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REGULAR PROJECTIVELY ANOSOV FLOWS WITH COMPACT LEAVES

by Takeo NODA[(1)]

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1. Introduction and the statement of the result.

In [ET], Eliashberg and Thurston investigated relations between codimension 1 foliations and contact structures on 3-manifold, using the theory of *confoliations*. They proved that on a closed oriented 3-manifold every codimension 1 foliation of class C^r $(r \geq 2)$ except for the product foliation on $S^2 \times S^1$ can be C^0-approximated by a positive (and a negative) contact structure.

As a special case of such approximations, they defined a *linear deformation* of a foliation, that is, a 1-parameter family $\{\xi_t\}$ of plane fields defined by 1-forms α_t satisfying

$$\alpha_0 \wedge \mathrm{d}\alpha_0 = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}(\alpha_t \wedge \mathrm{d}\alpha_t)_{|t=0} \neq 0$$

for any point of the manifold.

A typical example of linear deformations is given by Anosov foliations. Y. Mitsumatsu observed in [Mi1] that both of the weak unstable and stable foliations of an Anosov flow are linearly deformed into a positive and a negative contact structure which intersect transversely at the tangent space of the flow, where such a pair of contact structures is called a *bi-contact*

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structure. However, a bi-contact structure is not always induced by an Anosov flow, so he defined a *projectively Anosov flow* so as to be equivalent to the flow on the intersection of a bi-contact structure (it is also equivalent to a *conformally-Anosov flow* defined in [ET]).

A non-singular flow ϕ^t on a closed oriented 3-manifold M is a projectively Anosov flow if there exist a Riemannian metric on M, a transversely oriented continuous splitting $\mathcal{E}^u \oplus \mathcal{E}^s$ of $TM/T\phi$ which is invariant under the action of ϕ^t, and a positive real number C such that

$$\frac{\|\mathrm{d}\phi^t(v^u)\|}{\|\mathrm{d}\phi^t(v^s)\|} \geq \mathrm{e}^{Ct}\frac{\|v^u\|}{\|v^s\|}$$

holds for all $t \geq 0$, $v^u \in \mathcal{E}^u$ and $v^s \in \mathcal{E}^s \setminus 0$.

The invariant subbundles $\mathcal{E}^u$ and $\mathcal{E}^s$ naturally induce invariant plane fields E^u and E^s on M. Like the Anosov cases, these plane fields are continuous and integrable, but unlike the Anosov cases the integral manifolds may not be determined uniquely in general. From a viewpoint of linear deformations of foliations, it is important to study the cases where the plane fields E^u and E^s are smooth. We call such a projectively Anosov flow *regular.* In the case of a regular projectively Anosov flow, the plane fields E^u and E^s determine smooth codimension 1 foliations $\mathcal{F}^u$ and $\mathcal{F}^s$, which are called the *unstable foliation* and the *stable foliation*, respectively. One of the greatest differences from Anosov flows is the fact that these foliations may contain compact leaves, which are necessarily homeomorphic to tori.

Anosov flows with smooth invariant foliations are classified by Ghys [Gh1], [Gh2]. They are either the suspension flows of hyperbolic diffeomorphisms of T^2 or quasi-Fuchsian flows on Seifert manifolds. As to projectively Anosov flows, some classification results on several manifolds have been known. The author studied in [Nd] the regular projectively Anosov flows on T^2-bundles over S^1 whose unstable or stable foliation contains a compact leaf and proved that such flows are decomposed into components called *$T^2 \times I$-models.* T. Tsuboi and the author investigated in [NT] the regular projectively Anosov flows without compact leaves on T^2-bundles over S^1 with hyperbolic monodromy and the unit tangent bundle of a closed surface and showed that such flows are actually Anosov flows. In particular, these results complete the classification of the regular projectively Anosov flows on T^2-bundles over S^1. They are either the unions of $T^2 \times I$-models or the suspension Anosov flows. In the recent paper [Ts],

T. Tsuboi has proved that if the unstable and stable foliations of a regular projectively Anosov flow on a closed oriented Seifert manifold contain no compact leaves then the flow is isotopic to a quasi-Fuchsian flow.

The main result of this paper is a classification theorem on Seifert manifolds in the case the invariant foliations contain compact leaves. Hence this completes the classification on Seifert manifolds.

THEOREM 1.1. — *Let M be a closed oriented Seifert manifold and ϕ^t a regular projectively Anosov flow with unstable foliation $\mathcal{F}^u$ and stable foliation $\mathcal{F}^s$. Suppose that one of these foliations contains a compact leaf. Then M is homeomorphic to the 3-torus and ϕ^t can be represented as a finite union of $T^2 \times I$-models.*

Remark 1.2. — The regularity of projectively Anosov flows means by definition that the invariant foliations are of class $\mathcal{C}^\infty$, but in the proof of Theorem 1.1 we use only that they are of class $\mathcal{C}^2$.

