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LOCALLY CONNECTED EXCEPTIONAL MINIMAL SETS OF SURFACE HOMEOMORPHISMS

by Andrzej BIŚ (*), Hiromichi NAKAYAMA (**) and Paweł WALCZAK (***)

In 1977, Fathi and Herman [5] proposed the following problem: Does there exist a $\mathcal{C}^\infty$ diffeomorphism f of a compact manifold such that f admits a minimal set which is locally homeomorphic to neither a Euclidean space nor the product of a Euclidean space and the Cantor set?

Certainly, there is no $\mathcal{C}^\infty$ diffeomorphism of the circle with such a minimal set. Although Handel [8] constructed a $\mathcal{C}^\infty$ diffeomorphism of a 2-dimensional manifold with such a minimal set (in fact, it is a pseudo-circle), their problem still indicates some intrinsic property in dynamical systems, in particular in 2-dimensional case. However it is difficult to treat the condition that a minimal set is not locally homeomorphic to the product of a Euclidean space and the Cantor set.

In this paper, we replace this condition by the local connectivity of a minimal set and examine the topological types of minimal sets for homeomorphisms of closed orientable surfaces. The condition of local connectivity appears in topological dynamics in a natural way either as a property of the space carrying the dynamics or as a property of minimal sets which is either assumed, proved or disproved. For example, the results of [17] show that a wide class of homogeneous flows admits no locally connected minimal sets while Kim [11] has shown that locally connected minimal sets of flows of compact separable metric spaces reduce to either single

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points or circles whenever they have cohomological dimension (with respect to Alexander-Spanier cohomology with coefficients in a principal ideal domain) ≤ 1. Our results here begin with the following

THEOREM 1. — *Let f be a homeomorphism of a closed orientable surface Σ different from the torus T^2. If a minimal set $\mathfrak{M}$ of f is locally connected, then $\mathfrak{M}$ is either a finite set or a finite disjoint union of simple closed curves.*

THEOREM 2. — *Let f be a homeomorphism of T^2. If there exists a locally connected minimal set $\mathfrak{M}$ which is neither finite, nor a finite disjoint union of simple closed curves, nor the whole T^2, then $\mathfrak{M}$ is the unique minimal set of f. This set $\mathfrak{M}$ satisfies the following conditions 1)–5), where $\{U_i\}_{i=1,2,\dots}$ denotes the family of all the connected components for the complement of $\mathfrak{M}$:*

1) *Each U_i is the interior of an embedded disc ($i = 1, 2, \dots$).*
2) *$\{\overline{U_i}\}_{i=1,2,\dots}$ is a null sequence (i.e. the diameter of U_i tends to 0 as $i \to \infty$).*
3) *$\overline{U_i}$ intersects $\overline{U_j}$ at most at one point when $i \neq j$, and their intersection (if non-empty) consists of a locally separating point of $\mathfrak{M}$.*
4) *There is no finite chain $U_{i_1}, U_{i_2}, \cdots, U_{i_n}$ $(n > 1)$ such that $\overline{U_{i_j}} \cap \overline{U_{i_{j+1}}} \neq \emptyset$ $(j = 1, 2, \dots, n-1)$ and $\overline{U_{i_1}} \cap \overline{U_{i_n}} \neq \emptyset$.*
5) *$\mathfrak{M}$ is connected.*

If, instead of conditions 3) and 4) of Theorem 2, we assume that $\{\overline{U_i}\}$ consists of mutually disjoint sets, then $\mathfrak{M}$ appears to be homeomorphic to the Sierpiński T^2-set, which is obtained from T^2 by removing the interiors of a null sequence of mutually disjoint closed discs whose union is dense in T^2 (compare [2]). Thus we obtain the following:

COROLLARY 1. — *Let f be a homeomorphism of T^2. Any locally connected minimal set without a locally separating point either is finite, or coincides with the whole T^2, or is homeomorphic to the Sierpiński T^2-set.*

The next (and last) result here shows that the assumption of absence of locally separating points cannot be deleted from Corollary 1.

THEOREM 3. — *There exists a homeomorphism of T^2 having a locally connected minimal set which admits a locally separating point and is not a finite disjoint union of simple closed curves.*

In §1, we will construct a homeomorphism of T^2 satisfying the condition of Theorem 3 by pinching holes of the Sierpiński T^2-set. Theorems 1 and 2 are proved in §3 and §4 respectively. In order to establish these theorems, we will show in §2 (Lemma 2) the non-existence of cut points in any connected minimal set.

Note that by results of Chu [3] the construction similar to that of Theorem 3 cannot be performed for flows (or actions of arbitrary connected topological groups).

1. Pinching Sierpiński T^2-sets.

Let X be a compact metric space and S be a subset of X. As usually, we denote by ∂S the frontier of S and by $\operatorname{int} S$ its interior. Furthermore, $\operatorname{diam} S$ denotes the diameter of S, i.e., the smallest upper bound for the distances of points in S. A countable collection $\{S_i\}_{i=1,2,\dots}$ of subsets S_i is called a *null sequence* if, for each $\varepsilon > 0$, only finitely many of the sets S_i have diameter greater than ε (see [4]). In other words, $\lim_{i\to\infty} \operatorname{diam} S_i = 0$.

A point z of S is called a *cut point* of S if $S \setminus \{z\}$ is not connected in S. Also, a point z of a subset S is called *locally separating* if there exists a connected neighbourhood U of z in S such that $U \setminus \{z\}$ is not connected. Finally, let us recall that a subset S is *locally connected* if, for any point z of S and any neighbourhood U of z in S, one can find a connected neighbourhood of z contained in U.

Let f be a homeomorphism of X. A non-empty subset $\mathfrak{M}$ of X is called *minimal* if $\mathfrak{M}$ is closed, invariant under f (i.e., $f(\mathfrak{M}) = \mathfrak{M}$) and minimal with respect to the inclusion among all non-empty closed f-invariant sets. By Zorn Lemma, any homeomorphism of a compact metric space has a minimal set. When the whole X is a minimal set, the homeomorphism f is called *minimal.* Then all its orbits are dense.

Typical examples of minimal homeomorphisms of surfaces are minimal translations of the torus T^2 defined as follows: Let α and β be irrational numbers such that α/β is also irrational. A homeomorphism f of T^2 defined by

$$f(x, y) = (x + \alpha, y + \beta)$$

for $x, y \in \mathbb{R} / \mathbb{Z}$ is minimal and called a *minimal translation* of T^2.

Whyburn [18] showed that the Sierpiński curve (called also Sierpiński carpet) can be characterized as a subset of the sphere S^2 obtained by removing the interiors of a null sequence of mutually disjoint closed discs whose union is dense in S^2. His arguments can be also applied to such subsets of the torus T^2 (and arbitrary closed manifolds, [2]). Thus we may define the *Sierpiński T^2-set* as a subset of T^2 obtained by removing the interiors of a null sequence of mutually disjoint closed discs whose union is dense in T^2.

Aarts and Oversteegen [1] constructed a homeomorphism of the Sierpiński curve with a dense orbit. They inserted mutually disjoint discs into S^2 and extended a homeomorphism of S^2 with a dense orbit to a homeomorphism of S^2 with the union of inserted discs invariant. This construction can be performed also in the case of a minimal translation of T^2 (see [2] for a detailed description), so we can obtain in this way a homeomorphism f of T^2 with a minimal set $\mathfrak{M}$ homeomorphic to the Sierpiński T^2-set and such that the family $\{f^n(U)\}_{n\in\mathbb{Z}}$ consists of mutually disjoint sets for any connected component U of $T^2\setminus\mathfrak{M}$. By suitable use of this homeomorphism, we will construct soon a homeomorphism of T^2 satisfying the conditions of Theorem 3.

