ANNALES

DE

L'INSTITUT FOURIER

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Tome 54, nº 4 (2004), p. 875-885.

<http://aif.cedram.org/item?id=AIF_2004__54_4_875_0>

Ann. Inst. Fourier, Grenoble
54, 4 (2004), 875–885

PARTIALLY DEFINED COCYCLES AND THE MASLOV INDEX FOR A LOCAL RING

by **Amedeo MAZZOLENI**

1. Cocycles in general position.

DEFINITION 1. — *Let G be a group. Let Y be a subset of G. We say that Y is 0-dense if $Y \neq \emptyset$. Let $m \geqslant 1$. We say that Y is m-dense if*

$$(g_1 \cdot Y) \cap \ldots \cap (g_m \cdot Y) \neq \emptyset$$

for all $g_1, \ldots, g_m \in G$.

EXAMPLE 2. — *Let G be a topological group. If U is an open dense subset of G, then U is m-dense for all $m \geqslant 0$.*

Proof. — This follows from

1. the set $g \cdot U$ is an open dense set, for $g \in G$;
2. the intersection of two open dense sets is an open dense set. □

LEMMA 3. — *Let Y be an m-dense subset of G. Then there exists $(g_1, \ldots, g_m) \in Y^m$ such that $g_i g_{i+1} \cdots g_{i+j} \in Y$, for $1 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant m - i$.*

Keywords: Cocycle – m-dense – Simplicial set – Lagrangian – Transversal – Sympletic group.
Math. classification: 20J06 – 11E08.

Proof. — We prove the lemma by induction on m. The lemma is true if $m = 0$ or $m = 1$.

We suppose that $m > 1$. By the induction hypothesis there is $(g_1, \dots, g_{m-1})$ in Y^{m-1} such that the product $g_i g_{i+1} \cdots g_{i+j} \in Y$, for $1 \leqslant i \leqslant m-1$ and $0 \leqslant j \leqslant m-1-i$. We choose $\widetilde{g}_m \in Y \cap (g_1 \cdot Y) \cap \dots \cap (g_1 g_2 \cdots g_{m-1} \cdot Y)$. We let $g_m = (g_1 g_2 \cdots g_{m-1})^{-1} \widetilde{g}_m$. We have that $\widetilde{g}_m \in (g_1 g_2 \cdots g_{i-1} \cdot Y) \cap (g_1 g_2 \cdots g_{m-1} \cdot Y)$, for $2 \leqslant i \leqslant m-1$. Hence $g_i g_{i+1} \cdots g_m \in Y$, for $1 \leqslant i \leqslant m$. This proves the lemma. □

Let $m \geqslant 1$. We assume that Y is an m-dense subset of G. Let $1 \leqslant n \leqslant m$. We let $Y^n_{\text{gen}} = \{(g_1, \dots, g_n) \in Y^n \mid g_i \cdots g_{i+j} \in Y \text{ for } 1 \leqslant i \leqslant n \text{ and } 0 \leqslant j \leqslant n-i\}$.

Let B be an abelian group with trivial G-action. We consider the complex (of groups)

$$0 \longrightarrow B \xrightarrow{0} C^1_Y \xrightarrow{d^1} C^2_Y \xrightarrow{d^2} \cdots \xrightarrow{d^{m-1}} C^m_Y$$

where $C^n_Y = \mathrm{Map}(Y^n_{\text{gen}}, B)$ and

$$d^{n-1}(f)(g_1, g_2, \dots, g_{n-1}) = f(g_2, \dots, g_{n-1}) - f(g_1 g_2, \dots, g_{n-1}) + \cdots \\ \cdots + (-1)^{n-1} f(g_1, g_2, \dots, g_{n-2}).$$

Definition 4. — *Let $0 \leqslant n \leqslant m-1$. An element of $\ker d^n$ is called n-cocycle for Y. We denote by $H^n_Y(G, B)$ the group $\ker d^n / \operatorname{im} d^{n-1}$.*

Theorem 5. — *Let $m \geqslant 1$. We assume that Y is a $2m$-dense subset of G. Let $0 \leqslant n \leqslant m-1$. Then the natural embedding $Y^n_{\text{gen}} \to G^n$ induces an isomorphism between $H^n(G, B)$ and $H^n_Y(G, B)$. Moreover, if c is an n-cocycle for Y, then there is an n-cocycle $\bar{c}$ such that its restriction to Y^n_{gen} is c.*

