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STRATIFICATION THEORY FROM THE NEWTON POLYHEDRON POINT OF VIEW

by OULD M. ABDERRAHMANE*

A stratification of a variety V is an expression of V as the disjoint union of a locally finite set of connected analytic manifolds, called strata, such that the frontier of each stratum is the union of a set of lower-dimensional strata. The most important notion in stratification theory is the regularity condition between strata. The notion of (w)-regularity introduced by Verdier in [15] plays a very important role in the study of algebraic and analytic varieties. Moreover, he showed that the (w)-regularity condition implies the Whitney (b)-regularity condition. The (c)-regularity, defined by K. Bekka in [2], is weaker than the Whitney (b)-regularity, and he showed that the (c)-regularity condition implies topological triviality. In this paper, we will investigate these regularity conditions relative to a Newton filtration in terms of the defining equations of the strata. The article is organized as follows. In Section 1 we present a characterization for Bekka's (c)-regularity condition. Next we give a criterion for regularity conditions in terms of the defining equations of the strata, following [1] we introduce a pseudo-metric adapted to the Newton polyhedron in Section 2. Using this construction we obtain versions relative to the Newton filtration of the Fukui-Paunescu Theorem (Theorem 4 below). In this approach it is possible to consider a version relative to a Newton filtration of the (w)-regularity condition. We show that this

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condition implies the (c)-regularity condition. In Section 3, using the criterion of the regularity condition given in Section 2, we prove that the J. Damon and T. Gaffney condition in ([5], Theorem 1) implies the (w)-regularity condition related to the Newton polyhedron.

Since complex varieties can be considered as real varieties, we shall only consider the real case.

Notation. — To simplify the notation, we will adopt the following conventions: for a function $g(x,t)$, we denote by ∂g the gradient of g and by $\partial_x g$ the gradient of g with respect to the variables x. For a non zero vector v of $\mathbb{R}^n$, we denote by $L(v)$ the line spanned by v. Also, let $\mathbb{R}^n_+ = \{(x_1,\dots,x_n) \in \mathbb{R}^n,$ each $x_i \geqslant 0,\ i=1,\dots,n\}$ and $\mathbb{Q}^n_+ = \mathbb{Q}^n \cap \mathbb{R}^n_+$, $\mathbb{Z}^n_+ = \mathbb{Z}^n \cap \mathbb{Q}^n_+$.

Let $\varphi, \psi \colon (\mathbb{R}^n,0) \to (\mathbb{R},0)$ be two functions. We say that $|\varphi(x)| \lesssim |\psi(x)|$ if there exists a constant C such that $|\varphi(x)| \leqslant C\,|\psi(x)|$. We write $|\varphi| \sim |\psi|$ if $|\varphi(x)| \lesssim |\psi(x)|$ and $|\psi(x)| \lesssim |\varphi(x)|$. Finally, $|\varphi(x)| \ll |\psi(x)|$ when x tends to x_0 means $\lim_{x\to x_0} \frac{\varphi(x)}{\psi(x)} = 0$.

1. Stratification.

In this section, we recall some definitions about stratification. The stratification theory has been introduced by H. Whitney [16] and R. Thom [13].

Let M be a smooth manifold, and let X, Y be smooth submanifolds of M such that $Y \subseteq \overline{X}$ and $X \cap Y = \emptyset$.

(i) (Whitney (a)-regularity)
(X,Y) is (a)-regular at $y_0 \in Y$ if:
for each sequence of points $\{x_i\}$ which tends to y_0 such that the sequence of tangent spaces $\{T_{x_i}X\}$ tends in the Grassman space of (dim X)-planes to some plane τ, then $T_{y_0}Y \subset \tau$. We say (X,Y) is (a)-regular if it is (a)-regular at any point $y_0 \in Y$.

(ii) (Bekka (c)-regularity)
Let ρ be a smooth non-negative function such that $\rho^{-1}(0) = Y$. (X,Y) is (c)-regular at $y_0 \in Y$ for the control function ρ if:
for each sequence of points $\{x_i\}$ which tends to y_0 such that the sequence of tangent spaces $\{\mathrm{Ker}\, d\rho(x_i) \cap T_{x_i}X\}$ tends in the Grassman space of (dim $X - 1$)-planes to some plane τ, then $T_{y_0}Y \subset \tau$. (X,Y)

is (c)-regular at y_0 if it is (c)-regular for some control function ρ. We say (X,Y) is (c)-regular if it is (c)-regular at any point $y_0 \in Y$.

1.1. A criterion for (c)-regularity.

We suppose now that $M = \mathbb{R}^{n+m}$ and $0 \in Y \subset \overline{X} - X$(the regularity conditions are defined locally). Modulo an analytic transformation of $\mathbb{R}^{n+m}$ near 0, if necessary, we may assume that Y coincides with its tangent space T_0Y. Let $(x,t) = (x_1,\dots,x_n,t_1,\dots,t_m)$ denote a system of coordinates of $\mathbb{R}^{n+m}$. For notational convenience we also use $x_{n+s} = t_s$. We assume that

$$Y = \{(x,t) \in \mathbb{R}^{n+m} \mid x_1 = \dots = x_n = 0\}.$$

Then we can characterize (c)-regularity as follows:

THEOREM 1.— *The pair (X,Y) is (c)-regular at 0 for the control function ρ if and only if (X,Y) is (a)-regular at 0 and $|\partial_t(\rho_{|X})_{(x,t)}| \ll |\text{grad } (\rho_{|X})_{(x,t)}|$ as $(x,t) \in X$ and $(x,t) \to 0$.*

The following proof is inspired by the proof of Bekka-Koike ([3], Theorem 2.4)

Proof.— At first, we have the following equality:

$$T_{(x,t)}X = (\operatorname{Ker} d\rho(x,t) \cap T_{(x,t)}X) \oplus K_{(x,t)},$$

where $K_{(x,t)} = (\operatorname{Ker} d\rho(x,t) \cap T_{(x,t)}X)^{\perp} \cap T_{(x,t)}X = L(\partial(\rho_{|X})_{(x,t)})$ i.e., a line spanned by the gradient of the function $\rho_{|X}$.