Remark 1.3. — M. Asaoka has given recently in [A] a complete classification of regular and non-degenerate projectively Anosov flows and proved that such a flow is either an Anosov flow or a finite union of $T^2 \times I$-models. Here, a flow is *non-degenerate* if all periodic orbits are hyperbolic, so flows with an invariant torus where the restricted flow is conjugate to a rational linear flow are excluded.

This paper is organized as follows. In Section 2, we review some known results about regular projectively Anosov flows and introduce $T^2 \times I$-models, the fundamental examples.

In Section 3, we see that each leaf of the invariant foliations is either vertical tori or horizontal. To do this, we classify the foliations on Seifert manifolds with compact leaves which are incompressible tori.

In Section 4, we prove that the compact leaves of the unstable foliation and those of the stable foliation have no intersection. Therefore the underlying Seifert manifold is decomposed into compact Seifert manifolds bounded by vertical closed leaves of the invariant foliations.

In Section 5, we prove that each component bounded by closed leaves is isotopic to a $T^2 \times I$-model. To show this, we study the leaf spaces of the lifted foliations of the invariant foliations on the universal covering.

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2. Regular projectively Anosov flows.

Let M be a closed oriented 3-manifold and ϕ^t a regular projectively Anosov flow on M with the unstable foliation $\mathcal{F}^u$ and the stable foliation $\mathcal{F}^s$.

By the following result, we can see that each compact leaf of $\mathcal{F}^u$ and $\mathcal{F}^s$ is an incompressible torus in M.

PROPOSITION 2.1 (see [Mi2], [Nd]). — *The unstable and stable foliations of a regular projectively Anosov flow on a compact oriented 3-manifold do not contain Reeb components.*

Together with Novikov's theorem [Nv] (see also [Ta], for example), we can deduce the following.

COROLLARY 2.2. — *There is no regular projectively Anosov flow on S^3 and on $S^2 \times S^1$.*

Remark 2.3. — The assumption of regularity makes an essential restriction to the topology of M. Indeed, as Mitsumatsu mentioned in [Mi2] and [Mi3], we can see by the approximation theorem of Eliashberg and Thurston [ET] and Hardorp's theorem [H] any closed oriented 3-manifold admits a bi-contact structure and therefore a projectively Anosov flow.

We introduce a fundamental example of a projectively Anosov flow whose unstable and stable foliations contain compact leaves. This example is called a $T^2 \times I$*-model.*

Example 2.4 ($T^2 \times I$-model). — Let (x, y, z) be coordinates in $T^2 \times I$. Take two linear 1-forms $\omega_u = p_u \,\mathrm{d}x + q_u \,\mathrm{d}y$ and $\omega_s = p_s \,\mathrm{d}x + q_s \,\mathrm{d}y$ on T^2 such that $\omega_u \wedge \omega_s > 0$. Consider two foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ defined by the following two 1-forms on $T^2 \times I$

$$\alpha_u = (1 - z)\,\mathrm{d}z + f_u(z)\omega_u, \quad \alpha_s = z\,\mathrm{d}z + f_s(z)\omega_s,$$

where $f_u, f_s \colon [0,1] \to [0,1]$ are orientation preserving and reversing diffeomorphisms of an interval, respectively.

It is proved in [Nd] that a flow tangent to the intersection of the foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ is a projectively Anosov flow and $\mathcal{F}^u$ and $\mathcal{F}^s$ are the unstable and stable foliations (see Figure 1).

Note that $\{z = 0\}$ and $\{z = 1\}$ are compact leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$, so this model is not an Anosov flow. Remark that it is essential that $f_u'(0)$ and $f_s'(1)$ do not vanish, which implies that the linear holonomy groups along the compact leaves are non-trivial.

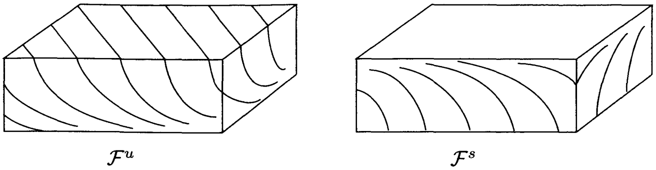

Figure 1

Gluing together a finite collection of such models, we can construct an example of a projectively Anosov flow on a T^2-bundle over S^1. Since the linear holonomy groups along the compact leaves are non-trivial, we have to take the same ω_u and ω_s for each model to make the resulting foliations differentiable. It follows that the monodromy of the bundle must contain at least two invariant directions and therefore it is isotopic to the identity or a hyperbolic automorphism.

In fact, we know that such examples on T^2-bundles are the only examples with compact leaves.

THEOREM 2.5 (see [Nd]). — *Let M be a T^2-bundle over S^1 and ϕ^t a regular projectively Anosov flow on M. If one of the invariant foliations of ϕ^t has a compact leaf, then ϕ^t is represented as a finite union of $T^2 \times I$-models. In particular, M is homeomorphic to either T^3 or T^2-bundle over S^1 with hyperbolic monodromy.*

3. Foliations on Seifert manifolds.

In order to understand the topology of the invariant foliations, we classify foliations on Seifert manifolds with compact leaves which are incompressible tori.

