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ORBIFOLDS, SPECIAL VARIETIES AND CLASSIFICATION THEORY: AN APPENDIX

by Frédéric CAMPANA

Contents.

The present text is a technical appendix to [Ca04], giving with proofs the results used there concerning geometric quotients, Zariski regularity, and S-stability.

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1. Geometric quotients of complex spaces.

1.1. Introduction.

This first part is a polished version of a first draft, partly based on the exposition of this first draft given in the master thesis of A. Höring ([H03]), whom I thank for his authorisation to use his text here. The aim of this first part of the present text is to prove the following result.

THEOREM 1.1. — *Let X be a compact connected normal complex space and let $S \subset \mathcal{C}(X)$, the Chow Scheme of X, be a covering family for X. Note $R(S)$ the equivalence relation on X induced by S, for which $x, y \in X$ are equivalent iff contained in a connected union of a finite number of the members of the family of analytic cycles of X parametrised by S.*

Then there exists a fibration $q_S : X - \to X_S$ such that its general fibre is an equivalence class for $R(S)$. Furthermore q_S is almost holomorphic and unique up to equivalence of meromorphic fibrations. The map q_S is called the S-quotient of X.

A covering family $S \subset \mathcal{C}(X)$ must be understood as in Def. 1.8, i.e. its irreducible components are compact, the generic members of the family S are irreducible and the projection from the incidence graph to X is surjective.

Theorem 1.1 was proved in [Ca81] in the case where S is supposed to be irreducible. The general case however easily reduces to this special case. We follow very closely the approach of [Ca81], simplified in one step. A similar result holds in the algebraic category, as carefully shown by a different approach in [De01, ch.5], see also [Ko96].

The normality of X is essential, as shown by the examples given in [Ca81]. By way of constrast, the compactness assumption of X and of the components of S can be weakened to the X-properness of the incidence graph of each of the components of S (see [Ca94]).

We now shortly sketch the proof of Theorem 1.1: it is obtained from the study of the increasing sets $R_n(x) \subset X$ consisting of all $y \in X$ which can be connected to a given generic $x \in X$ by a connected union of $n \geqslant 0$ members of the family S.

Easy arguments (given in Section 1.4, based on Remmert's proper image theorem and the compactness of the irreducible components of S)

show that these sets are analytic closed in X (assuming S to have only finitely many components, a case to which one can easily reduce the general case).

In a first step (Section 1.5) we begin by treating the easiest case: when the support of a generic cycle parametrized by S is already the fibre of an almost-holomorphic fibration $q : X - \to S$ (see notations below in 1.1 for this notion). We show that in this case q is already the S-quotient.

We next solve the case when the family is "stationary" (ie: for generic $x \in X$, the dimension of $R_2(x)$ is the same as that of $R_1(x)$), by showing that the $R_1(x)'s$, for $x \in X$, are the family of fibres of an *almost-holomorphic* meromorphic map.

In Section 1.6 we solve the main case, where S is irreducible, by reducing to the preceding "stationary" case. This reduction step is a consequence of the fact that the dimension of $R_n(x)$ is obviously bounded. We show that when this dimension becomes stationary, the family $R_n(x)$, for $x \in X$, is in fact a "stationary" family.

This is the crucial step of the proof. It unfortunately requires preliminary technical irreducibility criteria both of global and local (analytic) nature, exposed at the beginning of Section 1.6.

Finally in Section 1.7 we solve the general case, by an easy reduction to the case where S is irreducible. This reduction consists in suitably composing the quotients obtained component-by-component of S.

The proof allows to find (finite) bounds on the length of chains needed to connect points in one generic equivalence class, which (slightly) improve the ones shown in [K-M-M92]. This is exposed in Section 1.8.

For reader's convenience, we started by recalling in Section 1.2 the few basic facts that we use from the theory of Chow-Schemes in the analytic context .

The remaining sections prove Theorem 1.1.

Notations, fibrations, almost holomorphic maps:

For general facts on complex analytic spaces, we refer to [KK83]. Complex spaces are reduced and of finite dimension. The topology considered is, unless otherwise specified, the metric (analytic) toplogy. Otherwise, it is the Zariski topology. For meromorphic maps we use the terminology defined in 1.2. The abbreviation "wlog", for: "without loss of generality" will be frequently used. We also recall the following terminology from [Ca04].

Definition 1.2 [Ca04]. — *Let $f : X - \to Y$ be a surjective meromorphic map between normal compact irreducible complex spaces and let $\Gamma \subset X \times Y$ be the closure of the graph of f.*

$$\begin{tikzcd} X \arrow[r, dashed, "(id{,}f)"] \arrow[dr, dashed, "f"'] & \Gamma \arrow[d, "p_Y"] \\ & Y \end{tikzcd}$$

1. *Let $p_Y : \Gamma \to Y$ be the restriction of the second projection to Γ, then we define $p_Y(\Gamma)$ as the image of f. By consequence f is said to be surjective if $Y = p_Y(\Gamma)$. This definition is equivalent to the fact that $f|_U$ is dominant, where $U \subset X$ is a Zariski open set such that f is defined on U.*
2. *We define the indeterminacy locus I_f to be the set of points $x \in X$ where the fibre of the first projection $p_X^{-1}(x) \subset \Gamma$ is not a singleton. Define furthermore $f(I_f) = p_Y(p_X^{-1}(I_f))$:*

   Then f is said to be almost holomorphic if $f(I_f) \neq Y$.
3. *For $y \in Y$, define $p_X(p_Y^{-1}(y))$ as the (Chow-theoretic) fibre of f over y. If $y \notin f(I_f)$, this definition coincides with the usual definition of a fibre.*
4. *For $U \subset X$, define the image of U by f to be: $f(U) := p_Y(p_X^{-1}(U))$. Of course, if U does not meet the indeterminacy locus this coincides with the usual definition.*

Definition 1.3 [Ca04]. — *A fibration $f : X - \to Y$ is a surjective meromorphic map between irreducible compact complex spaces such that the generic fibre of f is irreducible. A fibration is said to be (almost) holomorphic, if so is the map.*

Another fibration $f' : X' - \to Y'$ is said to be equivalent to f if there exist bimeromorphic maps $u : X - \to X'$ and $v : v : Y - \to Y'$ such that $f' \circ u = v \circ f$. We say that f' is a model or representative of f.

1.2. Generalities on the cycle space.

In this paragraph we recall some basic notions about the cycle space that parametrizes compact cycles with multiplicities for a given complex space. We give a brief presentation of this so-called Cycle Space (or Chow

Scheme in the algebraic case) following the original paper [Ba75, Ch. 0-II] and the expository [C-94, Ch. VIII]. A description of the topology of the cycle space can be found in [SGAN82]. Lemmas 1.9-1.12 expose elementary properties of covering families that will be of importance in the following sections.

DEFINITION 1.4 [Ba75]. — *Let X be a complex space and $d \in \mathbb{N}$ an integer. A d-cycle on X is a finite linear combination $Z = \sum n_i Z_i$ where the n_i's are nonnegative integers and the Z_i's compact irreducible analytic subsets of X of pure dimension d which are pairwise distinct. The support of Z, denoted $|Z|$ is the union of the (reduced) Z_i's. The integer n_i is called the multiplicity of Z_i in the cycle Z.*

The set of all d-cycles is denoted $\mathcal{C}_d(X)$ and the set of all cycles (the union of all $\mathcal{C}_d(X)$ for $d \in \mathbb{N}$) is denoted $\mathcal{C}(X)$. We call $\mathcal{C}(X)$ the cycle space or Chow scheme, or Barlet space of X.

We fix the following notation: If S is a complex space parametrizing a family of cycles $(Z_s)_{s\in S}$, then we note Z_s the cycle with multiplicities, $|Z_s| \subset X$ its support and $[Z_s]$ the point in $\mathcal{C}(X)$ corresponding to Z_s. If S is embedded in $\mathcal{C}(X)$, we identify occasionally $s \in S$ and $[Z_s] \in \mathcal{C}(X)$.

We shall define the notion of an analytic family $(Z_s)_{s\in S}$ of d-cycles of X parametrised by an analytic space S only when S is normal, because a simple geometric description of this notion can be given in this situation, which is the only one we shall use. Moreover, one can always reduce to this special case by normalising the parameter space.

DEFINITION 1.5 ([Ba75, Chapt. 1, Thm 1]). — *Let S be a normal complex space and $(Z_s)_{s\in S}$ be a family of d-cycles of X parametrized by S (ie: for each $s \in S$, Z_s is an element of $\mathcal{C}_d(X)$). Let*

$$|G_S| := \{(s,x)|\ x \in |Z_s|\} \subset S \times X$$

The set $|G_S|$ is called the incidence graph of the family S. We note p_S and p_X the restrictions of the first and second projections of $S \times X$ to G_S.

Then this family is said to be analytic if:

1. *the incidence graph $|G_S|$ is a closed analytic subset of $S \times X$.*
2. *the restriction of the first projection p_S of $S \times X$ to $|G_S|$ is proper, surjective and its fibres have pure dimension d.*

3. *for any irreducible component $|G^j_S|$ of $|G_S|$, there exists a positive integer n^j such that for s generic in $S^j := p_S(|G^j_S|)$ all irreducible components of $|Z_s|$ contained in $|G^j_S|$ have multiplicity n^j. (The closed analytic cycle $G_S = \sum_j n^j G^j_S$ is called the graph of the analytic family parametrized by S).*

4. *For any $s \in S$, any j, and any local multisection $\sigma : S' \to |G^j_S|$, defined on a small open neighborhood S' of s in S, if the image of σ meets Z_s at a single point x, contained in a unique irreducible component Z^i_s of Z_s, the multiplicity of $|Z^i_s|$ in Z_s is $m.n^j$, where m is the degree of the restriction of p_X to the image of σ.*

Remark.— Let us comment briefly on condition 4. in the above definition: when $|G_S|$ is given, and also the n^j 's as above, the multiplicities on each fibre $|Z_s|$, for any $s \in S$ can be uniquely determined, so as to satisfy this condition if S is normal. This is the content of the second assertion of 1.7 below.