This result will be proved in Section 3. A consequence of this theorem is the following corollary:

Corollary 6. — *Let G be a topological group. Let U be an open dense subset of G. Then the natural embedding $U^n_{\text{gen}} \to G^n$ induces an isomorphism between $H^*(G, B)$ and $H^*_U(G, B)$. Moreover, if c is an n-cocycle for U, then there is an n-cocycle $\bar{c}$ such that its restriction to U^n_{gen} is c.*

2. The generalized Mayer-Vietoris sequence.

DEFINITION 7. — *Let X be a CW-complex. We say that X is -1-acyclic if $X \neq \emptyset$. Let $k \geqslant 0$. We say that X is k-acyclic if X is -1-acyclic and $\widetilde{H}_n(X) = 0$, for all $0 \leqslant n \leqslant k$. We say that X is acyclic if it is k-acyclic for all $k \in \mathbb{N}$.*

Let X be a CW-complex which is the union of a family of non-empty subcomplexes X_α, where α ranges over some totally ordered index set I. Let K be the abstract simplicial complex whose vertex set is I and whose simplices are the non-empty finite subsets J of I such that the intersection $\cap_{\alpha \in J} X_\alpha$ is non empty. We denote by $K^{(p)}$ the set of the p-simplices of K. Then (cf. [1] 166–167).

PROPOSITION 8. — *We have a spectral sequence E such that*

$$E^1_{p,q} = \bigoplus_{J \in K^{(q)}} H_p\Big(\bigcap_{\alpha \in J} X_\alpha\Big) \Rightarrow H_{p+q}(X).$$

Let K be a simplicial set. Recall that $\overline{K}$, the geometric realization of K, is a CW-complex. Moreover $H_*(K) = H_*(\overline{K})$. We say that K is k-acyclic if $\overline{K}$ is k-acyclic. The following corollary is a consequence of the Proposition 8.

COROLLARY 9. — *Let K be a simplicial set which is the union of a family of non-empty simplicial subsets K_α, where α ranges over some index set I. Let $k \geqslant -1$. We suppose that $K_{\alpha_1} \cap K_{\alpha_2} \cap \ldots \cap K_{\alpha_n}$ is $k-n+1$-acyclic for all $1 \leqslant n \leqslant k+2$ and for all $\{\alpha_1, \ldots, \alpha_n\} \subset I$. Then K is k-acyclic.*

3. Proof of Theorem 5.

Let X be a subset of the group G. We first assume that $1 \in Y$. We let $X^0_Y = X$. Let $n \geqslant 1$. We let $X^n_Y = \{(g_0, \ldots, g_n) \in X^{n+1} \mid g_i^{-1} g_j \in Y$ for all $i < j\}$. The two following assertions are straightforward.

1. $\partial_i(g_0, \ldots, g_n) = (g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n) \in X^{n-1}_Y$, for all $(g_0, \ldots, g_n) \in X^n_Y$ and for $0 \leqslant i \leqslant n$.
2. $s_i(g_0, \ldots, g_n) = (g_0, \ldots, g_{i-1}, g_i, g_i, g_{i+1}, \ldots, g_n) \in X^{n+1}_Y$, for $0 \leqslant i \leqslant n$ and for all $(g_0, \ldots, g_n) \in X^n_Y$.

We consider the simplicial set $K_Y(X)$ whose n-simplices are the $(g_0,\dots,g_n)\in X_Y^n$, the face operators are the ∂_i's and the degenerency operators are the s_i's. (*)

LEMMA 10. — *Let $k\geqslant 0$. Let $X,Y\subset G$ such that $1\in Y$. Assume that*

$$X\cap(g_1\cdot Y)\cap\ldots\cap(g_{2k}\cdot Y)\neq\emptyset$$

for all $g_1,\dots,g_{2k}\in X$. Then $K_Y(X)$ is $(k-1)$-acyclic.

Proof. — We prove the lemma by induction on k.

If $k=0$ then $X\neq\emptyset$. Hence $K_Y(X)$ is -1-acyclic and the lemma is true.