($\Rightarrow$) Let (x_i,t_i) be a sequence of points X which tends to 0 such that $T_{(x_i,t_i)}X$ tends to some (dim X)-dimensional space τ. Taking a subsequence if necessary we can suppose that $\operatorname{Ker} d\rho(x_i,t_i) \cap T_{(x_i,t_i)}X$ tends to some (dim $X - 1$)-dimensional space τ' and $K_{(x_i,t_i)}$ tends to some one-dimensional space L. By Bekka (c)-regularity $\{0\} \times \mathbb{R}^m \subset \tau'$. Since $\operatorname{Ker} d\rho(x_i,t_i) \cap T_{(x_i,t_i)}X \subset T_{(x_i,t_i)}X$ and $K_{(x_i,t_i)}$ is orthogonal to $\operatorname{Ker} d\rho(x_i,t_i) \cap T_{(x_i,t_i)}X$, we have $\{0\} \times \mathbb{R}^m \subset \tau$ and L is orthogonal to $\{0\} \times \mathbb{R}^m$ which means (X,Y) is (a)-regular at 0 and $| \partial_t(\rho_{|X})_{(x_i,t_i)} | \ll | \partial(\rho_{|X})_{(x_i,t_i)} |$.

($\Leftarrow$) Let (x_i,t_i) be a sequence of points X which tends to 0 such that $\operatorname{Ker} d\rho(x_i,t_i) \cap T_{(x_i,t_i)}X$ tends to some (dim $X - 1$)-dimensional space τ.

When passing to a subsequence one can suppose that all the $T_{(x_\imath,t_\imath)}X$ have the same dimension (dim X), and that this sequence of space converges to some space τ' and $K_{(x_\imath,t_\imath)}$ tends to some one-dimensional space L. By the Whitney (a)-regularity $\{0\} \times \mathbb{R}^m \subset \tau'$. Since $\mid \partial_t(\rho_{|X})_{(x_\imath,t_\imath)} \mid \ll \mid \partial(\rho_{|X})_{(x_\imath,t_\imath)} \mid$, which implies $L \subset \mathbb{R}^n \times \{0\}$, L is orthogonal to $\{0\} \times \mathbb{R}^m$. Hence we have $\{0\} \times \mathbb{R}^m \subset \tau$.

This completes the proof of the theorem. □

1.2. Ratio test conditions and (w)-regularity.

For X, Y as above, we say X is (r)-regular (resp. (w)-regular) over Y at 0, if for any unit vector v tangent to Y

$$|\pi_p(v)|\,|(x,t)| \ll |x| \text{ as } p = (x,t) \in X \text{ and } (x,t) \to 0$$

(resp. $|\pi_p(v)| \lesssim |x|$ when $p = (x,t) \in X$ near 0) where π_p denotes the orthogonal projection of R^{n+m} to the normal space of X at $p \in X$. We can find a lot of information about this in [6, 8, 14].

Let $F\colon (\mathbb{R}^n \times \mathbb{R}^m, \{0\} \times \mathbb{R}^m) \to (\mathbb{R}^p, 0)$ be an analytic map-germ. We denote by V_F the variety of the zero locus of F. One can note that $\Sigma(V_F) = \{F^{-1}(0) - \{0\} \times \mathbb{R}^m,\ \{0\} \times \mathbb{R}^m\}$ gives a stratification of V_F around $\{0\} \times \mathbb{R}^m$. Hereafter, we will assume that

$$X = F^{-1}(0) - \{0\} \times \mathbb{R}^m \quad\text{and}\quad Y = \{0\} \times \mathbb{R}^m.$$

Setting $F := (F_1, \dots, F_p)$, assume that the Jacobi matrix of F has rank k on X near 0, where $k \leqslant p$ is the codimension of X in $\mathbb{R}^{n+m}$. We note that the normal space to X is generated by the gradient of the functions F_j $(j = 1, \dots, p)$ at each $P \in X$ near 0. Let us recall some definitions and notations, used by Fukui and Paunescu in [6].

Let $j_1 \dots, j_k$ be integers with $1 \leqslant j_1 < \cdots < j_k \leqslant p$. We set $J = \{j_1, \dots, j_k\}$, $F_J = (F_{j_1}, \dots, F_{j_k})$ and

$$dF_J = dF_{j_1} \wedge \cdots \wedge dF_{j_k}, \quad \text{where } dF_j = \sum_{i=1}^{n+m} \frac{\partial F_j}{\partial x_i} dx_i,$$

$$d_xF_J = d_xF_{j_1} \wedge \cdots \wedge d_xF_{j_k}, \quad \text{where } d_xF_j = \sum_{i=1}^{n} \frac{\partial F_j}{\partial x_i} dx_i,$$

and we define d^xF_J by $dF_J = d_xF + d^xF_J$.

For $I \subset \{1,\dots,n\}$, $S \subset \{1,\dots,m\}$, $J \subset \{1,\dots,p\}$ with $\#I + \#S = \#J = k$, we set $\frac{\partial F_J}{\partial(x_I,t_S)}$ to be the Jacobian of F_J with respect to the variables $x_i\,(i \in I)$, and $t_s\,(s \in S)$. When $S = \emptyset$, we simply denote it by $\frac{\partial F_J}{\partial x_I}$. We then define $\|dF\|$, $\|d_xF\|$ and $\|d^xF\|$ by the following formulae:

$$\|dF\|^2 = \sum_J \|dF_J\|^2 \quad \text{where } \|dF_J\|^2 = \sum_{I,S} \left|\frac{\partial F_J}{\partial(x_I,t_S)}\right|^2,$$

$$\|d_xF\|^2 = \sum_J \|d_xF_J\|^2 \quad \text{where } \|d_xF_J\|^2 = \sum_{I} \left|\frac{\partial F_J}{\partial x_I}\right|^2,$$

$$\|d^xF\|^2 = \sum_J \|d^xF_J\|^2 \quad \text{where } \|d^xF_J\|^2 = \sum_{I,S:\,S\neq\emptyset} \left|\frac{\partial F_J}{\partial(x_I,t_S)}\right|^2.$$

For a matrix M we denote by $|M|$ the absolute value of its determinant.

Then we have a simple criterion for the regularity conditions of $\Sigma(V_F)$ as follows:

THEOREM 2. — *For X, Y as above, we have the following equivalences*

(i) *(X,Y) is (a)-regular at 0 if and only if $\|d^xF\| \ll \|dF\|$ when $(x,t) \to 0$ on X.*

(ii) *(X,Y) is (r)-regular at 0 if and only if $\|d^xF\|\,|(x,t)| \ll |x|\,\|d_xF\|$ when $(x,t) \to 0$ on X.*

(iii) *(X,Y) is (w)-regular at 0 if and only if $\|d^xF\| \lesssim |x|\,\|d_xF\|$ holds on X near 0.*

(iv) *(X,Y) is (c)-regular at 0 for the function ρ if and only if $\|d^xF\| \ll \|dF\|$ and $|\partial_t\rho_{|X}| \ll \frac{\|dF\wedge d\rho\|}{\|dF\|}$ as $(x,t) \in X$, $(x,t) \to 0$.*

Here, $\|dF \wedge d\rho\|^2 = \sum_J \|dF_J \wedge d\rho\|^2$.