One property of analytic families should be mentioned (which is part of the definition, in general): if $(s, x) \in |G_S|$, the graph of such an analytic family, then shrinking S near s, there exists a finite proper ramified covering map $g : U \to S \times B$, defined on some open neighborhood U of (s, x) in $|G_S|$, where B is an open polydisc in $\mathbb{C}^d$, such that $p_S = h \circ g$, if $h : S \times B \to S$ is the first projection.

The main result here is:

THEOREM 1.6 [Ba75].— *The definition of an analytic family (for arbitrary S) defines a contravariant functor F^d_X from the category of complex spaces to the category of sets, $F^d_X(S)$ being simply the set of analytic families of compact d-dimensional cycles parametrized by S. Morphisms are defined by base change over the parameter space.*

This functor is representable, i.e. there exists a complex space $\mathcal{C}_d(X)$ and an isomorphism of functors $F^d_X \longleftrightarrow Mor(-, \mathcal{C}_d(X))$.

The following geometric version of Hironaka's flattening theorem is of constant use:

THEOREM 1.7 ([Ba79], see also [C-P94]).— *Let $G \subset S \times X$ be an irreducible compact analytic subset such that the restriction $p : G \to S$ is surjective.*

There exists a unique meromorphic map $f : S- \to \mathcal{C}(X)$ sending a generic $s \in S$ to the reduced cycle of X with support $p^{-1}(s)$. In particular, the image of f is compact since S is.

If moreover the fibres of p are all of same dimension, and if S is normal, then f is holomorphic.

We call f the (Chow-theoretic) fibre map of p.

1.3. Covering families.

Notation: If $s \in S$ and Z_s is the corresponding cycle on X, then the restriction of p_X to $p_S^{-1}(s) = \{(s,x) | \, x \in |Z_s|\}$ is an embedding with image $|Z_s|$. This identification motivates to note $p_S^{-1}(s) =: \widehat{Z_s}$.

DEFINITION 1.8.— *Let X be a compact connected normal complex space. Then $S \subset \mathcal{C}(X)$ is said to be a covering family of X if the following conditions are satisfied:*

1. *S is an at most countable disjoint union of compact irreducible subvarieties $S_i \subset \mathcal{C}(X)$.*
2. *If $s \in S_i$ is a generic point, then Z_s is irreducible and reduced, this for any irreducible component S_i of S.*
3. *X is the union of all $|Z_s|$'s for $s \in S$.*

Remarks. —

1. Note that our condition on X implies in particular that X is irreducible. Furthermore as generic cycles are supposed to be reduced, the multiplicities n^j will be equal to one. So we shall not distinguish between the incidence graph $|G_S|$ and the graph G_S. In the following we write G_S for the incidence graph.
2. We could replace conditions 2)-3) equivalently by: the restrictions of the projection p_S (resp. p_X) to the incidence graph has irreducible reduced generic fibres (resp. is surjective).
3. If S is a covering family, then at least one of its irreducible components is a covering family. Conditions 1) and 2) are obvious and for 3) consider that by compactness of the components of S, the incidence graph of every irreducible component of S is itself compact, so its image by p_X in X is either X or a proper analytic subset. As X is irreducible, Baire's category theorem yields the result.

4. The condition of compactness of the irreducible components will be critical for our proof. By [Li75], it is always satisfied for compact Kähler manifolds.

5. We will always suppose S to be normal. If S is not normal, let $d : S' \to S$ be its normalization, then the morphism d corresponds to an analytic family of n-cycles parametrized by S' (use the functoriality in 1.6). In fact, this family contains the same cycles as S, but the same cycle Z_s will appear several times, if $s \in S$ is not a normal point. This also shows that normalizing does not change the equivalence relation $R(S)$ induced on X (cf. Section 1.4).

The following easy lemma will play a crucial role at some key steps of the construction. Especially important is the irreducibility property stated in the second assertion.

Proposition 1.9. — *If S is a covering family of X and S_i is any of its irreducible components, then*

1. *Z_s is connected for $s \in S$,*
2. *G_{S_i} (the incidence graph of S_i) is irreducible and compact for every i.*

Proof. — The first assertion is an immediate consequence of the analytic Zariski's Main theorem, combined with Stein factorisation.

The second assertion will be proven using the next few easy lemmas. Fix an irreducible component S_i. By 1.12, p_{S_i} is open. By the irreducibility Lemma 1.10 there is a dense Zariski open subset $S_i^* \subset S_i$ such that $G^* := p_{S_i}^{-1}(S_i^*) \subset G_{S_i}$ is irreducible. By the density Lemma 1.11, G^* is dense in G_{S_i}, so G_{S_i} is also irreducible. The compactness follows from the properness of p_{S_i} stated in 1.5. □

Remark. — Note that the proof of the second assertion also holds for every irreducible compact analytic subset of S.

Lemma 1.10 (Irreducibility lemma). — *Let X and T be complex spaces, with T irreducible. Assume X has only finitely many irreducible components. Let $\varphi : X \to T$ be a proper surjective holomorphic map, the generic fibre of which is irreducible.*

Then, there exists a dense Zariski open subset $T^* \subset T$, such that $\varphi^{-1}(T^*)$ is irreducible.

Proof. — There is a unique irreducible component of X, say X_0, mapped surjectively onto T, otherwise the fibre of φ over the generic point of T were not irreducible. Now the other irreducible components of X are mapped by φ to finitely many proper closed analytic subsets of T. Just choose T^* to be contained in their complement. □

LEMMA 1.11. — *Let $(Z_s)_{s\in S}$ be an analytic family of cycles for X, a compact connected normal complex space, parametrized by a reduced complex space S. Let $S^* \subset S$ be a dense Zariski open subset of S and G_S (resp. G_{S^*}) the incidence graph of the family S (resp. S^*), with their topology induced as subsets of $S \times X$. Then G_{S^*} is dense in G_S (for both topologies: metric and Zariski).*

Proof. — As the metric topology is finer than the Zariski topology, it is sufficient to show the assertion in the first case. By Lemma 1.12, p_S is an open mapping, so the preimage in G_S of the dense S^* is dense.

PROPOSITION 1.12. — *In the situation of lemma 1.11, let $p_S : G_S \to S$ be the restriction of the projection on the first factor. Then p_S is an open mapping for the analytic topology.*

Proof. — Let $(s, x) \in |G_S|$, the graph of such an analytic family, be any point. Then shrinking S near s, there exists a finite proper ramified covering map $g : U \to S \times B$, defined on some open neighborhood U of (s, x) in $|G_S|$, where B is an open polydisc in $\mathbb{C}^d$, such that $p_S = h \circ g$, if $h : S \times B \to S$ is the first projection. (This is part of the definition of an analytic family. (See [Ba75, Chapt. 1], or the remark following 1.5). This obviously implies the stated openness of p_S. □

Remark. — An equivalent statement is: if $f : X \to S$ is a surjective proper holomorphic map from X, purely $n + d$-dimensional, to S, normal and purely n-dimensional, then f is an open map, if all fibres of f are d-dimensional. The similar statement is standard for flat maps, but we could (unfortunately) not find this statement in the literature.

1.4. S-equivalence.

DEFINITION 1.13. — *Let $S \subset \mathcal{C}(X)$ be a covering family for X. For $s_1, \ldots, s_n \in S$ we say that $Z_{s_1}, \ldots, Z_{s_n}$ form an n-chain of S if the union* *of their supports is connected. We say that the n-chain is ordered if* $|Z_{s_j}|$ *meets* $|Z_{s_{j+1}}|$ *for* $j = 1, \ldots, n-1$.

Two points $x, x' \in X$ *are called* S*-equivalent iff* x *and* x' *can be joined by an n-chain for some* $n \in \mathbb{N}$ *, depending on* x, x'*. In which case we say that* x *and* x' *are n-equivalent.*

As every point x *in* X *is connected to itself by a 1-chain, this defines an equivalence relation* $R(S)$ *on* X.

We say X *is* S*-connected if* $R(S)$ *has a single equivalence class (i.e. any two points can be connected by some n-chain). A subset of* X *is* S*-saturated if it is the union of* $R(S)$*-equivalence classes.*

Remarks. —

1. Note that we did not suppose S to be irreducible, yet we suppose as usual for covering families that its irreducible components are compact. Furthermore the equivalence relation $R(S)$ depends only on the supports of the cycles on X, the structure of an analytic family does not appear in the definition. Arbitrary base changes on S do not change this equivalence relation. In particular, the normalization of S does not change $R(S)$ (cf. Def. 1.8).
2. If Z_s is a member of the family S , that does not meet any other member of the family S (ie: $|Z_s| \cap |Z_{s'}| \neq \varnothing \Rightarrow s = s'$), then $|Z_s|$ is an equivalence class for $R(S)$. For example, if $f : X \to S$ is a flat proper surjective connected holomorphic map with X smooth (normal is sufficient), and if $(Z_s)_{s \in S}$ is the family of its fibres, the equivalence relation $R(S)$ is of course the one having the fibres of f as equivalence classes. We will use this (trivial) example of an equivalence class in 1.5.
3. We note

$$R^{\infty}(S) = R(S) = \{(x, y) \in X \times X |\ x \overset{S}{\sim} y\} \subset X \times X$$

the graph of the equivalence relation. In the same way, for $n \geqslant 1$ we set

$$R_n(S) = \{(x, y) \in X \times X |\ x \text{ and } y \text{ are } n\text{-equivalent }\}.$$

Set, by definition, $R_0(S) := \Delta_X$, the diagonal of $X \times X$ and note that $R(S) = \cup_n R_n(S)$. In the following, we fix a covering family S and replace $R(S)$ by R and $R_n(S)$ by R_n.

