \frac{1}{j}, jv\right)$ is plurisubharmonic on Ω, continuous on $\bar{\Omega}$ and equal to zero on $\partial\Omega$.

Also, since, u_j is the upper envelope of continuous functions, u_j is lower semicontinuous. We claim that u_j is upper semicontinuous. We have

$$u_j = \sup_{j \leqslant m}\left(u_{r_m} - \frac{1}{m}, mv\right) = \sup_{j \leqslant m}\left[\max\left(u_{r_m}, \; mv + \frac{1}{m}\right) - \frac{1}{m}\right]$$

$$\leqslant \max\left\{\max_{j \leqslant m < k}\left[\max\left(u_{r_m}, \; mv + \frac{1}{m}\right) - \frac{1}{m}\right], \max\left(u_{r_k}, \; kv + \frac{1}{k}\right)\right\}$$

since $\max(u_{r_k},\ kv+\frac{1}{k})$ is decreasing in k. Finally, since

$$\max\left\{\max_{j\leqslant m<k}\left[\max\left(u_{r_m},\ mv+\frac{1}{m}\right)-\frac{1}{m}\right],\ \max(u_{r_k},\ kv+\frac{1}{k})\right\}\searrow u_j,$$

$$k\to+\infty$$

it follows that each u_j is upper semicontinuous. It is a consequence of Stokes theorem that $\int_\Omega(dd^cu_j)^n\leqslant j^n\int_\Omega(dd^cv)^n<+\infty$.   □

Remark. — Note that the u_j:s, used in the proof above have compactly supported Monge-Ampère measures.

Remark. — There is always an exhaustion function in $\mathcal{E}_0\cap C^\infty(\Omega)$. Cf. [10]. For the definition of $\mathcal{E}_0$ see next section.

Remark. — In the case when Ω is a so called B-regular domain, arbitrary continuous boundary data has been studied in [19].

3. Integration by parts.

In this section, we study "integration by parts" which is an essential tool in this paper.

We first recall the definition of the class $\mathcal{E}_0$, introduced in [7]; $\mathcal{E}_0(\Omega)$ is the convex cone of bounded plurisubharmonic functions φ with $\lim_{z\to\xi}\varphi(z)=0$, $\forall\xi\in\partial\Omega$ and $\int_\Omega(dd^c\varphi)^n<+\infty$.

LEMMA 3.1. — $C_0^\infty(\Omega)\subset\mathcal{E}_0\cap C(\bar\Omega)-\mathcal{E}_0\cap C(\bar\Omega)$.

Proof. — Choose $0>\psi\in\mathcal{E}_0$ and let $\chi\in C_0^\infty(\Omega)$ be given. Choose m so large that

$$\chi+m|z|^2\in PSH(\Omega).$$

Let $a<\inf\chi<\sup_\Omega(|\chi|+m|z|^2)<b$ and define

$$\varphi_1=\max(\chi+m|z|^2-b,\ M\psi)$$

where M is so large that $M\psi<a-b$ on the support of χ. Then $\varphi_1\in\mathcal{E}_0$ and so is $\varphi_2=\max(m|z|^2-b,\ M\psi)$. This proves the lemma since $\chi=\varphi_1-\varphi_2$.   □

THEOREM 3.2. — Suppose $u,v\in PSH^-(\Omega)$, $u\not\equiv0$, $\lim_{z\to\xi}u(z)=0$, $\forall\xi\in\partial\Omega$, and T a positive and closed current of bidegree (n-1,n-1). Then

$dd^c u \wedge T$ is a well-defined positive measure on Ω. Furthermore, if

$$\int v \, dd^c u \wedge T > -\infty$$

then $dd^c v \wedge T$ is also a well-defined positive measure on Ω and

$$\int v \, dd^c u \wedge T \leqslant \int u \, dd^c v \wedge T.$$

The following lemma is well-known, cf. [13].

LEMMA 3.3. — If $u \in PSH^-(\Omega)$, T a positive and closed current of bidegree (n-1,n-1) and if $u \in L^\infty(\Omega \setminus K)$ where K is a compact subset of Ω, then $dd^c u \wedge T$ is a well-defined positive measure on Ω.

Proof. — Note first that if $\chi \in C_0^\infty(\Omega)$ then,

$$\int g dd^c \chi \wedge T, g \in C_0^\infty(\Omega)$$

is a Radon measure with mass zero.

Therefore, $\int w dd^c \chi \wedge T$ is a positive number when

$$0 \leqslant \chi \in C_0^\infty(\Omega'), w \in PSH \cap L^\infty(\Omega')$$

for every $\Omega' \subset\subset \Omega$ any strictly pseudoconvex domain containing K. In particular, the conclusion of Lemma 3.3 holds true if K is empty. Choose $u_j^1, u_j^2 \in PSH \cap C^\infty$ in a neighborhood of $\bar{\Omega}'$:

$$\begin{aligned} u_j^1 &\searrow u, & j \to +\infty \\ u_j^2 &\searrow \max\big(u, \inf_{\xi \in \Omega \setminus K} u(\xi)\big), & j \to +\infty \end{aligned}$$

and $u_j^1 = u_j^2$ near $\partial\Omega'$.

Then $\int_{\Omega'} dd^c(u_j^1 - u_j^2) \wedge T = 0$ since T is closed. Also, if $0 \leqslant \chi \in C_0^\infty(\Omega'), \chi \equiv 1$ near $\mathrm{supp}(u_j^1 - u_j^2)$ then

$$\int_{\Omega'} (dd^c u_j^1 - dd^c u_j^2) \wedge T = \int \chi dd^c u_j^1 \wedge T - \int u_j^2 dd^c \chi \wedge T.$$

So since $\lim \int u_j^2 dd^c \chi \wedge T$ exists, so does $\lim \int_{\Omega'} \chi dd^c u_j^1 \wedge T$. Therefore, $dd^c u \wedge T$ is a well-defined positive measure on Ω.   □

Proof of Theorem 3.2. — Suppose $u, v \in PSH(\Omega) \cap C(\bar{\Omega})$, $u = v = 0$ on $\partial\Omega$ and that

$$\int v \, dd^c u \wedge T > -\infty.$$

Then, by Theorem 3.3 in [12]

$$-\infty < \int_\Omega v\, dd^c u \wedge T \leqslant \int_{\{u<-\varepsilon\}} v\, dd^c u \wedge T \leqslant \int_{\{u<-\varepsilon\}} (u+\varepsilon)dd^c v \wedge T$$

so if we denote by χ_ε the characteristic function of $\{u \leqslant -\varepsilon\}$, we have that $(u+\varepsilon)\chi_\varepsilon$ decreases to u when ε decreases to zero.