Let us recall some basic properties of Seifert manifolds (for details, see [O], [Sc]). A *trivial fibered solid torus* is a $D^2 \times S^1$ with the product foliation by circles $\{x\} \times S^1$ for $x \in D^2$ and a *fibered solid torus* can be obtained from a trivial fibered solid torus by cutting it open along $D^2 \times \{y\}$ for some $y \in S^1$, rotating q/p of a full turn, and glueing back together. A *Seifert manifold* is a compact 3-manifold M with a decomposition into disjoint circles, called *fibers*, such that each fiber has a neighborhood which consists entirely of fibers and is isomorphic to a fibered solid torus. A fiber of a Seifert manifold is called *regular* if its fibered neighborhood is trivial and *singular* otherwise. By the equivalent relation of identifying the points in a fiber, we obtain the *base* of the Seifert manifold, which is a compact 2-orbifold.

A surface embedded in a Seifert manifold is called *horizontal* if it is transverse to fibers at each point and *vertical* if it is a union of regular fibers. Note that a vertical surface must be a torus, a Klein bottle, or an annulus.

Foliations without compact leaves on a circle bundle over a surface are studied in [Th] and [L], which are generalized to those on Seifert manifolds in [EHN], [Ma], [Br1].

THEOREM 3.1. — *Let M be a closed oriented Seifert manifold whose base is not a torus or Klein bottle without singular point. Suppose that $\mathcal{F}$ is a codimension 1 transversely orientable $\mathcal{C}^r$-foliation ($r \geq 2$) on M without compact leaves. Then $\mathcal{F}$ is isotoped to a foliation by horizontal leaves.*

Remark 3.2. — Brittenham [Br1] proved this theorem for Seifert manifolds whose bases are S^2 with three singular points and the isotopies are given as $\mathcal{C}^0$-isotopies. In the other cases, the isotopies can be taken to be of class $\mathcal{C}^r$.

Now let us study foliations with compact leaves which are incompressible tori. If the underlying Seifert manifold is simultaneously a T^2-bundle over S^1 then we may assume that all compact leaves are isotopic to fibers of the bundle, after changing the bundle structures if necessary. Cutting the manifold along a compact leaf, we obtain a foliation on $T^2 \times I$ which is tangent to the boundary. Such foliations are classified in [MR], so we exclude T^2-bundles.

THEOREM 3.3. — *Let M be a closed oriented Seifert manifold which cannot be covered by a T^2-bundle over S^1, and $\mathcal{F}$ a codimension 1 transversely oriented foliation on M. Suppose $\mathcal{F}$ contains compact leaves each of which is an incompressible torus in M. Then $\mathcal{F}$ can be isotoped so that each leaf is either vertical or horizontal.*

This can be proved by a simple application of the following result by Brittenham. Here, a *Reeb sublamination* is a sublamination on $I \times (S^1 \times I)$ which contains at least one of the non-compact leaves of the foliation defined by a Reeb component on $S^1 \times I$ crossed with I, or its non-orientable analogues. For definitions and notations concerning essential laminations, see [GO].

THEOREM 3.4 (see [Br2]). — *Let M be an orientable compact Seifert manifold with non-empty boundary, and $\mathcal{L}$ an essential lamination in M, which is either transverse to, or contains as a leaf, each boundary component of M. Then, possibly after splitting $\mathcal{L}$ open along a finite number of leaves, either $\mathcal{L}$ can be isotoped so that each leaf is either vertical or horizontal, or it has finitely many Reeb sublaminations with horizontal boundary.*

Proof of Theorem 3.3. — Let L be a compact leaf of $\mathcal{F}$. By the theorem of Jaco (see [J], Theorem IV. 34), which characterizes two-sided incompressible surfaces embedded in Seifert manifolds, L satisfies one of the following cases:

(a) L is a fiber in a fibration of M as a T^2-bundle over S^1.

(b) $M = M_1 \cup M_2$ where $M_1 \cap M_2 = \partial M_1 = \partial M_2 = L$ and M_i ($i = 1, 2$) is homeomorphic to $K^2 \widetilde{\times} I$, a twisted I-bundle over the Klein bottle.

(c) L is a vertical torus.

We only have to consider the case (c) because in the other cases M is covered by a T^2-bundle. So each compact leaf is isotopic to a vertical one. We can decompose M by a finite collection of compact leaves $\{L_j\}$ into compact Seifert manifolds with non-empty boundary so that the restriction of $\mathcal{F}$ on each component is tangent to the boundary.

Let W be such a component, then we can split $\mathcal{F}|_W$ along a finite collection of leaves to yield an essential lamination $\mathcal{L}$. By Theorem 3.4, $\mathcal{L}$ is isotoped so that each leaf is either vertical or horizontal. Furthermore, we may assume the interstitial I-bundle $W \setminus \mathcal{L}$ is fibered by arcs which are either horizontal or vertical. Then we can crush each fiber to a point and retrieve $\mathcal{F}$ with a desired isotopy. □

Remark 3.5. — The isotopy given by the proof above is not differentiable in general, but we can prove this theorem also by a similar argument as in [L], where the isotopy has the same differentiability as that of the given foliation.