We assume that $k>0$. Let $g\in X$ and denote by K_g the simplicial subset of $K_Y(X)$ whose the n-simplices are the $(g_0,\dots,g_n)\in X_Y^n$ such that $g=g_0$ or $(g,g_0,\dots,g_n)\in X_Y^{n+1}$. Clearly $K_Y(X)=\bigcup_{g\in X}K_g$. Let $g_1,\dots,g_m\in X$ such that $g_i\neq g_j$ for $i\neq j$. We let $K_{g_1,\dots,g_m}=K_{g_1}\cap\ldots\cap K_{g_m}$. We will prove that $K_{g_1,\dots,g_m}$ is $(k-m)$-acyclic, for $1\leqslant m\leqslant k+1$ and for $(g_1,\dots,g_m)\in X^m$.

The geometric realization of K_g is a cone, hence K_g is acyclic. Let $2\leqslant m\leqslant k+1$. Let $g_1,\dots,g_m\in X$ such that $g_i\neq g_j$, for $i\neq j$. We put $\overline{X}=X\cap(g_1\cdot Y)\cap\ldots\cap(g_m\cdot Y)$ and $\overline{X}_Y^n=\{(g_0,\dots,g_n)\in\overline{X}^{n+1}\mid g_i^{-1}g_j\in Y$ for all $i<j\}$. Then $K_{g_1,\dots,g_m}=K_Y(\overline{X})$, the simplicial set whose the n-simplices are the $(g_0,\dots,g_n)\in\overline{X}_Y^n$. Let $h_1,\dots,h_{2(k-m+1)}\in\overline{X}$. Then

$$\overline{X}\cap(h_1\cdot Y)\cap\ldots\cap(h_{2(k-m+1)}\cdot Y)\neq\emptyset,$$

since $m+2(k-m+1)\leqslant 2k$.

Hence, by induction hypothesis, $K_{g_1,\dots,g_m}$ is $(k-m)$-acyclic. From Corollary 9 follows that $K_Y(X)$ is $(k-1)$-acyclic. This proves the lemma. □

Now we assume that $1\notin Y$. We let $X_Y^0=X$. Let $n\geqslant 1$. We let $X_Y^n=\{(g_0,\dots,g_n)\in X^{n+1}\mid g_i^{-1}g_j\in Y$ for all $i<j\}$. Note that

1. If $i\neq j$, then $g_i\neq g_j$, for all $(g_0,\dots,g_n)\in X_Y^n$.
2. $\partial_i(g_0,\dots,g_n)=(g_0,\dots,g_{i-1},g_{i+1},\dots,g_n)\in X_Y^{n-1}$, for all $(g_0,\dots,g_n)\in X_Y^n$ and for $0\leqslant i\leqslant n$.

It follows from (1) and (2) that there is a simplicial set $\overline{K}_Y(X)$ whose the non degenerate n-simplices are the $(g_0,\dots,g_n)\in X_Y^n$ and the face operators are the ∂_i's defined above.

Note that $\overline{K}_Y(X) = K_{Y'}(X)$, where $Y' = Y \cup \{1\}$ (see (*)).

LEMMA 11. — *Let $k \geqslant 0$. Let $X, Y \subset G$ such that $1 \notin Y$. We assume that*

$$X \cap (g_1 \cdot Y) \cap \ldots \cap (g_{2k} \cdot Y) \neq \emptyset$$

for all $g_1, \ldots, g_{2k} \in G$. Then $\overline{K}_Y(X)$ is $(k-1)$-acyclic.

Proof. — We have that $\overline{K}_Y(X) = K_{Y'}(X)$, where $Y' = Y \cup \{1\}$. Clearly

$$X \cap (g_1 \cdot Y') \cap \ldots \cap (g_{2k} \cdot Y') \neq \emptyset$$

for all $g_1, \ldots, g_{2k} \in G$. Hence this lemma is a consequence of Lemma 10.□