Hence,

$$\int_\Omega v\, dd^c u \wedge T \leqslant \int_\Omega u\, dd^c v \wedge T$$

and in the same way, using that $-\infty < \int u\, dd^c v \wedge T$ we find that

$$\int_\Omega u\, dd^c v \wedge T = \int_\Omega v dd^c u \wedge T.$$

To complete the proof of Theorem 3.2, we use Theorem 2.1 and choose

$$u_j, v_j \in \mathcal{E}_0 \cap C(\bar\Omega),$$

$$u_j \searrow u, \ v_j \searrow v, \ j \to +\infty.$$

Now, since

$$\int v\, dd^c u \wedge T > -\infty.$$

we have by dominated convergence that

$$\lim_{\varepsilon \to 0} \int \max(v, -\varepsilon)\, dd^c u \wedge T = 0.$$

Then, by Lemma 3.3 and what we have proved so far,

$$\int v_k dd^c u \wedge T = \lim_j \int v_k dd^c u_j \wedge T$$

$$= \lim_j \int u_j dd^c v_k \wedge T = \int u dd^c v_k \wedge T$$

so since u is upper semicontinuous we get

$$\int v dd^c u \wedge T \leqslant \int u dd^c v \wedge T$$

which proves Theorem 3.2. □

COROLLARY 3.4. — *Suppose*

$$u, v \in PSH(\Omega), \quad \lim_{z \to \xi} u(z) = \lim_{z \to \xi} v(z) = 0,$$

$\forall \xi \in \partial\Omega$ *and that T is a positive closed current of bidegree (1,1). If*

$$\int v\, dd^c u \wedge T > -\infty \quad then$$

$$\int u\, dd^c v \wedge T > -\infty \quad and$$

$$\int v\, dd^c u \wedge T = \int u\, dd^c v \wedge T.$$

Remark. — See also [6], Theorem 5.7.

Remark. — In Section 4, we will extend the result of Corollary 3.4.

4. Definition of the complex Monge-Ampère operator.

Using decreasing sequences of $C^\infty-$ smooth plurisubharmonic functions, it is known from [2] how to define $(dd^c u)^n$ when u is plurisubharmonic and locally bounded which is a special case of plurisubharmonic functions satisfying a certain growth condition studied in [1]. In these cases, as well as in the class $\mathcal{E}_p$, studied in [7], the comparison principle is valid.

Leaving this principle behind, $(dd^c u)^n$ is well-defined for u bounded near the boundary of its domain of definition, [13] and [16].

We are now going to consider a definition which covers all these cases.

DEFINITION 4.1. — *Assume $u \in PSH^-(\Omega)$. We say that $u \in \mathcal{E}(\Omega)$ if to every $z_0 \in \Omega$ there is a neighborhood ω of z_0 in Ω and a decreasing sequence $h_j \in \mathcal{E}_0(\Omega)$ such that $h_j \searrow u$ on ω and $\sup_j \int_\Omega (dd^c h_j)^n < +\infty$.*

Remark. — Since $\mathcal{E}_0(\Omega)$ is a convex cone, so is $\mathcal{E}$. See Section 2 in [7].

THEOREM 4.2. — *Suppose $u^p \in \mathcal{E}(\Omega), 1 \leqslant p \leqslant n$. If $g_j^p \in \mathcal{E}_0(\Omega)$ decreases to u^p, $j \to +\infty$, then $dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n$ is weak*-convergent and the limit measure does not depend on the particular sequences $(g_j^p)_{j=1}^\infty$.*

Proof. — Suppose first that $\sup_j \int (dd^c g_j^p)^n < +\infty$. Then, for $h \in \mathcal{E}_0(\Omega)$, $\int h dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n$ is a decreasing sequence by Corollary 3.4 and since

$$\int h (dd^c g_j^p)^n \geqslant \left(\inf_\Omega h \right) \sup_j \int (dd^c g_j^p)^n > -\infty,$$

$\lim_{j \to \infty} \int h dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n$ exists for all $h \in \mathcal{E}_0$.

By Lemma 3.1, $dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n$ is weak*-convergent.

If v_j^p is another sequence decreasing to u^p, we get, again by Corollary 3.4,

$$\int h dd^c v_j^1 \wedge dd^c v_j^2 \wedge \ldots \wedge dd^c v_j^n = \int v_j^1 dd^c h \wedge dd^c v_j^2 \wedge \ldots \wedge dd^c v_j^n \geqslant$$

$$\int u^1 dd^c h \wedge dd^c v_j^2 \wedge \ldots \wedge dd^c v_j^n = \lim_{s_1 \to +\infty} \int g_{s_1}^1 dd^c h \wedge dd^c v_j^2 \wedge \ldots \wedge dd^c v_j^n$$

$$= \lim_{s_1 \to +\infty} \int v_j^2 dd^c h \wedge dd^c g_{s_1}^1 \wedge \ldots \wedge dd^c v_j^n \geqslant \ldots$$

$$\geqslant \lim_{s_1 \to +\infty} \lim_{s_2 \to +\infty} \ldots \lim_{s_n \to +\infty} \int h dd^c g_{s_1}^1 \wedge \ldots \wedge dd^c g_{s_n}^n$$

$$\geqslant \lim_{s \to +\infty} \int h dd^c g_s^1 \wedge dd^c g_s^2 \wedge \ldots \wedge dd^c g_s^n.$$

Therefore, $\lim_{j \to +\infty} \int h dd^c v_j^1 \wedge dd^c v_j^2 \wedge \ldots \wedge dd^c v_j^n$ exists and is minorized by

$$\lim_{j \to +\infty} \int h dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n.$$

But this is a symmetric situation so we conclude that the limits are equal.

To complete the proof it remains to remove the restriction

$$\sup_j \int (dd^c g_j^p)^n < +\infty.$$

Let K be a given compact subset of Ω, cover K with finitely many W_q, $q = 1, \ldots, N$ as in the definition of $\mathcal{E}$. Let $\{h_j^{pq}\}_{j=1}^\infty$, $1 \leqslant q \leqslant N, 1 \leqslant p \leqslant n$ be the corresponding h_j^p:s and put $w_j^p = \sum_{q=1}^N h_j^{pq}$.

Then $w_j^p \in \mathcal{E}_0, w_j^p \leqslant g_j^p$ on $\bigcup_{q=1}^N W_q$, and $\sup_j \int_\Omega (dd^c w_j^p)^n < +\infty$. Thus, if we define $v_j^p = \max(g_j^p, w_j^p)$, $\sup_j \int_\Omega (dd^c v_j^p)^n < +\infty$ and $v_j^p = g_j^p$ near K. $\qquad \square$

DEFINITION 4.3. — *For $u^p \in \mathcal{E}, 1 \leqslant p \leqslant n$, we define $dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n$ to be the limit measure obtained in Theorem 4.2.*

DEFINITION 4.4. — *Consider a class $\mathcal{K} (= \mathcal{K}(\Omega)) \subset PSH^-(\Omega)$ such that:*

(1) If $u \in \mathcal{K}$, $v \in PSH^-(\Omega)$ then $\max(u, v) \in \mathcal{K}$.