Now let us consider the invariant foliations of a regular projectively Anosov flow. Assume that at least one of them has a compact leaf, which is an incompressible torus. Then the projectively Anosov property makes some restrictions on the foliation in the neighborhood of each compact leaf. Comparing the holonomies of $\mathcal{F}^u$ and $\mathcal{F}^s$ along the orbits on a compact leaf, we can easily obtain the following.

LEMMA 3.6. — *Let ϕ^t be a regular projectively Anosov flow on a compact oriented 3-manifold and $\mathcal{F}^\sigma$ the unstable or stable foliation of ϕ^t. Suppose that $\mathcal{F}^\sigma$ contains a closed leaf L. Then the linear holonomy group of L is non-trivial, i.e., there exists a closed curve γ on L with linear holonomy $\neq 1$.*

If the linear holonomy group is non-trivial, the germ of the foliation is determined on both sides of the compact leaf.

LEMMA 3.7. — *Let M be a closed Seifert manifold and ϕ^t a regular projectively Anosov flow on M. Suppose that the unstable or stable foliation $\mathcal{F}^\sigma$ contains a vertical compact leaf L. Then if a leaf of $\mathcal{F}^\sigma$ in the neighborhood of L is vertical (resp. horizontal), all non-compact leaves in the neighborhood are vertical (resp. horizontal).*

Proof. — By the previous lemma, the linear holonomy groups of the compact leaves of $\mathcal{F}^\sigma$ are non-trivial, so compact leaves are isolated.

It is proved in [MR] that the germ of an isolated compact leaf homeomorphic to T^2 is topologically conjugate to a foliation defined by a 1-form:

$$\Omega = \mathrm{d}z + \psi(z)(p\,\mathrm{d}x + q\,\mathrm{d}y),$$

where (x, y, z) are coordinates of $T^2 \times [0, \varepsilon)$, $\psi(z)$ is an increasing function satisfying $\psi(0) = 0$, and $(p, q) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. This tells us that a leaf of $\mathcal{F}^\sigma$ near a vertical compact leaf is vertical (resp. horizontal) if and only if the linear holonomy of the compact leaf along a fiber is equal to 1 (resp. $\neq 1$). This completes the proof. □

The invariant foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ are characterized by the following proposition.

PROPOSITION 3.8. — *Let M be a closed oriented Seifert manifold and $\mathcal{F}^\sigma$ the unstable or stable foliation of a regular projectively Anosov flow ϕ^t on M. Suppose that $\mathcal{F}^\sigma$ contains a compact leaf. Then the number of the compact leaves is finite and possibly after changing the Seifert fibration structures of M, $\mathcal{F}^\sigma$ is isotopic to a foliation such that all compact leaves are vertical, and that all non-compact leaves are horizontal.*

Proof. — If M is covered by a T^2-bundle then Theorem 2.5 implies that M is actually the 3-torus and $\mathcal{F}^\sigma$ satisfies the required properties.

Suppose M cannot be covered by a T^2-bundle. Since $\mathcal{F}^\sigma$ is a transversely oriented codimension 1 smooth foliation and all compact leaves are incompressible tori by Proposition 2.1, we can apply Theorem 3.3. In particular, each compact leaf is a vertical one.

Let U be the set of all horizontal non-compact leaves. We may suppose $U \neq \emptyset$. Indeed, if all leaves of $\mathcal{F}^\sigma$ is vertical then $\mathcal{F}^\sigma$ is a foliation defined by the pull-back of a foliation on the base space of an S^1-bundle over a closed surface. However, such a foliation can exist only on manifolds which are covered by T^2-bundles over S^1.

The set U is open and its boundary $B = \partial U$ in M consists of vertical leaves. Take a fiber γ of M in U which is transverse to $\mathcal{F}^\sigma$ and let $N(\gamma)$ be its tubular neighborhood. Then we can take a properly embedded horizontal surface S in $M \setminus N(\gamma)$ such that S is transverse to each vertical leaf of $\mathcal{F}^\sigma$ and ∂S is transverse to $\mathcal{F}^\sigma|_{N(\gamma)}$. The intersection with $\mathcal{F}^\sigma$ induces a smooth foliation $F(S)$ with singularities on S.

The set $B \cap S$ is closed and saturated by non-singular leaves of $F(S)$. By the theorem of Schwarz [Sch], each minimal set in $B \cap S$ is a closed leaf, so it is a component of the intersection of a compact leaf of $\mathcal{F}^\sigma$ and S. By Lemma 3.7, U lies on the both sides of each compact leaf in B. Therefore B consists of a finite collection of vertical compact leaves. Thus the union $B \cup U$ is an open and closed set in M, that is, it coincides the whole M. We have thus proved the proposition. □

4. Intersection of compact leaves.

PROPOSITION 4.1. — *Let M be a closed oriented Seifert manifold and ϕ^t a regular projectively Anosov flow with unstable foliation $\mathcal{F}^u$ and stable foliation $\mathcal{F}^s$. Then the compact leaves of $\mathcal{F}^u$ and those of $\mathcal{F}^s$ do not intersect.*

Proof. — By Proposition 3.8, each of $\mathcal{F}^u$ and $\mathcal{F}^s$ has only finitely many compact leaves and can be reformed by isotopy into a foliation such that all compact leaves are vertical tori and all non-compact leaves are horizontal.