Remark 1. — A $\mathcal{C}^{3-\varepsilon}$ diffeomorphism of T^2 with a minimal Sierpiński T^2-set has been constructed by McSwiggen [13] for any $\varepsilon > 0$. To get it, he chooses an Anosov diffeomorphism of T^3 and modifies in a suitable way the first return map of a global cross section of the unstable foliation.

A diffeomorphism of T^2 is called *of Denjoy type* if it is semiconjugate to a minimal translation by a continuous map h such that $h^{-1}(x)$ is a single point for all but countably many x. Norton [15] and Norton-Sullivan [16] showed the non-existence of $\mathcal{C}^3$ diffeomorphisms of Denjoy type under certain conditions.

Proof of Theorem 3. — Let f be the above mentioned homeomorphism of T^2 with a minimal set $\mathfrak{M}$ homeomorphic to the Sierpiński T^2-set and such that the sets $f^n(U)$, $n\in\mathbb{Z}$, are mutually disjoint for any connected component U of $T^2\setminus\mathfrak{M}$.

Let $\{U_i\}_{i=1,2,\dots}$ denote the family of all the connected components of $T^2\setminus\mathfrak{M}$. Let us choose a properly embedded (i.e., such that the intersection $\ell\cap\partial U_1$ coincides with the pair of end points of ℓ) arc ℓ contained in $\bar{U}_1$. Since $\{\bar{U}_i\}_{i=1,2,\dots}$ is a null-sequence, $\operatorname{diam} f^n(\ell)$ converges to 0 as $n\to\pm\infty$. Let us define the equivalence relation $\sim$ by $z_1\sim z_2$

(where $z_1, z_2 \in T^2$) whenever either $z_1 = z_2$ or both, z_1 and z_2, are contained in $f^n(\ell)$ for some $n \in \mathbb{Z}$ (Figure 1). Let $\pi : T^2 \to T^2/\sim$ denote the quotient map. The family of the closed sets $\{f^n(\ell)\}$ and the points of $T^2 \setminus \bigcup_{n\in\mathbb{Z}} f^n(\ell)$ forms a so called *decomposition with respect to* $\{f^n(\ell)\}_{n\in\mathbb{Z}}$. This decomposition is shrinkable (in the sense of [4], see also [2] and [14]), and therefore $T^2/\sim$ is homeomorphic to T^2 by Theorem 6 in [4], p. 28. (Certainly, π itself is not a homeomorphism but just a *near homeomorphism*, i.e., it can be approximated by homeomorphisms in the sense described in [4].) Let us define a homeomorphism g of $T^2/\sim$ by $g(\pi(z)) = \pi f(z)$.

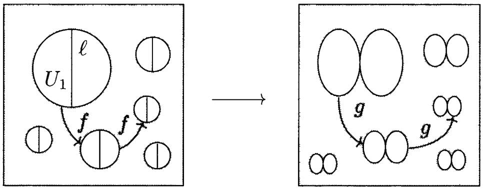

Figure 1. *Pinching process*

We will show that $\pi(\mathfrak{M})$ is a minimal set of g. Suppose that K is a non-empty closed g-invariant set contained in $\pi(\mathfrak{M})$. Then $\pi^{-1}(K) \cap \mathfrak{M}$ is closed, f-invariant and contained in $\mathfrak{M}$. Thus the set $\pi^{-1}(K) \cap \mathfrak{M}$ is either empty or coincides with $\mathfrak{M}$. In the first case, $\pi^{-1}(K)$ would be contained in $\bigcup_{i=1}^{\infty} U_i$. Since the sets $f^n(U_i)$ are mutually disjoint, the ω-limit set of a point of $\pi^{-1}(K)$ would be disjoint from $\bigcup_{i=1}^{\infty} U_i$, a contradiction. Thus $\pi^{-1}(K) \cap \mathfrak{M}$ coincides with $\mathfrak{M}$, and $\pi^{-1}(K)$ contains $\mathfrak{M}$. Therefore, $K = \pi(\pi^{-1}(K))$ coincides with $\pi(\mathfrak{M})$ and this implies that $\pi(\mathfrak{M})$ is a minimal set of g indeed.

Next, we show that $\pi(\mathfrak{M})$ is locally connected. Let p be a point of $\pi(\mathfrak{M})$ and U a neighbourhood of p in T^2. First, suppose that p does not belong to $\pi(\bigcup_{n\in\mathbb{Z}} f^n(\ell))$. Let q denote the unique point such that $\pi(q) = p$. Since $\mathfrak{M}$ is locally connected, there is a neighbourhood V of q in T^2 such that the intersection $V \cap \mathfrak{M}$ is connected and contained in $\pi^{-1}(U) \cap \mathfrak{M}$. By one of the properties of Sierpiński T^2-sets (see [18]), ∂V can be further assumed to be disjoint from $\bigcup_{n\in\mathbb{Z}} f^n(\ell)$. Then $\pi^{-1}\pi(V)$ is equal to V. In fact, if r is a point of $\pi^{-1}\pi(V)$, then $\pi(r)$ lies in $\pi(V)$, and there is a point z of V such that $\pi(r) = \pi(z)$. If $r = z$, then obviously r lies in V. On the other hand, if $r \neq z$, then there is $n \in \mathbb{Z}$ such that both, r and z, belong to $f^n(\ell)$. Since $f^n(\ell)$ is disjoint from ∂V, r belongs also to V. Thus the set $\pi^{-1}\pi(V)$ is contained in V and, consequently, $\pi^{-1}\pi(V) = V$. Therefore,

$\pi(V)$ is open in T^2. Moreover, $\pi(V) \cap \pi(\mathfrak{M})$ coincides with $\pi(V \cap \mathfrak{M})$ because of the following: If $z_1 \in V$ and $z_2 \in \mathfrak{M}$ are such that $\pi(z_1) = \pi(z_2)$ and $z_1 \neq z_2$, then there is $n \in \mathbb{Z}$ such that both, z_1 and z_2, belong to $f^n(\ell)$, and hence z_2 lies in V. This implies the required equality

$$\pi(V) \cap \pi(\mathfrak{M}) = \pi(V \cap \mathfrak{M}).$$

Thus $\pi(V) \cap \pi(\mathfrak{M})$ is a connected neighbourhood of p in $\pi(\mathfrak{M})$, which is contained in U. Next, consider the case when p is a point of $\pi(\bigcup_{n \in \mathbb{Z}} f^n(\ell))$. Let j denote the integer such that p is contained in $\pi(f^j(\ell))$, and q_1 and q_2 – the end points of $f^j(\ell)$. We can choose neighbourhoods V_i $(i = 1, 2)$ of q_i in T^2 such that $V_i \cap \mathfrak{M}$ is contained in $\pi^{-1}(U) \cap \mathfrak{M}$, $V_i \cap \mathfrak{M}$ is connected and

$$\partial V_i \cap \Big(\bigcup_{n \neq j} f^n(\ell) \Big) = \emptyset.$$

Let $W = V_1 \cup V_2$. Then $\pi(W) \cap \pi(\mathfrak{M})$ $\big(= \bigcup_{i=1}^2 \pi(V_i \cap \mathfrak{M}) \big)$ is a connected neighbourhood of p in $\pi(\mathfrak{M})$ contained in U by the same reason as above. This shows that the set $\pi(\mathfrak{M})$ is locally connected indeed. (The local connectedness of $\pi(\mathfrak{M})$ can also be shown by general arguments: Since π is the quotient map, it is continuous, and the continuous image of a locally connected compact connected set is locally connected (Theorem 5 in [12], p. 257). This argument was communicated to the authors by A. Koyama and T. Yagasaki.)