We consider the complex $C = (C_n, \delta_n)_{n \geqslant -1}$, where

1. $C_{-1} = \mathbb{Z}$,
2. $C_0 = \mathbb{Z}G$,
3. for $n \geqslant 1$, C_n is the free abelian group generated by the elements of $G_Y^n = \{(g_0, \ldots, g_n) \in G^{n+1} \mid g_i^{-1} g_j \in Y \text{ for all } i < j\}$,
4. $\delta_0 : C_0 \to C_{-1}$ is the augmentation map,
5. for $n \geqslant 1$, $\delta_n : C_n \to C_{n-1}$ is defined by
$$\delta_n(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i \partial_i (g_0, \ldots, g_n).$$

COROLLARY 12. — *Let $m \geqslant 1$. Let Y be a $2m$-dense subset of G. Then $H_n(C) = 0$ for all $n \leqslant m-1$.*

Proof. — This corollary is a consequence of Lemma 10 and Lemma 11. □

Proof of Theorem 5. — Let $0 \leqslant n \leqslant m-1$. The complex C defined above is a complex of G-modules, where the G-action is defined by $g \cdot (g_0, \ldots, g_k) = (gg_0, \ldots, gg_k)$. Then C_k is free with basis $\{(1, g_1, \ldots, g_1 \ldots g_k) \mid (g_1, \ldots, g_k) \in Y_{\text{gen}}^k\}$, for $k \leqslant 2m$. This means that there is $(\overline{C}_k)_{k \geqslant 0}$ a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ such that $\overline{C}_{n+1} = C_{n+1}$. Hence $H_Y^n(G, B)$ is isomorphic to $H^n(G, B)$. Clearly the isomorphism is induced by the natural embedding $Y_{\text{gen}}^n \to G^n$. This proves part one.

We now prove the second part of the theorem. We consider an n-cocycle $\bar{c}$ and an n-cocycle for Y c such that the class of the restriction of

$\bar{c}$ to Y^n_{gen} in $H^n_Y(G,B)$ is the same of the class of c. There exists $f \in C^{n-1}_Y$ such that $\bar{c} = c + d^{n-1}(f)$. But $\mathrm{Hom}(G^{n-1}, B)$ maps onto C^{n-1}_Y. This means that there exists $\bar{f}$ in $\mathrm{Hom}(G^{n-1}, B)$ which maps to f. It then follows that the n-cocycle c', defined by $c'(g_1, g_2) = c(g_1, g_2) - \bar{f}(g_1) - \bar{f}(g_1) + \bar{f}(g_1 g_2)$, maps to c. □

COROLLARY 13. — *Let Y be a $2m$-dense subset of G. Let $0 \leqslant n \leqslant m-1$. We consider two n-cocycles c, c'. We suppose that there exists $g \in G$ such that*

$$c(g_1, \ldots, g_n) = c'(g g_1 g^{-1}, \ldots, g g_n g^{-1}),$$

for all $(g_1, \ldots, g_n) \in Y^n_{\mathrm{gen}}$. Then c and c' are cohomological equivalent.

Proof. — Let $n \leqslant m-1$. The set gYg^{-1} is a $2m$-dense subset of G. The map $r_g : G \to G$ defined by $r_g(h) = ghg^{-1}$ induces two homomorphisms $i_g : H^n(G,B) \to H^n(G,B)$, $i_g : H^n_Y(G,B) \to H^n_{gYg^{-1}}(G,B)$ and the following commutative diagramm

$$\begin{array}{ccc} H^n(G,B) & \xrightarrow{i_Y} & H^n_Y(G,B) \\ i_g \downarrow & & \downarrow i_g \\ H^n(G,B) & \xrightarrow{i_{gYg^{-1}}} & H^n_{gYg^{-1}}(G,B), \end{array}$$

where i_Y and $i_{gYg^{-1}}$ denote the isomorphisms induced by the natural embeddings $Y^n_{\mathrm{gen}} \to G^n$ and $(gYg^{-1})^n_{\mathrm{gen}} \to G^n$. Note that $i_g : H^n(G,B) \to H^n(G,B)$ is the identity map. This proves the corollary. □

4. An application.

In the second part of this paper we give an application of Theorem 5.