LEMMA 4.2. — *A closed orbit c of ϕ^t is isotopic to a regular fiber of the Seifert manifold M if and only if there exist a compact leaf L^u of $\mathcal{F}^u$ and a compact leaf L^s of $\mathcal{F}^s$ such that c is contained in $L^u \cap L^s$.*

Proof. — It is obvious that the intersection of vertical compact leaves consists of closed orbits isotopic to regular fibers.

Suppose that c is a closed orbit of ϕ^t isotopic to a fiber and that c is contained in a non-compact leaf L of $\mathcal{F}^u$ or $\mathcal{F}^s$. Since L can be reformed by isotopy to a horizontal surface, the projection $\pi\colon M \to B$ of the Seifert manifold to its base defines a covering map of L to its image $\pi(L)$ and any closed curves on L can be regarded as lifts of closed curves on B. However, $\pi(c)$ is a null homotopic closed curve on B and therefore c is a null homotopic closed curve on L. It is a contradiction. $\square$

Suppose that a compact leaf L_1 of $\mathcal{F}^s$ intersects a compact leaf of $\mathcal{F}^u$. Then $\phi^t|_{L_1}$ has a finite number of closed orbits isotopic to fibers. A region in a leaf bounded by adjacent two closed orbits is called a *slope component* if the direction of these closed orbits coincides and a *Reeb component* otherwise (see Figure 2).

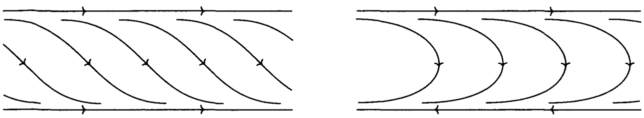

Figure 2

Let A_1 be such an annulus on L_1 and c_1 and c_2 closed orbits such that $\partial A_1 = c_1 \cup c_2$. Then there exists a compact leaf L_2 of $\mathcal{F}^u$ containing c_2 and we can take on L_2 a closed orbit c_3 adjacent to c_2 and an annulus A_2 bounded by $c_2 \cup c_3$. Iterating this procedure, we can take a sequence of A_i's and c_i's. The finiteness of the compact leaves implies that we can find some integer k such that $c_{2k+1} = c_1$ and $A_{2k+1} = A_1$ (see Figure 3).

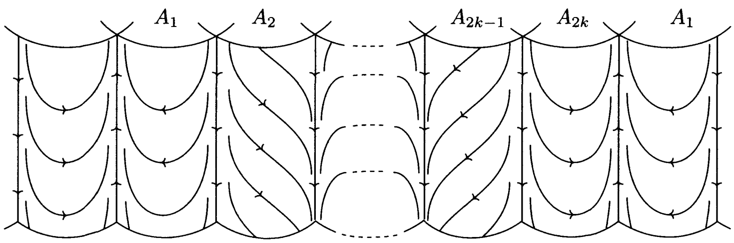

Figure 3

Let $\{(A_i, c_i)\}_{1\leq i\leq 2k}$ be such a chain of annuli and closed orbits. Note that A_i is contained in $\mathcal{F}^s$ if i is odd and in $\mathcal{F}^u$ if i is even. Let h_i^u and h_i^s be the linear holonomies of $\mathcal{F}^u$ and $\mathcal{F}^s$, respectively, along a closed orbit c_i

with orientation. Then projectively Anosov property implies $h_i^s > h_i^u$ for all i. In the following, indices are defined by mod $2k$.

CLAIM (see Figure 4). — 1) *For odd i, $h_i^s = h_{i+1}^s$ if the orientations of c_i and c_{i+1} coincide and $h_i^s = (h_{i+1}^s)^{-1}$ otherwise.*

2) *For even i, $h_i^u = h_{i+1}^u$ if the orientations of c_i and c_{i+1} coincide and $h^u = (h_{i+1}^u)^{-1}$ otherwise.*

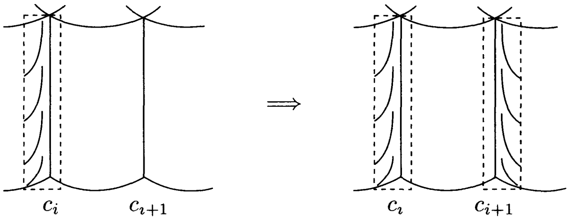

Figure 4

This is obvious from the fact that two isotopic closed orbits have the same linear holonomy.