LEMMA 1.14. — *Let X be a compact connected normal complex space and $S \subset \mathcal{C}(X)$ an irreducible compact covering family for X. Note $G_S \subset S \times X$ the incidence graph and $p_S : G_S \to S$ (resp. $p_X : G_S \to X$) the restriction of the first (resp. second) projection. For any $n \geqslant 1$, let $p'_n : R_n \to X$ (resp. $p''_n : R_n \to X$) be the restriction of the first (resp. second) projection of $X \times X$ on X to R_n. Then*

1. *R_1 is irreducible and compact.*
2. *$R_{n+1} = (p'_n \times p''_1)[(p''_n \times p'_1)^{-1}(\Delta_X)]$*
3. *$R_n \subset X \times X$ is Zariski closed and compact.*

Proof. — 1) We follow [Ca81]: Let $(\Gamma_s)_{s\in S} := (Z_s \times Z_s)_{s\in S}$ be the analytic family of compact cycles in $X \times X$ parametrized by S. Then for generic s the cycle Γ_s is irreducible and reduced, so we apply 1.9 to obtain that the incidence graph $\Gamma \subset S \times X \times X$ is irreducible and compact. Let $q_i : \Gamma \to X$ be the restriction of the i-th projection ($i = 2, 3$) to Γ, then $R_1 = (q_2 \times q_3)(\Gamma)$ is irreducible and compact.

2) We have:

$$(p''_n \times p'_1)^{-1}(\Delta_X) = \{((y,x),(x,z)) |\ x \in X \text{ ,} y \text{ and } x \text{ are } n\text{-equiv.,}$$
$$z \text{ and } x \text{ are 1-equiv.}\}$$

In particular, y and z can be connected by an $(n+1)$-chain, which allows to conclude, as $(p'_n \times p''_1)((y,x),(x,z)) = (y,z)$. For the other inclusion suppose y and z to be $(n+1)$-equivalent (but not n-equivalent, otherwise there is nothing to show) and $s_1, \dots, s_{n+1} \in S$ such that $|Z_{s_1}| \cup \ldots \cup |Z_{s_{n+1}}|$ is connected and contains y and z. Suppose wlog this chain to be ordered (otherwise permute the indices s_j), then $y \in |Z_{s_1}|$ and $z \in |Z_{s_{n+1}}|$. As the cycles are connected, we have, for $x \in |Z_{s_1}| \cap |Z_{s_2}|$, that x and y (resp. z) are 1-equivalent (resp. n-equiv.) and $(p'_n \times p''_1)((y,x),(x,z)) = (y,z)$.

3) Complex spaces are separated, so Δ_X is closed. Then $T := (p''_n \times p'_1)^{-1}(\Delta_X) \subset X \times X \times X \times X$ is closed, so compact for the metric topology. It is also Zariski closed, as $p''_n \times p'_1$ is holomorphic. So the restriction $p'_n \times p''_1 : T \to X \times X$ is a holomorphic mapping between compact complex spaces. By Remmert's proper mapping theorem, its image is an analytic subset of $X \times X$, its compactness being obvious.

DEFINITION 1.15. — *Let R^0 be an irreducible component of R_n. Then R^0 is said to be significant if it contains Δ_X.*

In the following we note R_n^{Δ} the union of the significant irreducible components of R_n and $d_n := \dim(R_n^{\Delta})$. Because $R_n \subset R_{n+1}$, we have: $R_n^{\Delta} \subset R_{n+1}^{\Delta}$, and so the sequence d_n is non-decreasing and bounded by the dimension of $X \times X$, and so becomes eventually stationary.

Remarks. —

1. Let $\sigma : X' \to X$ the blow-up of a point P on a smooth projective surface. Then define two points on X' to be equivalent if their images in X coincide. The graph of this equivalence relation is $X' \times_X X' \subset X' \times X'$. More precisely, if $E = \sigma^{-1}(P)$ then $X' \times_X X' = \Delta_{X'} \cup (E \times E)$. So the only significant component is the diagonal itself.
2. Let X be smooth projective surface and $f : X \to C$ a morphism onto a smooth curve and $s : C \to X$ a section of f, note $C' \subset X$ its image. Let R be the equivalence relation on X induced by $R_1' = (X \times_C X) \cup (C' \times C') \subset X \times X$ (two points are 1-equivalent if they are both in the same fibre or both in C'). It is then clear that any two points can be connected by a 3-chain, so $R = X \times X$; X consists of single equivalence class. If we delete the non-significant component $C' \times C'$ from R_1', two points are n-equivalent iff they are in the same fibre, so $R = X \times_C X$, each fibre is an equivalence class for R. We see that the non-significant component has important influence on the equivalence classes.

The following lemma shows that, by way of contrast, for an *irreducible* covering family, the non-significant components have no influence on the graph of the equivalence relation.

LEMMA 1.16 (Infinity lemma). — *Let X be a compact connected normal complex space. For any symmetric set $\Delta_X \subset A \subset X \times X$ (i.e. $(x, y) \in A \Leftrightarrow (y, x) \in A$) note $R^{\infty}(A) \subset X \times X$ the graph of the equivalence relation induced by the reflexive, symmetric relation on X with graph A.*

Let $S \subset \mathcal{C}(X)$ be a normal irreducible covering family for X. Then with the notations of 1.13 and 1.15, we have for any $n \geqslant 1$:

$$R^{\infty}(R_1(S)) = R^{\infty}(R_n^{\Delta}(S)) = R^{\infty}(R_n(S)).$$

Proof. — As $A \subset B$ implies $R^{\infty}(A) \subset R^{\infty}(B)$, we obtain

$$R^{\infty}(R_1(S)) \subset R^{\infty}(R_n^{\Delta}(S)) \subset R^{\infty}(R_n(S)).$$

(The irreducibility of S assures $R_1(S) = R_1^\Delta(S) \subset R_n^\Delta(S)$). We are left to show that $R^\infty(R_n(S)) \subset R^\infty(R_1(S))$. Yet by definition $R_n(S) \subset R^\infty(S) = R^\infty(R_1(S))$ and $R^\infty(R^\infty(A)) = R^\infty(A)$.

1.5. The stationary case.

If S is an irreducible covering family of the complex space X and G_S the incidence graph, we have the following basic diagram:

$$\begin{tikzcd} G_S \arrow[r, "p_X"] \arrow[d, "p_S"] & X \\ S & \end{tikzcd}$$

We will start with an easy case.

PROPOSITION 1.17. — *Let S be irreducible, assume $p_X : G_S \to X$ is a proper modification, that $p_S : G_S \to S$ is proper surjective, with generic fibre irreducible, and assume that $q := p_S \circ (p_X)^{-1} : X - \to S$ is almost-holomorphic. The generic fibres of q (ie: the generic Z_s) are then $R(S)$-equivalence classes, and q is the S-quotient of the family S.*

Remark. — The hypothesis that q is almost-holomorphic is essential, here. Consider indeed the case in which $X = \mathbb{P}^2$, and $(Z_s)_{s \in \mathbb{P}^1}$ is the family of lines through a given point $a \in \mathbb{P}^2$.

Proof. — If Z_s is a regular fibre of q (ie: if Z_s does not meet the indeterminacy locus of q), then Z_s does not meet any other member of the family S', if $\sigma : S - \to \mathcal{C}(X)$ is the fibre map, and if $S' \subset \mathcal{C}(X)$ is its image. (See the part c) of proof of 1.19 below for details). □

DEFINITION 1.18. — *Let X be a compact connected normal complex space, S an irreducible covering family for X and G_S the incidence graph. Then S is said to be stationary if*

1. $p_X : G_S \to X$ *is a modification*
2. $\dim(R_2^\Delta(S)) = d_2 = d_1 = \dim(R_1^\Delta(S))$

THEOREM 1.19. — *Let X be a compact connected normal complex space and $S \subset \mathcal{C}(X)$ an irreducible stationary covering family for X, note*

$G_S \subset S \times X$ the incidence graph and $p_X : G_S \to X$ (resp. $p_S : G_S \to S$) the restriction of the first (resp. second) projection to G_S.

Then $q_S = p_S \circ p_X^{-1} : X - \to S$ is an almost holomorphic fibration which is the S-quotient of X.

Proof. — We show by contradiction that q_S is an almost holomorphic fibration and suppose that a generic fibre of q_S meets the indeterminacy locus I_{q_S}. By construction this is equivalent to say that the fibre meets the indeterminacy locus $I_{p_X^{-1}}$. Moreover, for $s \in S$, $q_S^{-1}(s) = |Z_s|$.

a) We will obtain a contradiction in the following way: For $j = 1, 2$ let $q_j : R_j^{\triangle} \to X$ be the restriction of the second projection of $X \times X$ on X to $R_j^{\triangle}$. As p_X is a modification there is a Zariski open dense subset X^* of X, such that every point $x \in X^*$ lies in exactly one cycle $|Z_s|$ parametrized by S (in particular p_X^{-1} is defined on X^*). Hence $\dim(q_1^{-1}(x)) = \dim(|Z_s|) = n$.

Claim: $\dim(q_2^{-1}(x)) \geqslant \dim(|Z_s|) + 1$.

Assuming this, as x is generic in X, it follows that:

$$\dim(R_2^{\triangle}) \geqslant \dim(|Z_s|) + \dim(X) + 1 > \dim(|Z_s|) + \dim(X) = \dim(R_1^{\triangle}).$$

This clearly contradicts the fact that S is stationary.

b) We are left to show the above claim:

As $p_X^{-1}(X^*)$ is Zariski open (hence dense) and p_S open by 1.12, the set $p_S(p_X^{-1}(X^*))$ is open dense in S. So if we fix $x \in X^*$, we may suppose that for $s = q_S(x)$ there exists an irreducible Zariski open neighborhood $S^* \subset S$ such that $Z_{s'}$ is irreducible and reduced for all $s' \in S^*$.

By our hypothesis $|Z_s|$ meets $I_{p_X^{-1}}$, so let $y \in |Z_s| \cap I_{p_X^{-1}}$. By the analytic version of Zariski's main theorem, as X is normal $p_X^{-1}(y)$ is connected, so of positive dimension everywhere. We choose a curve $C \subset p_X^{-1}(y)$ that contains (s, y) and is locally irreducible at this point. As $C \subset S \times \{y\}$, $p_S(C)$ has dimension 1 and is locally irreducible in s. Up to restricting a bit further we may suppose that $S^* \cap p_S(C)$ is irreducible.