Finally, we shall show that our set $\pi(\mathfrak{M})$ has a locally separating point. Let z_1 and z_2 denote the end points of ℓ. For any $i = 1, 2$, there exists a neighbourhood V_i of z_i in T^2 such that $V_i \cap \mathfrak{M}$ is connected, $\partial V_i \cap (\bigcup_{n \neq 0} f^n(\ell)) = \emptyset$ and $V_1 \cap V_2 \cap \mathfrak{M} = \emptyset$. Therefore, $\pi(V_1 \cup V_2) \cap \pi(\mathfrak{M})$ is a connected neighbourhood of $p = \pi(\ell)$ in $\pi(\mathfrak{M})$ (by the same argument as that in the proof of local connectedness of $\pi(\mathfrak{M})$). Moreover, $\pi(V_1) \setminus \{p\}$ and $\pi(V_2) \setminus \{p\}$ are disjoint open subsets of $\pi(\mathfrak{M})$. Therefore, $\pi(\ell)$ separates $\mathfrak{M}$ locally. □

Remark 2. — A point which is not contained in $\pi(\bigcup_{n \in \mathbb{Z}} f^n(\ell))$ is not locally separating. Thus the minimal set $\pi(\mathfrak{M})$ is a locally connected continuum (i.e., a compact connected set) which is not homogeneous and admits a minimal homeomorphism. Another one-dimensional continuum which is not homogeneous and admits a minimal homeomorphism was introduced in Theorem 14.24 in [7]; that continuum is not locally connected (compare also [6]).

Remark 3. — In the proof of Theorem 3, we inserted just one properly embedded arc into the closure of a connected component of the minimal set. We can modify this construction easily by inserting a null-sequence of infinitely many pairwise disjoint properly embedded arcs in there.

Remark 4. — Aarts and Oversteegen [1] showed that the Sierpiński curve admits no minimal homeomorphism while Kato [9] proved that the Sierpiński curve admits no expansive homeomorphism (compare [1] again). Our article is in fact strongly stimulated by these papers.

2. Cut points of minimal sets.

In this section, we provide some general properties of minimal sets for homeomorphisms of arbitrary compact metric spaces. Although there exists a compact metric space which is not homogeneous but admits a minimal homeomorphism (see Remark 2 in §1), such minimal sets enjoy 'homogeneity' of certain kind.

Throughout the paper, the following simple observation will be used for several times.

LEMMA 1. — *Let $\mathfrak{M}$ be a connected minimal set of a homeomorphism f of a compact metric space. Then there is no non-empty compact proper subset K of $\mathfrak{M}$ such that $K, f(K), \dots, f^n(K)$ $(n \geq 0)$ are mutually disjoint and either $f^{n+1}(K)$ is contained in K or $f^{n+1}(K)$ contains K.*

Proof. — First, we consider the case when $f^{n+1}(K)$ is contained in K. If such a compact set K exists, then its ω-limit set

$$\omega(K) = \bigcap_{k\geq 0} \overline{\bigcup_{j\geq k} f^j(K)}$$

is compact, f-invariant and contains $\bigcap_{j\geq 0} f^{(n+1)j}(K)$ $(\neq \emptyset)$. Hence $\mathfrak{M}$ coincides with $\omega(K)$. On the other hand, $\omega(K)$ is contained in $K \cup f(K) \cup \dots \cup f^n(K)$, which is also contained in $\mathfrak{M}$. Thus we have

$$\mathfrak{M} = K \cup f(K) \dot{\cup} \dots \cup f^n(K).$$

However, this contradicts the assumption $K \neq \mathfrak{M}$ when $n = 0$ and that of connectedness of $\mathfrak{M}$ when $n > 0$.

One can complete the proof by replacing f with f^{-1} and $f^{n+1}(K)$ with K in the case when $f^{n+1}(K)$ contains K. □

LEMMA 2. — *Let $\mathfrak{M}$ be a connected minimal set of a homeomorphism f of a compact metric space. Then $\mathfrak{M}$ has no cut points.*

Proof. — Assume that $\mathfrak{M}$ has a cut point z. Certainly, each of the points $f^n(z)$, $n \in \mathbb{Z}$, cuts $\mathfrak{M}$ as well. By definition, $\mathfrak{M} \setminus \{z\}$ consists of two non-empty sets V_1 and V_2 such that both of them are open in $\mathfrak{M}$. Let K_i $(i = 1, 2)$ denote $V_i \cup \{z\}$. Then K_i's $(i = 1, 2)$ are closed in $\mathfrak{M}$ (just because V_i's are open). Furthermore, K_i's are connected. In fact, if one of them, say K_1, were not connected, then there would exist two disjoint closed subsets A_1 and A_2 of $\mathfrak{M}$ such that $K_1 = A_1 \cup A_2$ and $z \in A_1$. Then the sets $A_1 \cup K_2$ and A_2 would be disjoint and closed in $\mathfrak{M}$ contradicting the connectedness of $\mathfrak{M}$. Thus we have two continua K_1 and K_2 contained in $\mathfrak{M}$ and such that $K_1 \cap K_2$ consists of the single point z, $K_1 \cup K_2 = \mathfrak{M}$, and both, K_1 and K_2, contain more than one point.

We claim that either $f(K_1) \cap K_1 = \emptyset$ or $f(K_2) \cap K_2 = \emptyset$. Since $\mathfrak{M}$ contains at least three points, z is not fixed by f. Hence z does not belong to $f(K_1) \cap f(K_2)$ because $f(K_1) \cap f(K_2) = f(K_1 \cap K_2) = \{f(z)\} \neq \{z\}$. First, suppose that $z \notin f(K_2)$. Then $f(K_2) \cap K_1$ and $f(K_2) \cap K_2$ are disjoint closed sets. Since $f(K_2)$ is connected, $f(K_2)$ is contained in either $K_1 \setminus \{z\}$ or $K_2 \setminus \{z\}$. The second possibility is excluded by Lemma 1. Thus $f(K_2) \subset K_1 \setminus \{z\}$, and hence $f(K_2) \cap K_2 = \emptyset$. In the same way, $f(K_1) \cap K_1 = \emptyset$ if $f(K_1)$ does not contain z. Thus one can always find $i \in \{1, 2\}$ for which $f(K_i) \cap K_i = \emptyset$. In the following, we will assume, without loss of generality, that $f(K_1)$ does not intersect K_1.