Proof. — Since (i), (ii) and (iii) have already been obtained in [6], we only have to prove (iv). Indeed, following ([6], lemma 1.4), one get that the orthogonal projection π of $v \in T_{(x,t)}M$ to the tangent space $T_{(x,t)}X$ is expressed by the following form:

$$\pi(v) = \sum_{i=1}^{n+m} \frac{\sum_J \langle dF_J \wedge dx_i, dF_J \wedge v\rangle}{\|dF\|^2}\,\frac{\partial}{\partial x_i}. \tag{1.1}$$

Since $\partial\rho_{|X} = \pi(\partial\rho)$, we can easily see that $\langle\partial\rho_{|X}, \partial\rho\rangle = \frac{\| dF\wedge d\rho\|^2}{\|dF\|^2}$, but $\partial\rho = \partial\rho_{|X} + \partial\rho_{|N}$ (where N denotes the normal space to X), which implies

$$(1.2) \qquad |\partial\rho_{|X}|^2 = \langle\partial\rho_{|X}, \partial\rho\rangle = \frac{\| dF \wedge d\rho\|^2}{\|dF\|^2}.$$

Hence, we can deduce from Theorem 1 that (iv) holds. □

We next state one sufficient condition for (c)-regularity.

Corollary 3. — *Suppose that $\partial_t\rho = 0$, then X is (c)-regular over Y at 0, if*

$$(1.3) \qquad \|d^x F\| \ll \frac{\| dF \wedge d\rho\|}{|\partial\rho|} \quad \text{as } (x,t) \in X,\ (x,t) \to 0.$$

Note that when $p = k = 1$, this inequality is a necessary condition for (c)-regularity.

Proof. — It is trivial that (1.3) implies (X,Y) is (a)-regular at 0. We first remark, by (1.1) the following equality:

$$\begin{aligned}\partial_{t_j}\rho_{|X} &= \frac{\sum_J \langle dF_J \wedge dt_j, dF_J \wedge d\rho\rangle}{\|dF\|^2} \frac{\partial}{\partial t_j}\\ &= \frac{\sum_{i=1}^n \frac{\partial\rho}{\partial x_i} \sum_J \langle dF_J \wedge dt_j, dF_J \wedge dx_i\rangle}{\|dF\|^2} \frac{\partial}{\partial t_j}.\end{aligned}$$

Then, by Cauchy-Schwartz inequality, we have

$$|\partial_{t_j}\rho_{|X}| \lesssim \frac{|\partial\rho|\ \|d^x F\|}{\|dF\|} \quad \textit{for } j = 1, \ldots, m.$$

We now assume (1.3). We then have $|\partial_t\rho_{|X}| \ll \frac{\| dF\wedge d\rho\|}{\|dF\|}$ as $(x,t) \in X$, $(x,t) \to 0$. It follows from the equivalence in (iv) of Theorem 2 that (X,Y) is (c)-regular at 0. □

2. (w)-regularity and (c)-regularity relative to the Newton filtration.

Let us recall some basic definitions and properties of the Newton filtration (see [1, 5, 7] for details). Let $\mathcal{A} \subset \mathbb{Q}_+^n$. A Newton polyhedron

$\Gamma_+(\mathcal{A}) \subset \mathbb{R}^n$ is defined by {the convex closure of $\mathcal{A} + \mathbb{R}^n_+$}. The Newton boundary of $\mathcal{A}$, $\Gamma(\mathcal{A})$ is the union of the compact faces of $\Gamma_+(\mathcal{A})$. We let $\mathcal{F}(\mathcal{A})$ denote the union of the top dimensional faces of $\Gamma(\mathcal{A})$. The Newton vertex $\mathrm{Ver}(\mathcal{A})$ is defined by $\{\alpha : \alpha \text{ is vertex of } \Gamma(A)\}$. $\mathcal{A}$ is called convenient if the intersection of $\Gamma_+(\mathcal{A})$ with each coordinate axis is non-empty. Throughout, we suppose that $\mathcal{A}$ is convenient.

From the Newton polyhedron, we construct the Newton filtration. We first observe that by the convenience assumption on $\mathcal{A}$, any face $F \in \mathcal{F}(\mathcal{A})$, $\dim F = n-1$. So let w^F be the unique vector of $\mathbb{Q}^n_+$ such that $F = \{\, b \in \Gamma_+(\mathcal{A}) \,:\, \langle b, w^F\rangle = 1 \,\}$. We can suppose that the vertices of $\mathcal{A}$ are sufficiently close to the origin so that all the $w^F \in \mathbb{Z}^n_+$. We will suppose henceforth that $\mathcal{A}$ satisfies this property. Then, we construct the following map $\phi\colon \mathbb{R}^n_+ \to \mathbb{R}_+$. The restriction of ϕ to each cone $C(F)$(where $C(F)$ denotes the cone of half-rays emanating from 0 and passing through F) is defined as follows:

$$\phi_{|C(F)}(\alpha) = \langle \alpha\,,\, w^F\rangle, \qquad \text{for all } \alpha \in C(F).$$

We extend this map to $\mathbb{R}^n_+$ as follows:

$$\phi(\alpha) = \min\left\{ \langle\, \alpha\,,\, w^F\rangle \;:\; F \in \mathcal{F}(\mathcal{A}) \,\right\}, \qquad \text{for all } \alpha \in \mathbb{R}^n_+. \tag{2.1}$$

The map ϕ is linear on each cone $C(F)$ (where $F \in \mathcal{F}(\mathcal{A})$), and the value of ϕ along each point over $\Gamma(\mathcal{A})$ is equal to 1 and $\phi(\mathbb{Z}^n_+) \subset \mathbb{Z}_+$. This is called the Newton filtration induced by $\mathcal{A}$.

For any monomial x^α, we define $\mathrm{fil}(x^\alpha) = \phi(\alpha)$. This extends to a filtration on the ring $\mathcal{C}_n$ of analytic function germs $: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ (via Taylor expansion) by defining

$$\mathrm{fil}\left(\sum c_\alpha x^\alpha\right) = \min\{\phi(\alpha) \;:\; c_\alpha \neq 0\}. \tag{2.2}$$

We denote the set of g with $\mathrm{fil}(g) \geqslant l$ in $\mathcal{C}_n$ by $\mathcal{A}_l$. The number $\mathrm{fil}(g)$ will be also called the level of g with respect to $\mathcal{A}$.