(2) If $u \in \mathcal{K}, \varphi_j \in PSH^- \cap L_{loc}^\infty, \varphi_j \searrow u, j \to +\infty$, then $\left((dd^c \varphi_j)^n\right)_{j=1}^\infty$ is weak*-convergent.

Remark. — We will now see that $\mathcal{E}$ is the largest class for which (1) and (2) holds true. Thus, we may say that $\mathcal{E}$ is optimal in this sense.

THEOREM 4.5. — *The class $\mathcal{E}$ has property 1. and 2. of Definition 4.4. Conversely, if $\mathcal{K}$ meets the requirements of Definition 4.4, then $\mathcal{K} \subset \mathcal{E}$.*

Proof. — Suppose $u \in \mathcal{E}$. Then (1) holds true by Theorem 3.2. To prove (2), suppose $\varphi_j \in PSH^-(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, $\varphi_j \searrow u$, $j \to +\infty$. If $\varphi \in \mathcal{E}_0$, then $g_j = \max(\varphi_j, m_j\varphi) \in \mathcal{E}_0$ and since $g_j \searrow u \in \mathcal{E}$, $(dd^c g_j)^n$ is weak*-convergent by Theorem 4.2. Here, (m_j) is any sequence, decreasing to $-\infty$. Hence $(dd^c \varphi_j)^n$ is weak*-convergent and therefore (2) follows.

Conversely, suppose $u \in \mathcal{K}$, ω open and relatively compact in Ω. By Theorem 2.1, we can find $h_j \in \mathcal{E}_0 \cap C(\bar{\Omega})$, $h_j \searrow u$ on Ω, $j \to +\infty$.

Define

$$\tilde{h}_j = \sup\{v \in PSH^-(\Omega), \, v \leqslant h_j \text{ on } \omega\}$$

Then $\tilde{h}_j \in PSH(\Omega) \cap L^\infty$, $\text{supp}(dd^c\tilde{h}_j)^n \subset \bar{\omega}$ and $\tilde{h}_j \searrow u$ on ω, $\tilde{h}_j$ decreases on Ω, and $\tilde{h}_j \geqslant u$ everywhere on Ω.

Now, $u \in \mathcal{K}$ and so is $\tilde{h} = \lim_{j \to +\infty} \tilde{h}_j$ since $\tilde{h} \geqslant u$. Therefore, by (2), $(dd^c\tilde{h}_j)^n$ is weak*-convergent and since $\text{supp}(dd^c\tilde{h}_j)^n \subset \bar{\omega} \subset\subset \Omega$ it follows that $\sup_j \int_\Omega (dd^c\tilde{h}_j)^n < +\infty$ and we have proved that $u \in \mathcal{E}$. □

Remark. — Note that $PSH^-(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \subset \mathcal{E}$.

DEFINITION 4.6. — *We denote by $\mathcal{F}(\Omega)$ the subclass of functions u in $\mathcal{E}(\Omega)$ such that there exists a decreasing sequence $u_j \in \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ on Ω and $\sup_j \int_\Omega (dd^c u_j)^n < +\infty$.*

Remark. — It follows from Corollary 3.4 and Theorem 4.2 that integration by parts is allowed in $\mathcal{F}$.

Remark. — It follows from the proof of Theorem 4.5 that every $u \in \mathcal{E}(\Omega)$ is locally in $\mathcal{F}(\Omega)$; to every $u \in \mathcal{E}(\Omega)$ and every w, open and relatively compact in Ω, there is a $u_w \in \mathcal{F}(\Omega)$ with $u = u_w$ in w.

5. The class $\mathcal{F}$.

In this section we to study $\mathcal{F}$ and prove some inequalities, a generalized comparison principle and a decomposition of $(dd^c u)^n$, $u \in \mathcal{F}$.

PROPOSITION 5.1. — Suppose $u^p \in \mathcal{F}(\Omega)$, $1 \leqslant p \leqslant n$ and $h \in PSH^-(\Omega)$. If $g_j^p \in \mathcal{E}_0(\Omega)$ decreases to u^p, $j \to +\infty$, then

$$\lim_{j \to +\infty} \int h \, dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n = \int h \, dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n.$$

Proof. — Since Ω is open, and since $u^p \in \mathcal{F}(\Omega)$, $1 \leqslant p \leqslant n$, $dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n$ converges weak* to $dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n$ and we have

$$+\infty > \lim_{j \to +\infty} \int dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n \geqslant \int dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n.$$

If h is in $\mathcal{E}_0 \cap C$ then, by the proof of Theorem 4.2

$$\int h \, dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n = \lim_{j \to +\infty} \int h \, dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n.$$

Suppose now $h \in PSH^-(\Omega)$ and that $\int h \, dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n$ is finite.

For each j, choose $h_j \in \mathcal{E}_0 \cap C$, decreasing to h, q_j and s_j such that

$$\int -h \, dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n \leqslant \frac{1}{j} - \int h_j \, dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n$$

$$\leqslant \frac{2}{j} - \int h_j \, dd^c g_{q_j}^1 \wedge \ldots \wedge dd^c g_{q_j}^n \leqslant \frac{2}{j} - \int h \, dd^c g_{q_j}^1 \wedge dd^c g_{q_j}^2 \wedge \ldots \wedge dd^c g_{q_j}^n$$

$$\leqslant \frac{4}{j} - \int h_{s_j} \, dd^c g_{q_j}^1 \wedge dd^c g_{q_j}^2 \wedge \ldots \wedge dd^c g_{q_j}^n$$

$$\leqslant \frac{4}{j} - \int h_{s_j} \, dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n$$

$$\leqslant \frac{4}{j} - \int h \, dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n.$$

Also, if $\int h \, dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n = -\infty$, then $\lim_{j \to +\infty} \int h \, dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n = -\infty$. $\qquad \square$

COROLLARY 5.2. — Suppose $u^p \in \mathcal{F}(\Omega)$, $1 \leqslant p \leqslant n$ and $h \in PSH^-(\Omega)$. If $g_j^p \in \mathcal{E}_0(\Omega)$ decreases to u^p, $j \to +\infty$ and

$$\int -h \, dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n < +\infty$$

then $hdd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n$ tends weak* to $hdd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n$, $j \to +\infty$.