Now, using the same argument as above inductively, we will show that all the sets $K_1, f(K_1), f^2(K_1), \ldots$ are mutually disjoint. Suppose that $K_1, f(K_1), \ldots, f^n(K_1)$ $(n \geq 1)$ are mutually disjoint but $f^{n+1}(K_1)$ intersects $f^m(K_1)$ for some m $(0 \leq m \leq n)$. Then $f^{n+1-m}(K_1)$ intersects K_1. Since $K_1, f(K_1), \cdots, f^n(K_1)$ are mutually disjoint, $f^{n+1}(K_1)$ intersects K_1 $(m = 0)$. If $f^{n+1}(K_1)$ does not contain the cut point z, then the connected set $f^{n+1}(K_1)$ is contained in K_1, what contradicts Lemma 1. Thus the point z is contained in $f^{n+1}(K_1)$. Since z is not a periodic point, z has to lie in $f^{n+1}(K_1 \setminus \{z\})$. Then $f^{n+1}(K_2)$ does not contain z. Thus the set $f^{n+1}(K_2)$ is contained either in $K_1 \setminus \{z\}$ or in $K_2 \setminus \{z\}$. In the second case, $f^{n+1}(K_1)$ contains K_1, what contradicts Lemma 1. Thus $f^{n+1}(K_2)$ has to be contained in $K_1 \setminus \{z\}$. In particular, $f^{n+1}(K_2 \setminus \{z\})$ is contained in $K_1 \setminus \{z\}$. Since $f(K_1)$ is disjoint from K_1, we obtain that $f^{n+1}(K_2 \setminus \{z\}) \cap f(K_1) = \emptyset$. This implies that $f^n(K_2 \setminus \{z\}) \cap K_1 = \emptyset$, and hence $f^n(K_1)$ contains K_1, what contradicts the assumption. Thus all the sets $K_1, f(K_1), f^2(K_1), \ldots$ are mutually disjoint indeed. (Let us remark that all the family $\{f^k(K_1)\}_{k\in\mathbb{Z}}$ consists of mutually disjoint sets just because $f^{k_1}(K_1) \cap f^{k_2}(K_1) = f^{k_1}(K_1 \cap f^{k_2-k_1}(K_1)) = \emptyset$ if $k_2 > k_1$.)

Since $f^k(K_1 \setminus \{z\})$ $(k \in \mathbb{Z})$ are non-empty open sets, the complement of the union $\bigcup_{k\in\mathbb{Z}} f^k(K_1 \setminus \{z\})$ is f-invariant, closed and different from $\mathfrak{M}$, and hence it has to be empty. In other words, the family $\mathcal{A} = \{f^k(K_1 \setminus \{z\})\}_{k\in\mathbb{Z}}$ covers $\mathfrak{M}$. Since $\mathfrak{M}$ is compact, some finite subfamily of $\mathcal{A}$ covers $\mathfrak{M}$. Since, as was observed before, the sets $f^k(K_1 \setminus \{z\})$, $k \in \mathbb{Z}$ are mutually disjoint, this contradicts the assumption that $\mathfrak{M}$ is connected. □

Next we provide some properties of locally connected continua without cut points. These will be used frequently in the proofs of Theorems 1 and 2 in §3 and §4.

LEMMA 3. — *Let $\mathfrak{M}$ be a connected and locally connected compact metric space without cut points. If there exist a compact subset K of $\mathfrak{M}$ and an arc ℓ_1 in $\mathfrak{M}$ such that K contains at least two points and $K \cap \ell_1$ consists of a single point z, then there is an arc ℓ_2 in $\mathfrak{M}$ such that one of its end points is z, the other end point w of ℓ_2 is contained in $K \setminus \{z\}$, and $\ell_2 \setminus \{z, w\}$ is disjoint from K (Figure 2).*

Hereafter, by an arc we mean an injective continuous image of a closed interval.

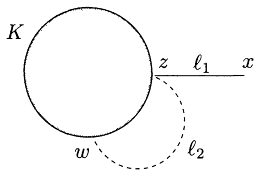

Figure 2.

Proof. — Let x denote the end point of ℓ_1 different from z. Denote by V_1 the path-connected component of $\mathfrak{M} \setminus \{z\}$ containing x. Let $V_2 = \mathfrak{M} \setminus (V_1 \cup z)$. Then $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = \mathfrak{M} \setminus \{z\}$, by definition. By Mazurkiewicz-Moore-Menger Theorem (see [12], p. 254), any complete locally connected metric space is locally arcwise connected, therefore our set $\mathfrak{M}$ is locally arcwise connected (and arcwise connected too). Thus V_1 and V_2 are open in $\mathfrak{M} \setminus \{z\}$. Since $\mathfrak{M}$ is assumed to have no cut points, the set V_2 has to be empty. Let us choose an arbitrary point p of K, $p \neq z$. Certainly, p belongs to V_1. Thus we can find an arc ℓ_3 joining points p and x

and contained in $\mathfrak{M} \setminus \{z\}$. Denote by β the connected component of $\ell_1 \setminus \ell_3$ containing z. The end point q of β, $q \neq z$, is connected with a point of K by a subarc ℓ_4 of ℓ_3 which intersects K only at its end points. Let ℓ_2 denote the union of β and ℓ_4. The arc ℓ_2 satisfies the conditions of Lemma 3. □

COROLLARY 2. — *If $\mathfrak{M}$ is a connected and locally connected compact metric space which is not a single point and has no cut points, then $\mathfrak{M}$ contains a simple closed curve.*

Proof. — Let x and y be two distinct points of $\mathfrak{M}$, ℓ an arc joining x and y in $\mathfrak{M}$ and z a point of ℓ different from x and y. Denote by ℓ_1 the subarc of ℓ between z and x, and by K the subarc of ℓ between z and y. There exists an arc ℓ_2 in $\mathfrak{M}$ satisfying the conditions of Lemma 3. The union $\ell \cup \ell_2$ contains a simple closed curve (which is obviously contained in $\mathfrak{M}$). □

3. Complements of minimal sets.

In order to consider locally connected minimal sets, it is important to examine topological properties of their complements. In this section, we will prove some facts (Lemmas 4 and 5) concerning simple closed curves in the boundary of such complements. These facts will be used to prove Theorem 1 in the final part of this section.

LEMMA 4. — *Let f be a homeomorphism of a closed orientable surface Σ with a connected and locally connected minimal set $\mathfrak{M}$. Let U be a connected component of $\Sigma \setminus \mathfrak{M}$. If there exists a simple closed curve C contained in the frontier ∂U of U and satisfying the following conditions (Figure 3) :*

1) *$\Sigma \setminus C$ consists of two disjoint connected open sets V_1 and V_2,*

2) *V_1 contains U, and*

3) *V_2 is disjoint from $\mathfrak{M}$,*

then $\mathfrak{M}$ coincides with C. (In particular, $U = V_1$.)

Proof. — Assume that $\mathfrak{M}$ does not coincide with C. Let p be a point of $\mathfrak{M} \cap V_1$. Since $\mathfrak{M}$ is connected, there exists an arc γ_1 of $\mathfrak{M}$ joining p and a point q of C such that $\gamma_1 \setminus \{q\}$ is disjoint from C. Applying Lemma 3 to $\overline{V}_2$ and γ_1 as K and ℓ_1 respectively, we can find a properly embedded arc ℓ_1 of $\overline{V}_1$ contained in $\mathfrak{M}$.

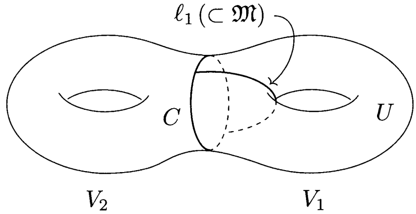

Figure 3.

We claim that $V_1 \setminus \ell_1$ is connected. In fact, if $V_1 \setminus \ell_1$ consists of two disjoint open sets W_1 and W_2, then both $C \cap \overline{W}_1$ and $C \cap \overline{W}_2$ are non-empty arcs with common end points (just because the end points of ℓ_1 cut C into two arcs and V_1 is connected). On the other hand, either W_1 or W_2, say W_1 contains U. Then C is contained in $\overline{W}_1$ because $C \subset \partial U$. However this contradicts the condition $C \cap \overline{W}_2 \neq \emptyset$. Thus $V_1 \setminus \ell_1$ is connected indeed.