CLAIM (see Figure 5). — *Let A be an annulus on a stable leaf and let $\partial A = c \cup c'$. If $1 \geq h_c^s > h_c^u$ then A is a Reeb component and $h_{c'}^s > h_{c'}^u \geq 1$.*

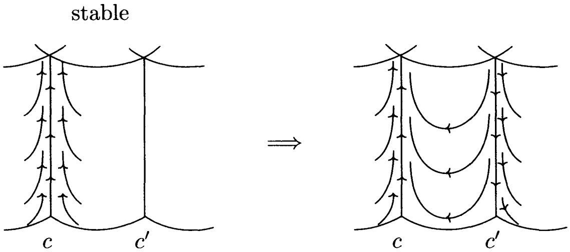

Figure 5

CLAIM (see Figure 6). — *Let A be an annulus on an unstable leaf and let $\partial A = c \cup c'$. If $h_c^s > h_c^u \geq 1$ then A is a Reeb component and $1 \geq h_{c'}^s > h_{c'}^u$.*

These two claims are deduced from the fact that it holds $h_c^\sigma \leq 1$ if and only if $h_{c'}^\sigma \geq 1$ for $\sigma = s, u$, whether A is a slope component or a Reeb component.

Let us apply these claims in our case. Since A_1 is contained in a stable leaf, it holds that $h_1^u \geq 1$ or $h_2^u \geq 1$, so $h_1^s > h_1^u \geq 1$ or $h_2^s > h_2^u \geq 1$

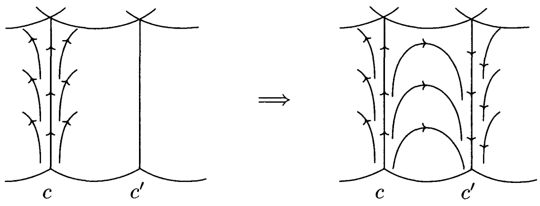

Figure 6

respectively. By applying Claim 2 and 3 repeatedly, we can see that all A_i's are Reeb components. Furthermore, by Claim 1,

- $h_i^s h_{i+1}^s = 1$ if i is odd,
- $h_i^u h_{i+1}^u = 1$ if i is even.

Therefore we obtain

$$h_1^s h_2^s \cdots h_{2k}^s = h_1^u h_2^u \cdots h_{2k}^u = 1.$$

It contradicts the projectively Anosov property that $h_i^s > h_i^u$ for all i. □

5. Regular projectively Anosov flows bounded by closed leaves.

In this section, we complete the proof of Theorem 1.1. By Proposition 3.8 and Proposition 4.1 we can decompose M by the compact leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$ into a finite collection compact Seifert manifolds, each of which is bounded by a disjoint union of compact leaves and contains no other compact leaves in the interior. Let W be such a component and we will see that (W, ϕ^t) is conjugate to a $T^2 \times I$-model.

To do this, we study the leaf spaces of the unstable and stable foliations. This technique is investigated for Anosov flows in [Gh1], [Ba] and [F], for examples, and for projectively Anosov flows in [NT].

Let $\widetilde{W}$ be the universal covering space of W and $\widetilde{\phi}^t, \widetilde{\mathcal{F}^u}, \widetilde{\mathcal{F}^s}$ be the induced flow and foliations on $\widetilde{W}$. The *leaf spaces* are the 1-dimensional spaces Q^u and Q^s defined by $Q^u = \widetilde{W}/\widetilde{\mathcal{F}^u}$ and $Q^s = \widetilde{W}/\widetilde{\mathcal{F}^s}$, respectively. Note that they are not Hausdorff in general. Proposition 2.1 implies that $\widetilde{\mathcal{F}^u}$ and $\widetilde{\mathcal{F}^s}$ are foliations by planes and therefore Q^u or Q^s is Hausdorff if and only if $\widetilde{\mathcal{F}^u}$ or $\widetilde{\mathcal{F}^s}$ is conjugate to the product foliation. The projections

$p^u: \widetilde{W} \to Q^u$ and $p^s: \widetilde{W} \to Q^s$ are $\pi_1(W)$-equivariant and the juxtaposition map of projections:

$$p = (p^u, p^s): \widetilde{M} \to Q^u \times Q^s$$

is a $\pi_1(W)$-equivariant submersion. Let W_0 be the interior of W and define also the analogues $\mathcal{F}_0^u$, $\mathcal{F}_0^s$, $\widetilde{W_0}$, $\widetilde{\mathcal{F}_0^u}$, $\widetilde{\mathcal{F}_0^s}$, Q_0^u, Q_0^s.

LEMMA 5.1. — *Each of the foliations $\widetilde{\mathcal{F}_0^u}$ and $\widetilde{\mathcal{F}_0^s}$ is conjugate to the product foliation.*

Proof. — We shall discuss $\widetilde{\mathcal{F}_0^u}$ then the same argument works for $\widetilde{\mathcal{F}_0^s}$.

Suppose first that ∂W coincides the union of the compact leaves of $\mathcal{F}^u$. Then each leaf of $\mathcal{F}_0^u$ is horizontal by Proposition 3.8, we can take a regular fiber γ so that its lift $\widetilde{\gamma}$ in $\widetilde{W_0}$ intersects all leaves of $\widetilde{\mathcal{F}_0^u}$. Hence we only have to prove that each leaf of $\widetilde{\mathcal{F}_0^u}$ intersects $\widetilde{\gamma}$ at just one point. Suppose on the contrary that a leaf $\widetilde{L}$ intersects $\widetilde{\gamma}$ at two points p, q. Then we can construct a loop by two arcs connecting p to q on $\widetilde{L}$ and on $\widetilde{\gamma}$, but a slight deformation of the loop yields a null homotopic transversal to $\widetilde{\mathcal{F}_0^u}$. It is a contradiction.