Proof. — Since $-h$ is lower semicontinuous so is $-h\chi$ for all $0 \leqslant \chi \in C_0^\infty(\Omega)$. Hence

$$\liminf_{j \to +\infty} \int -h\chi dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n \geqslant \int -h\chi dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n$$

but

$$\lim_{j \to +\infty} \int hdd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n = \int hdd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n.$$

□

The following lemma is proved in [4].

LEMMA 5.3. — If $\varphi \in PSH^- \cap L^\infty(\Omega)$ and $\psi \in \mathcal{F}$ then

$$\int (-\psi)^n (dd^c \varphi)^n \leqslant n!(\sup(-\varphi))^{n-1} \int -\varphi(dd^c \psi)^n.$$

We are going to prove another type of inequality, where we do not need to control the sup-norm; Theorem 5.5 and Corollary 5.6.

LEMMA 5.4. — Suppose $u_1, u_2 \in \mathcal{E}_0, 1 \leqslant p, q < n$ and $T = -hT_1$ where $h, g_1, \ldots, g_{n-p-q} \in \mathcal{E}_0$ and where $T_1 = dd^c g_1 \wedge \ldots \wedge dd^c g_{n-p-q}$. Then

$$\int (dd^c u_1)^p \wedge (dd^c u_2)^q \wedge T$$

$$\leqslant \left[\int (dd^c u_1)^{p+q} \wedge T \right]^{p/p+q} \left[\int (dd^c u_2)^{p+q} \wedge T \right]^{q/p+q}$$

Proof. — 1. We first prove the statement in the lemma when p=q=1.

$$\int dd^c u_1 \wedge dd^c u_2 \wedge T = \int -hdd^c u_1 \wedge dd^c u_2 \wedge T_1$$

$$= \int -u_1 dd^c u_2 \wedge dd^c h \wedge T_1$$

$$= \int du_1 \wedge d^c u_2 \wedge dd^c h \wedge T_1$$

$$\leqslant \left[\int du_1 \wedge d^c u_1 \wedge dd^c h \wedge T_1 \right]^{1/2} \left[\int du_2 \wedge d^c u_2 \wedge dd^c h \wedge T_1 \right]^{1/2}$$

$$= \left[\int -u_1 dd^c u_1 \wedge dd^c h \wedge T_1 \right]^{1/2} \left[\int -u_2 dd^c u_2 \wedge dd^c h \wedge T_1 \right]^{1/2}$$

$$= \left[\int -h(dd^c u_1)^2 \wedge T_1 \right]^{1/2} \left[\int -h(dd^c u_2)^2 \wedge T_1 \right]^{1/2}$$

2. Assuming the lemma is proved when $p + q \leqslant m$, we prove it for $p + q \leqslant m + 1$. However, we first prove:

$$\int (dd^c u_2)^{p+q} \wedge dd^c u_1 \wedge T$$

$$\leqslant \left[\int (dd^c u_2)^{p+q+1} \wedge T \right]^{p+q/p+q+1} \left[\int (dd^c u_1)^{p+q+1} \wedge T \right]^{1/p+q+1}.$$

For

$$\int (dd^c u_2)^{p+q} \wedge dd^c u_1 \wedge T = \int (dd^c u_2)^{p+q-1} \wedge dd^c u_1 \wedge dd^c u_2 \wedge T$$

$$\leqslant \left[\int (dd^c u_2)^{p+q} \wedge dd^c u_2 \wedge T \right]^{p+q-1/p+q} \left[\int (dd^c u_1)^{p+q} \wedge dd^c u_2 \wedge T \right]^{1/p+q}$$

$$= \left[\int (dd^c u_2)^{p+q+1} \wedge T \right]^{p+q-1/p+q} \left[\int (dd^c u_1)^{p+q} \wedge dd^c u_2 \wedge T \right]^{1/p+q}$$

$$\leqslant \left(\int (dd^c u_2)^{p+q+1} \wedge T \right)^{p+q-1/p+q} \left[\left(\int (dd^c u_1)^{p+q+1} \wedge T \right)^{p+q-1/p+q} \right.$$

$$\left. \left(\int (dd^c u_2)^{p+q} \wedge (dd^c u_1) \wedge T \right)^{1/p+q} \right]^{1/p+q}$$

Therefore,

$$\int (dd^c u_2)^{p+q} \wedge dd^c u_1 \wedge T$$

$$\leqslant \left[\int (dd^c u_2)^{p+q+1} \wedge T \right]^{p+q/p+q+1} \left[\int (dd^c u_1)^{p+q+1} \wedge T \right]^{1/p+q+1}.$$

Using this, we have

$$\int (dd^c u_1)^{p+1} \wedge (dd^c u_2)^q \wedge T = \int (dd^c u_1)^p \wedge (dd^c u_2)^q \wedge dd^c u_1 \wedge T$$

$$\leqslant \left[\int (dd^c u_1)^{p+q} \wedge dd^c u_1 \wedge T \right]^{p/p+q} \left[\int (dd^c u_2)^{p+q} \wedge dd^c u_1 \wedge T \right]^{q/p+q}$$

$$\leqslant \left[\int (dd^c u_1)^{p+q+1} \wedge T \right]^{p/p+q} \left[\left(\int (dd^c u_2)^{p+q+1} \wedge T \right)^{p+q/p+q+1} \right.$$

$$\left. \left(\int (dd^c u_1)^{p+q+1} \wedge T \right)^{1/p+q+1} \right]^{q/p+q}$$

$$= \left[\int (dd^c u_1)^{p+q+1} \wedge T \right]^{p+1/p+q+1} \left[\int (dd^c u_2)^{p+q+1} \wedge T \right]^{q/p+q+1}$$

and the lemma is proved. □

THEOREM 5.5. — Suppose $u_1, \ldots, u_n \in \mathcal{F}$ and $h \in \mathcal{E}_0$. Then

$$\int -h dd^c u_1 \wedge \ldots \wedge dd^c u_n$$

$$\leqslant \left(\int -h(dd^c u_1)^n \right)^{1/n} \ldots \left(\int -h(dd^c u_n)^n \right)^{1/n}.$$

Proof. — Using the definition of $\mathcal{F}$ and Proposition 5.1, we see that it is enough to consider the case when $u_1, \ldots, u_n \in \mathcal{E}_0$. We can then use Lemma 5.4:

$$\int -hdd^c u_1 \wedge (dd^c u_2)^{n-1} \leqslant \left(\int -h(dd^c u_1)^n \right)^{1/n} \left(\int -h(dd^c u_2)^n \right)^{n-1/n}$$

so the theorem is true if $u_2 = \ldots = u_n = u$.

Assume the theorem is true for $u_{p+1} = \ldots = u_n = u$ and suppose $u_{p+2} = \ldots u_n = u$.