Let A be a local commutative ring such that $2 \in A^*$. Let $\mathfrak{M}$ denote the maximal ideal of A and $K = A/\mathfrak{M}$. Let V be a free A-module of dimension $2n$ with a non-degenerate alternating form φ. For a subset W of V, we set

$$W^\perp = \{v \in V \mid \varphi(v,w) = 0 \text{ for all } w \in W\}.$$

A direct summand of V is called *subspace* and a *Lagrangian* for V is a subspace W of dimension n such that $W = W^\perp$. Let X denote the set of the Lagrangians in V. Let $L_1, L_2 \in X$. We say that L_1 is *transversal* to L_2, denoted $L_1 \pitchfork L_2$, if $L_1 + L_2 = V$.

We let $\mathrm{Sp}(V)$ the symplectic group of (V, φ), that is

$$\mathrm{Sp}(V) = \{\alpha \in \mathrm{GL}(V) \mid \varphi\big(\alpha(x), \alpha(y)\big) = \varphi(x,y) \text{ for all } x, y \in V\}.$$

Let W be a submodule of V. We let $\overline{W} = W \otimes_A K$ and $\overline{\varphi} : \overline{V} \times \overline{V} \to K$ denote the non-degenerate alternating form induced by φ. Finally $\overline{X}$ denotes the set of the Lagrangians in $\overline{V}$. We have

LEMMA 14. — *Let $\{v_1, \ldots, v_{2n}\}$ be a basis of V. Then there exists a basis $\{u_1, \ldots, u_{2n}\}$ of V such that $\varphi(v_i, u_j) = \delta_{ij}$.*

Proof. — The space V' denotes the dual of V. Then $d_\varphi : V \to V'$ defined by $d_\varphi(x) = \varphi(-, x)$ is an isomorphism because φ is non-degenerate. We consider the dual basis $\{z_1, \ldots, z_{2n} \in V'\}$ of $\{v_1, \ldots, v_{2n}\}$ and we let $u_i = d_\varphi^{-1}(z_i)$. Then $\delta_{ij} = z_i(v_j) = d_\varphi d_\varphi^{-1}(z_i)(v_j) = \varphi(v_j, u_i)$. □

COROLLARY 15. — *Let $v_1, \ldots, v_n \in V$ such that $\overline{v}_1, \ldots, \overline{v}_n$ are linear independents in $\overline{V}$. Then there exists $\{u_1, \ldots, u_n\}$ a subset of V such that $\varphi(v_i, u_j) = \delta_{ij}$. Moreover, if L_2 is a Lagrangian of V transversal to $L_1 \in X$ and $\{v_1, \ldots, v_n\}$ is a basis of L_1, then there exists a basis $\{w_1, \ldots, w_n\}$ of L_2 such that $\varphi(v_i, w_j) = \delta_{ij}$.*

Proof. — We prove only the second part of the corollary. We consider $\{v_1, \ldots, v_n\}$ a basis of L_1 and $\{v_{n+1}, \ldots, v_{2n}\}$ a basis of L_2. There is a basis $\{w_1, \ldots, w_{2n}\}$ of V such that $\varphi(v_i, w_j) = \delta_{ij}$. This means that $w_1, \ldots, w_n \in L_2^\perp$. But $L_2 = L_2{}^\perp$, hence $\{w_1, \ldots, w_n\}$ is a basis of L_2. □

COROLLARY 16. — *X maps onto $\overline{X}$.*

Proof. — Let $\{\overline{v}_1, \ldots, \overline{v}_n \in \overline{V}\}$ be a basis of $\overline{L}$, a Lagrangian for $\overline{V}$. We consider $\{v_1, \ldots, v_n\}$ a lift of $\{\overline{v}_1, \ldots, \overline{v}_n\}$ in V and $m = \max\{k \mid \varphi(v_i, v_j) = 0 \text{ for all } 1 \leqslant i, j \leqslant k\}$. We prove the corollary by induction on $n - m$.

If $n - m = 0$, then the corollary is true.