Let q and q' denote the end points of ℓ_1 and r be a point of C different from q and q'. Since $\mathfrak{M}$ is a minimal set, the orbit starting from q accumulates at r. Furthermore, q is a branch point of $C \cup \ell_1$. Thus the image of a neighbourhood of q in $\mathfrak{M}$ by this orbit cannot be contained in C. Therefore, condition 3) of our lemma implies that, arbitrarily close to r, there exists a point of $\mathfrak{M}$, which lies in V_1 but not in C. Since $\mathfrak{M}$ is locally connected, there exists a small arc close to r and contained in $\mathfrak{M}$ which intersects C only at one of its end points. Applying Lemma 3 we get an arc ℓ_2 contained in $\mathfrak{M}$ and such that ℓ_2 intersects $C \cup \ell_1$ only at its end points and one of the end points is contained in C. Then $V_1 \setminus (\ell_1 \cup \ell_2)$ is connected because, if not, two sides of ℓ_2 would be contained in distinct connected components of $V_1 \setminus (\ell_1 \cup \ell_2)$ one of which contains U, therefore ∂U could not contain C as above.

Proceeding inductively, we obtain infinitely many arcs $\ell_1, \ell_2, \ldots$ in $\mathfrak{M}$ such that ℓ_i intersects $C \cup \ell_1 \cup \ell_2 \cup \ldots \cup \ell_{i-1}$ only at its end points and $V_1 \setminus (\ell_1 \cup \ell_2 \cup \ldots \cup \ell_i)$ is connected for all $i = 1, 2, \ldots$.

Finally, choose a regular neighbourhood R of $C \cup (\ell_1 \cup \ell_2 \cup \ldots \cup \ell_i)$. Let

$$\Sigma_1 = \overline{V_2 \cup R} \quad \text{and} \quad \Sigma_2 = \overline{\Sigma \setminus \Sigma_1}.$$

Then the Euler characteristic $\chi(\Sigma_2)$ of Σ_2 is smaller than or equal to 2 (just because Σ_2 is connected). The Mayer-Vietoris sequence yields

$$\chi(\Sigma_1) = \chi(\Sigma_1 \cup \Sigma_2) - \chi(\Sigma_2) \geq (2 - 2g) - 2 \geq -2g,$$

where g is the genus of Σ. On the other hand,

$$\chi(\Sigma_1) \leq 1 - i.$$

Since this is impossible for a sufficiently large i, $\mathfrak{M}$ coincides with C. □

In the case when C does not separate Σ, we need some additional consideration because U may accumulate at C from both sides.

LEMMA 5. — *Let again f be a homeomorphism of a closed orientable surface Σ with a connected and locally connected minimal set $\mathfrak{M}$. Let U be a connected component of $\Sigma \setminus \mathfrak{M}$. If ∂U coincides with $\mathfrak{M}$ and there exists a simple closed curve C contained in ∂U such that $\Sigma \setminus C$ is connected (Figure 4), then $\mathfrak{M}$ coincides with C.*

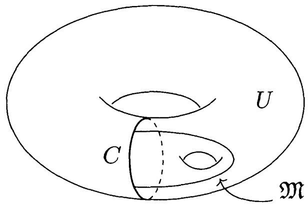

Figure 4.

Proof. — Assume that $\mathfrak{M}$ does not coincide with C. As before, we will construct by induction infinitely many arcs $\ell_1, \ell_2, \ldots$ in $\mathfrak{M}$ such that ℓ_i intersects $C \cup \ell_1 \cup \ldots \cup \ell_{i-1}$ at its end points and $\Sigma \setminus (C \cup \ell_1 \cup \ldots \cup \ell_i)$ is connected for $i = 1, 2, \ldots$.

The first step of induction (existence of ℓ_1) follows from Lemma 3 as in the proof of Lemma 4.

Suppose that $\ell_1, \ell_2, \ldots, \ell_n$ $(n \geq 1)$ satisfy the above conditions. Let S_n denote $C \cup \ell_1 \cup \ldots \cup \ell_n$. Since S_n has finitely many branch points, there is an arc $\ell \subset \mathfrak{M}$ such that ℓ intersects S_n only at its end points and one of the end points of ℓ is contained in $C \setminus (\ell_1 \cup \ldots \cup \ell_n)$. Suppose that $\Sigma \setminus (S_n \cup \ell)$ is not connected. Let V_1 and V_2 denote the connected components of $\Sigma \setminus (S_n \cup \ell)$ such that U is contained in V_1. Then ∂U $(= \mathfrak{M})$ is contained in $\overline{V}_1$, and hence V_2 is disjoint from $\mathfrak{M}$. Since ℓ intersects C at $C \setminus (\ell_1 \cup \ldots \cup \ell_n)$, the set $(C \cap \overline{V}_2) \setminus (\ell_1 \cup \ell_2 \cup \ldots \cup \ell_n)$ contains a non-empty open arc α. Then a one-sided neighbourhood of α in V_2 is disjoint from U. Next, let us choose as above another arc ℓ' of $\mathfrak{M}$ such that ℓ' intersects S_n only at its end points, one of them being contained in α. Since V_2 is disjoint from $\mathfrak{M}$, ℓ' is disjoint from V_2. Suppose also that $\Sigma \setminus (S_n \cup \ell')$ is not connected. Let V_1' and V_2' denote the connected components of $\Sigma \setminus (S_n \cup \ell')$ such that U is contained in V_1'. Then V_2' is disjoint from U as above.

By the same argument as for ℓ, $C \cap \overline{V_2'}$ contains a non-empty open arc α' contained in α such that one of the end points of α' is an end point of ℓ'. Since ℓ' is disjoint from V_2, both sides of α' are disjoint from U. Since this contradicts the assumption that C is contained in ∂U, we obtain that either $\Sigma \setminus (S_n \cup \ell)$ or $\Sigma \setminus (S_n \cup \ell')$ is connected so we can put either $\ell_{n+1} = \ell$ (in the first case) or $\ell_{n+1} = \ell'$ (in the second one). As promised, induction provides infinitely many arcs $\ell_1, \ell_2, \ldots$ such that each ℓ_i intersects $C \cup \ell_1 \cup \ldots \cup \ell_{i-1}$ at end points and all the sets $\Sigma \setminus (C \cup \ell_1 \cup \ldots \cup \ell_i)$ are connected.

One can complete the proof by the same arguments as those in the final step of the proof of Lemma 4. □

Proof of Theorem 1. — Let $\mathfrak{N}$ be a connected component of the minimal set $\mathfrak{M}$. Certainly, any connected component of a locally connected space is open. Hence the complement of $\bigcup_{n \in \mathbb{Z}} f^n(\mathfrak{N})$ is an f-invariant closed set, and has to be empty. In other words, $\{f^n(\mathfrak{N})\}_{n \in \mathbb{Z}}$ is an open covering of $\mathfrak{M}$. By the compactness of $\mathfrak{M}$, $\mathfrak{M}$ coincides with the union of a finite subfamily of $\{f^n(\mathfrak{N})\}_{n \in \mathbb{Z}}$. Consequently, there exists $j \in \mathbb{Z}$ such that the equality $f^j(\mathfrak{N}) = \mathfrak{N}$ holds. We assume that j_0 is the least positive integer j satisfying this equality. By the minimality of $\mathfrak{M}$, $\mathfrak{M}$ coincides with $\bigcup_{n=0}^{j_0-1} f^n(\mathfrak{N})$, and $\mathfrak{N}$ is a minimal set of f^{j_0}.