Now we introduce the control functions associated to $\mathcal{A}$ as follows:

$$\rho(x) = \Bigg(\sum_{\alpha \in \mathrm{Ver}(\mathcal{A})} x^{2p\alpha}\Bigg)^{\frac{1}{2p}} \quad \text{and} \quad \overline{\rho}(x) = \sum_{\alpha \in \mathrm{Ver}(\mathcal{A})} x^{2p\alpha}, \tag{2.3}$$

where p a positive integer. Moreover if p is big enough (it suffices, for example, that $p\alpha \in \mathbb{Z}^n_+$), $\overline{\rho}$ will be C^w.

Note that for an element $g = \sum c_\alpha x^\alpha \in \mathcal{C}_n$, the support of g is $\mathrm{supp}(g) = \{\alpha : c_\alpha \neq 0\}$; it is clear that $g \in \mathcal{A}_l$ if and only if $\mathrm{supp}(g) \subset \Gamma_+(l\mathcal{A})$ which is also equivalent to $|g| \lesssim \rho^l$ (see [1, 5] for details). Thus $\mathcal{A}_l$ can be written as

$$\mathcal{A}_l = \{\, g \in \mathcal{C}_n \,:\, \mathrm{supp}(g) \subset \Gamma_+(l\mathcal{A})\} = \{\, g \in \mathcal{C}_n \,:\, |g| \lesssim \rho^l\}. \tag{2.4}$$

We say that an analytic function germ $g \in \mathcal{C}_n$ is an $\mathcal{A}$-form of degree d if $\mathrm{supp}(g) \subset \Gamma(d\mathcal{A})$ (i.e., $g \in \mathcal{A}_d \setminus \mathcal{A}_{d+1}$). Furthermore, for $f \in \mathcal{C}_n$, we denote the Taylor expansion of $f(x)$ at the origin by $\sum_\nu c_\nu\, x^\nu$. Setting

$$H_j(x) = \sum_{\nu \in \Gamma(j\mathcal{A})} c_\nu x^\nu, \quad j \in \mathbb{Z}_+,$$

we can write $f(x) = \sum_j H_j(x)$ (Newton filtration), where H_j is $\mathcal{A}$-form of degree j. Also if $\#\mathcal{F}(\mathcal{A}) = 1$, we can replace the Newton filtration associated with $\mathcal{A}$ by the weighted filtration associated to w^F. Moreover, if $w^F = (1, \dots, 1)$, this Newton filtration coincides with the usual filtration.

2.1. Compensation factor.

Let $\rho_i \colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a continuous function. We say that ρ_i is the ith compensation factor associated with $\mathcal{A}$ if for each $g \in \mathcal{C}_n$, we have that $|\rho_i \partial_{x_i} g| \lesssim \rho^{\mathrm{fil}(g)}$. Next we give some examples of compensation factors associated with $\mathcal{A}$.

(i) Here, we have the trivial example for the compensation factors, given by

$$\rho_i(x) = x_i \quad \text{for} \quad i = 1, \dots, n.$$

(ii) Let $L_j = L(x_j)$ denote the x_j-axis. We then put $\alpha^j = L_j \cap \Gamma(\mathcal{A})$ for $j = 1, \dots, n$ (the axial vertices of $\Gamma(\mathcal{A})$). We define the weight of the variable x_i, $\mathcal{A}(i) = \mathcal{A}(x_i) = \max\{\, w_i^F : F \in \mathcal{F}(\mathcal{A})\}$. We may introduce the compensation factors as follows:

$$\rho_i(x) = \Bigg(x_i^{\frac{2p}{\mathcal{A}(i)}} + \sum_{\alpha \in \mathrm{Ver}(\mathcal{A}) \setminus \{\alpha^i\}} x^{2p\,\alpha} \Bigg)^{\frac{\mathcal{A}(i)}{2p}}, \quad i = 1, \dots, n.$$

It is easy to check that these functions ρ_i are compensation factors associated with $\mathcal{A}$ (see [1, 11] for details).

(iii) The following compensation factors are inspired by the work of Damon-Gaffney in [5]. For all integers $l \geqslant 0$, we let

$$R_{l,i} = \{\alpha \in \mathbb{Q}_+^n \ : \ \langle \alpha\, , w^F \rangle \geqslant l + w_i^F, \ \forall F \in \mathcal{F}(\mathcal{A})\} \quad \text{for } i = 1, \ldots, n.$$

We may introduce the compensation factors as follows:

$$\rho_{l,i}(x) = \Big(\sum_{\alpha \in Ver(R_{l,i})} \frac{x^{2\alpha}}{\rho^{2l}} \Big)^{\frac{1}{2}}, \quad i = 1, \ldots, n.$$

It is easy to see that for any integers $l \geqslant 0$, we have that $\rho_{l,i}(x) \lesssim \rho^{m_i}(x)$, where $m_i = \min_{F \in \mathcal{F}(\mathcal{A})} \{w_i^F\}$, which implies that $\rho_{l,i}$ is continuous at the origin. On the other hand, by the construction of $\rho_{l,i}$ we can deduce that $|\rho_{l,i} \partial_{x_i} g| \lesssim \rho^{\mathrm{fil}(g)}$ for all $g \in \mathcal{C}_n$. Hence, we get that these functions $\rho_{l,i}$ are compensation factors associated with $\mathcal{A}$.

Observation.— We should note that in the case where $\# \mathcal{F}(\mathcal{A}) = 1$ (i.e., weighted filtration associated with $w = (w_1, \ldots, w_n)$), the natural choice of compensation factor is that given by L. Paunescu in [10] as follows:

$$\rho_i = \rho^{w_i} \ \text{ for } \ i = 1, \ldots, n.$$

Moreover, for any other compensation factors $\xi_1, \ldots, \xi_n$ associated with the weighted filtration, we have that $\xi_i \lesssim \rho^{w_i}$, $i = 1, \ldots, n$. Unfortunately, in the general case we have not succeeded in finding the best compensation factors $\rho_1, \ldots, \rho_n$ such that for any other compensation factors $\xi_1, \ldots, \xi_n$, we have that $\xi_i \lesssim \rho_i$. However, for each $\gamma \in \mathbb{Q}_+^n$ such that the monomial x^γ is ith compensation factor, we have $|x^\gamma| \lesssim \rho_{l,i}$, where $\rho_{l,i}$ are the compensation factors defined in (iii).