By assumption the fibres of p_S over $S^* \cap p_S(C)$ are irreducible, so by the Irreducibility Lemma 1.10 applied to $p_S| : p_S^{-1}(p_S(C)) \to p_S(C)$ we obtain a Zariski open set $U \subset p_S(C)$ such that $p_S^{-1}(U)$ is irreducible of dimension $\dim(|Z_s|) + 1$. As x is generic, we may suppose $s \in U$, and to ease notations that $U = p_S(C)$. It follows that $W := p_X(p_S^{-1}(p_S(C))) \subset X$

is irreducible of dimension $\dim(|Z_s|) + 1$, as it contains $|Z_s|$ as a proper subset.

We show that $W \times \{x\} \subset R_2^{\triangle}$. Indeed if y' and y'' in W are generic, note $Z_{s'}$ (resp. $Z_{s''}$) the unique cycle containing y' (resp. y'') and $s' \in p_S(C)$ (resp. $s'' \in p_S(C)$). Then $Z_{s'}$ and $Z_{s''}$ contain y by construction. So y' and y'' can be joined by a 2-chain. Furthermore as $|Z_s| \subset W$, we have $(x, x) \in W \times \{x\}$, as x is generic this allows to conclude. It follows in particular that the fibre $q_2^{-1}(x)$ contains $W \times \{x\}$ hence is of dimension at least $\dim(|Z_s|) + 1$. This completes the proof of the claim.

c) We show that q_S is a fibration and the S-quotient: As seen in the first part, the generic fibre of q_S is a cycle Z_s, which is by definition irreducible. We saw that q_S is almost holomorphic, so there exists a Zariski open $S^* \subset S$, such that for $s \in S^*$, the fibre $q_S^{-1}(s) = |Z_s|$ yields $|Z_s| \cap I_{p_X^{-1}} = \varnothing$. Suppose there exists $x \in |Z_s|$ and $s' \in S, s' \neq s$ such that $x \in |Z_{s'}|$. Then $p_X^{-1}(x)$ is not a singleton, so $x \in I_{p_X^{-1}}$, a contradiction. So $|Z_s| \cap |Z_{s'}| = \varnothing$ for $s \neq s'$ and by the remark after 1.13, $|Z_s| = q_S^{-1}(s)$ is an S-equivalence class. □

1.6. The irreducible case.

We will now show Theorem 1.1 in the case where S is irreducible, otherwise said, that S is an irreducible covering family of X.

The proof (Thm 1.24) consists in a reduction to the stationary case. For this, we will construct a stationary irreducible covering family S' that induces the same equivalence relation on X and then apply Theorem 1.19.

The generic member of this stationary family consists of the set $R_n(x)$ of all $y \in X$ which are n-equivalent to x, for $x \in X$ generic, and n sufficiently large, but independent of x. The main point of the proof consists in showing that this $R_n(x)$ is irreducible. This needs some technical criteria for irreducibility, both of local and global nature, which we expose now, starting with the standard:

LEMMA 1.20. — *Let V be a complex space and $W \subset V$ a complex subspace. Suppose V to be locally irreducible at W (ie: at its generic point).*

For any desingularization $d : V' \to V$ of V, the fibres of d over W are connected.

Proof. — If V is normal along W, the analytic form of Zariski's main theorem gives the stated result.

In the general case, let $d' : V'' \to V$ be the normalization of V and let $d'' : V' \to V''$ be a desingularization of V''. Then $d := d' \circ d''$ is a desingularization of V. We saw that d'' has connected fibres, so d has connected fibres if d' is bijective.

We show this property: The claim is local on the base, so fix a $v \in W \subset V$, now d' is a finite map, and for each point $v" \in d'^{-1}(v)$, there is a unique local irreducible component of $V"$ through $v"$, and this component is mapped bimeromorphically onto the unique local irreducible component of V through v. Thus, $d'^{-1}(v)$ consists of the single point $v"$, hence d' is bimeromorphic.

We now come to the second and main technical lemma:

Lemma 1.21 [Ca81]. — *Let V be a complex space, W an irreducible complex space. Let $g : V \to W$ be a surjective holomorphic map and $\sigma : W \to V$ a holomorphic section with image $W' := \sigma(W)$. If V is locally irreducible at the generic point of W' then for $w \in W$ generic:*

1. *The fibre $V_w = g^{-1}(w)$ is locally irreducible at $\sigma(w)$.*
2. *If g is proper and V irreducible then V_w is irreducible.*

Proof. — This result was shown in [Ca81] using analytic methods and cycle space theory. We shall give here a short proof based on the existence of desingularizations[(1)].

Before starting the proof, let us indeed observe that if V is smooth, the first property follows from Sard's theorem, together with the analytic lower-semicontinuity of the rank of holomorphic map. And that the second property follows from the Stein factorisation. We shall now reduce to this situation using a desingularisation of V.

Proof of 1) As the assertion concerns only generic fibres, we suppose W to be smooth and connected. Let $d : V' \to V$ be a desingularization of V as in 1.21, i.e. the fibres of d over W' are connected. Note $W'' :=$

[(1)] As T. Peternell indicated to me, this result can be deduced from [B-F93], by combining theorems 1.1 (applied to property (1.5.10)), and 2.1. Because the proofs of [B-F93] are quite involved, we give the alternative easy proof above (using desingularization, however).

$d^{-1}(W') \subset V'$ and $g' = g \circ d : V' \to W$. Then as V' is smooth, there exists a Zariski open dense $W^* \subset W$ such that $g'| : g'^{-1}(W^*) \to W^*$ is smooth (this stronger version of Sard's lemma follows directly from the usual one and the analyticity of maps under consideration). So for $w \in W^*, g'^{-1}(w)$ is smooth and there is exactly one connected component of $g^{-1}(w)$ that intersects W'', this by the preceding lemma.

Now assertion 1) follows directly, as $g'^{-1}(w)$ is smooth and connected near the connected W'', so irreducible. This implies that $V_w = d(g'^{-1}(w))$ is irreducible at $\sigma(w)$.

$$\begin{tikzcd}
& V' \arrow[d, "d"] \arrow[ddl, "\overline{g}"'] & W'' \arrow[l] \arrow[d, "d|"] \arrow[ddll, bend right=60, "\overline{g}|"'] \\
& V \arrow[d, "g"] & W' \arrow[l] \\
\overline{W} \arrow[r, "h"] \arrow[rru, bend right=60, "h'"'] & W \arrow[ru, "\sigma"'] &
\end{tikzcd}$$

Proof of 2) Keeping the same notations, we see that $g' : V' \to W$ is a proper map between nonsingular varieties, so the Stein factorization of g' exists, let $\overline{g} : V' \to \overline{W}$ be with connected fibres and $h : \overline{W} \to W$ finite such that $g' = h \circ \overline{g}$. As in the first part suppose $g'| : g'^{-1}(W^*) \to W^*$ to be a smooth map, i.e. that its fibres $g'^{-1}(w)$ are smooth. We will show that they are also connected, so irreducible, this implies that $V_w = d(g'^{-1}(w))$ is irreducible.

As $g'^{-1}(w) = (h \circ \overline{g})^{-1}(w) = \overline{g}^{-1}(h^{-1}(w))$ and the fibres of $\overline{g}$ are connected, we see that $g'^{-1}(w)$ is connected if and only if $h^{-1}(w)$ is a singleton. So we have to show that h is one-to-one. Let $h' := \sigma \circ h : \overline{W} \to W'$ then h is one-to-one exactly if h' is one-to-one, as the section σ is injective. Let $d| : W'' \to W'$ and $\overline{g}| : W'' \to W'$ be the restrictions of d and $\overline{g}$ respectively. By hypothesis, $d|$ has connected fibres, yet $d|^{-1}(w) = \overline{g}|^{-1}(h'^{-1}(w))$ (see diagram) for $w \in W'$. So h' is one-to-one. □

We shall need the last small technical lemma:

LEMMA 1.22. — *Let $h : W \to R$ be an holomorphic map between complex analytic spaces, with W irreducible. Then some irreducible component of R contains $h(W)$.*

Proof. — Otherwise, for each irreducible component R' of R, $h^{-1}(R')$ is of empty interior and closed in W. This contradicts Baire's category theorem, because W is covered by countably many such sets. □

We now come to the crucial irreducibility lemma wich allows to reduce the case S irreducible to the case S stationary.

LEMMA 1.23 [Ca81]. — *Let $S \subset \mathcal{C}(X)$ be an irreducible compact covering family and $R_n = R_n(S) \subset X \times X$ the graph of n-chains induced by S on X (cf 1.4). Note $p : R_n^{\triangle} \to X$ the restriction of the first projection to the union of its significant components, and for $x \in X$, let $R_n^{\triangle}(x)$ be its fibre over x.*

Suppose $d_n := \dim(R_n^{\Delta}) = \dim(R_{2n}^{\Delta}) =: d_{2n}$. Then

1. *$R_n^{\triangle}$ is symmetric,*
2. *$R_n^{\triangle}(x)$ is irreducible and locally irreducible at (x,x) for a generic $x \in X$,*
3. *$R_n^{\triangle}$ is irreducible and locally irreducible at (x,x) for a generic $x \in X$.*

Proof. — 1) is obvious, since R_n is symmetric.

2) follows from 3) and Lemma 1.21 applied to the proper map $p : R_n^{\triangle} \to X$ and its natural holomorphic section $\sigma : X \to \Delta_X$.

3) It is clearly sufficient to show that $R_n^{\triangle}$ is locally irreducible at the generic point (x,x) of the diagonal, because then $R_n^{\triangle}$ will be irreducible. (Because otherwise, two of its components, both containing the diagonal, would provide distinct local irreducible components at each point of the diagonal).