By assumption, the Euler characteristic of Σ is different from zero. Hence, f has a periodic point (see, for example, [10], p. 330, Exercise 8.6.2), denoted here by p. Let m denote its period. Then p is a fixed point of f^{mj_0}. Since $\mathfrak{N}$ is connected and minimal for f^{j_0}, $\mathfrak{N}$ is – by Theorem 2.28 in [7] – minimal also for f^{mj_0}. Thus we have only to show that any connected and locally connected minimal set $\mathfrak{M}$ of a homeomorphism f of a closed orientable surface Σ with a fixed point p coincides with either a single point or a simple closed curve.

Suppose that $\mathfrak{M}$ is not a single point. Then p cannot belong to $\mathfrak{M}$. Let U be a connected component of $\Sigma \setminus \mathfrak{M}$ containing p. Then $f(U)$ coincides with U, and hence $f(\partial U)$ coincides with ∂U. Since ∂U is a closed invariant set, our minimal set $\mathfrak{M}$ coincides with ∂U too. By Corollary 2, $\mathfrak{M}$ contains a simple closed curve C. If C does not separate Σ, then $\mathfrak{M}$ coincides with C by Lemma 5. If C separates Σ, i.e. $\Sigma \setminus C$ is the union of two disjoint open sets V_1 and V_2 (without loss of generality, we may suppose that U is contained in V_1), then V_2 is disjoint from $\mathfrak{M}$ (just because $\mathfrak{M}$ coincides with ∂U $(\subset \overline{V_1})$), and again $\mathfrak{M}$ coincides with C, this time by Lemma 4. □

4. Homeomorphisms of the torus.

Let f be a homeomorphism of T^2 and U a connected component of the complement of its minimal set $\mathfrak{M}$. If $f(U)$ coincides with U, then ∂U is closed and f-invariant, therefore ∂U coincides with $\mathfrak{M}$. Corollary 2 obtained in the course of proof of Theorem 1 yields the existence of a simple closed curve contained in ∂U in this case. In the case when the sets $f^n(U)$, $n \in \mathbb{Z}$, are mutually disjoint, this argument does not work. In order to find a simple closed curve in ∂U, we need several preparatory facts (Lemmas 6, 7 and 8).

LEMMA 6. — *Let f be a homeomorphism of T^2 and let $\mathfrak{M}$ be a connected and locally connected minimal set which is neither a single point nor a simple closed curve. Then, for any connected component U of $T^2 \setminus \mathfrak{M}$, its saturation $\{f^n(U)\}_{n\in\mathbb{Z}}$ consists of mutually disjoint sets.*

Proof. — Let us assume on the contrary that the sets $\{f^n(U)\}_{n\in\mathbb{Z}}$ are not mutually distinct. Then there is $k \neq 0$ such that $f^k(U) = U$. Since $\mathfrak{M}$ is connected, $\mathfrak{M}$ is also a minimal set of f^k. Now ∂U is a closed set invariant under f^k. Thus $\partial U = \mathfrak{M}$. By Corollary 2, $\mathfrak{M}$ contains a simple closed curve C, and furthermore $\mathfrak{M} = C$ by Lemmas 4 and 5 as in the proof of Theorem 1. This contradicts the assumption. □

LEMMA 7. — *If $\mathfrak{M}$ is a compact connected subset of the torus which has infinitely many complementary connected components U_i, and if $\mathfrak{M}$ is locally connected, then $\{\overline{U}_i\}_{i=1,2,\dots}$ is a null-sequence.*

Proof. — Assume that $\{\overline{U}_j\}_{j=1,2,\dots}$ is not a null sequence. Then there exists $\varepsilon > 0$ such that infinitely many of $\overline{U}_j$'s have diameter greater than ε. Denote by $\{V_i\}_{i=1,2,\dots}$ the collection of all such U_j's. Denote by d the standard metric on T^2 and, for each $i \in \mathbb{N}$, choose points x_i and y_i of V_i such that $d(x_i, y_i) > \varepsilon$, and an arc γ_i in V_i joining x_i and y_i. Let z_i be a point of γ_i such that $d(x_i, z_i) > \frac{1}{3}\varepsilon$ and $d(y_i, z_i) > \frac{1}{3}\varepsilon$. Passing several times to a subsequence (if necessary), we may assume that $\{x_i\}$, $\{y_i\}$ and $\{z_i\}$ converge (as $i \to \infty$) to points x, y and z, respectively (Figure 5).

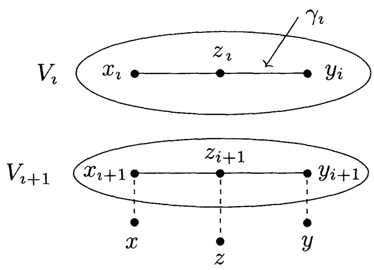

Figure 5.

Then we have $d(x,z) \geq \frac{1}{3}\varepsilon$ and $d(y,z) \geq \frac{1}{3}\varepsilon$. Furthermore, z is contained in $\mathfrak{M}$ because, if not, z would belong to some connected component U_k of $T^2 \setminus \mathfrak{M}$, and – since U_k contains at most one point of the sequence $\{z_i\}$ – this sequence would not be able to converge to z. Since $\{x_i\}$ and $\{y_i\}$ converge to x and y respectively, there is $N > 0$ such that $d(x_i, x) < \frac{1}{6}\varepsilon$ and $d(y_i, y) < \frac{1}{6}\varepsilon$ if only $i \geq N$. Let

$$D = \left\{w;\ d(w,z) \leq \frac{1}{6}\varepsilon\right\}.$$

The points x_i and y_i do not lie in D when $i \geq N$. By the local connectivity of $\mathfrak{M}$, there exists a neighbourhood W of z in T^2 such that W is contained in D and $W \cap \mathfrak{M}$ is path-connected. Replacing eventually W by its connected component containing z we may assume that W itself is connected as well. Let L be an integer greater than N and such that z_i lies in W whenever $i \geq L$. For $j = L, L+1, L+2$, the points x_j and y_j are not contained in D, and hence we can choose properly embedded arcs β_j contained in $D \cap \gamma_j$ and passing through z_j. The arcs $\beta_L, \beta_{L+1}, \beta_{L+2}$ split D into four closed discs. Among these discs, there are two whose boundaries consist of two arcs chosen from $\{\beta_L, \beta_{L+1}, \beta_{L+2}\}$ and two others contained in ∂D. Since z does not belong to the boundaries of these two discs, one of them (denoted by Δ from now) does not contain z. Without loss of generality, we can assume that $\partial\Delta$ consists of β_L, β_{L+1} and two arcs contained in ∂D, and furthermore that β_{L+1} is closer to z than β_L (i.e., z and β_L are contained in different components of $D \setminus \beta_{L+1}$).

We claim that $W \cap \mathfrak{M} \cap \operatorname{int}\Delta \neq \emptyset$. Indeed, since W is connected, there exists an arc α_1 in W joining z and z_L. Then α_1 intersects β_{L+1}. Thus there exists a subarc α_2 of α_1 properly embedded in Δ and such that one of the end points is contained in β_L while the other one in β_{L+1}. Since $\beta_L \subset V_L$ and $\beta_{L+1} \cap V_L = \emptyset$, one can find a point q of α_2 contained in the intersection of ∂V_L and $\operatorname{int}\Delta$.

Since $q \in \mathfrak{M} \cap W$, q can be connected to z by a path ℓ in $\mathfrak{M} \cap W$. However, this is impossible since such an ℓ would intersect β_{L+1} which is disjoint from $\mathfrak{M} \cap W$. Therefore, $\{\overline{U}_i\}_{i=1,2,\dots}$ is a null sequence indeed. □

Any disc is contained in a Janiszewski space without a cut point, which is homeomorphic to S^2 (see [12]). By Theorem 4 in [12], §61, II, we have the following.