Then

$$\int -hdd^c u_1 \wedge \ldots \wedge dd^c u_{p+1} \wedge (dd^c u)^{n-p-1}$$

$$\leqslant \left(\int -h(dd^c u_{p+1})^{n-p} \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_p \right)^{1/n-p}$$

$$\left(\int -h(dd^c u)^{n-p} \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_p \right)^{n-p-1/n-p}$$

$$\leqslant \left[\left(\int -h(dd^c u_1)^n \right)^{1/n} \ldots \right.$$

$$\left. \left(\int -h(dd^c u_p)^n \right)^{1/n} \left(\int -h(dd^c u_{p+1})^n \right)^{n-p/n} \right]^{1/n-p}$$

$$\left[\left(\int -h(dd^c u_1)^n \right)^{1/n} \ldots \left(\int -h(dd^c u_p)^n \right)^{1/n} \right.$$

$$\left. \left(\int -h(dd^c u)^n \right)^{n-p/n} \right]^{n-p-1/n-p}$$

$$\leqslant \left(\int -h(dd^c u_1)^n \right)^{1/n} \ldots \left(\int -h(dd^c u_p)^n \right)^{1/n}$$

$$\left(\int -h(dd^c u_{p+1})^n \right)^{1/n} \left(\int -h(dd^c u)^n \right)^{n-p-1/n}.$$

The theorem is proved. □

COROLLARY 5.6. — *Suppose $u_1, \ldots, u_n \in \mathcal{F}$. Then*

$$\int dd^c u_1 \wedge \ldots \wedge dd^c u_n \leqslant \left(\int (dd^c u_1)^n \right)^{1/n} \ldots \left(\int (dd^c u_n)^n \right)^{1/n}.$$

COROLLARY 5.7. — *Suppose $u \in \mathcal{E}(\Omega)$ and $x \in \Omega$. Then*

$$2\pi \nu_u(x) \leqslant ((dd^c u)^n(\{x\}))^{1/n}$$

where $\nu_u(x)$ is the Lelong number of u at x.

Proof. — We first assume that $u \in \mathcal{F}(B), x = 0$. Note that

$$\lim_{s \to 0} \int_{B(0,s)} dd^c u \wedge (dd^c \log |z|)^{n-1} = (2\pi)^n \nu_u(0).$$

so, for $1 \leqslant r$, using Theorem 5.5,

$$(2\pi)^n \nu_u(x) \leqslant \int - \max(\log|z|/r, -1) dd^c u \wedge (dd^c \log|z|)^{n-1}$$

$$\leqslant \left[\int - \max(\log|z|/r, -1)(dd^c u)^n \right]^{1/n}$$

$$\left[\int - \max(\log|z|/r, -1)(dd^c \log|z|)^n \right]^{(n-1)/n}$$

$$= (2\pi)^{n-1} \left(\int - \max(\log|z|/r, -1)(dd^c u)^n \right)^{1/n}.$$

If we now let r tend to $+\infty$, we get the desired conclusion. In the general case, we replace $\log|z|$ by the pluricomplex Greenfunction with pole at x. $\square$

Remark. — It follows from Corollary 5.7 that if $u \in \mathcal{E}$, then $\{\nu_u(x) > 0\}$ is discrete.

THEOREM 5.8. — *Suppose E is a pluripolar set in Ω. Then there is an $h \in \mathcal{F}_1$ such that $E \subset \{h = -\infty\}$.*

Proof. — Recall the definition of $\mathcal{F}_p$ from [7]: $u \in \mathcal{F}$ is in $\mathcal{F}_p$ if $\sup \int (-u_j)^p (dd^c u_j)^n$ is finite where u_j is as in Definition 4.6. Choose a sequence of relatively compact subsets θ_j of Ω such that every point of E is in all but finitely many θ_j and $\int (dd^c h_{\theta_j})^n < \frac{1}{j}$ where h_{θ_j} denotes the relative extremal plurisubharmonic function. By Lemma 3.9 in [7], there is a subsequence θ_{K_j} such that $\sum h_{\theta_{K_j}} \in \mathcal{E}_1$. By Corollary 5.6 we can select a subsequence of this subsequence, denoted by h_j, such that $\sum h_j \in \mathcal{F}$. Hence $\sum h_j \in \mathcal{F}_1$ and obviously $\sum h_j = -\infty$ on E. $\square$

EXAMPLE 5.9. — *The function $\log|z_2|$ is not in $\mathcal{E}(B)$ where B is the unit ball in $\mathbb{C}^2$. The classical energy of $\varphi \in \mathcal{E}_0$ equals*

$$16 \int_B -\varphi(dd^c\varphi) \wedge dd^c(1-|z|^2) = 16 \int (1-|z|^2)(dd^c\varphi)^2.$$

Since the classical energy of $\log|z_2|$ is locally unbounded, it follows from the remark after Definition 4.6 that $\log|z_2|$ is not an element of $\mathcal{E}(B)$. Using this idea, a computation, performed in [8], shows that $-(-\log|z_2|)^v$ is in $\mathcal{E}(B)$ if and only if $0 < v < 1/2$.

LEMMA 5.10. — *Suppose $u \in \mathcal{E}$. Then*

$$\frac{(dd^c u)^n}{(1-u)^{2n}} \leqslant (dd^c \frac{u}{1-u})^n$$

Proof. — We can assume that $u \in \mathcal{F}$. Choose $u_j \in \mathcal{E}_0 \cap C(\Omega)$ decreasing to $u, j \to \infty$. Then $\frac{u_j}{1-u_j}$ decreases to $\frac{u}{1-u} \in \mathcal{F} \cap L^\infty(\Omega)$ and a simple calculation shows that

$$\frac{(dd^c u_j)^n}{(1-u_j)^{2n}} \leqslant (dd^c \frac{u_j}{1-u_j})^n.$$

Now, $(dd^c \frac{u_j}{1-u_j})^n$ tends weakly to $(dd^c \frac{u}{1-u})^n$ and to prove the lemma, it is enough to prove that $\frac{(dd^c u_j)^n}{(1-u_j)^{2n}}$ tends weakly to $\frac{(dd^c u)^n}{(1-u)^{2n}}$, $j \to +\infty$.

Since $\frac{1}{(1-u)^{2n}} - 1 \in PSH^-(\Omega)$ we can use Corollary 5.2 and find, for every fixed p,

$$\left(\frac{1}{(1-u)^{2n}} - 1\right)(dd^c u)^n = \lim_{j\to\infty} \left(\frac{1}{(1-u)^{2n}} - 1\right)(dd^c u_j)^n$$

$$\leqslant \lim_{j\to\infty} \left(\frac{1}{(1-u_j)^{2n}} - 1\right)(dd^c u_j)^n \leqslant \lim_{j\to\infty} \left(\frac{1}{(1-u_p)^{2n}} - 1\right)(dd^c u_j)^n$$

$$= \left(\frac{1}{(1-u_p)^{2n}} - 1\right)(dd^c u)^n.$$

Letting p tend to ∞, we get the desired conclusion.   □

THEOREM 5.11. — *Assume that μ is a positive measure on Ω. Then there is a $\psi \in \mathcal{E}_0$ and a function $0 \leqslant f \in L^1_{\text{loc}}((dd^c\psi)^n)$ such that*

$$\mu = f(dd^c\psi)^n + \nu$$

where ν is carried by a pluripolar set.