Let $n - m \geqslant 1$. We choose $u_1, \ldots, u_n \in V$ such that $\varphi(v_i, u_j) = \delta_{ij}$. We put $\widetilde{v}_i = v_i$, if $i \neq m+1$ and $\widetilde{v}_{m+1} = v_{m+1} - \sum_{i=1}^m \varphi(v_i, v_{m+1}) u_i$. Clearly $\{\widetilde{v}_1, \ldots, \widetilde{v}_n\}$ is a lift of $\{\overline{v}_1, \ldots, \overline{v}_n\}$ because $\varphi(v_i, v_{m+1}) \in \mathfrak{M}$ for all $1 \leqslant i \leqslant m$. Moreover $\varphi(\widetilde{v}_i, \widetilde{v}_j) = 0$ for all $1 \leqslant i, j \leqslant m+1$. This proves the corollary. □

COROLLARY 17. — *$\mathrm{Sp}(V)$ acts transitevely on X.*

Proof. — Let $L_0, L_1 \in X$. There are $\overline{L}_2, \overline{L}_3 \in \overline{X}$ such that $\overline{L}_0 \pitchfork \overline{L}_2$

and $\overline{L}_1 \pitchfork \overline{L}_3$. Let L_0, L_1, L_2, L_3 be lifts of $\overline{L}_0, \overline{L}_1, \overline{L}_2, \overline{L}_3$ in X. Clearly $L_0 \pitchfork L_2$ and $L_1 \pitchfork L_3$. We choose $\{v_1, \dots, v_{2n}\}$ and $\{u_1, \dots, u_{2n}\}$ two basis of V such that $\{v_1, \dots, v_n\} \subset L_0$, $\{v_{n+1}, \dots, v_{2n}\} \subset L_1$, $\{u_1, \dots, u_n\} \subset L_2$, $\{u_{n+1}, \dots, u_{2n}\} \subset L_3$ and $\varphi(v_i, v_{n+j}) = \varphi(u_i, u_{n+j}) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant n$. Now we consider $\alpha \in \mathrm{GL}(V)$ such that $\alpha(v_i) = u_i$, for $1 \leqslant i \leqslant 2n$. Clearly $\alpha \cdot L_0 = L_1$ and $\varphi\big(\alpha(x), \alpha(y)\big) = \varphi(x, y)$ for all $x, y \in V$. Hence $\alpha \in \mathrm{Sp}(V)$. □

Now we consider $(L_1, L_2, L_3) \in X^3$ such that $L_\imath \pitchfork L_j$ for $i \neq j$. We define $\psi : L_1 \oplus L_2 \oplus L_3 \to V$ by $\psi(v_1, v_2, v_3) = v_1 + v_2 + v_3$. Then ψ is surjective and $\mathcal{K}_{123} = \ker \psi$ is free of dimension n. We define the quadratic form $q : \mathcal{K}_{123} \to A$ by $q(v_1, v_2, v_3) = \varphi(v_1, v_2)$. Then q is a non-degenerate quadratic form and the *Maslov index* of (L_1, L_2, L_3), denoted by $m(L_1, L_2, L_3)$, is the class of q in $W(A)$.

In comparison with [3], we do not define the Maslov index for all (L_1, L_2, L_3) in X^3, but, using theorem 5, we obtain (Theorem 24) an extension

$$0 \longrightarrow I^2(A) \longrightarrow \widetilde{\mathrm{Sp}(V)} \longrightarrow \mathrm{Sp}(V) \longrightarrow 1$$

as in Theorem 2.2 of [3].

PROPOSITION 18. — *Let $(L_0, L_1, L_2, L_3) \in X^4$ such that $L_\imath \pitchfork L_\jmath$ for $i \neq j$. Then $m(L_1, L_2, L_3) - m(L_0, L_2, L_3) + m(L_0, L_1, L_3) - m(L_0, L_1, L_2) = 0$.*

Proof. — The proof is exactly the same as in the proof of Proposition 1.2 of [3]. □

LEMMA 19. — *Let A be a local ring such that $|A/\mathfrak{M}| \geqslant m$. Then, given m Lagrangians $L_0, L_1, \dots, L_m$, there exists a Lagrangian L such that $L \pitchfork L_i$, for $0 \leqslant i \leqslant m$.*

Proof. — It follows from Corollary 16 that we just need to prove this lemma when $A = K$ a field.

Assume the dimension of V is 2. Then K has more than m 1-dimensional subspaces and the lemma is true. We prove the lemma by induction on $\dim V$.

We show that there exists $v \in V$, $v \notin \cup_{i=0}^m L_i$. This is proved if $|K| = \infty$. Suppose $|K| = q$. Then a space of dimension l has cardinality q^l. This means that $|\cup_{i=0}^m L_i| \leqslant (m+1)q^m < q^{2m} = |V|$.