LEMMA 8. — *Let X be a free of cut points and locally connected continuum contained in a disc D and such that ∂D is contained in X. Let U be a connected component of $D \setminus X$. Then U is the interior of a disc.*

LEMMA 9. — *Let f be a homeomorphism of T^2 and U a connected component of the complement of a connected and locally connected minimal set $\mathfrak{M}$ of f. If $\mathfrak{M}$ is neither a single point nor a simple closed curve, then U is the interior of a disc.*

Proof. — By Corollary 2, $\mathfrak{M}$ contains a simple closed curve C. By Lemma 3, there is an arc ℓ of T^2 which is contained in $\mathfrak{M}$ and intersects C only at its end points. Then the θ-curve (compare [12], p. 328) $C \cup \ell$ of T^2 belongs to one of the three types according to the number of the connected components of its complement (which is always smaller than or equal to three).

First, we consider the case when $T^2 \setminus (C \cup \ell)$ is connected. Then the manifold obtained by cutting T^2 along $C \cup \ell$ becomes a closed disc D_1 after pasting a circle to $T^2 \setminus (C \cup \ell)$ along the boundary. Since $C \cup \ell$ is not the whole $\mathfrak{M}$, $\operatorname{int} D_1$ contains a point p of $\mathfrak{M}$. Let d_1 denote the distance between ∂D_1 and p. By the minimality of $\mathfrak{M}$, for any point q of ∂U, there is $n \in \mathbb{Z}$ such that the distance between $f^n(q)$ and p is smaller than $\frac{1}{3}d_1$. Furthermore, by Lemmas 6 and 7, we may assume that $\operatorname{diam} f^n(U)$ is also smaller than $\frac{1}{3}d_1$. Hence $f^n(U)$ is contained in the interior of D_1. Thus $f^n(U)$ is a connected component of $D_1 \setminus \mathfrak{M}$ such that $\overline{f^n(U)}$ is disjoint from ∂D_1. By Lemma 8, there is a disc D_2 in $\operatorname{int} D_1$ such that $\operatorname{int} D_2 = f^n(U)$. The disc D_2 remains embedded in T^2 after pasting the boundary of D_1 along $C \cup \ell$. Thus U is the interior of the disc $f^{-n}(D_2)$ contained in T^2.

Next, we assume that $T^2 \setminus (C \cup \ell)$ has three connected components. Then the manifold obtained by cutting T^2 along $C \cup \ell$ consists of two discs Σ_1 and Σ_2 and a one-punctured torus Σ_3. The discs Σ_1 and Σ_2 are adjacent by an arc. If the interiors $\operatorname{int} \Sigma_1$ and $\operatorname{int} \Sigma_2$ are disjoint from $\mathfrak{M}$, then the intersection $\Sigma_1 \cap \Sigma_2$ contains a closed arc α "isolated" in $\mathfrak{M}$ (in the sense, that $\mathfrak{M}$ is locally homeomorphic to an arc in a neighbourhood of any point of α different from end points). By the minimality of $\mathfrak{M}$, $\{f^n(\alpha)\}_{n\in\mathbb{Z}}$ covers $\mathfrak{M}$. Furthermore, by the compactness of $\mathfrak{M}$, finitely many of the sets $f^n(\alpha)$, $n \in \mathbb{Z}$ cover $\mathfrak{M}$. Therefore, $\mathfrak{M}$ is a simple closed curve, which contradicts the assumption. Thus either $\operatorname{int}\Sigma_1$ or $\operatorname{int}\Sigma_2$ contains a point p of $\mathfrak{M}$. By the same argument as in the case when $T^2 \setminus (C \cup \ell)$ is connected, we can show that there exists $n \in \mathbb{Z}$ such that $f^n(U)$ is contained in either Σ_1 or Σ_2, so that $\overline{f^n(U)} \cap (C \cup \ell) = \emptyset$, and hence U is the interior of a disc as above.

Finally, we consider the case when $T^2 \setminus (C \cup \ell)$ has two connected components. Then the manifold obtained by cutting T^2 along $C \cup \ell$ consists of a disc Σ_1 and an annulus Σ_2. Replacing C by $\partial\Sigma_1$ if necessary, we may assume that C itself bounds Σ_1. If $\operatorname{int}\Sigma_1$ intersects $\mathfrak{M}$, then U is the interior of a disc by the same argument as in the case when $T^2 \setminus (C \cup \ell)$ is connected. Thus we may assume that $\operatorname{int}\Sigma_1$ is disjoint from $\mathfrak{M}$. The orbit starting from a branch point of $C \cup \ell$ accumulates at a point of $C \setminus \ell$, and there is a small arc in $\Sigma_2 \cap \mathfrak{M}$ such that one of the end points is contained in $C \setminus \ell$. By Lemma 3, there is an arc ℓ' of $\mathfrak{M}$ such that ℓ' intersects $C \cup \ell$ only at its end points, one of them being contained in $C \setminus \ell$. If $\Sigma_2 \setminus (\ell \cup \ell')$ is connected, then the manifold obtained by cutting Σ_2 along $\ell \cup \ell'$ is a disc adjacent to Σ_1 (Figure 6 (a)).

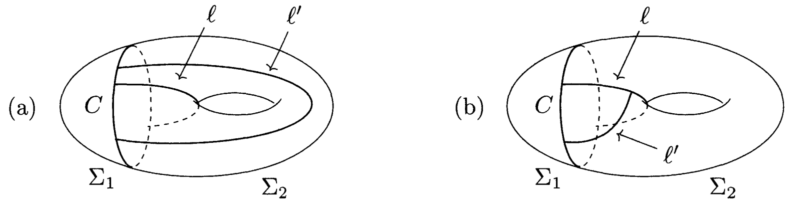

Figure 6.

On the other hand, if $\Sigma_2 \setminus (\ell \cup \ell')$ is not connected, then the manifold obtained by cutting Σ_2 along $\ell \cup \ell'$ is the union of a disc and an annulus (Figure 6 (b)). Here this disc is also adjacent to the disc Σ_1. In both cases, U is the interior of a disc according to the same argument as in the case when $T^2 \setminus (C \cup \ell)$ has three connected components. □

LEMMA 10. — *Let f be a homeomorphism of T^2 and $\mathfrak{M}$ be a connected and locally connected minimal set of f. Let $\{U_i\}_{i=1,2,\ldots}$ denote the family of the connected components for the complement of $\mathfrak{M}$. If $\mathfrak{M}$ is neither a single point nor a simple closed curve nor the whole T^2, then*

1) *$\overline{U}_i$ intersects $\overline{U}_j$ at most at one point when $i \neq j$, and the intersection $\overline{U}_i \cap \overline{U}_j$ consists of a locally separating point of $\mathfrak{M}$ (if non-empty)*;

2) *there is no finite chain $U_{i_1}, U_{i_2}, \dots, U_{i_n}$ $(n > 1)$ such that $\overline{U_{i_j}} \cap \overline{U_{i_{j+1}}} \neq \emptyset$ $(j = 1, 2, \dots, n-1)$ and $\overline{U_{i_1}} \cap \overline{U_{i_n}} \neq \emptyset$.*