Now we fix the compensation factors ρ_i for $i = 1, \ldots, n$ relative to the Newton filtration, and consider the singular metric of $M = \mathbb{R}^{n+m}$ defined by

$$\langle \rho_i(x) \frac{\partial}{\partial x_i}\, , \rho_j(x) \frac{\partial}{\partial x_j} \rangle = \delta_{i,j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$
$$\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial t_j} \rangle = 0 \text{ and } \langle \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \rangle = \delta_{i,j}.$$

Here, $(x, t) = (x_1, \ldots, x_n, t_1, \ldots, t_p)$ denotes a system of coordinates of $\mathbb{R}^{n+m}$. By elementary calculation we have

$$\langle dx_{i_1} \wedge \cdots \wedge dx_{i_k}\, , dx_{i_1} \wedge \cdots \wedge dx_{i_k} \rangle = \rho_I := \rho_{i_1} \cdots \rho_{i_k}. \tag{2.5}$$

2.2. (w)-regularity associated with $\mathcal{A}$.

Let $F: (\mathbb{R}^n \times \mathbb{R}^m, \{0\} \times \mathbb{R}^m) \to (\mathbb{R}^p, 0)$ be analytic. We next assume that

(2.6) $Y = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}^m : x_1 = \cdots x_n = 0\}$ and $X = F^{-1}(0) - Y.$

Setting $F := (F_1, \dots, F_p)$, assume that the Jacobi matrix of F has rank k on X near 0, where $k \leqslant p$ is the codimension of X in $\mathbb{R}^{n+m}$. We note that the normal space of X is generated by the gradient of the functions F_j $(j = 1, \dots, p)$ at each $P \in X$ near 0. Following [6], we define $\|dF\|_{\mathcal{A}}$, $\|d_x F\|_{\mathcal{A}}$, $\|d^x F\|_{\mathcal{A}}$ and $D_{\mathcal{A}}(\ell)$ by the following formulae:

(2.7)

$$\|dF\|_{\mathcal{A}}^2 = \sum_J \|dF_J\|_{\mathcal{A}}^2 \quad \text{where } \|dF_J\|_{\mathcal{A}}^2 = \sum_{I,S} \left(\rho_I \left| \frac{\partial F_J}{\partial(x_I, t_S)} \right| \right)^2,$$

$$\|d_x F\|_{\mathcal{A}}^2 = \sum_J \|d_x F_J\|_{\mathcal{A}}^2 \quad \text{where } \|d_x F_J\|_{\mathcal{A}}^2 = \sum_{I} \left(\rho_I \left| \frac{\partial F_J}{\partial x_I} \right| \right)^2,$$

$$\|d^x F\|_{\mathcal{A}}^2 = \sum_J \|d^x F_J\|_{\mathcal{A}}^2 \quad \text{where } \|d^x F_J\|_{\mathcal{A}}^2 = \sum_{I,S:\, S \neq \emptyset} \left(\rho_I \left| \frac{\partial F_J}{\partial(x_I, t_S)} \right| \right)^2$$

and

$$(2.8) \qquad D_{\mathcal{A}}(\ell) = \sum_J \sum_{I,S:\, \#S = \ell} \left(\rho_I \left| \frac{\partial F_J}{\partial(x_I, t_S)} \right| \right)^2. \quad \text{Here } \rho_I = \prod_{i \in I} \rho_i.$$

We first remark that $\langle dF\,,\, dF \rangle = \|dF\|_{\mathcal{A}}^2$ and $\langle d_x F\,,\, d_x F \rangle = \|d_x F\|_{\mathcal{A}}^2$.

Now using the above construction, we state the version relative to the Newton filtration of the Fukui-Paunescu Theorem ([6], Theorem 2.1).

THEOREM 4.— *The following conditions are equivalent*

(i) $D_{\mathcal{A}}(m) \lesssim D_{\mathcal{A}}(m-1) \lesssim \cdots \lesssim D_{\mathcal{A}}(1) \lesssim D_{\mathcal{A}}(0)$ *holds on* X *near* 0.

(ii) $\|d^x F\|_{\mathcal{A}} \lesssim \|d_x F\|_{\mathcal{A}}$ *holds on* X *near* 0.

(iii) *For any* $C1$*-functions* φ_j $(j = 1, \dots, p)$ *near* 0, *and* $s = 1, \dots m$,

$$\left| \sum_{j=1}^{p} \varphi_j \frac{\partial F_j}{\partial t_s} \right| \lesssim \sum_{i=1}^{n} \rho_i \left| \sum_{j=1}^{p} \varphi_j \frac{\partial F_j}{\partial x_i} \right| \quad \text{holds on } X \text{ near } 0.$$

(iv) *For* $J \subset \{1,\dots,p\}$, $I = \{i_1,\dots,i_{k-1}\} \subset \{1,\dots,n\}$ *with* $1 \leqslant i_1 < \cdots < i_{k-1} \leqslant n$, $s = 1,\dots,m$,

$$\rho_I \left| \frac{\partial F_J}{\partial(x_I, t_s)} \right| \lesssim \|d_x F\|_{\mathcal{A}} \quad \textit{holds on } X \textit{ near } 0.$$

(v) *For* $J \subset \{1,\dots,p\}$, $i = 1,\dots,n$, $s = 1,\dots,m$,

$$|\langle dF_J \wedge dx_i\,,\, dF_J \wedge dt_s\rangle| \lesssim \rho_i \|d_x F\|^2_{\mathcal{A}} \quad \textit{holds on } X \textit{ near } 0.$$

(vi) *For some positive* $C1$*-functions* ϕ_J *on* X *with* $J \subset \{1,\dots,p\}$, $i = 1,\dots,n$, $s = 1,\dots,m$,

$$\left| \sum_J \phi_J \langle dF_J \wedge dx_i\,,\, dF_J \wedge dt_s\rangle \right| \lesssim \rho_i \sum_J \phi_J \|d_x F\|^2_{\mathcal{A}} \quad \textit{holds on } X \textit{ near } 0.$$

Proof. — The proof is similar to that of Fukui-Paunescu in [6]; it is enough to replace the $\|x\|_w^{w_i}$ (resp. $\|x\|_w^{w_I}$) in the proof of Theorem 2.1 [6] by the ρ_i (resp. ρ_I). □

We say that X is (w)-regular over Y at 0 with respect to $\mathcal{A}$ (or $w^{\mathcal{A}}$-regular), if one of the above equivalent conditions holds. When $\#\mathcal{F}(\mathcal{A}) = 1$, we find that $\rho_i(x) = \rho^{w_i^F}(x)$ for $i = 1,\dots,n$, hence our $(w^{\mathcal{A}})$-regularity reduces to the weighted (w)-regularity (see [6]). Moreover, if $w^F = (1,\cdots,1)$, these coincide with the usual (w)-regularity (Verdier's regularity).