Let $V_1 = v^\perp$ and $\overline{V}_1 = V_1/\langle v\rangle$. Let $\overline{L}_i$ be the image of $L_i \cap V_1$ in $\overline{V}_1$. Then $\{\overline{L}_i \mid 0 \leqslant i \leqslant m\}$ are Lagrangians in $\overline{V}_1$. By induction on the dimension of V, there is a Lagrangian $\overline{L}$ in $\overline{V}_1$ such that $\overline{L} \pitchfork \overline{L}_i$, for $0 \leqslant i \leqslant m$. We consider L the subspace of V_1 of dimension n such that $L/\langle v\rangle = \overline{L}$. Then L is a Lagrangian in V and $L \pitchfork L_i$, $0 \leqslant i \leqslant m$. □

Corollary 20. — *Let A be a local ring such that $|A/\mathfrak{M}| \geqslant m$. We fix $L_0 \in X$ and we consider $Y_{L_0} = \{g \in \mathrm{Sp}(V) \mid g \cdot L_0 \pitchfork L_0\}$. Then Y_{L_0} is m-dense.*

Proof. — We first remark that, if $L_1 \pitchfork L_2$, then $g \cdot L_1 \pitchfork g \cdot L_2$, for $g \in \mathrm{Sp}(V)$ and $L_1, L_2 \in X$. Let $g_1, \dots, g_m \in \mathrm{Sp}(V)$. By the previous lemma there is an $L \in X$ transversal to $g_i \cdot L_0$, for $1 \leqslant i \leqslant m$. We choose $g \in Sp(V)$ such that $g \cdot L_0 = L$. Then $g \cdot L_0 \pitchfork g_i \cdot L_0$, $1 \leqslant i \leqslant m$. This means that $g_i^{-1} g \in Y_{L_0}$. But $g = g_i g_i^{-1} g$, hence $g \in (g_1 \cdot Y_{L_0}) \cap \ldots \cap (g_m \cdot Y_{L_0})$. □

Now we fix $L_0 \in X$ and define $c : (Y_{L_0})^2_{\mathrm{gen}} \to W(A)$ as follows:

$$c(g_1, g_2) = m(L_0, g_1 \cdot L_0, g_1 g_2 \cdot L_0).$$

Proposition 21. — *Let A be a local ring such that $|A/\mathfrak{M}| \geqslant 6$. Then c is a 2-cocycle for Y_{L_0} which defines a central extension*

$$(\star) \qquad 0 \longrightarrow W(A) \longrightarrow \widetilde{\mathrm{Sp}(V)} \longrightarrow \mathrm{Sp}(V) \longrightarrow 1$$

This extension is independent of the choice of L_0.

Note that $A/\mathfrak{M}$ is a field. Hence $|A/\mathfrak{M}| \geqslant 6$ implies that $|A/\mathfrak{M}| \geqslant 7$.

Proof. — Let $L_1, L_2, L_3 \in X$ such that $L_i \pitchfork L_j$, for $i \neq j$. We remark that $m(L_1, L_2, L_3) = m(g \cdot L_1, g \cdot L_2, g \cdot L_3)$, for $g \in G$. It then follows that c is a 2-cocycle for Y_{L_0}. Hence, using Theorem 5 and Corollary 20, we see that c induces $(\star)$.

We are now left with proving that $(\star)$ is independent of the choice of L_0.

Let $L_1 \in X$. We consider c', the 2-cocycle for Y_{L_1} defined by

$$c'(g_1, g_2) = m(L_1, g_1 \cdot L_1, g_1 g_2 \cdot L_1).$$

We choose $g \in G$ such that $g \cdot L_0 = L_1$. Let $(g_1, g_2) \in (Y_{L_1})^2_{gen}$. We have that

$$\begin{aligned} c'(g_1, g_2) &= m(L_1, g_1 \cdot L_1, g_1 g_2 \cdot L_1) = m(g \cdot L_0, g g^{-1} g_1 g \cdot L_0, g g^{-1} g_1 g_2 g \cdot L_0) \\ &= m(L_0, g^{-1} g_1 g \cdot L_0, g^{-1} g_1 g_2 g \cdot L_0) = c(g^{-1} g_1 g, g^{-1} g_2 g). \end{aligned}$$