We now assume by contradiction that I_1, I_2 are two local irreducible components of $R_n^{\triangle}$ at a generic point (x,x) of the diagonal. We can assume, wlog that $\dim(I_1) = d_n$.

The respective sets of reducible points of I_1, I_2 are analytically constructible and do by construction not contain Δ_U . So if $p_j : I_j \to U$ is the restriction of the j-th projection to I_j $(j = 1,2)$, and $\sigma_j : U \to I_j$ its natural section via the diagonal, then I_j is irreducible at the generic point of Δ_U. So by Lemma 1.21 we see that the fibres $I_j(x) := p_j^{-1}(x)$ are irreducible and locally irreducible at (x,x) for $x \in U$ generic. To limit notations we identify $I_j(x) \subset \{x\} \times X$ and its image in X, for $j = 1,2$.

The second projection $p_2| : I_1 \to U$ induces a U-space structure on I_1, and similarly for $p_1| : I_2 \to U$. So we can define the fibred product

$$W := I_1 \times_U I_2 = \{((x,y),(y,z)) \mid x \in U \ , \ y \in I_1(x) \ , \ z \in I_2(y)\}.$$

We next define $h : W \to R_{2n}$, by : $h((x,y),(y,z)) := (x,z)$. This is well-defined, as (x,y) can be joined by an n-chain and so do (y,z).

Claim: We can suppose wlog that W is irreducible.

Proof: Let $q : I_1 \times_U I_2 \to I_1$ be the projection on the first factor of the fibered product, then the generic fibre $q^{-1}((x,y)) \cong I_2(y)$ is irreducible. I_1 itself is irreducible, so by the Irreducibility Lemma 1.10 there exists a Zariski open set $I_1^* \subset I_1$ such that $q^{-1}(I_1^*)$ is irreducible. In particular $U^* := p_1(I_1^*)$ is Zariski dense in U. We conclude that $q^{-1}(I_1^*)$ is irreducible and contains Δ_{U^*}. Yet as $x \in U$ generic, we may suppose that $x \in U^*$ and up to restricting to U^*, that $I_1 \times_U I_2$ is irreducible. □

We now show that this implies $h(W) \subset R_{2n}^{\triangle}$. Indeed, for x in U, we have $((x,x),(x,x)) \in W$, so $\Delta_U \subset h(W)$. Because W is irreducible, $h(W)$ is contained in one irreducible component R^0 of R_{2n}, by Lemma 1.22 above.

Suppose that R^0 is not significant, then $R^0 \cap \Delta_X$ is a proper analytic subset of the irreducible Δ_X, so of empty interior in Δ_X. In particular, it does not contain the relatively open $\Delta_U \subset \Delta_X$. Thus R^0 is significant, and $h(W) \subset R_{2n}^{\triangle}$.

Moreover, $h(W)$ contains

$$I_1 = h(\{((x,y),(y,y)) \mid x \in U, y \in I_1(x)\})$$

and

$$I_2 = h(\{((x,x),(x,z)) \mid x \in U, z \in I_2(x)\}).$$

Summing up, we see that $h(W) \subset R_{2n}^{\triangle}$ is contained in some local irreducible component R' of $R_{2n}^{\triangle}$, and contains I_1 as a proper subset, as $I_1 \neq I_2$. This is a contradiction, because it implies, by the choice of n:

$$d_{2n} \geqslant \dim(R') > \dim(I_1) = d_n = d_{2n}.$$

(We implicitely used here the following standard property: if $A \subset B$ are irreducible complex analytic spaces, with A closed in B, and if $A \neq B$, then: $\dim(B) > \dim A$). □

To conclude the proof of Theorem 1.1 for an irreducible family S, it is thus now sufficient, due to Theorem 1.19, to prove:

THEOREM 1.24. — *Let X be a compact connected normal complex space and S an irreducible covering family for X. Then there exists an irreducible stationary covering family S' for X such that $R(S) = R(S')$.*

Remark. — This theorem implies the existence of the S-quotient in the case where S is supposed to be irreducible, the uniqueness following from the universal property 1.25. Indeed by Theorem 1.19 the S'-quotient exists, yet as the equivalence classes for S and S' coincide, this is also the S-quotient.

Proof. — The sequence $R_n^\triangle \subset X \times X$ is totally ordered and nondecreasing, so there exists some $n \in \mathbb{N}$ such that $\dim(R_n^\triangle) = d_n = d_{2n} = \dim(R_{2n}^\triangle)$. Fix such a n and let $p : R_n^\triangle \to X$ be the restriction of the first projection. Then by Lemma 1.23 the generic fibre is irreducible, so p is a fibration. By [C-P94, p. 331, prop. 2.20], this induces a unique meromorphic map $g : X - \to \mathcal{C}(X)$ defined by $x \mapsto R_n^\triangle(x)$ on the Zariski open dense subset of X consisting of points where the fibre of p is irreducible, reduced and of minimal dimension. We note S' the image of g. Because X is compact and irreducible, S' is also compact and irreducible by Theorem 1.7. Let $G_{S'} \subset S' \times X$ be the incidence graph of S' then the image of $p_X : G_{S'} \to X$ is dense (it contains the points where g is defined). But $G_{S'}$ is compact, so the continuous p_X is surjective. We conclude that S' is an irreducible covering family.

We show the following two claims:

1. $R(S) = R(S')$,
2. S' is stationary.

Proof of 1) By prop. 1.14, $R_1^\triangle(S')$ is irreducible and so, by Lemma 1.23, $R_n^\triangle(S)$ is irreducible, too. For $x \in X$ generic, we have: $R_n^\triangle(S)(x) = R_1^\triangle(S')(x)$ by construction of S'. So the generic fibres of the first projections $p : R_1^\triangle(S') \to X$ and $p' : R_n^\triangle(S) \to X$ coincide, hence $R_1^\triangle(S')$ and $R_n^\triangle(S)$ are bimeromorphic. The bimeromorphic map between them is just the restriction of the identity map on $X \times X$, so admits a bijective extension. So we even have $R_1^\triangle(S') = R_n^\triangle(S) \subset R(S)$. Apply now the infinity Lemma 1.14 to both sides to obtain $R(S') = R(S)$.

Proof of 2) By part 1), $R_1(S') = R_1^{\triangle}(S') = R_n^{\triangle}(S) \subset R_n(S)$, so $R_2(S') \subset R_{2n}(S)$, in particular $R_2^{\triangle}(S') \subset R_{2n}^{\triangle}(S)$. This implies $\dim(R_2^{\triangle}(S')) \leqslant d_{2n} = d_n = \dim(R_1^{\triangle}(S'))$.

So we are left to show that $p_X : G_{S'} \to X$ is a modification. Let $B \subset X$ be an analytic subset such that $g| : X \setminus B \to \mathcal{C}(X)$ is a holomorphic map. Then $S^* := \mathrm{im}(g|) \subset S'$ is a Zariski open irreducible subset of S, and we note $G'_{S^*} := G_{S'} \cap (S^* \times (X \setminus B)) \subset G_{S'}$ (this is a subset of the incidence graph of S^*) which is Zariski open and dense in $G_{S'}$. Then $p_X| : G'_{S^*} \to X \setminus B$ is holomorphic and the following diagram commutes (because $g \circ p = g \circ p'$, if $p' : R_n^{\triangle} \to X$ is the second projection, this because $R_n^{\triangle}$ is symmetric; this is the main point of this part of the argument):

$$\begin{tikzcd} G'_{S^*} \arrow[r, "p_X|"] \arrow[dr, "p_{S^*}"'] & X \setminus B \arrow[d, "g|"] \\ & S^* \end{tikzcd}$$

For $x \in X \setminus B$, we have $p_X|^{-1}(x) \subset S^* \times \{x\}$. So if the fibre contains (s, x) and (s', x) then $s = p_{S^*}((s, x)) = g(p_X|(s, x)) = g(p_X|(s', x)) = p_{S^*}((s', x)) = s'$ by the commutativity of the diagram. Summing up we see that $p_X|$ is holomorphic and one-to-one, so it is a biholomorphism. As X and $G_{S'}$ are reduced, we obtain that p_X is a modification. $\square$

1.7. Proof of Theorem 1.1.

We shall now, as in [Ca99], consider the set of (equivalence classes of) almost-holomorphic fibrations $f : X - \to Y$ of which the general fibre is contained in some S-equivalence class (depending on that fibre). We denote by $\mathcal{F}(X, S)$ this set of (equivalence classes) of fibrations.

If the S-quotient exists, it will belong to $\mathcal{F}(X, S)$, and as its general fibres are of maximal possible dimension (=dimension of an S-equivalence class) its base space will be of minimal dimension among base spaces of fibrations in $\mathcal{F}(X, S)$.

The proof now reduces to show that the inverse is true: If we choose a fibration in $\mathcal{F}(X, S)$ with base space of minimal dimension, it is the S-quotient. The main argument is Lemma 1.29 which makes extensive use of the irreducible case.

We start with a preliminary easy observation:

Proposition 1.25 (Universal Property of the S-Quotient). — *Let X be a compact connected normal complex space and $S \subset \mathcal{C}(X)$ an irreducible compact covering family for X. Let $q_S : X - \to X_S$ be the S-quotient and $\pi : X - \to Y$ be an almost holomorphic fibration on Y, a compact normal complex space such that the general fibre is contained in an S-equivalence class. Then there exists a unique almost holomorphic map $g : Y - \to X_S$ such that $q_S = g \circ \pi$.*

Remark. — This universal property shows that the S-quotient is unique up to meromorphic equivalence and completes the proof of Theorem 1.1 in the irreducible case. Note that the proof uses only the compactness of X_S and not the irreducibility of S, hence remains valid for the case where the covering family S is not irreducible nor compact, as X_S will stay compact (cf. proof of Thm 1.1).

Proof. — Uniqueness is obvious: For $y \in Y$ general, note $\mathcal{R}_y$ the S-equivalence class containing the fibre $\pi^{-1}(y)$ then $g(y) := q_S(\mathcal{R}_y)$ is the only possible choice. This defines g on a dense set.