We shall prove the following theorem.

THEOREM 5. — *For* X, Y *as above, if* (X,Y) *is* $(w^{\mathcal{A}})$*-regular, then* (X,Y) *is* (c)*-regular for the control function* $\overline{\rho}$ *(we recall that* $\overline{\rho}(x) = \sum_{\alpha \in \mathrm{Ver}(\mathcal{A})} x^{2p\alpha}$*).*

REMARK 6. — *The converse of the theorem is false in general: (Kuo's example [8])*

$$F(x,y,t) = y^2 - tx^2 - x^5, \quad X = \{y^2 = tx^2 + x^5\} - \{0\} \times \mathbb{R} \ \textit{ and } Y = \{0\} \times \mathbb{R}.$$

We consider the usual filtration $(\mathcal{A} = \{(1,0);(0,1)\})$*. It is easy to see that* (X,Y) *is* (c)*-regular at* 0 *for the control function* $\overline{\rho}(x,y) = x^2 + y^2$*, but that* (X,Y) *is not Verdier* (w)*-regular at* 0 *(see [14] for details).*

As an immediate corollary we have

COROLLARY 7. — *Let $f_t\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$, $t\in\mathbb{R}^m$ be a family of weighted homogeneous polynomials defining an isolated singularity at the origin. We set $F(x,t)=f_t(x)$, then the stratification $\Sigma(V_F)$ is (c)-regular.*

(we again recall that $\Sigma(V_F)=\{F^{-1}(0)-\{0\}\times\mathbb{R}^m,\ \{0\}\times\mathbb{R}^m\}$)

Proof. — Let us put $X=F^{-1}(0)-\{0\}\times\mathbb{R}^m$ and $Y=\{0\}\times\mathbb{R}^m$. Consider the weighted filtration associated with $\mathcal{A}=\{(\frac{1}{w_1},0,\cdots,0),\ldots,(0,\cdots,0,\frac{1}{w_n})\}$ such that f_t is a weighted homogeneous polynomial with the weight $w=(w_1,\cdots,w_n)\in\mathbb{Z}^n_+$. Now from the Theorem 5, it is enough to show that (X,Y) is $(w^{\mathcal{A}})$-regular, that is,

$$(2.9)\qquad |\partial_t F|\lesssim\|d_xF\|_{\mathcal{A}}\ \text{ holds on } X \text{ near } Y.$$

Since f_t defines an isolated singularity at the origin, we can see that $\|d_xF\|^2_{\mathcal{A}}=\sum_{i=1}^n(\rho^{w_i}\frac{\partial F}{\partial x_i})^2$ is not zero outside the origin, and this implies our inequality. □

COROLLARY 8. — *Let $f_t\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$, $t\in\mathbb{R}^m$ be a real analytic family non-degenerate (in the sense of Kouchnirenko [7]) and $\Gamma(f_t)=\Gamma(f_0)$, then the stratification $\Sigma(V_F)$ is (c)-regular.*

Proof. — By standard argument, based on the curve selection lemma, we can see that

$$|\partial_t F|\lesssim\sum_{\alpha\in\mathrm{Ver}(\Gamma(f_0))}|x^\alpha|\lesssim\sum_{i=1}^n|x_i\frac{\partial F}{\partial x_i}|.$$

Therefore, (X,Y) is $(w^{\mathcal{A}})$-regular for any Newton filtration. In particular, (X,Y) is usual (w)-regular (Verdier's regular). □

Before starting the proofs of the above results, we will first illustrate these results with several examples.

EXAMPLE 9 (Briançon-Speder family [4]). — *Let $f_t\colon(\mathbb{R}^3,0)\to(\mathbb{R},0)$, $t\in J=[-1,1]$, be a family of weighted homogeneous polynomials defined by*

$$f_t(x,y,z)=z^5+t\,y^6z+x\,y^7+x^{15}.$$

We set $F(x,t)=f_t(x)$, $Y=\{0\}\times J$ and $X=F^{-1}(0)-Y$. It is easy to check that $|\partial_tF|\lesssim\|d_xF\|_{\mathcal{A}}$ holds on X near 0, where $\mathcal{A}=$

$\{(1,0,0),\ (0,\frac{1}{2},0),\ (0,0,\frac{1}{3})\}$. *Thus, by Theorem 5, we have that* (X,Y) *is* (c)*-regular for the function* $\overline{\rho}(x,y,z) = x^{12} + y^6 + z^4$. *(It is well known that* f_t *is not Whitney regular and not usual* (w)*-regular).*

EXAMPLE 10 (Oka family [9]). — *Let* $f_t\colon (\mathbb{R}^3,0) \to (\mathbb{R},0)$, $t \in J = [-1,1]$, *be a family of polynomial functions defined by*

$$f_t(x,y,z) = x^8 + y^{16} + z^{16} + t\,x^5 z^2 + x^3 y\, z^3.$$

We set $F(x,t) = f_t(x)$, $Y = \{0\} \times J$, $X = F^{-1}(0) - Y$ *and*

$$\mathcal{A} = \left\{ (\frac{1}{2},0,0),\ (0,1,0),\ (0,0,1),\ (\frac{5}{16},0,\frac{1}{8}) \right\}.$$

It is not hard to see that the inequality $|\partial_t F|^2 \lesssim \|d_x F\|_{\mathcal{A}}^2 = \sum_{i=1}^n (\rho_i \frac{\partial F}{\partial x_i})^2$ *holds on* X *near* Y, *where* ρ_i *denotes the* i*th compensation factor of type (ii) as defined in 2.1. It follows from Theorem 5 that* (X,Y) *is* (c)*-regular for the control function* $\overline{\rho}(x,y,z) = x^{16} + y^{32} + z^{32} + x^{10} z^4$.

2.3. Proof of Theorem 5.

In order to show this theorem we need the following lemma.

LEMMA 11.

(1) $\|d\overline{\rho}\|_{\mathcal{A}} \lesssim \overline{\rho}(x)$, x *near* 0,

(2) $\overline{\rho} \ll \frac{\|dF\wedge d\overline{\rho}\|}{\|dF\|}$ *when* $(x,t) \to 0$ *on* X.