Hence the proposition follows from Corollary 13. □

In the last part of this paper we will prove that c can be reduced to

$$\overline{c} : (Y_{L_0})^2_{\text{gen}} \to I^2(A).$$

We consider the map $t : Y_{L_0} \to W(A)$, defined by $t(g) = \langle \text{id}_n \rangle$, where id_n denotes the bilinear space (A^n, ι_n) defined by

$$\iota_n\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) = x_1 y_1 + \dots + x_n y_n.$$

Let $(g_1, g_2) \in (Y_{L_0})^2_{\text{gen}}$. We put $c'(g_1, g_2) = c(g_1, g_2) - t(g_1) - t(g_2) + t(g_1 g_2)$.

LEMMA 22. — *c' is a 2-cocycle for Y_{L_0} and $c'\big((Y_{L_0})^2_{\text{gen}}\big) \subset I(A)$.*

Let $L, L_0 \in X$ such that $L \pitchfork L_0$. We choose $B = \{v_1, \dots, v_n\}$ a basis of L and $B_0 = \{u_1, \dots, u_n\}$ a basis of L_0. $M\big((L, B), (L_0, B_0)\big)$ denotes the matrix $(r_{ij}) = -\varphi(v_i, u_j)$. The matrix $M\big((L, B), (L_0, B_0)\big)$ is in $GL_n(A)$ because $L \pitchfork L_0$.

PROPOSITION 23. — *Let $(L_1, L_2, L_3) \in X^3$ such that $L_i \pitchfork L_j$ for $i \neq j$. We choose $B_1 = \{v_1, \dots, v_n\}$ a basis of L_1, $B_2 = \{u_1, \dots, u_n\}$ a basis of L_2 and $B_3 = \{w_1, \dots, w_n\}$ a basis of L_3. Then*

$$\partial(m(L_1, L_2, L_3)) = (-1)^{n(n-1)/2} \cdot \overline{\det}(M_{23}) \cdot \overline{\det}(M_{13})^{-1} \cdot \overline{\det}(M_{12}),$$

where M_{ij} denotes the matrix $M\big((L_i, B_i), (L_j, B_j)\big)$,the map $\partial : W(A) \to A^*/(A^*)^2$ denotes the signed determinant and $\overline{\det}$ denotes the homomorphism between $\text{GL}_n(A)$ and $A^*/(A^*)^2$ induced by the determinant.

Proof. — The proof is exactly the same as the first part of the proof of Proposition 2.1 of [3]. □

We fix $L_0 \in X$. Let $B_0 = \{v_i \mid 1 \leqslant i \leqslant n\}$ be a basis of L_0. Then $g \cdot B_0 = \{g \cdot v_i \mid 1 \leqslant i \leqslant n\}$ is a basis of $g \cdot L_0$, for $g \in \text{Sp}(V)$. We consider the map $t_{L_0} : Y_{L_0} \to I(A)$, defined by

$$t_{L_0}(g) = \Big\langle \det\Big(M\big((L_0, B_0), (g \cdot L_0, g \cdot B_0)\big)\Big), (-1)^{n(n-1)/2} \Big\rangle.$$

Let $(g_1, g_2) \in (Y_{L_0})^2_{\text{gen}}$. We let $\overline{c}(g_1, g_2) = c'(g_1, g_2) - t_{L_0}(g_1) - t_{L_0}(g_2) + t_{L_0}(g_1 g_2)$.

THEOREM 24. — *Let A be a local ring such that $|A/\mathfrak{M}| \geqslant 7$. Then $\overline{c}$ is a 2-cocycle for Y_{L_0} which induces a central extension*

$$0 \longrightarrow I^2(A) \longrightarrow \widetilde{\text{Sp}(V)} \longrightarrow \text{Sp}(V) \longrightarrow 1.$$

Acknowledgments. — I would like to thank Dr. Gael Collinet for the helpful discussions and for having send me his thesis [2]. In his paper I have in particular found many useful informations about the study of the acyclicity of semi-simplicial sets.

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Manuscrit reçu le 13 novembre 2003,
accepté le 13 janvier 2004.

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