Proof. — First, we will show that $\overline{U}_i$ intersects $\overline{U}_j$ at most at one point when $i \neq j$. Assume that $\overline{U}_i \cap \overline{U}_j$ contains two points p_1 and p_2. Since $\overline{U}_i$ and $\overline{U}_j$ are discs, there is an arc γ_1 (resp. γ_2) contained in $\overline{U}_i$ (resp. $\overline{U}_j$) such that both γ_1 and γ_2 join p_1 and p_2 and, furthermore, γ_1 (resp. γ_2) intersects ∂U_i (resp. ∂U_j) only at its end points. Since $U_i \cap U_j = \emptyset$, the union $\gamma_1 \cup \gamma_2$ is a simple closed curve, denoted hereafter by C. By Lemmas 6 and 7, $\operatorname{diam} f^n(\overline{U}_i)$ and $\operatorname{diam} f^n(\overline{U}_j)$ converge to 0 as $n \to \infty$. Thus there is $N > 0$ such that $f^N(C)$ bounds a disc D_1. Let z_ℓ $(\ell = 1, 2)$ be a point of γ_ℓ different from p_1 and p_2. Since $f^N(z_1)$ (resp. $f^N(z_2)$) is a point of $f^N(U_i)$ (resp. $f^N(U_j)$), the interior $\operatorname{int} D_1$ intersects $f^N(U_i)$ and $f^N(U_j)$. Therefore, there is a point q of $\operatorname{int} D_1$ which belongs also to $\mathfrak{M}$. Let d_1 denote the distance between ∂D_1 and q. Since $\{\overline{U}_k\}_{k=1,2,\dots}$ is a null sequence and $\mathfrak{M}$ is a minimal set, there is an integer L such that

$$\max\bigl\{\operatorname{diam} f^L(\overline{U}_i), \operatorname{diam} f^L(\overline{U}_j), d(f^L(p_1), q)\bigr\} \leq \frac{d_1}{4}\cdot$$

Then, $f^L(\overline{U}_i \cup \overline{U}_j)$ is contained in $\operatorname{int} D_1$ just because $f^L(p_1)$ belongs to $f^L(\overline{U}_i) \cap f^L(\overline{U}_j)$. In particular, $f^L(C)$ is contained in $\operatorname{int} D_1$. The simple closed curve $f^L(C)$ bounds a disc D_2 contained in $\operatorname{int} D_1$. Now the boundary of $f^{L-N}(D_1)$ coincides with ∂D_2. If $f^{L-N}(D_1)$ were different from D_2, then our surface would be homeomorphic to the sphere, but this is not the case. Therefore, $f^{L-N}(D_1) = D_2$ and, in particular, $f^{L-N}(D_1) \subset \operatorname{int} D_1$. Thus $f^{L-N}(D_1 \cap \mathfrak{M})$ is contained in $D_1 \cap \mathfrak{M}$. By Lemma 1, $\mathfrak{M}$ is not a minimal set of f^{L-N}. In particular, $\mathfrak{M}$ is not a minimal set of f, what contradicts our assumption and shows that $\overline{U}_i$ intersects $\overline{U}_j$ at most at one point (when $i \neq j$).

Next, we assume that there exists a finite chain $U_{i_1}, U_{i_2}, \dots, U_{i_n}$ $(n > 1)$ such that $\overline{U_{i_j}} \cap \overline{U_{i_{j+1}}} \neq \emptyset$ $(j = 1, 2, \dots, n-1)$ and $\overline{U_{i_1}} \cap \overline{U_{i_n}} \neq \emptyset$. Then, joining suitable arcs properly embedded in $\overline{U_{i_j}}$, we obtain a simple closed curve C contained in $\bigcup_{j=1,2,\dots,n} \overline{U_{i_j}}$ (Figure 7). By the same argument as above, this is impossible. This yields our condition 2).

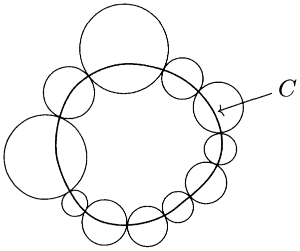

Figure 7.

Finally we will show that the intersection of $\overline{U}_i$ and $\overline{U}_j$ consists of a locally separating point (if non-empty). Let z denote the unique point of $\overline{U}_i \cap \overline{U}_j$. Choose an arc α contained in $\overline{U}_i \cup \overline{U}_j$ and such that $\alpha \cap (\partial U_i \cup \partial U_j) = \{z\}$, $\alpha \cap U_i \neq \emptyset$ and $\alpha \cap U_j \neq \emptyset$. Let V be a neighbourhood of z in T^2 which is cut by α into two pieces. Then the pathwise connected component W of $\mathfrak{M} \cap V$ containing z is a neighbourhood of z such that $W \setminus \{z\}$ is not connected. Thus, z is locally separating. □

Proof of Theorem 2. — First we will show that $\mathfrak{M}$ is connected. Assume that this is not the case. Let $\mathfrak{N}$ be a connected component of $\mathfrak{M}$. By arguments of the proof of Theorem 1, there is $N > 1$ such that $f^N(\mathfrak{N}) = \mathfrak{N}$ and $f^i(\mathfrak{N}) \neq \mathfrak{N}$ $(i = 1, 2, \ldots, N-1)$, and furthermore, $\mathfrak{N}$ is a minimal set of f^N. Let $g = f^N$. Denote by V the connected component of $T^2 \setminus \mathfrak{N}$ containing $f(\mathfrak{N})$. Since $f^{N+1}(\mathfrak{N}) = f(\mathfrak{N})$, both, $g(V)$ and V, contain $f(\mathfrak{N})$. But, $g(V)$ is also a connected component of $T^2 \setminus \mathfrak{N}$. Thus $g(V) = V$ and hence $g(\partial V) = \partial V$. Therefore, ∂V is a g-invariant closed set contained in $\mathfrak{N}$. Since $\mathfrak{N}$ is a minimal set of g, ∂V coincides with $\mathfrak{N}$. By Lemma 9, ∂V is a simple closed curve. Since (as in proof of Theorem 1) $\mathfrak{M}$ has finite number of connected components, this contradicts our assumptions on the structure of our minimal set $\mathfrak{M}$. Therefore, $\mathfrak{M}$ is connected indeed.

By Lemmas 7, 9 and 10, the conditions 1), 2), 3) and 4) of Theorem 2 are satisfied. The remaining problem is the uniqueness of minimal sets. Suppose that there exists another minimal set $\mathfrak{M}'$. Then one of its connected components has to be contained in some connected component U of the complement of $\mathfrak{M}$. By Lemma 6, all the sets of $\{f^n(U)\}_{n\in\mathbb{Z}}$ are mutually disjoint. Then the orbit starting from a point of the intersection $\mathfrak{M}' \cap U$ never approaches to this point again, a contradiction. □

Remark 5. — A connected component of $\bigcup_{i=1}^{\infty} \overline{U}_i$ can be invariant under f. Moreover, it is possible that the union $\bigcup_{i=1}^{\infty} \overline{U}_i$ is just connected.

Such an example was communicated to the authors by Takashi Tsuboi: Let g be a minimal translation of T^2. We choose a point x and a (straight) segment ℓ joining x and $g(x)$. Inserting mutually disjoint discs along the orbit $\{g^n(x)\}_{n\in\mathbb{Z}}$ as in §1, we obtain a homeomorphism h of T^2 whose minimal set $\mathfrak{M}$ is homeomorphic to the Sierpiński T^2-set. Then the arcs corresponding to $\{g^n(\ell)\}_{n\in\mathbb{Z}}$ are contained in $\mathfrak{M}$ and are mutually disjoint. Thus the decomposition with respect to these arcs is shrinkable. By the same argument as in the proof of Theorem 3, we can collapse these arcs and obtain a homeomorphism f such that $\bigcup_{i=1}^{\infty} \bar{U}_i$ is connected and invariant (where, as before, U_i's are connected components of the complement of the minimal set).

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