Proof. — We first recall that:

$$\|d\overline{\rho}\|_{\mathcal{A}}^2 = \sum_{i=1}^n \left(\rho_i \frac{\partial \overline{\rho}}{\partial x_i}(x) \right)^2 .$$

Therefore, (1) is a simple consequence of the construction of the compensation factors and the control functions.

Let us observe that, by (1.2) we have $|\partial \overline{\rho}_{|X}| = \frac{\|dF\wedge d\overline{\rho}\|}{\|dF\|}$. On the other hand, $\partial\overline{\rho} = \partial\overline{\rho}_{|X} + \partial\overline{\rho}_{|N}$ (where N denotes the normal space to X). Since N is generated by the gradients of F_j ($j = 1,\dots,p$), we have that $\partial\overline{\rho}_{|X} = \partial\overline{\rho} + \eta_1 \partial F_1 \cdots + \eta_p \partial F_p$. After this, (2) in the lemma, follows from the following more general proposition.

PROPOSITION 12. — *Let* $f\colon(\mathbb{R}^n,0)\to(\mathbb{R}^r,0)$, $g\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$ *be two germs of analytic maps, setting* $f := (f_1,\dots,f_r)$. *Then there exists a real constant* C *such that for* $p\in f^{-1}(0)$, *and sufficiently close to the origin,*

$$(2.10)\qquad |g(p)|\leqslant C\mid p\mid \inf_{(\eta_1,\dots,\eta_r)\in\mathbb{R}^r}|\eta_1\partial f_1(p)+\cdots+\eta_r\partial f_r(p)+\partial g(p)|.$$

We note that if $r=1$, one finds Theorem 1.1 of Adam Parusiński [12]. Moreover, the proof of this proposition is similar to that of Theorem 1.1 in [12] (we omit the details). □

Now we are ready to prove Theorem 5. We assume that (X,Y) is $(w^{\mathcal{A}})$-regular at 0. By inequality (iii) in Theorem 4, we have

$$(2.11)\qquad \left|\frac{\partial F_J}{\partial(x_I,t_S)}\right|\lesssim\sum_{i=1}^n\rho_i\left|\frac{\partial F_J}{\partial(x_I,t_{\hat S},x_i)}\right|\quad\text{on } X \text{ near } 0,$$

where $\hat S\subset S$ such that $\#\hat S=\#S-1$. Thus we obtain $\|d^xF\|\ll\|dF\|$ when $(x,t)\to 0$ on X (i.e., (X,Y) is (a)-regular at 0), and so by Theorem 2, we only have to prove that:

$$(2.12)\qquad |\partial_t\overline{\rho}_{|X}|\ll\frac{\|dF\wedge d\overline{\rho}\|}{\|dF\|}\quad\text{as } (x,t)\in X,\ (x,t)\to 0.$$

We first remark, by (1.1) the following equality:

$$|\partial_{t_\eta}\overline{\rho}_{|X}|=\left|\sum_J\sum_{I,S}\frac{\frac{\partial(F_J,t_\eta)}{\partial(x_I,t_S,t_\eta)}\ \frac{\partial(F_J,\overline{\rho})}{\partial(x_I,t_S,t_\eta)}}{\|dF\|^2}\right|,$$

and hence

$$(2.13)\qquad |\partial_{t_\eta}\overline{\rho}_{|X}|\lesssim\left|\sum_J\sum_{I,S}\frac{\frac{\partial(F_J,\overline{\rho})}{\partial(x_I,t_S,t_\eta)}}{\|dF\|}\right|.$$

According to the inequality in (iii) of Theorem 4, we have

$$\left|\frac{\partial(F_J,\overline{\rho})}{\partial(x_I,t_S,t_\eta)}\right|\lesssim\sum_{i=1}^n\rho_i\left(\left|\frac{\partial(F_J,\overline{\rho})}{\partial(x_I,t_S,x_i)}\right|+\left|\frac{\partial\overline{\rho}}{\partial x_i}\right|\left|\frac{\partial F_J}{\partial(x_I,t_S,)}\right|\right).$$

Thus, we obtain

$$\left| \frac{\partial(F_J, \overline{\rho})}{\partial(x_I, t_S, t_\eta)} \right| \lesssim \|d\overline{\rho}\|_{\mathcal{A}} \, \|dF\| + \sum_{i=1}^{n} \rho_i \| \, dF \wedge d\overline{\rho}\|$$

and, using (2.13), we obtain

$$|\partial_t \overline{\rho}_{|X}| \lesssim \|d\overline{\rho}\|_{\mathcal{A}} + \sum_{i=1}^{n} \rho_i \frac{\| \, dF \wedge d\overline{\rho}\|}{\|dF\|} \quad \text{on } X \text{ near } 0. \tag{2.14}$$

It follows from Lemma 11 that (2.12) holds. This completes the proof of Theorem 5.

3. The Damon-Gaffney condition and (c)-regularity.

In this section we describe some definitions and notations used by Damon-Gaffney in [5].

Given a Newton filtration $\mathcal{A}$ as above. We extend this filtration on the ring $\mathcal{C}_{x,t}$ of formal power series in the variables $x_1, \dots, x_n \, ; \, t_1, \dots, t_m$ around the origin by defining

$$\operatorname{fil}(\sum_{\nu} c_\nu(t) x^\nu) = \min\{\phi(\nu) \; : \; c_\nu(t) \neq 0\}. \tag{3.1}$$

Let $g = \sum_\nu c_\nu(t) x^\nu$ be a series in $\mathcal{C}_{x,t}$, the support of g, denoted by $\operatorname{supp}(g)$, is the set of points $\nu \in \mathbb{Z}_+^n$ such that $c_\nu(t) \neq 0$. We denote the set of g with $\operatorname{fil}(g) \geqslant l$ in $\mathcal{C}_{x,t}$ by $\mathcal{A}_{l,x,t}$. It is not difficult to see the following equality:

$$\mathcal{A}_{l,x,t} = \{ \, g \in \mathcal{C}_{x,t} \; : \; \operatorname{supp}(g) \subset \Gamma_+(l\mathcal{A})\} = \{ \, g \in \mathcal{C}_{x,t} \; : \; |g| \lesssim \rho^l \}. \tag{3.2}$$