Secondly, assume that ∂W contains a compact leaf L^s of $\mathcal{F}^s$. Let us see that the restricted foliation F of $\mathcal{F}^u$ to L^s can be isotoped so that each leaf is horizontal. Since F cannot contain a vertical leaf by Lemma 4.2, we only have to show that F does not contain a Reeb component on an annulus. If there exists, the two closed leaves on the boundary lie on the same leaf of $\mathcal{F}^u$, so their linear holonomies of F coincide. However it contradicts the projectively Anosov property (see Theorem 4.1 of [Nd]).

By the similar argument in the proof of Theorem 3.3, $\mathcal{F}_0^u$ can be isotoped to a horizontal foliation. The rest part of the argument is the same as above. □

Then Theorem 1.1 follows from the theorem below. This theorem does not assume any more that the underlying manifold is a Seifert manifold.

THEOREM 5.2. — *Let W be a compact oriented 3-manifold with non-empty boundary and ϕ^t a regular projectively Anosov flow on W with unstable foliation $\mathcal{F}^u$ and stable foliation $\mathcal{F}^s$. Suppose that ∂W is the disjoint union of all closed leaves in $\mathcal{F}^u$ and $\mathcal{F}^s$ and that in the universal covering $\widetilde{W_0}$ of the interior W_0 of W, each of the lifted foliations $\widetilde{\mathcal{F}_0^u}$ and $\widetilde{\mathcal{F}_0^s}$ is conjugate to the product foliation. Then W is homeomorphic to $T^2 \times I$ and ϕ^t is a $T^2 \times I$-model.*

We will prove this theorem by several lemmas.

In the case leaf spaces are Hausdorff, the following result is known (see [NT]).

PROPOSITION 5.3. — *Let ϕ^t be a regular projectively Anosov flow on a 3-manifold M. Assume that the lifted stable foliation $\widetilde{\mathcal{F}^s}$ on the universal covering space $\widetilde{M}$ is diffeomorphic to the product foliation of $\mathbb{R}^3$. Then the leaf space of the lifted orbit foliation $\widetilde{\phi}$ restricted to each leaf $\widetilde{L^u}$ of the lifted unstable foliation $\widetilde{\mathcal{F}^u}$ is Hausdorff.*

Applying this proposition for $\widetilde{\mathcal{F}_0^s}$ and $\widetilde{L} \in \widetilde{\mathcal{F}^u}$, we obtain the following lemma:

LEMMA 5.4. — *Let $\widetilde{L}$ be a leaf of $\widetilde{\mathcal{F}^u}$. Then its image $p(L)$ is Hausdorff and therefore homeomorphic to an open interval.*

The same holds for the leaves of $\widetilde{\mathcal{F}^s}$.

LEMMA 5.5. — *If two distinct points p, q in Q^σ ($\sigma = u$ or s) are non-separable from each other then both of them are in $Q^\sigma \setminus Q_0^\sigma$.*

Proof. — Let p, q be such points. Since Q_0^σ is Hausdorff by assumption, at least one of them, say p, lies in $Q^\sigma \setminus Q_0^\sigma$. Let L_p be a corresponding closed leaf of $\mathcal{F}^\sigma$ in W then the action of $\pi_1(L_p) \cong \mathbb{Z} \oplus \mathbb{Z}$ on Q^σ fixes p and maps q to points which are non-separable from p. If q is in Q^σ then the restricted action on Q_0^σ fixes q, but it is impossible because the fundamental group of a leaf in $\mathcal{F}_0^\sigma$ does not admit an abelian subgroup of rank 2. □

LEMMA 5.6. — *The image $p(\widetilde{W})$ is Hausdorff.*

We use the following lemma in the proof.

LEMMA 5.7. — *Let ϕ^t be a regular projectively Anosov flow on a compact oriented 3-manifold and $\mathcal{F}^u$ the unstable foliation of ϕ^t. Suppose that L is a closed leaf in $\mathcal{F}^u$ and that the restricted flow $\phi^t|_L$ is conjugate to the suspension of a diffeomorphism of S^1. Then each orbit contained in ϕ-saturated neighborhood of L is attracted to L as $t \to +\infty$.*

The proof is obvious since the linear holonomy group of L is non-trivial by Lemma 3.6.