Furthermore, if $\mu = (dd^c u)^n$ for a function $u \in \mathcal{F}$ then ν is carried by $\{u = -\infty\}$.

Proof. — Using the Radon-Nikodym theorem, the first part follows from Theorem 6.3 in [7]. By Lemma 5.10,

$$\frac{(dd^c u)^n}{(1-u)^{2n}} \leqslant \left(dd^c \frac{u}{1-u}\right)^n.$$

In particular, $\frac{(dd^c u)^n}{(1-u)^{2n}}$ has no mass on pluripolar sets. So if

$$(dd^c u)^n = f(dd^c\psi)^n + \nu$$

we have that $\frac{\nu}{(1-u)^{2n}}$ equals zero so ν is carried by $\{u = -\infty\}$.   □

The following theorem, usually refered to as the comparison principle, was proved in [3] and generalized to $\mathcal{F}_p$ in [7].

THEOREM 5.12. — If $u, v \in PSH \cap L^\infty(\Omega)$, $\lim_{z \to \zeta}\big(u(z) - v(z)\big) = 0$, $\forall \zeta \in \partial\Omega$ and $(dd^c u)^n \leqslant (dd^c v)^n$ on Ω, then $u \geqslant v$ on Ω.

DEFINITION 5.13. — We denote by $\mathcal{F}^a$ the subclass of functions $\varphi \in \mathcal{F}$ such that $(dd^c\varphi)^n$ vanishes on all pluripolar sets.

LEMMA 5.14. — Assume that μ is a positive measure on Ω. If $\mu(\Omega) < +\infty$ and if μ vanishes on all pluripolar sets, then there exists a uniquely determined function $\varphi \in \mathcal{F}^a$ such that $(dd^c\varphi)^n = \mu$.

Proof. — It follows from Theorem 5.11 that there is a $\psi \in \mathcal{E}_0$ and a function $0 \leqslant f \in L^1((dd^c\psi)^n)$ such that $\mu = f(dd^c\psi)^n$. By [15] (see also [7]), there is a unique solution $g^j \in \mathcal{E}_0$ to $(dd^c g^j)^n = \min(f, j)(dd^c\psi)^n$ and it follows from Theorem 5.12 that g^j is a decreasing sequence. We put $g = \lim_{j \to +\infty} g^j$ and it follows from Lemma 5.3 that g is plurisubharmonic and therefore in $\mathcal{F}$. We will now prove that g is uniquely determined; assume $\varphi \in \mathcal{F}^a$ with $(dd^c\varphi)^n = \mu$, we will prove that $\varphi = g$.

Let s_j be a sequence of natural numbers and $K_j \subset\subset \Omega$ a fundamental sequence of compacts of Ω with h_{K_j} continuous such that

$$\int [(1 + h_{K_j}) - \max(\varphi/s_j, -1)](dd^c\varphi)^n \leqslant 1/j$$

which is possible by monotone convergence since $(dd^c\varphi)^n$ has no mass on pluripolar sets. Furthermore, using Proposition 5.1, we know that

$$\lim_{j \to +\infty} \int (dd^c \max(\varphi, s_j h_{K_j}))^n = \int (dd^c\varphi)^n.$$

Let $s \geqslant s_j$ and write $d_j = -h_{K_j} + \max(\varphi/s_j, h_{K_j})$. We have that $0 \leqslant d_j \leqslant \chi_{\{\varphi > s_j h_{K_j}\}}$ and $\chi_{\{\varphi > s_j h_{K_j}\}}(dd^c \max(\varphi, sh_{K_j}))^n$ is independent of s by Lemma 5.4 in [7]. This means that

$$0 \leqslant d_j(dd^c \max(\varphi, sh_{K_j}))^n \leqslant \chi_{\{\varphi > s_j h_{K_j}\}}(dd^c \max(\varphi, sh_{K_j}))^n$$

$$= \chi_{\{\varphi > s_j h_{K_j}\}}(dd^c \max(\varphi, s_j h_{K_j}))^n \leqslant (dd^c \max(\varphi, sh_{K_j}))^n$$

so letting $s \to +\infty$, we find, using Corollary 5.2,

$$d_j(dd^c\varphi)^n \leqslant \chi_{\{\varphi > s_j h_{K_j}\}}(dd^c \max(\varphi, s_j h_{K_j}))^n \leqslant (dd^c\varphi)^n.$$

Combining these inequalities, we get

$$d_j \frac{\min(f,p)}{f}(dd^c \max(\varphi, s_j h_{K_j}))^n \leqslant \frac{\min(f,p)}{f}(dd^c\varphi)^n = (dd^c g^p)^n$$

$$\leqslant \frac{\min(f,p)}{f}(dd^c \max(\varphi, s_j h_{K_j}))^n + (1-d_j)\frac{\min(f,p)}{f}(dd^c\varphi)^n.$$

Solve for $v_j^p \in \mathcal{E}_0$, $(dd^c v_j^p)^n = (1-d_j)\frac{\min(f,p)}{f}(dd^c\varphi)^n$.

Then it follows from Theorem 5.12 that

$$\max(\varphi, s_j h_{K_j}) + v_j^p \leqslant g^p \leqslant w_j^p$$

where

$$w_j^p \in \mathcal{E}_0, (dd^c w_j^p)^n = d_j \frac{\min(f,p)}{f}(dd^c(\max(\varphi, s_j h_{K_j})))^n.$$

Put $v_j = \lim_{p\to+\infty} v_j^p$. Since $\int (dd^c v_j)^n \leqslant 1/j$, it follows that v_j tends to zero, $j \to +\infty$ so it remains to prove that $w_j = \lim_{p\to+\infty} w_j^p$ tends weakly to $\varphi, j \to +\infty$.

Solve for $t_j \in \mathcal{E}_0$, $(dd^c t_j)^n = (1-d_j)(dd^c(\max(\varphi, s_j h_{K_j})))^n$.