Existence: Let $f = (\pi, q_S) : X - \to Y \times X_S$ and $\Gamma \subset Y \times X_S$ be the closure of its image. The projections $p_Y : \Gamma \to Y$ and $p_{X_S} : \Gamma \to X_S$ are surjective proper holomorphic mappings. We will show that p_Y is a modification, then $g := p_{X_S} \circ (p_Y)^{-1}$ allows to conclude.

$$\begin{tikzcd} X \arrow[r, dashed, "q_S"] \arrow[d, dashed, "\pi"'] \arrow[dr, dashed, "f"] & X_S \\ Y & \Gamma \arrow[l, "p_Y"] \arrow[u, "p_{X_S}"'] \end{tikzcd}$$

Note I_{q_S} (resp. I_f) the indeterminacy locus of q_S (resp. f). A general fibre $\pi^{-1}(y)$ is in a S-equivalence class, so is contracted by q_S. As q_S and f are almost holomorphic, it meets neither I_{q_S} nor I_f. Now, $\pi^{-1}(y) = f^{-1}(p_Y^{-1}(y))$ is contracted by q_S, hence by f. So $p_Y^{-1}(y) = f(f^{-1}(p_Y^{-1}(y)))$ is a point, so there exists a open neighborhood $U \subset Y$ of y such that $p_Y| : p_Y^{-1}(U) \to U$ is finite. As the general points are dense in Y, p_Y is finite.

But Γ is irreducible as it is the image of the closure of the graph of f in $X \times Y \times X_S$ by the projection $p_Y \times p_{X_S}$. So if $Y^* \subset Y$ is a Zariski open set such that for $y \in Y^*, \pi^{-1}(y)$ meets neither I_π nor I_{q_S}, then $\Gamma^* = p_Y^{-1}(Y^*)$ is Zariski open and dense in Γ. Note $X^* := f^{-1}(\Gamma^*)$, then the restrictions of π and f to X^* are holomorphic. π is a fibration, so the generic fibres are irreducible, hence connected. As $\pi = p_Y \circ f$ and for $y \in Y^*$, $\pi^{-1}(y) \subset X^*$, this implies that the restriction of p_Y to Γ^* is injective, hence a holomorphic inverse $p_Y^{-1} : Y^* \to p_Y^{-1}(Y^*)$ exists. As p_Y is finite, it extends to a meromorphic map on Y.

We are left to show that g is almost holomorphic. For $s \in X_S$ generic, $g^{-1}(s) = \pi(q_S^{-1}(s))$, yet the generic fibre of q_S is contained in X^*, so its image by π is in the dense Y^*. □

We now proceed to construct step-by-step the S-quotient in the general case:

Definition 1.26. — *A fibration $f : X - \to Y$ induces an equivalence relation on X in the following way: for $y \in Y$, let $f^{-1}(y)$ be the (Chow-theoretic) fibre as defined in 1.2. By definition two points x, x' in X are 1-equivalent if there exists a $y \in Y$ such that $x, x' \in f^{-1}(y)$. As every point x is connected to itself, the graph of 1-chains $R_1(f) \subset X \times X$ is symmetric and contains the diagonal, hence induces an equivalence relation on X (as in 1.16), whose graph will be denoted $R(f) \subset X \times X$.*

Definition 1.27. — *Let S be a not necessarily irreducible covering family for a compact connected normal complex space X. A fibration $f : X - \to Y$ on a compact connected normal complex space Y is subordinate to S if a general fibre of f is contained in an S-equivalence class. We note $\mathcal{F}(X, S)$ the (non-empty) set of S-subordinate almost holomorphic fibrations of X.*

Remark. — The fibration f is thus subordinate to S iff $R^1(f) \subset R(S)$, or equivalently, if $R(f) \subset R(S)$.

Definition 1.28. — *For S a covering family for X, let $S_i \subset S$ be an irreducible compact component. Note G_{S_i} the incidence graph of S_i and $p_X^i : G_{S_i} \to X$ the projection to X. For $f \in \mathcal{F}(X, S)$, we say that S_i is f-covering if $f \circ p_X^i : G_{S_i} - \to Y$ is surjective.*

Remark. — If S_i is a covering family for X, it is f-covering. In fact, S_i is f-covering if and only if a generic fibre of f meets some cycle parametrized by S_i.

The following lemma provides a "constructive" method to find the S-quotient in finitely many (at most $\dim(X)$) effective steps. Start with an element of $\mathcal{F}(X,S)$ (for example id) and an f-covering irreducible component S_i of S. Then there are two possibilities: a generic cycle for S_i is contained in an f-fibre, then go on with the next f-covering irreducible component, if any. If this is not the case, construct an irreducible covering family S' and its quotient map $q_{S'}$. Then $q_{S'}$ is in $\mathcal{F}(X,S)$ and we replace f by $q_{s'}$. Continue until all f-covering irred. components satisfy the first case, then f is the S-quotient (cf. proof of Thm. 1.1 below).

Lemma 1.29. — *Let $f : X - \to Y \in \mathcal{F}(X,S)$ and S_i an f-covering irreducible component of S. Then either:*

1. *$R(S_i) \subset R(f)$, or,*
2. *there exists an irreducible compact covering family S' of X such that the S'-quotient $q_{S'} : X - \to X_{S'}$ satisfies the following properties,*
3. *$\exists\, g : Y - \to X_{S'}$ an almost holomorphic mapping such that $q_{S'} = g \circ f$ (in other words: f is subordinate to S').*
4. *$\dim(X_{S'}) < \dim(Y)$*
5. *$R(S_i) \subset R(S')$*
6. *$R(S') \subset R(S)$*

Proof. — Assume $R(S_i) \not\subset R(f)$; then $R_1(S_i) \not\subset R_1(f)$ (cf. proof of the infinity lemma). So for a generic $s \in S_i$, we have $\dim(f(|Z_s|)) > 0$, otherwise the irreducible cycle $|Z_s|$ would be contained in some fibre of f. Because S_i is f-covering, a generic cycle $(Z_s)_{s\in S_i}$ is not contained in the indeterminacy locus I_f nor in any exceptional fibre of the fibration f (i.e. fibres of dimension larger than that of the generic fibre).

In this case set $|Z_s^*| = |Z_s| \setminus I_f$ and define $f(|Z_s|) := \overline{f(|Z_s^*|)}$ to be the Zariski closure. By construction $f(|Z_s|)$ is an irreducible compact cycle in Y and up to restricting to a Zariski open set $S_i^* \subset S_i$ we can suppose the family of cycles to be equidimensional. The same argument as in the proof of Lemma 1.30 below, shows that $(f(|Z_s|))_{s\in S_i^*}$ is an analytic family in Y. If $\Gamma \subset S_i \times Y$ denotes the closure of the incidence graph of this family, then the generic fibres of the projection of Γ on S_i are equidimensional, and by [C-P94], or [Ba79], induce a meromorphic map $\varphi : S_i - \to \mathcal{C}(Y)$. Note $\overline{S_i^*}$ the normalization of the closure of the image, then $\overline{S_i^*}$ parametrizes an analytic family on Y. It is even an irreducible compact covering family as S_i is f-covering.

If we restrict S_i^* a bit further, an application of Lemma 1.30 shows that $Z'_s := f^{-1}(f(|Z_s|))$ for $s \in S_i^*$ defines an analytic family in X. As before, passing by the closure of the graph, we get a meromorphic map from S_i to $\mathcal{C}$(X). Normalizing its image, we obtain an irreducible analytic family S', which is compact by Theorem 1.7, the generic member of which is some Z'_s, constructed above. This is a covering family, since it contains the dense family $(Z'_s)_{s \in S_i^*}$.

Let $q_{S'} : X - \rightarrow X_{S'}$ be the S'-quotient of X. By construction, a generic fibre of f is contained in some Z'_s (which is an S'-chain of length one). It follows that a general fibre of f is contained in a S'-equivalence class and we use the above universal property of the S'-quotient to obtain an almost holomorphic $g : Y - \rightarrow X_{S'}$ such that $q_{S'} = g \circ f$.

$$\begin{tikzcd} X \arrow[r, dashed, "q_{S'}"] \arrow[d, dashed, "f"'] & X_{S'} \\ Y \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

We check the other properties stated:

a) A generic $q_{S'}$-fibre is an S'-equivalence class, so contains at least one $|Z'_s|$. By hypothesis its image $f(|Z'_s|) = f(|Z_s|)$ is of positive dimension, so a generic fibre of g is positive dimensional, hence $\dim(Y) > \dim(X_{S'})$.

b) By construction of S', a generic S_i-chain of length one is contained in a similar S'-chain. So, $R_1(S_i) \subset R_1(S')$ by irreducibility of both sides, and we conclude by the infinity lemma.

c) It is sufficient to show that $R_1(S') \subset R(S)$ (by the infinity lemma, again). By a density and irreducibility argument, we are left to show that $Z'_s = f^{-1}(f(|Z_s|))$ is contained in some S-equivalence class, for $s \in S_i^*$ generic.

However, f is subordinate to S, and so for $y \in f(|Z_s|)$ generic, the fibre $f^{-1}(y)$ is contained in one S-equivalence class, depending on y. But each such fibre of f contains some element of the length one S_i-chain $|Z_s|$. As S_i is a subset of S, $|Z_s|$ is in a single S-equivalence class, and so, any $x \in f^{-1}(f(|Z_s|))$ generic there belongs to the S-equivalence class containing $|Z_s|$. Thus also $R_1(S') \subset R(S)$, as claimed. $\square$

In the course of the preceding proof, we used the following:

Lemma 1.30. — *Let S be an irreducible covering family for Y and $f : X - \rightarrow Y$ an almost holomorphic fibration between compact connected normal complex spaces. Then there exists a Zariski open set $S^* \subset S$ such that*

$$Z'_s := \overline{f^{-1}(|Z^*_s|)}\ \forall\ s \in S^*$$

defines an analytic family of cycles on X, where $|Z^*_s| = |Z_s| \setminus f(I_f)$ and the closure is taken in the Zariski topology.