We say that level $\mathcal{A}_l$ of the Newton filtration is fit if all the vertices of $\phi^{-1}(l)$ are lattice points of $\mathbb{R}_+^n$. This says that $l \operatorname{Ver}(\mathcal{A}) = \operatorname{Ver}(l\mathcal{A}) \in \mathbb{Z}_+^n$ (because of the linearity of the Newton filtration on cones). For $\mathcal{A}_l$ which is fit, we let

$$\operatorname{ver}(A_l) = \{x^\beta \; : \; \beta \text{ is a vertex of } \phi^{-1}(l)\} = \{x^{l\alpha} \; : \; \alpha \in \operatorname{Ver}(\mathcal{A})\}. \tag{3.3}$$

We also let

$$\mathcal{V}_{l,x,t} = \left\{ \zeta \in \mathcal{A}_{l+1,x,t}\{\partial/\partial x_i\} \; : \; \zeta(\mathcal{A}_{k,x,t}) \subset \mathcal{A}_{l+k,x,t} \right\}, \tag{3.4}$$

with $\mathcal{A}_{l+1,x,t}\{\partial/\partial x_i\}$ denoting the $\mathcal{A}_{l+1,x,t}$-module generated by the $\partial/\partial x_i$, $i=1,\dots,n$. Finally, for an element $g \in \mathcal{C}_{x,t}$, we let $\mathcal{V}_{l,x,t}(g) = \{\zeta(g) : \zeta \in \mathcal{V}_{l,x,t}\}$.

Now we can announce the Damon-Gaffney Theorem.

THEOREM 13 (Damon-Gaffney [5]). — *Let $f\colon (\mathbb{R}^{n+m},0) \to (\mathbb{R},0)$ be an analytic deformation of a germ $f_0\colon (\mathbb{R}^n,0) \to (\mathbb{R},0)$ (i.e.,$f \in \mathcal{C}_{x,t}$). Then a sufficient condition that f be a topologically trivial deformation is that there exists a fit $\mathcal{A}_l$ so that*

$$\mathrm{ver}(\mathcal{A}_l) \cdot \frac{\partial f}{\partial t_j} \subset \mathcal{V}_{l,x,t}(f),\ j=1,\dots,m. \tag{3.5}$$

We will call condition (3.5) the Damon-Gaffney condition. Next, our principal goal will be to show that this condition implies a (w)-regularity condition relative to the Newton filtration, hence, these deformations will, in fact, satisfy the Bekka condition.

Given an analytic function $f \in \mathcal{C}_{x,t}$, we define

$$\Sigma_f(\mathbb{R}^n \times \mathbb{R}^m) = \{\, \mathbb{R}^n \times \mathbb{R}^m - f^{-1}(0),\ f^{-1}(0) - \{0\} \times \mathbb{R}^m,\ \{0\} \times \mathbb{R}^m \,\},$$

which gives a stratification of $\mathbb{R}^n \times \mathbb{R}^m$ around $\{0\} \times \mathbb{R}^m$. Then, we have

THEOREM 14. — *For $f \in \mathcal{C}_{x,t}$, if there is a positive integer l such that*

$$\mathrm{ver}(\mathcal{A}_l) \cdot \frac{\partial f}{\partial t_j} \subset \mathcal{V}_{l,x,t}(f),\ j=1,\dots,m \ \text{(The Damon-Gaffney condition)},$$

then the stratification $\Sigma_f(\mathbb{R}^n \times \mathbb{R}^m)$ is (c)-regular.

Proof. — Let us put $\mathrm{ver}(\mathcal{A}_l) = \{x^\alpha\}$ then we get the following expression:

$$x^\alpha \frac{\partial f}{\partial t_j} = \sum_{i=1}^n \xi_{ij}^{(\alpha)} \frac{\partial f}{\partial x_i} = \xi_j^{(\alpha)}(f),$$

and summing over $x^\alpha \in \mathrm{ver}(\mathcal{A}_l)$ we obtain

$$\Big(\sum_{\alpha \in \mathrm{Ver}(l\mathcal{A})} |x^\alpha| \Big) \left| \frac{\partial f}{\partial t_j} \right| \lesssim \sum_{i=1}^n \Big(\sum_{\alpha \in \mathrm{Ver}(l\mathcal{A})} |\xi_{ij}^{(\alpha)}| \Big) \left| \frac{\partial f}{\partial x_i} \right|. \tag{3.6}$$

Since $\mathrm{Ver}(l\,\mathcal{A}) = l\,\mathrm{Ver}(\mathcal{A})$, which means $\rho^l \sim \sum_{\alpha\in\mathrm{Ver}(l\mathcal{A})} |x^\alpha|$. Then we let

$$\xi_i' = \sum_{j=1}^{m} \sum_{\alpha\in\mathrm{Ver}(l\mathcal{A})} \rho^{-l}|\xi_{ij}^{(\alpha)}| \ \text{ for } i = 1,\ldots,n.$$

It follows from (3.6) that $|\partial_t f|^2 \lesssim \sum_{i=1}^n (\xi_i' \frac{\partial f}{\partial x_i})2$, and so by Theorem 5, it is sufficient to show that these ξ_i' are compensation factors associated with $\mathcal{A}$. Indeed, for any $g \in \mathcal{C}_n$, we have from the filtration properties of the $\xi_{ij}^{(\alpha)}$ that

$$\mathrm{fil}(\xi_{ij}^{(\alpha)}(g)) = \mathrm{fil}(\xi_{ij}^{(\alpha)}\partial_{x_i} g) \geqslant \mathrm{fil}(g) + l$$

which means

$$|\xi_{ij}^{(\alpha)}\partial_{x_i} g)| \lesssim \rho^{l+\mathrm{fil}(g)}.$$

Therefore, for $i = 1,\ldots,n$,

$$|\xi_i'\partial_{x_i} g| \lesssim \rho^{\mathrm{fil}(g)}.$$

This completes the proof of the Theorem □

REMARK 15. — *We observe that* $\zeta = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} \in \mathcal{V}_{l,x,t}$ *if and only if* $supp(\xi_i) \subset R_{l,i}$ *(we recall that* $R_{l,i} = \{\alpha \in \mathbb{Q}_+^n \ : \ \langle\alpha\,,\,w^F\rangle \geqslant l + w_i^F\,,\ \forall\, F \in \mathcal{F}(\mathcal{A})\}$*) which is also equivalent to* $|\xi_i| \lesssim \sum_{\alpha\in Ver(R_{l,i})} |x^\alpha|$*. Hence, the Damon-Gaffney condition implies a* $(w^{\mathcal{A}})$*-regularity condition with* $\rho_{l,i}$ *as compensation factors, where* $\rho_{l,i}$ *denotes the* i*th compensation factor of type (iii) as defined in 2.1.*

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