Then $w_j + t_j \leqslant \max(\varphi, s_j h_{K_j}) \leqslant w_j$ and

$$\int (dd^c t_j)^n = \int (1-d_j)(dd^c(\max(\varphi, s_j h_{K_j})))^n$$

$$\leqslant \int (1-d_j^2)(dd^c\varphi)^n \leqslant 2\int (1-d_j)(dd^c\varphi)^n$$

$$= 2\int (1 + h_{K_j} - \max(\varphi/s_j, h_{K_j}))(dd^c\varphi)^n \leqslant 2/j.$$

Hence t_j tends weakly to zero, $j \to +\infty$ so w_j tends weakly to φ which completes the proof of Lemma 5.14. □

THEOREM 5.15. — If $u \in \mathcal{F}^a$ and $v \in \mathcal{E}$ with $(dd^c v)^n \geqslant (dd^c u)^n$ then $v \leqslant u$.

Proof. — Without loss of generality we can assume that $v \in \mathcal{F}$. We know that $(dd^c u)^n = h(dd^c\psi)^n$ for some $\psi \in \mathcal{E}_0, 0 \leqslant h \in L^1((dd^c\psi)^n)$ and

$$(dd^c v)^n = f(dd^c w)^n + \nu$$

for some $w \in \mathcal{E}_0, 0 \leqslant f \in L^1((dd^c w)^n)$ and where ν is carried by a pluripolar set. Since $(dd^c v)^n \geqslant (dd^c u)^n$, we can assume that $\psi = w$ and thus $h \leqslant f$. By Lemma 5.14, it is enough to show that for the unique solution g to

$(dd^c g)^n = f(dd^c w)^n$ we have that $v \leqslant g$. Let K be any non-empty compact subset of Ω. We have already used that $-h_K + \max(v/s, h_K) \leqslant \chi_{\{v > sh_K\}}$ so using Corollary 5.2 and Lemma 5.4 in [7], we find

$$(-h_K + \max(v/s, h_K))(dd^c v)^n$$
$$= \lim_{j \to +\infty} (-h_K + \max(v/s, h_K))(dd^c \max(v, jh_K))^n$$
$$\leqslant \lim_{j \to +\infty} \chi_{\{v > sh_K\}}(dd^c \max(v, jh_K))^n \leqslant \chi_{\{v \geqslant sh_K\}}(dd^c \max(v, sh_K))^n.$$

Hence $(-h_K + \max(v/s, h_K))(dd^c v)^n \leqslant \chi_{\{v > sh_K\}}(dd^c \max(v, sh_K))^n$ so $(-h_K + \max(v/s, h_K))(dd^c v)^n$ has no mass on pluripolar sets (Which we already know from Theorem 5.11). Therefore, if $g_{s,K}$ is the unique solution in $\mathcal{E}_0$ to

$$(dd^c g_{s,K})^n = (-h_K + \max(v/s, h_K))(dd^c v)^n$$

then $g_{s,K} \geqslant \max(v, sh_K)$ by the comparison principle and

$$(dd^c g_{s,K})^n = \big(-h_K + \max(v/s, h_K)\big)(dd^c v)^n$$
$$= (-h_K + \max(v/s, h_K))f(dd^c w)^n = (-h_K + \max(v/s, h_K))(dd^c g)^n.$$

Lemma 5.14 gives that $g_{s,K}$ decreases to g when s tends to $+\infty$ and K increases to Ω. Thus, $v = \lim_{s \to +\infty} \max(v, sh_K) \leqslant g$. $\qquad \square$

Remark. — It was shown in [3] that $(dd^c)^n$ is continuous under increasing sequences in L^∞_{loc} and it can be shown that $(dd^c)^n$ is also continuous under increasing sequences in $\mathcal{F}$. In [20] it was shown that $(dd^c)^n$ is continuous under sequences in $PSH \cap L^\infty_{\text{loc}}$, converging in capacity and in [9], this was generalized to sequences in $\mathcal{F}$.

6. The Dirichlet problem in $\mathcal{F}$.

The Dirichlet problem for $(dd^c)^n$ on $PSH \cap L^\infty$ was studied in [2], [11] and [15] and on $\mathcal{E}_p$ in [7]. Here, we consider the Dirichlet problem in $\mathcal{F}$.

LEMMA 6.1. — *Suppose $\psi \in \mathcal{E}_0, v \in \mathcal{F}$ where $(dd^c v)^n$ is carried by a pluripolar set. Then there is a $g \in \mathcal{F}$ with*

$$(dd^c g)^n = (dd^c \psi)^n + (dd^c v)^n.$$

Proof. — By assumption and Theorem 5.11 we can assume that $(dd^c v)^n$ is carried by $\{v = -\infty\}$.

Choose $v_j \in \mathcal{E}_0 \cap C(\bar{\Omega}), v_j \searrow v, j \to +\infty$. Choose an increasing sequence of compact sets $K_j \subset \{v = -\infty\}$ such that

$$\frac{1}{2j} + \int_{K_j} (dd^c v)^n > \int_\Omega (dd^c v)^n$$

and then t_j such that $K_j \subset \{v_{t_j} < -j^3\}$. This is possible since K_j is compact and $K_j \subset \{v = -\infty\}$. Then

$$\frac{1}{2j} + \int_{\{v_{t_j} < -j^3\}} (dd^c v)^n > \int_\Omega (dd^c v)^n$$

and since $\{v_{t_j} < -j^3\}$ is open,

$$\frac{1}{j} + \int_{\{v_{t_j} < -j^3\}} (dd^c v_{s_j})^n > \int_\Omega (dd^c v)^n$$

for s_j large enough. But v_{t_j} is decreasing so

$$\frac{1}{j} + \int_{\{v_{s_j} < -j^3\}} (dd^c v_{s_j})^n \geqslant \int_\Omega (dd^c v)^n.$$

Choose now $\chi_j \in C_0^\infty(\{v_{s_j} < -j^2\}), 0 \leqslant \chi_j \leqslant 1$ such that $\chi_j \equiv 1$ on $\{v_{s_j} \leqslant -j^3\}$. To simplify notation, we write v_j for v_{s_j}.

Solve

$$(dd^c \omega_j)^n = (1 - \chi_j)(dd^c v_j)^n, \ \omega_j \in \mathcal{E}_0$$
$$(dd^c g_j)^n = (dd^c \psi)^n + (dd^c v_j)^n, \ g_j \in \mathcal{E}_0.$$

Define

$$g^j(z) = \sup\{\varphi(z), \varphi \in \mathcal{E}_0, (dd^c \varphi)^n \geqslant (dd^c \psi)^n, \varphi \leqslant v_j\}$$

Then we define g by $g^j \searrow g, j \to +\infty$. By the comparison principle we have $g_j \leqslant g^j$ and we claim that

$$g^j + \frac{v_j}{j} + \omega_j \leqslant g_j \quad \text{if} \quad -j < \inf_\Omega \psi.$$

For

(1) $g^j + \frac{v_j}{j} + \omega_j = g_j = 0$ on $\partial\Omega$.