Proof. — The support of a generic cycle Z_s is not contained in the image of the indeterminacy locus $f(I_f)$ (cf. 1.2). Note that for $y \notin f(I_f)$ the usual and the Chow-theoretic fibre coincide, so no distinction will be necessary in the following.

As Z_s is irreducible, $|Z^*_s| = |Z_s| \setminus f(I_f)$ is an irreducible relatively Zariski open set in $|Z_s|$. As f is a fibration, we may apply the remark after the irreducibility lemma to obtain that for a generic s, $f^{-1}(|Z^*_s|)$ is open and irreducible in X. Let $S^* \subset S$ be the Zariski open set where this is satisfied and the Zariski closure $Z'_s := \overline{f^{-1}(|Z^*_s|)}$ is of dimension $\dim(|Z_s|) + \dim(X) - \dim(Y)$.

We want to show that $(Z'_s)_{s \in S^*}$ is an analytic family. But S^* being normal, [Ba75, par. 1, Thm. 1] states that if the incidence graph $G'_{S^*} = \{(s, x) \mid x \in |Z'_s|\} \subset S^* \times X$ is analytic and the projection $p_{S^*} : G'_{S^*} \rightarrow S^*$ is proper surjective with equidimensional fibres, the family Z'_s is analytic. The family $(Z_s)_{s \in S^*} \subset Y$ being analytic, its incidence graph

$$G_S = \{(s, x) \mid x \in |Z_s|\} \subset S^* \times Y$$

is analytic and the meromorphic map $(id, f) : G'_{S^*} - \rightarrow G_{S^*}$ is holomorphic on a dense subset V of G'_{S^*} (for s fixed, (id, f) is holomorphic on $\{s\} \times f^{-1}(|Z^*_s|)$). Being by definition the preimage of the analytic space G_{S^*}, V is locally analytic, so its Zariski closure G'_{S^*} is locally analytic and closed, hence analytic. As X is compact, p_{S^*} is proper surjective and by the condition on the dimension imposed above, its fibres are equidimensional. □

Proof of Theorem 1.1. —

Let $f : X - \rightarrow Y$, $f \in \mathcal{F}(X, S)$ be an almost holomorphic fibration such that $\dim(Y)$ is minimal and note $(S_i)_{i \in I}$ the irreducible components of S. We show that f is the S-quotient. Define

$$J := \{i \in I \mid S_i \text{ is not} f\text{-covering}\}$$

For $i \in J$, by definition of f-covering, $f \circ p^i_X(G_{S_i})$ is analytic of codimension 1 or more in every point, and so:

$E := \cup_{i \in J}(f \circ p^i_X(G_{S_i}))$ is contained in a countable union of proper analytic subsets of Y.

If $F = f^{-1}(y)$ is a *general* fibre such that $y \notin (f(I_f) \cup E)$, then F is an $R(f)$-equivalence class as it does not meet I_f, so has no intersection with other f-chains of length one.

Now on the one hand, by Lemma 1.29, for $i \in (I \setminus J)$, we have $R(S_i) \subset R(f)$, so if Z_s is a cycle parametrized by $S_i, i \in (I \setminus J)$, we have either $|Z_s| \subset F$, or $F \cap |Z_s| = \varnothing$. On the other hand if Z_s a cycle parametrized by $S_i, i \in J$, as F is general, we have $F \cap |Z_s| = \varnothing$.

It follows that F is S-saturated, as it is the union of the $R(S)$-equivalence classes generated by the cycles it contains. As f is supposed to be S-subordinate, F is contained in a single S-equivalence class.

We conclude that a general f-fibre is an $R(S)$-equivalence class. □

1.8 An estimate of the length of S-chains.

The above proof of Theorem 1.1 easily gives an estimate of the length of the S-chains connecting two general S-equivalent points of X (see [K-M-M92] and [De01] for similar results).

Proposition 1.31. — *Let S be an irreducible covering family of a normal variety X. Let $q_S : X - \to X_S$ be the S-quotient of X and F a regular fibre of q_S (ie: F is assumed to be contained in the locus where q_S is regular, or holomorphic. The existence of regular fibres is equivalent to q_S being almost-holomorphic). Then any two points of F are joined by an S-chain of length at most $l := 2^{f-r}$, where $r := \dim(Z_s)$, and $f := \dim(X) - \dim(X_S)$ is the dimension of a generic fibre of q_S. In particular, $l \leqslant 2^{f-1}$, unless S is the family of points of X.*

Proof. — Assume the sequence $d_1, d_2, d_4, \ldots, d_{2^k}$ is strictly increasing, while $d_{2^k} = d_{2^{k+1}}$. Thus $d_{2^k} \geqslant d_1 + k$. From the proof of 1.1, we see that $d_1 = m + r$, while $d_{2^k} = m + f$, if $m := \dim(X)$. Thus $m + f = d_{2^k} \geqslant d_1 + k = m + r + k$, and $k \leqslant f - r$. Hence the claim, since by construction, two points of F are connected by a T-chain of length 2^k. □

Remark. — The proofs given here work also when S has all of its components S_i covering, if r denotes the infimum of the corresponding r_i's.

So, in this case, we always get: $l \leqslant 2^{f-1}$. The estimate is then in this case slightly sharper than the one $l \leqslant 2^f - 1$ given in [K-M-M92], [De01]. The estimate $l \leqslant 2^f - 1$ can easily be deduced but apparently not improved in the general case from 1.31, by the reduction process described in the previous section.

In the general case, we have:

PROPOSITION 1.32 ([De01], [K-M-M92]). — *Any two points of a regular fibre F' of $q : X - \to X_{S'}$ can be connected by an S'-chain of length at most $l' \leqslant 2^b - 1$, if $b = \dim(X) - \dim(X_{S'})$ is the dimension of F'.*

(Recall from 1.31 that a fibre of an holomorphic fibration is said to be regular if it does not meet the indeterminacy locus of that fibration, or equivalently: is contained in the locus where q is holomorphic).

Proof. — Using the same notations as in the proof of 1.29 above, we just need to show that if this estimates holds for f in place of $q_{S'}$, it holds also for $f' := q_{S'}$, where $R(q_{S'})$ is (the equivalence relation) generated by $R^1(f)$ and $R^1(S_i)$, S' being constructed as in 1.29 from f and S_i, an irreducible f-covering family of X.

By 1.31, any two points of any regular fibre F' of f' are connected by an S'-chain of length $l'' \leqslant 2^{b'-r'} \leqslant 2^{b'-b-1}$, where $b' := \dim(F')$, $r' := \dim(Z'_{s'}) \geqslant b + 1$, with $b := \dim(F)$, and F any regular fibre of f. By the definition of $Z'_{s'}$, we immediately see that any two of its points are connected by an (S_i, f)-chain of length at most $2l + 1$, if any two points of any fibre of f can be connected by a f-chain of length l. By the induction hypothesis on f, we can take $l := (2^b - 1)$. So that any two points of any fibre of f' can be connected by an S'-chain of length at most $l' \leqslant l''(2l + 1) \leqslant 2^{b'-b-1}(2(2^b - 1) + 1) \leqslant 2^{b'} - 1$, as claimed. □

2. Zariski regularity.

We shall now introduce the notion of Z-regularity for subsets of arbitrary analytic spaces. This is a weak notion of countable constructibility which seems to be satisfied by the set of fibres enjoying an "analytic" property, for (essentially) arbitrary holomorphic maps between complex spaces.

DEFINITION 2.1. — *Let $A \subset S$, where S is a complex space. We say that A is Z-regular (in S) if for each irreducible Zariski closed subset T of S, $A \cap T$ either contains the general point of T, or is contained in a countable union of Zariski closed proper subsets of T. In the latter case, we say that A is of first Zariski category in T.*

There are many examples of this situation (actually it rather seems that counterexamples are unnatural in algebraic or analytic geometry).

THEOREM 2.2. — *Let $X \in \mathcal{C}$. Let $A_P(X) = A_P \subset \mathcal{C}(X)$ be the set of points t parametrising the set of irreducible reduced subvarieties V_t of X possessing a certain property P.*

Then $A_P(X) = A_P$ is Z-regular in $\mathcal{C}(X)$ if P is any one of the following properties.

1. *V_t is special.*
2. *V_t is a (possibly singular) rational curve.*
3. *V_t has algebraic dimension at least d, d any given nonnegative integer.*
4. *$\pi_1(\hat{V_t})_X \in \mathcal{G}$, where $\mathcal{G}$ is a class of (isomorphism classes) of groups, and $\pi_1(\hat{V_t})_X$ is the image in $\pi_1(X)$ of the fundamental group of the normalisation of V_t.*

Proof. — For 1 (resp. 3, 4), use [Ca04] and Proposition 2.3 below (resp. [Ca80'], [Ca99]). Assertion 2 is easy. □

Recall that a compact complex manifold X is said to be in Fujiki's class $\mathcal{C}$ if it is bimeromorphic to some compact Kähler manifold X' (depending on X).

PROPOSITION 2.3. — *Let P be a property such that if $f : X \dashrightarrow Y$ is any fibration such that $X \in \mathcal{C}$, then the set of irreducible fibres of f having property P is Z-regular (in Y). Then, for any $X \in \mathcal{C}$, $A_P(X)$ is Z-regular in $\mathcal{C}(X)$.*

Proof. — Let $T \subset \mathcal{C}(X)$ be irreducible Zariski closed. Let $V = V_T$ be the incidence graph of the family T. By [Ca80], $V \in \mathcal{C}$. Thus $A_P(X) \cap T$ is Z-regular in $\mathcal{C}(X)$, by the hypothesis made on P. □

PROPOSITION 2.4. — *Let $A \subset S$ be Z-regular. There exists a countable or finite family of Zariski closed irreducible subsets S_i of S such that:*

1. *for each i, $A_i := A \cap S_i$ contains the general point of S_i,*
2. *A is the union of the A_i's.*

The family S_i is moreover unique if it is irredundant. (That is, if there is no inclusion among the S_i's in the sense that if $i \neq j$, then S_i is not included in S_j, and conversely S_j is not included in S_i).