Proof of Lemma 5.6. — Clearly, two distinct points p, q in $p(\widetilde{W_0})$ or in the image of a leaf of $\widetilde{\mathcal{F}^u}$ or $\widetilde{\mathcal{F}^s}$ are separable. Suppose $p \in p(\widetilde{L_1})$

and $q \in p(\widetilde{L_2})$ where $\widetilde{L_1}, \widetilde{L_2} \in \widetilde{\mathcal{F}^u} \setminus \widetilde{\mathcal{F}^u_0}$. Let L_1 and L_2 be the projections in W of $\widetilde{L_1}$ and $\widetilde{L_2}$, respectively, and take ϕ-saturated neighborhood U_1 and U_2 of them. We can take their lifts $\widetilde{U_1}$ and $\widetilde{U_2}$ as $\widetilde{\phi}$-saturated neighborhoods of $\widetilde{L_1}$ and $\widetilde{L_2}$ in $\widetilde{W}$, then they are disjoint from each other by Lemma 5.7 and their images by p separate p and q. □

Taking parameterizations of Q^u_0 and Q^s_0 by $(0,1)$, we may assume by Lemma 5.6 that $p(\widetilde{M})$ is embedded in $[0,1] \times [0,1]$ with coordinates (x,y). Note that the image of each leaf of $\widetilde{\mathcal{F}^u}$ and $\widetilde{\mathcal{F}^s}$ is a vertical and a horizontal open interval, respectively.

LEMMA 5.8. — *The image $p(\widetilde{W})$ is homeomorphic to* $[0,1) \times [0,1)$.

Proof. — We may assume $\mathcal{F}^s$ has a closed leaf L in ∂W. Let N be a ϕ-saturated neighborhood of L in W and take the lifts $\widetilde{L}$ and $\widetilde{N}$ in $\widetilde{W}$, respectively. By Lemma 5.4 and 5.5, we may assume $p(\widetilde{L}) \subset [0,1] \times [0,1]$ is a horizontal open interval $\{a < x < b, y = 0\}$ and $p(\widetilde{N})$ lies in $y \geq 0$. Since $\{x = a\}$ and $\{x = b\}$ are invariant under the action of $\pi_1(L) \cong \mathbb{Z} \oplus \mathbb{Z}$, they do not intersect $p(\widetilde{W_0})$. Therefore $a = 0$ and $b = 1$. Thus we have seen that Q^u and Q^s are actually Hausdorff. In the y-coordinate, 0 and 1 are the only fixed points by the action of $\pi_1(L)$. Furthermore, the topology of $\mathcal{F}^u$ near L tells us that no other point is fixed by any non-trivial action of elements of $\pi_1(L)$.

By assumption $p(\widetilde{N}) \subset (0,1) \times [0,1)$. We claim that $p(\widetilde{N}) = (0,1) \times [0,1)$. For small $\varepsilon > 0$, $p(\widetilde{N})$ contains $(a,b) \times [0,\varepsilon]$, where (a,b) is an open interval whose image by the action of $\pi_1(L)$ covers $(0,1)$.

If there exist closed orbits of ϕ^t on L, the action f_α of the element $\alpha \in \pi_1(L)$ which corresponds to the closed orbits has fixed points in $(0,1)$. Take $(a',b') \subset (a,b)$ whose end points are fixed points. By replacing (a,b) with a larger one, we may assume that the orbit of (a',b') covers $(0,1)$. Since f_α acts on Q^s non-trivially, $(a',b') \times [0,1) = \bigcup f^n_\alpha\big((a',b') \times [0,\varepsilon]\big)$ is contained in $p(\widetilde{N})$. Then by the action of $\pi_1(L)$, we can see $p(\widetilde{N}) = (0,1) \times [0,1)$.

If there is no closed orbit on L then the restricted flow $\phi^t|_L$ is topologically isotopic to an irrational linear flow by the theorem of Denjoy [D]. Taking a sequence of closed curves which approximates the slope of the linear flow, we can see $p(\widetilde{N}) = (0,1) \times [0,1)$ also in this case.

We next observe the frontier. It is clear by Lemma 5.7 that $p(\widetilde{W})$ does not intersect $\{y = 1\}$. Let Ω be the ω-limit set of the orbits in N, which is non-empty, closed, and invariant by ϕ^t, and let $\widetilde{\Omega}$ be its lift. The image $p(\widetilde{\Omega})$ lies on the frontier of $p(\widetilde{N})$ because N does not contain a closed invariant set in the interior. Thus $p(\widetilde{\Omega})$ intersects $\{x = 0\} \cup \{x = 1\}$, so we may assume that $p(\widetilde{W})$ contains a point $(0, y_1)$. This point is in the image of a lift $\widetilde{L}'$ of a closed leaf L' in $\mathcal{F}^u$, therefore by applying the same argument with changing the roles of $\mathcal{F}^s$ and $\mathcal{F}^u$, we can see $p(\widetilde{W}) = [0,1) \times [0,1)$. $\square$

Thus $\widetilde{\phi}^t$ has the same orbit space as that of a $T^2 \times I$-model, the boundary ∂W consists of a union of two tori isotopic to each other, one of which is the closed leaf of $\mathcal{F}^u$ and the other is that of $\mathcal{F}^s$, and all orbits contained in W_0 are attracted to L^u (resp. L^s) as $t \to +\infty$ (resp. $t \to -\infty$). Hence (W, ϕ^t) is a $T^2 \times I$-model, by Proposition 5.15 of [Nd].

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