(2) On $\{\frac{v_j}{j} \leqslant \psi\}$ we have

$$g^j + \frac{v_j}{j} + \omega_j \leqslant v_j + \frac{v_j}{j} + \omega_j \leqslant v_j + \psi + \omega_j \leqslant g_j.$$

(3) On the open set $\{\frac{v_j}{j} > \psi\}$ we have for $-j < \inf \psi$

$$\frac{v_j}{j} > \psi > -j \Rightarrow v_j > -j^2 \quad \text{so} \quad \chi_j \equiv 0$$

on this set and

$$(dd^c g_j)^n = (dd^c \psi)^n + (dd^c v_j)^n = (dd^c \psi)^n + (1 - \chi_j)(dd^c v_j)^n$$

$$= (dd^c \psi)^n + (dd^c \omega_j)^n \leqslant (dd^c(g^j + \frac{v_j}{j} + \omega_j))^n$$

so $g^j + \frac{v_j}{j} + \omega_j \leqslant g_j$ by the comparison principle.

Integration by parts now gives that

$$\int \varphi(dd^c(g^j + \frac{v_j}{j} + \omega_j))^n \leqslant \int \varphi(dd^c g_j)^n \leqslant \int \varphi(dd^c g^j)^n, \ \forall \varphi \in \mathcal{E}_0.$$

Since $(dd^c g^j)^n \rightsquigarrow (dd^c g)^n$ and $(dd^c g_j)^n \rightsquigarrow (dd^c \psi)^n + (dd^c v)^n$, $j \to +\infty$, we have proved that $(dd^c g)^n = (dd^c \psi)^n + (dd^c v)^n$ if we prove that

$$\int \varphi(dd^c g^j)^{n-m} \wedge (dd^c \omega_j)^m \to 0, \ j \to +\infty, \ 1 \leqslant m \leqslant n.$$

But this follows from Theorem 5.5 so the proof of the lemma is complete. $\square$

THEOREM 6.2. — Suppose μ is a positive measure on Ω with finite total mass. Then $\mu = f(dd^c \psi)^n + \nu$ where $\psi \in \mathcal{E}_0, 0 \leqslant f \in L^1(dd^c \psi)^n)$ and ν is carried by a pluripolar set. If there is a $v \in \mathcal{F}$ with $(dd^c v)^n = \nu$ then there is a $g \in \mathcal{F}$ with $(dd^c g)^n = \mu$.

Proof. — It follows from Lemma 5.14 and Lemma 6.1 that for each j there is a $g_j \in \mathcal{F}$ with $(dd^c g_j)^n = \min(f, j)(dd^c \psi)^n + \nu$. It follows from the proof of Lemma 6.1 that $g_j \geqslant g_{j+1}$. Since $\int_\Omega (dd^c g_j)^n \leqslant \mu(\Omega) < +\infty$ it follows that $\lim_{j \to +\infty} g_j$ exists and is in $\mathcal{F}$. This completes the proof of the theorem. $\square$

Remark. — The theorem above generalizes results in [20] and [21].

BIBLIOGRAPHY

[1]   E. BEDFORD, Survey of pluripotential theory. Several complex variables, Proceedings of the Mittag-Leffler Inst. 1987-88. Edited by John Erik Fornaess, Mathematical Notes 38, Princeton University Press, (1994), 48–95.

[2]   E. BEDFORD and B.A. TAYLOR, The Dirichlet problem for a complex Monge-Ampère equation, Invent. Math., 37 (1976), 1–44.

[3]   E. BEDFORD and B.A. TAYLOR, A new capacity for plurisubharmonic functions, Acta Math., 149 (1982), 1–40.

[4]   Z. BLOCKI, Estimates for the complex Monge-Ampère operator, Bull. Pol. Acad. Sci. Math., 41 (1993), 151–157.

[5]   Z. BLOCKI, The complex Monge-Ampère operator in hyperconvex domains, Annali della Scuola Normale Superiore di Pisa 4, 23 (1996), 721–747.

[6]   M. CARLEHED, Potentials in pluripotential theory, Ann. de la Fac. Sci. de Toulouse, VII:3 (1999), 439–469.

[7]   U. CEGRELL, Pluricomplex energy, Acta Mathematica, 180:2 (1998), 187–217.

[8]   U. CEGRELL, Explicit calculation of a Monge-Ampère measure, Actes des rencontres d'analyse complexe, 25-28 Mars 1999. Edited by Gilles Raby and Frédéric Symesak. Atlantique. Université de Poitiers, 2000.

[9]   U. CEGRELL, Convergence in capacity, Isaac Newton Institute for Mathematical Sciences, Preprint Series NI01046-NPD, Cambridge, 2001.

[10] U. CEGRELL, Exhaustion functions for hyperconvex domains, Research reports, N° 10, 2001. Mid Sweden University.

[11] U. CEGRELL and S. KOLODZIEJ, The Dirichlet problem for the complex Monge-Ampère operator: Perron classes and rotation invariant measures, Michigan. Math. J., 41 (1994), 563–569.

[12] D. COMAN, Integration by parts for currents and applications to the relative capacity and Lelong numbers, Mathematica, tome 39(62) (1997), N° 1, 45–57.

[13] J.-P. DEMAILLY, Mesures de Monge-Ampère et mesures pluriharmoniques, Math. Z., 194 (1987), 519–564.

[14] N. KERZMAN and J.-P. ROSAY, Fonctions plurisousharmoniques d'exhaustion bornées et domaines taut, Math. Ann., 257 (1981), 171–184.

[15] S. KOLODZIEJ, The complex Monge-Ampère equation, Acta Mathematica, 180 (1998), 69–117.

[16] N. SIBONY, Quelques problèmes de prolongement de courants en analyse complexe, Duke Math. J., 52 (1985), 157–197.

[17] J. SICIAK, Extremal plurisubharmonic functions and capacities in $\mathbb{C}^n$, Sophia Kokyuroko in Mathematics, 1982.

[18] J.B. WALSH, Continuity of envelopes of plurisubharmonic functions, J. Math. Mech., 18 (1968), 143–148.

[19] F. WIKSTRÖM, Jensen measures and boundary values of plurisubharmonic functions, Ark. Mat., 39 (2001), 181–200.

[20] Y. XING, Complex Monge-Ampère equations with a countable number of singular points, Indiana Univ. Math. J., 48 (1999), 749–765.

[21] A. ZERIAHI, Pluricomplex Green functions and the Dirichlet problem for the Complex Monge-Ampère operator, Michigan Math. J., 44 (1997), 579–596.

Urban CEGRELL,
Umeå University
Department of Mathematics
S-901 87 Umeå (Sweden)
and
TFM
Mid Sweden University
S-851 70 Sundsvall (Sweden).
Urban.Cegrell@math.umu.se