The S_i's are then called the components of A.

Proof. — Let $\overline{A}$ be the Zariski closure of A in S, and let A' be an irreducible component of $\overline{A}$. Consider $A \cap A'$: it either contains the general point of A', or is contained in a countable union B of Zariski closed subsets of A'. In the first case, A' is a component of A. Proceed then in the same way with $\overline{A}$, but after removing its component A'. In the second case, remove A' from $\overline{A}$, and replace it by B. Repeat this process, observing that for each component of $\overline{A}$, we let at each step decrease the dimension of the remaining irreducible components. So the process ends in finitely many steps for each irreducible component of S, which shows the existence statement.

The uniqueness statement is easy to check: assume we have two irredundant families S_i and S'_j of components of A. It is sufficient to show that any S_i is contained in some S'_j. If not, $A \cap S_i$ is contained in the countable union of the $A \cap S'_j$. It is then of first Zariski category in S_i. This contradicts to the definition of a component of A. □

The notion of Z-regularity has applications to the construction of meromorphic quotients.

THEOREM 2.5. — *Let $X \in \mathcal{C}$ be normal. Let $A \subset \mathcal{C}(X)$ be Z-regular. Let $T := T(A)$ be the family of components of A, as defined in 2.4. (If T is not covering, we add the family of points of X). Let $q_A := q_T : X - \to X_T := X_A$ be the T-quotient of X. Let $t \in A$ such that V_t meets some general fibre F of q_T.*

Then V_t is contained in F. We call q_T the A-reduction of X.

Proof. — By the defining property of q_T, its general fibre F is an equivalence class for $R(T(A))$. In particular, any V_t, $t \in A$, is either disjoint from, or contained in F. □

Remark. — Two points in such a fibre are joined by a $T(A)$-chain, but not necessarily by an A-chain. Such a $T(A)$-chain turns out to be an A-chain if A is stable under specialisation, in the sense defined in the question just below for a family of special subvarieties of X.

For example, this is the case for properties 2, and 3 in 2.2 above. For property 2, this is simply because rational curves specialise to unions of such. For property 3, see [Ca80']. The case of property 1 is not presently known, but a positive answer to the following question is expected.

Question: Let $f : X- \to Y$ be a special fibration, with $X \in \mathcal{C}$. Is each irreducible component of X_y special, for each $y \in Y$?

An immediate application of 2.5 above, obtained explicitly in [Ca92] from [Ca81] by taking for A the family of rational curves, and in [K-M-M92] in the algebraic case by using their glueing lemma, is the following.

THEOREM 2.6. — *Let $X \in \mathcal{C}$ be normal. There exists an almost holomorphic map $r_X : X- \to R(X)$, called the rational quotient of X such that:*

1. *the fibres of r_X are rationally connected (ie any two of their points are contained in a connected union of finitely many rational curves),*
2. *if a rational curve C inside X meets some general fibre F of r_X, then C is contained in F.*

Moreover, A-reductions can be constructed in a relative situation as well.

THEOREM 2.7. — *Let $X \in \mathcal{C}$ be normal. Let $f : X \to Y$ be a holomorphic fibration. Let $A \subset \mathcal{C}(X)$ be Z-regular. Let $A_f \subset A$ be the set of all $t's \in A$ such that V_t is contained in some fibre of f (depending on t).*

Then:

1. *$A_f \subset \mathcal{C}(X)$ is also Z-regular.*

2. *If $q_{A_f} : X- \to X_{A_f}$ is the A_f-reduction of X, there exists a factorisation $h_{A_f} : X_{A_f}- \to Y$ such that $f = h_{A_f} \circ q_{A_f}$.*

3. *Moreover, for $y \in Y$ general, the restriction of q_{A_f} to X_y is the A_y-reduction of X_y, if $A_y := A_f \cap \mathcal{C}(X_y)$.*

The map q_{A_f} is called the A-reduction of f.

Proof. — The fact that A_f is Z-regular is obvious, because A is Z-regular, and A_f is the intersection of A with its Zariski closed subset $\mathcal{C}(X/Y)$ consisting of $t's$ such that V_t is contained in some fibre of f. We thus have a quotient q_{A_f} for A_f.

The natural map $f:(X)- \to Y$ now defines the factorisation $f = h_{A_f} \circ q_{A_f}$, because the generic equivalence classe for $T(A_f)$ is obviously contained in some fibre of f. (Recall indeed that, according to the notations introduced in 2.5, $T(A_f)$ stands for the family of *components* of A_f, defined in 2.4).

We shall now show that, for y general in Y, $(T(A_f))_y = T(A_y)$. This will immediately imply that $(X_y)_{A_y} = (X_y)_{(A_f)}$, and so the claim. (Recall that we denote by Z_y the fibre over y of a map $Q: Z \longrightarrow Y$, for $y \in Y$, and by Z_S the meromorphic quotient of Z by the equivalence relation generated by a family S of subvarieties $(S_i)_{i \in J}$ in $\mathcal{C}(Z)$).

Notice first that, by definition, $A_y = (A_f)_y$, for any y. Next, if $T'(A_f)$ is the union of all components of $T(A_f)$ surjectively mapped to Y by f_*, then $(T'(A_f))_y = (T(A_f))_y$ for y general in Y. It is thus sufficient to show that $(T'(A_f))_y = T(A_y)$ for y general in Y.

We shall now show this statement. First, we have, for y general, $A_y = (A_f)_y \subset (T'(A_f))_y$. This gives the inclusion: $T(A_y) \subset (T'(A_f))_y$.

To show the reverse inclusion, let T be a component of $T'(A_f)$. Then: $T_y \subset T(A_y)$ for general y. Indeed: $(T - (A_f \cap T))$ is contained in a countable union of Zariski closed proper subsets B_j of T (by Z-regularity). For each j, the set C_j of $y's \in Y$ such that $(B_j \cap T_y)$ has empty interior in T_y is Zariski closed, and proper in Y. If C is the union of the C_j's, and y is not in C, we conclude that $(T'(A_f))_y \subset T(A_y)$, as claimed. □

The main applications of 2.7 are to construct relative versions of the rational quotient, of the core, and other natural fibrations. For example, applying 2.7 to the family of rational curves on $X \in \mathcal{C}$, we get:

PROPOSITION 2.8. — *Let $f: X \to Y$ be a holomorphic fibration, with $X \in \mathcal{C}$, normal. Then f has a rational quotient, that is, a factorisation $f = R_f \circ r_f$, where $r_f: X- \to R(f)$ is a fibration which induces the rational quotient $r_{X_y}: X_y- \to R(X_y) = (R(f))_y$ for y general in Y.*

3. Stability.

DEFINITION 3.1. — *Let $A \subset \mathcal{C}(X)$ be Z-regular, for $X \in \mathcal{C}$, normal. We say that A is stable if it has the following two properties.*

(stab1) The general fibre of $q_T : X - \to X_T$ is in A if T is any covering irreducible component of A, in the sense of definition 2.4 above.

(stab2) If $V \subset X$ is a subvariety and $f : V - \to W$ is a fibration with general fibre V_w in A, and if there exists a subvariety $W' \subset V$ such that $[W'] \in A$, and $f(W') = W$, then $[V] \in A$. (Said otherwise: V is in A if it is fibered over W with fibers in A, and with a transversal W' also in A).

(By abuse of language, we say that a subvariety Z of X is in A if the point $[Z] \in \mathcal{C}(X)$ which parametrises this subvariety is in A).

Let us give some examples.

THEOREM 3.2. — *Let $X \in \mathcal{C}$, and let $A_P(X) \subset \mathcal{C}(X)$ be as in 2.2. Then:*

1. *A_P is stable, if P is the property of being special and X is smooth.*
2. *A_P is stable, if P is the propery that $\pi_1(\hat{V}_t)_X \in \mathcal{G}$, where $\mathcal{G}$ is a class of (isomorphism classes) groups which is stable in the sense of [Ca99]. (Such classes are the class of finite, or solvable groups, for example).*

The proof is given in [Ca04] for assertion 1, and in [Ca99] for assertion 2. □

The basic result about stability is the following.

THEOREM 3.3. — *Let $A \subset \mathcal{C}(X)$ be Z-regular and stable, for $X \in \mathcal{C}$, normal. Let $q_A : X - \to X_A$ be the A-reduction of X, as defined in 2.5. Then the general fibre of q_A is in A.*

Proof. — Because of [stab1], the statement is obviously satisfied for the quotient of X by any of the covering irreducible components of $T(A)$, and so for the first step of the construction of q_A, as exposed in 1.6.

To deal with the general case, we proceed by induction on the number of steps needed to construct q_T, with $T = T(A)$, this number k of steps being defined as in the proof of 1.32. We are thus reduced to show that, if $f : X - \to Y$ is an almost holomorphic fibration with its general fibre in A, and if $T' \subset T(A)$ is irreducible compact, and parametrises a family V_t of cycles of X such that V_t is irreducible, reduced and is in A for t general in T', and if, moreover, the family T' is f-covering and such that $dim(f(V_t)) > 0$, for t generic in T, the T"-quotient $q_{T"} : X - \to X_{T"}$ of X has general fibres in A, if $T" \subset \mathcal{C}(X)$ is defined as in the proof of Lemma 1.29. By the property [stab 1], it is sufficient to show that the general member V'_t of the family parametrised by $T"$ is in A for t general in T'. But this is immediate, by property [stab 2], since by its very definition, $f|V'_t : V'_t := f^{-1}(f(V_t)) - \to W := f(V_t)$ has general fibres in A, and has a subvariety W' (namely: $W' := V_t$)) which is in A, and is mapped surjectively onto W by $f|V'_t$. □

From 3.2 and 3.3, we immediately deduce:

COROLLARY 1. — *If $X \in \mathcal{C}$ is smooth, the general fibre of the core of X is special.*

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